

**STRONG STOP-LOSS CRITERIA: DEFINITION
AND APPLICATION TO RISK-MANAGEMENT**

**GRISELDA DEELSTRA
MARTINO GRASSELLI
PIERRE-FRANCOIS KOEHL**

**1998 GENERAL INSURANCE CONVENTION
AND
ASTIN COLLOQUIUM**

GLASGOW, SCOTLAND: 7-10 OCTOBER 1998

Strong stop-loss criteria: definition and application to risk-management.

Griselda Deelstra, CREST and ENSAE.

Martino Grasselli, CREST*

Pierre-François Koehl, CREST and ENSAE.

July 29, 1998

Abstract: In this paper, we define the concept of strong stop-loss domination and we use it for the obtention of bounds on the hedging prices of random variables. These hedging prices depend on the characteristics of the agent and in particular on her utility function, which is hard to estimate in practice. Our bounds have the advantage that they only depend on the characteristics of the financial market and of the random variable to hedge. Moreover, our interval is proved to be coherent with the equilibrium and it is tighter than the one obtained by the classical super-replication approach. At last, specifying the distributions of the financial assets' prices and the random variable to hedge, we compute the upper bound given by the strong stop-loss approach and we compare it with the super-replication bound.

1 Introduction

Over the last years, the hedging of random variables within an incomplete financial framework is a crucial topic in Finance and Insurance. In Finance, there are some famous papers devoted to the hedging of a risky position (see e.g. the references). More recently in Insurance, a debate has been started in France about the possible creation of pension funds and about the ways to use the financial markets to hedge the risks associated with the retirement

*Corresponding author. Mailing address: ENSAE, Tour J120, 3 Avenue Pierre-Larousse, 92245 Malakoff Cedex, FRANCE; tel: (33) 141 175 795; fax: (33) 141 173 853; E-mail: grasselli@ensae.fr

problem. In this paper, we provide a theoretical contribution by showing how to obtain bounds on the private values of risks to be hedged.

There are several ways to associate prices to random variables which are not priced by the financial market; the more popular ones are quadratic hedging and super-replication.

Quadratic hedging is introduced by Föllmer and Sondermann (1986) and extended by Föllmer and Schweizer (1991), Schweizer (1991,1992) and Duffie and Richardson (1991). Under the quadratic hedging approach, the price of any random variable is put equal to the L^2 -projection on the space of the random variables priced by the financial market. This approach is very practical since it provides a unique price which can be easily computed by using the theory of square-integrable random variables. Nevertheless, this method has an important weakness since the part of the random variable which is orthogonal to the space of the random variables priced by the financial market, is valorized by zero and this part (also called tracking error in literature) is precisely the risk that cannot be replicated by the financial markets.

Super-replication (see e.g. El Karoui and Quenez (1995)) proposes an interval of prices which contains the price of the random variable to be hedged. This interval is determined by an upper and lower bound which are defined as the infimum (resp. supremum) of the prices of the random variables priced by the financial market that dominate (resp are dominated by) the random variable to hedge almost surely. Unfortunately, this approach leads to very large intervals (see e.g. Soner, Shreve and Cvitanic (1995)).

Another method to determine prices in an incomplete framework is introduced by Hodges and Neuberger (1989), continued by Davis, Panas and Zariphopoulou (1993) and discussed by Karatzas and Kou (1996). Under this approach, the price of the random variable to be hedged, is valorized as the private value of the agent. Therefore, the proposed price depends on the characteristics of the agent and especially on its utility function, which is very hard to model in practice.

In this paper, we show how to determine bounds on the private value of any random variable which do not depend on the characteristics of the agents. The bounds only are depending on the characteristics of the financial market on one hand, and on the random variable to be hedged on the other hand. We prove that our bounds are strictly improving the bounds provided by super-replication.

We obtain those bounds by using a stochastic dominance approach. Stochastic dominance has been used by Levy (1985) in order to find bounds on

the prices of European options, but he needed a strong restriction on the financial portfolios of the agents. In our paper, we do not need such quite unrealistic hypotheses, but our approach uses a slightly stronger criterion in comparison with the second stochastic dominance criterion used by Levy (1985).

The paper is organized as follows: in section 2, we introduce the framework and the general notion of hedging that we use. Section 3 is devoted to the study of upper and lower bounds on the private values of any random variable, and this by using strong stop-loss dominance. We prove that the interval that we obtain, is included in the interval given by the super-replication approach. As an example, we compute the upper bound for the case of log-normal random variables in section 4. These bounds are compared with the ones determined by super-replication. Section 5 concludes the paper.

2 Framework and Notations

Throughout the paper, we work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume a financial market with two assets: a riskless asset and a risky asset with a price S_t at time t . In this paper, we consider only two dates: the initial date denoted by 0 and a time-horizon T , which we assume to be equal to 1. Only at the initial date 0, transactions are allowed and we denote the number of units of the risky asset chosen at date 0 by θ_0 . The rate of return on the riskless asset is supposed to be constant and is denoted by r . In the following, we assume that the financial market is able to supply any quantity of the two assets, and that the price of any portfolio is obtained linearly from the unitary prices.

Let L be the space of the integrable random variables. Then, a contingent claim attainable at date $T = 1$ is a random variable $X \in L$ such that:

$$\exists (x, \theta_0) \in \mathbb{R} \times \mathbb{R}, \quad X = x(1+r) + \theta_0(S_1 - S_0(1+r)). \quad (1)$$

We denote by $\mathcal{A}_T \subset L$ the subset of the contingent claims attainable at date T . The value of X is then by definition equal to x and is denoted by $V(X)$. From the no-arbitrage condition, $V(X)$ is well-defined and maps \mathcal{A}_T on \mathbb{R} .

In this paper, we study the use of the financial market in order to hedge the value of a non-attainable contingent claim at date T . We define our notion of hedging in a general setup. We assume that each agent a is characterized

by an initial wealth $x^a \in \mathbb{R}$, a risk $Y^a \in L$ and a Von Neumann-Morgenstern utility function u^a which is strictly concave and strictly increasing. Then, the optimization program of the agent is given by

$$\max_{\phi \in \mathbb{R}} \mathbb{E} u^a(X + Y^a)$$

under the constraint that $V(X) = x^a$. We denote by $\theta^a(x, Y)$ the optimal amount of units of risky assets associated to the initial wealth x and the risk Y ; and by $J(x, Y)$ the optimal value of $\mathbb{E} u^a(X + Y)$ associated to the optimal amount $\theta^a(x, Y)$.

Definition 1 *The hedging price of the random variable Y^a for the agent a is the real number $\pi^a(Y^a)$ such that $J(x^a, Y^a) = J(x^a + \pi^a(Y^a), 0)$*

The meaning of this definition is that $-\pi^a(Y^a)$ is the amount of money that the agent is willing to pay in order to remove the risk Y^a . This definition is not easy to manage, because it depends on the characteristics of the agent, and especially on the utility function u^a which is hard to estimate in practice. However if $Y^a \in \mathcal{A}_T$, then it is well-known that $\pi^a(Y^a) = V(Y^a)$, see e.g. Karatzas (1989) and Davis, Panas and Zariphopoulou (1993). In this way, the hedging price is an extension (depending on the agent) of V to the set L .

3 Strong stop-loss criterion: definition and use for hedging

In this section, we define strong stop-loss dominance and link this concept to first and second stochastic dominance and to P-a.c. dominance. These four binary relations on L lead to different ways to study the hedging price of a (non attainable) random variable $Y \in L$. We first show that by using strong stop loss dominance, we obtain in a natural way a quantity which is smaller than the P - a.e upper bound used in super-replication. We further prove that our proposed quantity has an important advantage over the first and second stochastic dominance analogues as it turns out to be an upper bound for the hedging price which is coherent with the equilibrium, whereas the first and second stochastic dominance approaches are not.

Afterwards, we concentrate on a lower bound of the hedging price of a (non-attainable) random variable $X \in L$. Therefore, we introduce a definition

using the strong stop-loss criterion, which is less natural than in the case of the upper bound, but which leads to analogous promising results.

We begin with introducing the definition of strong stop-loss dominance and we recall the notions of first and second stochastic dominance.

Definition 2 Let $(X, Y) \in L \times L$. We say that

X dominates Y in the sense of the strong stop-loss dominance, and we denote $X \succeq_{SSL} Y$, if and only if there exists a random variable ε such that $Y = X + \varepsilon$ \mathbb{P} -a.e. and $\mathbb{E}(\varepsilon | X) \leq 0$ \mathbb{P} -a.e.

X dominates Y in the sense of the First Stochastic Dominance, and we denote $X \succeq_{FSD} Y$, if and only if there exists $X' \in L$ such that X and X' are identically distributed and $X' \succ Y$ \mathbb{P} -a.e.

X dominates Y in the sense of the Second Stochastic Dominance, and we denote $X \succeq_{SSD} Y$, if and only if there exists $Y' \in L$ such that Y and Y' are identically distributed and $X \succ_{SSL} Y'$.

First and Second Stochastic Dominance are defined and discussed for instance in Huang-Litzenberger (1988) or Levy (1992). Their use in insurance is explained in for example Goovaerts et al. (1990). In fact, we found the inspiration for the name of our criterium in insurance, where Second Stochastic Dominance ordering is also called stop-loss ordering. The following Lemma is obvious and the proof is omitted.

Lemma 3 $\forall (X, Y) \in L \times L$,

- (i) $X \geq Y$ \mathbb{P} -a.e. implies $X \succeq_{FSD} Y$
- (ii) $X \geq Y$ \mathbb{P} -a.e. implies $X \succeq_{SSL} Y$
- (iii) $X \succeq_{FSD} Y$ implies $X \succeq_{SSD} Y$
- (iv) $X \succ_{SSL} Y$ implies $X \succeq_{SSD} Y$
- (v) $X \succ_{FSD} Y$ does not imply $X \succeq_{SSL} Y$
- (vi) $X \succeq_{SSL} Y$ does not imply $X \succeq_{FSD} Y$

The next Lemma provides an equivalent characterization of the *SSL*-Dominance:

Lemma 4 Let $(X, Y) \in L \times L$. Then $X \succeq_{SSL} Y$ if and only if $\mathbb{E}_x(Y | X) \leq X$ \mathbb{P} -a.e.

Proof. "If" part. Let us define $\varepsilon := Y - X$. Then $Y = X + \varepsilon$ \mathbb{P} -a.e. and $\mathbb{E}(\varepsilon | X) = \mathbb{E}(Y - X | X) = \mathbb{E}(Y | X) - X \leq 0$ \mathbb{P} -a.e.

"Only If" part. $Y = X + \varepsilon$ \mathbb{P} -a.e. and $\mathbb{E}(\varepsilon | X) \leq 0$ \mathbb{P} -a.e.; then $\mathbb{E}(\varepsilon | X) = \mathbb{E}(Y - X | X) = \mathbb{E}(Y | X) - X \leq 0$ \mathbb{P} -a.e.

We now want to determine upper bounds for the price of a random variable $Y \in L$, and this by using respectively \mathbb{P} -a.e. dominance, first and second stochastic dominance and strong stop-loss dominance. We therefore define the following quantities:

Definition 5 Let $Y \in L$. We define:

$$\bar{P}_{ac}(Y) := \inf\{V(X), X \in \mathcal{A}_T \text{ and } X \geq Y \text{ } \mathbb{P}\text{-a.e.}\}$$

$$\bar{P}_{FSD}(Y) := \inf\{V(X), X \in \mathcal{A}_T \text{ and } X \geq_{FSD} Y\}$$

$$\bar{P}_{SSD}(Y) := \inf\{V(X), X \in \mathcal{A}_T \text{ and } X \geq_{SSD} Y\}$$

$$\bar{P}_{SSL}(Y) := \inf\{V(X), X \in \mathcal{A}_T \text{ and } X \geq_{SSL} Y\}$$

with the convention $\inf\{\emptyset\} = +\infty$.

The quantity $\bar{P}_{ac}(Y)$ is known in the literature as the super-replication cost of Y . Thanks to Lemma 3, the following relations hold:

$$\begin{aligned} \bar{P}_{ac}(Y) &\geq \bar{P}_{FSD}(Y) \geq \bar{P}_{SSD}(Y) \\ \bar{P}_{ac}(Y) &\geq \bar{P}_{SSL}(Y) \geq \bar{P}_{SSD}(Y) \end{aligned}$$

We conclude that by using the strong stop-loss dominance, we obtain a quantity smaller than the super-replication upper bound. An example of the order of improvement will be given in section 4.

We now prove that \bar{P}_{ac} and \bar{P}_{SSL} are upper bounds for the hedging price which are coherent with the equilibrium. This means that for any equilibrium, the price of Y must be lower than $\bar{P}_{SSL}(Y)$, and thus a fortiori lower than $\bar{P}_{ac}(Y)$. This is in contrast with the quantities $\bar{P}_{FSD}(Y)$ and $\bar{P}_{SSD}(Y)$, which are no upper bounds coherent with the equilibrium as it is possible to find an equilibrium where the price of Y is larger than $\bar{P}_{FSD}(Y)$ (and consequently larger than $\bar{P}_{SSD}(Y)$).

We consider this result to be the main justification of our strong stop-loss dominance criterium, especially in comparison with the stochastic dominance ones.

Proposition 0 Let $X \in \mathcal{A}_T$ such that $X \geq_{SSL} Y^a$. Then $V(X) \geq \pi^a(Y^a)$.

Proof. Suppose $V(X) < \pi^a(Y^a)$. Let X^* be the optimal contingent claim chosen by the agent a in the presence of the risk Y^a ; and \hat{X} the optimal contingent claim chosen by the agent a when the risk Y^a is not present. By the definition of $\pi^a(Y^a)$, we must have

$$\hat{X} \sim_a X^* + Y^a \text{ and } V(\hat{X}) = V(X^*) + \pi^a(Y^a)$$

(where the notation \sim_a means that both portfolios are equivalent for the agent a). Since $X \succeq_{SSL} Y^a$, we get $X^* + X \succeq_a X^* + Y^a$. But $V(X^* + X) = V(X^*) + V(X) < V(X^*) + \pi^a(Y^a) = V(\hat{X})$. This contradicts the optimality of \hat{X} . ■

As an immediate consequence of this Proposition, we obtain that

$$\bar{P}_{SSL}(Y^a) \geq \pi^a(Y^a).$$

We stress that this result is very interesting since it yields an upper bound to the private value $\pi^a(Y^a)$, which depends only on Y^a and not on the other characteristics of the agent a .

We now turn our attention to the study of lower bounds. Following the literature on the pricing functionals, it would be natural to define the lower bound as minus the infimum of the contingent claims' values which dominate $-Y$ according to the strong stop-loss dominance criterion. However, this approach has to be rejected. Indeed, in order to prove the analogue of Proposition 6, the following property is necessary:

$$\text{For any agent } a \text{ and for all } Y \text{ and } Y' \text{ in } L, 0 \leq \pi^a(Y) \cdot \pi^a(-Y),$$

and it turns out to be false in general. Only in special cases like for example when $\pi^a(\cdot)$ is sublinear, the property is satisfied.

Another "natural" candidate would be the supremum of the contingent claims' values dominated by Y . Also this notion is not suited for obtaining the coherency with the equilibrium, because $Y \succ_{SSL} X$ does not imply $X' + Y \succeq_{SSL} X' + X$ for any $X' \in \mathcal{A}_T$.

Therefore, we choose the following definition of the lower bound for the hedging price of $Y \in L$:

Definition 7 For any $Y \in L$, let us define:

$$\hat{P}_{SSL}(Y) := \sup \{V(X), X \in \mathcal{A}_T \text{ and } \forall X' \in \mathcal{A}_T, X' + Y \succeq_{SSL} X' + X\}$$

This definition is quite *ad hoc* in the sense that it has been constructed in order to obtain the equilibrium compatibility with the hedging price. However, it yields an improvement in comparison with the super-replication criterion. Indeed, if we define:

$$\begin{aligned}\underline{P}_{oe}(Y) &:= \sup \{V(X), X \in \mathcal{A}_T \text{ and } Y \succeq_{oe} X\} \\ &= -\inf \{V(X), X \in \mathcal{A}_T \text{ and } X \succeq_{oe} -Y\}\end{aligned}$$

we obviously have: $\hat{P}_{SSL}(Y) \geq \underline{P}_{oe}(Y)$. So also the lower bound is better than the bounds proposed by super-replication. Moreover, we can now state the analogue of Proposition 6 for the lower bound:

Proposition 8 For all agents a , $\pi^a(Y^a) > \hat{P}_{SSL}(Y^a)$.

Proof. This is shown along the same lines as Proposition 6. ■

4 Hedging in a static framework

In this section we consider the case in which S_1 and R_1 are two correlated log-normal random variables such that

$$S_1 = S_0 \exp(\alpha - \frac{\sigma^2}{2} + \sigma W_1);$$

$$R_1 = R_0 \exp(\mu - \frac{\eta^2}{2} + \eta Z_1);$$

$W_1, Z_1 \sim N(0, 1)$, $\text{corr}(W_1, Z_1) = \rho$, with all parameters assumed to be constant.

This reflects the following idea: in a continuous time framework randomness is described by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, equipped with the filtration $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$. This filtration is assumed to be generated by a two dimensional standard brownian motion $W_t = (W_t, \bar{W}_t)$. The processes $(S_t)_{t \geq 0}$ and $(R_t)_{t \geq 0}$ are assumed to evolve stochastically according to the following stochastic differential equations:

$$dS_t = \alpha S_t dt + \sigma S_t dW_t$$

$$dR_t = \mu R_t dt + \eta R_t (\rho dW_t + \sqrt{1 - \rho^2} d\bar{W}_t) - \mu R_t dt + \eta R_t dZ_t$$

with $(\alpha, \sigma), (\mu, \eta) \in \mathbb{R} \times \mathbb{R}^+$ and where Z_t is a standard brownian motion with $\langle W_t, Z_t \rangle = \rho t$.

Suppose that the prices only are observable at date 0 and $T = 1$ and assume from now on that the riskless interest rate r is equal to zero. We recall that a contingent claim is attainable at date 1 if it is a random variable $X \in L$ such that:

$$\exists(x, \theta_0) \in \mathbb{R}^2, \quad X = x + \theta_0(S_1 - S_0).$$

We study the problem of using the financial market in order to hedge R_1 , the value at date 1 of the adapted process $(R_t)_{t \geq 0}$. Of course, if $\rho = 1$, $R_1 \in \mathcal{A}_1$, and the problem is trivial.

Remark 1 For an economic interpretation, let us consider a model with two agents: the employer and the employee. The employee works between dates 0 and T for the employer and receives a wage process $(R_t)_{0 \leq t \leq T}$, which is adapted w.r.t. the filtration \mathcal{F} , but in general it cannot be expressed as a linear combination of the financial assets. At his retirement date $T = 1$, he also receives a fixed amount B_T which is in fact a defined pension paid out at once at the date T . As the pension (usually) depends on the wage history at a fixed percentage, the problem of the employer, who has the charge of paying the contributions, is to hedge (such percentage of) R_T by using the financial market.

The aim here is to find the upper bound given by the SSL criterion for the price of R_1 , and to compare it to the super-replication bound.

As the SSL criterion involves a conditional expectation of two lognormal distributions, in order to proceed we first need the following well-known result:

Lemma 9 Let X, Y be two random variables with lognormal distribution; $X \sim \exp N(m_1, \sigma_1^2)$, $Y \sim \exp N(m_2, \sigma_2^2)$, $\text{corr}(X, Y) = \rho$. Then

$$\mathbb{E}[Y \mid X = w] = w^{\frac{\sigma_2^2}{\sigma_1^2}} \exp\left\{\frac{1}{2}\sigma_2^2(1 - \rho^2) + m_2 - \rho m_1 \frac{\sigma_2}{\sigma_1}\right\} \quad \mathbb{P}\text{-a.e., } \forall w \in \text{Im}(\Omega). \quad (2)$$

We recall the definition of the upper bound $\overline{P}_{SSL}(R_1)$ for the price of R_1 :

$$P_{SSL}(R_1) = \inf\{x \in \mathbb{R} : \exists \theta_0 \in \mathbb{R} \text{ such that } x + \theta_0(S_1 - S_0) \succeq_{SSL} R_1\}. \quad (3)$$

From Lemma 4 we have that $X \succeq_{SSL} R_1$ iff $\mathbb{E}[R_1 \mid X = w] \leq w$, $\forall w \in \text{Im}_X(\Omega)$, which can be rewritten as

$$\begin{aligned} \mathbb{E}[R_1 \mid X = w] &= \mathbb{E}[R_1 \mid x + \theta_0(S_1 - S_0) = w] \\ &\stackrel{(\theta_0 \neq 0)}{=} \mathbb{E}[R_1 \mid S_1 = S_0 + \frac{w-x}{\theta_0}] \\ &= \mathbb{E}[\exp N_2(\log R_0 + \mu - \frac{\sigma^2}{2}, \eta^2) \mid \exp N_1(\alpha - \frac{\sigma^2}{2}, \sigma^2) = 1 + \frac{w-x}{S_0\theta_0}] \\ &\stackrel{(1, \text{see Remark 5})}{=} (1 + \frac{w-x}{S_0\theta_0})^{\rho^2 \frac{1}{\bar{k}}}, \end{aligned}$$

where $\bar{k} = \exp[-\frac{\sigma^2}{2}(1 - \rho^2) + \mu - \frac{\sigma^2}{2} + \log R_0 - \rho^2(\alpha - \frac{\sigma^2}{2})]$.

We conclude that

$$X \succeq_{SSL} R_1 \Leftrightarrow (1 + \frac{w-x}{S_0\theta_0})^{\rho^2} \leq w\bar{k}, \quad \forall w \in \text{Im}_X(\Omega). \quad (4)$$

Our purpose is now to study this last inequality, in order to find θ_0 that minimizes (3).

If $\theta_0 = 0$, the inequality (4) is not defined, but in this case $\text{Im}_X(\Omega) = \{x\}$, and $\mathbb{E}[R_1 \mid X = w] = \mathbb{E}[R_1]$: as we look for the minimum x such that $\mathbb{E}[R_1 \mid X = x] - \mathbb{E}[R_1] \leq x$, the upper bound is given by

$$\overline{P}_{SSL}(R_1) = \mathbb{E}[R_1] = R_0 \exp[\mu]. \quad (5)$$

Let us now consider the case that $\theta_0 < 0$. Then $\text{Im}_X(\Omega) = \{w : 1 + \frac{w-x}{S_0\theta_0} > 0\} = (-\infty, x - S_0\theta_0)$, and the inequality (4) becomes

$$\underbrace{(1 + \frac{w-x}{S_0\theta_0})^{\rho^2}}_{\geq 0} \leq \underbrace{w\bar{k}}_{\leq 0 \text{ for } w \leq 0}$$

so that there are no solutions.

Suppose now $\theta_0 > 0$.

In this case $\text{Im}_X(\Omega) = \{w : 1 + \frac{w-x}{S_0\theta_0} > 0\} = (x - S_0\theta_0, +\infty)$.

If $x - S_0\theta_0 < 0$, then

$$\underbrace{\left(1 + \frac{w-x}{S_0\theta_0}\right)^{\rho^2}}_{\geq 0 \text{ for } w > x - S_0\theta_0} \leq \underbrace{wk}_{\leq 0 \text{ for } w \in (x - S_0\theta_0, 0)}$$

and therefore $x - S_0\theta_0 \geq 0$, from which we obtain that

$$\theta_0 \in [0, \frac{x}{S_0}].$$

We now have to distinguish some cases according to the sign of ρ .

If $\rho > 0$, let us denote $\delta := \frac{\sigma}{\rho\eta}$, $k := (\bar{k})^\delta$ and $\theta := S_0\theta_0$, so that

$$\begin{aligned} X \succ_{SSI} R_1 &\Leftrightarrow 1 + \frac{w-x}{\theta} \leq w^\delta k, \quad \forall w \in \text{Im}_X(\Omega) \\ &\Leftrightarrow \varphi(w) = \theta k w^\delta - w + x - \theta \geq 0, \quad \forall w \in (x - \theta, +\infty). \end{aligned} \quad (6)$$

If $\delta < 1$, there are no solutions as $\varphi(+\infty) = -\infty$.

If $\delta = 1$, $\varphi(w) = w(\theta k - 1) + x - \theta \geq 0$ in $(x - \theta, +\infty)$ iff

$$\varphi(x - \theta) > 0 \text{ and } \theta k - 1 \geq 0,$$

that is iff $\theta \geq \frac{1}{k}$ and $\theta \in [0, x]$, so the infimum of x is obtained at $\frac{1}{k}$:

$$\bar{F}_{SSI}(R_1) = \frac{1}{k} = R_0 \exp[\mu - \alpha], \text{ with } \frac{\sigma}{\rho\eta} = 1 \quad (7)$$

Finally, if $\delta > 1$, $\varphi(w) = \theta k w^\delta - w + x - \theta$.

In order to find when $\varphi(w) \geq 0$, it is enough to study its behaviour at the extremal points:

$$\varphi(+\infty) = +\infty,$$

$$\varphi(x - \theta) = \theta k (x - \theta)^\delta > 0,$$

$\varphi'(\bar{w}) = 0 \Leftrightarrow \bar{w} = (\frac{1}{\delta k})^{\frac{1}{s-1}}$, then $\varphi(w) \geq 0 \forall w \in (x - \theta, +\infty)$ iff $\bar{w} \geq x - \theta$ and $\varphi(\bar{w}) \geq 0$.

We remark that

$$\begin{aligned}\varphi(\bar{w}) &= \theta k (\frac{1}{\delta k \theta})^{\frac{s}{s-1}} - (\frac{1}{\delta k \theta})^{\frac{1}{s-1}} + x - \theta \\ &= \bar{w} \frac{1-\delta}{\delta} + x - \theta \geq 0 \Leftrightarrow x - \theta \geq \bar{w} \frac{\delta-1}{\delta}\end{aligned}$$

Therefore, $\bar{w} \geq x - \theta$ and $\varphi(\bar{w}) \geq 0$ iff $\bar{w}^{\frac{s-1}{s}} \leq x - \theta \leq \bar{w}$.

By rewriting $\varphi(\bar{w})$ as

$$f(\theta) := \varphi(\bar{w}) = -\beta \theta^{-\gamma} + x - \theta \stackrel{?}{>} 0, \text{ for some } \theta \in [0, x],$$

with $\beta = \frac{s-1}{\delta} (\frac{1}{\delta k})^{\frac{1}{s-1}} > 0$ and $\gamma = \frac{1}{s-1} > 0$, we can think about $\varphi(\bar{w})$ as a function of θ and apply the same argument:

$$f(0) = -\infty,$$

$$f(x) = -\beta x^{-\gamma} < 0.$$

There exists a $\theta \in [0, x]$ such that $f(\theta) \geq 0$ only if there exists a $\bar{\theta} \in (0, x)$ such that $f'(\bar{\theta}) = 0$ and $f(\bar{\theta}) \geq 0$.

As $f'(\theta) = 0$ at $\bar{\theta} = (\frac{1}{\beta\gamma})^{\frac{1}{\gamma+1}}$, then

$$\begin{aligned}f(\bar{\theta}) &= x - (\frac{1}{\beta\gamma})^{-\frac{\gamma}{\gamma+1}} - \beta (\frac{1}{\beta\gamma})^{-\frac{\gamma}{\gamma+1}} \\ &= x - (\frac{1}{\beta\gamma})^{-\frac{\gamma}{\gamma+1}} [1 + \frac{1}{\gamma}] \geq 0 \\ &\Leftrightarrow x \geq (\frac{1}{\beta\gamma})^{-\frac{\gamma}{\gamma+1}} [1 + \frac{1}{\gamma}].\end{aligned}$$

We easily see that the minimal x such that $f(\theta)$ remains positive is the upper bound given by

$$\begin{aligned}\bar{F}_{SSL}(R_1) &= (1 + \gamma) \gamma^{-\frac{\gamma}{\gamma+1}} \beta^{\frac{1}{\gamma+1}} - \frac{1}{k} \\ &= R_0 \exp[\frac{\eta^2}{2}(1 - \rho^2) + \mu - \frac{\eta^2}{2} - \rho \frac{\eta}{\sigma}(\alpha - \frac{\sigma^2}{2})], \text{ with } \frac{\sigma}{\rho\eta} > 1.\end{aligned}\tag{8}$$

Suppose now $\rho < 0$.

Since

$$\begin{aligned} X \succeq_{SSL} R_1 &\Leftrightarrow (1 + \frac{w-x}{\theta})\rho^2 \leq w\bar{k}, \quad \forall w \in (x - \theta, +\infty) \\ &\Leftrightarrow 1 + \underbrace{\frac{w-x}{\theta}}_{\substack{=0 \\ \text{for } w=(x-\theta)^+}} \geq \underbrace{\eta^2 k}_{>0} \end{aligned}$$

there are no solutions in this case.

Finally, let us consider $\rho = 0$.

Here $X \succeq_{SSL} R_1 \Leftrightarrow 1 \leq w\bar{k} \Leftrightarrow w \geq \frac{1}{\bar{k}} \Leftrightarrow x - \theta > \frac{1}{\bar{k}} \Leftrightarrow x \geq \theta + \frac{1}{\bar{k}}$,
such that the upper bound is given by

$$\bar{P}_{SSL}(R_1) = \frac{1}{\bar{k}} = R_0 \exp[\mu], \quad \text{with } \rho = 0 \quad (9)$$

Remark 2 The result in (9) is straightforward: when the two brownian motions are not correlated, the conditional expectation can be computed directly:

$$\mathbb{E}[R_1 | X = w] = \mathbb{E}[R_1] = R_0 \exp(\mu - \frac{\eta^2}{2} + \frac{\eta^2}{2}) = R_0 \exp(\mu).$$

We summarize these results in the next proposition:

Proposition 10 The upper bound for the price of R_1 consistent with the SSL criterion is given by

$$\begin{aligned} P_{SSL}(R_1) &= R_0 \exp[\mu - \alpha], & \text{when } \frac{\sigma}{\rho\eta} = 1 \\ \bar{P}_{SSL}(R_1) &= R_0 \exp[\frac{\alpha^2}{2}(1 - \rho^2) + \mu - \frac{\eta^2}{2} - \rho^2(\alpha - \frac{\sigma^2}{2})], & \text{when } \frac{\sigma}{\rho\eta} > 1 \text{ and } \rho \neq 0 \\ \bar{P}_{SSL}(R_1) &= R_0 \exp[\mu] & \text{otherwise.} \end{aligned}$$

It is clear that these form a significant improvement with respect to the super replication upper bound $\bar{P}_{ac}(R_1)$:

$$\begin{aligned}\bar{P}_{ac}(R_1) &= \inf\{x \in \mathbb{R} : \exists \theta_0 \in \mathbb{R} \text{ such that } x + \theta_0(S_1 - S_0) \geq R_1 \text{ P-a.e}\} \\ &= +\infty\end{aligned}\quad (10)$$

for $\rho \neq 1$, which follows immediately from the support $[0, +\infty)$ of log-normal random variables.

5 Conclusion

In an incomplete market setting, the pricing and hedging of risky positions is a difficult problem. Using super-replication, one obtains intervals for the hedging price of a risk, but these intervals turn out to be too large in general to be used in practice.

In order to determine tighter intervals, we have defined strong stop-loss dominance, which can be related with first and second order stochastic dominance. We have proved that we indeed obtain upper and lower bounds for the hedging price which are compatible with the equilibrium.

In some cases the use of strong stop-loss dominance improves the super-replication approach in a significant way. For example, in the case of log-normal random variables (which is an interesting case for pension funds), the super-replication yields $+\infty$ on upper bound, whereas our calculations provide explicit finite results.

In this paper we have concentrated on a static situation with only two dates of interest: an initial date and a time horizon. Further research has to be done in order to generalize this approach to a dynamic model by backward optimization. Another way would be to study the dynamic model only in the case of discrete finite-state cases like in Ritchken and Kuo (1988, 1989).

As mentioned above, we have started this study as we were looking at the retirement problem. A future line of research will be to translate our results back to a pension fund framework. We therefore intend to use estimated salary processes and index diffusions like the CAC 40, in order to study some scenarios of pension funding.

References

- [1] Davis M.H.A., V.G. Panas and T. Zariwopoulou (1993) "European option pricing with transaction costs", *SIAM Journal of Control and Optimization*, **31**, 470-493.
- [2] Duffie D. and H.R. Richardson (1991) "Mean-variance hedging in continuous time", *Annals of Applied Probability*, **1**, 1-15.
- [3] El Karoui N. and M.C. Quenez (1995) "Dynamic programming and pricing of contingent claims in an incomplete market", *SIAM Journal of Control and Optimization*, **33**, 29-66.
- [4] Föllmer H. and M. Schweizer (1991) "Hedging of contingent claims under incomplete information", in *Applied Stochastic Analysis*, Stochastic Monographs Vol 5, M.H.A. Davis and R.J. Elliot eds, Gordon and Breach, New-York.
- [5] Föllmer H. and D. Sondermann (1986) "Hedging of non redundant contingent claims. Contributions to mathematical economics," *In Honor of Gérard Debreu*, Hildenbrand-Mas-Colell editors, North-Holland, Amsterdam.
- [6] Goovaerts M., R. Kaas, E. van Hecrwarden and T. Bawelincx (1990) "Effective Actuarial Methods", North-Holland, Amsterdam.
- [7] Huang C.F. and R.H. Litzemberger (1988) "Foundations for Financial Economics", North-Holland.
- [8] Hodges S. and A. Neuberger (1989) "Optimal replication of contingent claims under transaction costs", *Review of Future Markets*, **8**, 222-234.
- [9] Karatzas I. (1989) "Optimisation problems in the theory of continuous trading", *SIAM Journal of Control and Optimization*, **27**, 1221-1259.
- [10] Karatzas I. and S. Kou (1996) "On the pricing of contingent claims under constraints", *Annals of Applied Probability*, **6**, 321-369.
- [11] Levy H. (1985) "Upper and lower bounds of put and call option value: Stochastic dominance approach," *Journal of Finance*, **40**, 1197-1217.

- [12] Levy H. (1992) "Stochastic dominance and expected utility: survey and analysis", *Management Science*, **38**, 555-593.
- [13] Ritchken P.H and S. Kuo (1988) "Option Bounds with Finite Revision Opportunities", *Journal of Finance*, **43**, 301-308.
- [14] Ritchken P.H and S. Kuo (1989) "On stochastic dominance and decreasing absolute risk averse option pricing bounds", *Management Science*, **35**, 51-59.
- [15] Schweizer M. (1991) "Option hedging for semimartingales," *Stochastic Processes and Applications*, **37** , 339-363.
- [16] Schweizer M. (1992) "Mean-variance hedging for general claims", *Annals of Applied Probability*, **2**, 171-179.
- [17] Soner H., S. Shreve and J. Cvitanic (1995) "There is no nontrivial hedging portfolio for option pricing with transaction costs", *Annals of Applied Probability*, **5**, 327-355.