EXAMINATIONS

September 2002

Subject 102 — Financial Mathematics

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

K G Forman
Chairman of the Board of Examiners
12 November 2002

EXAMINERS' COMMENT

Please note that differing answers may be obtained to those shown in the solutions depending on whether figures obtained form tables or form calculators are used in the calculations but candidates are not penalised for this.

However, candidates may be penalised where excessive rounding has been used, either during or at the end of the calculations, or where insufficient working is shown. Where candidates change the given period of convertibility of interest rates in answering questions (for example, question 11), they should keep sufficient significant figures in their working and interest rate values (say 5) to ensure that the final answer is not unduly rounded.

Candidates should be careful to state the units at the end of an answer (e.g. pounds, pence, per cent).

1 Government bills are short-dated securities (eg 3 months) issued by governments to fund their short term spending, borrowing and liquidity requirements.

They are issued at a discount and redeemed at par with no coupon.

They are mostly denominated in the domestic currency, although issues can be made in other currencies.

The yield on Government bills is typically quoted as a simple rate of discount for the term of the bill. For example, a 3 month bill may be quoted as being offered at a discount of 2%. This would mean that the initial investment required to buy the bill would be 2% less than the payment 3 months later.

Government bills tend to be secure and highly marketable, despite not being quoted.

They are often used as a benchmark risk-free short term investment.

Candidates also received credit for other relevant points made.

2 Present value of dividends, I, is

$$I = 1.5(v^{1/4} + v^{1/2} + v^{3/4})$$

Calculated at i' % where $1 + i' = (1.03)^2 \Rightarrow i' = 6.09\%$

so $I = 4.3693$

Hence, forward price, F, is

$$F = (12 - 4.3693)\ (1+i')^{\frac{10}{12}} \text{ at } 6.09\%$$

$$= 8.0160$$

$$= £8.02$$

3 (i) We want $i^{(4)}$ where

$1 + i = (1.005)^{12} \Rightarrow i = 0.061678.$

Hence, $\left(1+\dfrac{i^{(4)}}{4}\right)^4 = 1.061678$

$\Rightarrow i^{(4)} = 6.0301\%$

(ii) We want $i^{(4)}$ where

$1 + i = (1.12)^{½} \Rightarrow i = 0.058301$

Hence, $\left(1+\dfrac{i^{(4)}}{4}\right)^4 = 1.058301$

$\Rightarrow i^{(4)} = 5.7068\%$

4 Annual yield, i, to be used to value the dividends comes from:

$1 + i = (1.0125)^2(1.02)$

$\Rightarrow i = 4.5659\%$ p.a.

Value of dividends =

$$0.05\left(v^{\frac{10}{12}} + 1.03v^{1\frac{10}{12}} + (1.03)^2 v^{2\frac{10}{12}} + \ldots\right)$$

$$= 0.05\, v^{\frac{10}{12}}\left(\frac{1}{1-1.03v}\right)$$

$= £3.2169$

5 (i) $98 = 7a_{\overline{3}|} + 105v^3$

Try 10% RHS = 96.296
Try 9% RHS = 98.798

Interpolation gives:

$$i = 0.09 + \frac{98.798-98}{98.798-96.296} \times 0.01$$

$= 0.09319$

i.e. 9.319%

(ii) Let i_n = spot yield for term n

Then $98 = 112v_{i_1\%}$

$\Rightarrow i_1 = 14.286\%$

$98 = 7v_{i_1\%} + 112v_{i_2\%}^2$

$112(1 + i_2)^{-2} = 98 - 7 \times \dfrac{1}{1.14286}$

$\Rightarrow i_2 = 10.410\%$

$98 = 7v_{i_1\%} + 7v_{i_2\%}^2 + 112v_{i_3\%}^3$

Hence $112(1+i_3)^{-3} = 98 - 7\times\dfrac{1}{1.14286} - 7\times\dfrac{1}{(1.1041)^2}$

$\Rightarrow i_3 = 9.148\%$

Candidates lost marks in part (i) for interpolating or extrapolating over too wide a range or for extrapolating from a range too far away from the actual gross redemption yield.

In answering part (ii), several candidates calculated forward rates or gross redemption yields rather than spot rates.

6 (i) All amounts in £ millions

PV of outgo = $10\bar{a}_{\overline{2}|}$ at 10%

$\bar{a}_{\overline{2}|} = \dfrac{i}{\delta}a_{\overline{2}|} = 1.049206 \times 1.7355$

$= 1.82090$

$\Rightarrow$ PV of outgo = 18.209

Income is: 8 at the end of year 3, 7½ at the end of year 4, …, 3 at the end of year 13, 2 at the end of year 14 and 1 at the end of year 15.

PV of income = $v^2 \times 0.5 \times (Da)_{\overline{10}|} + v^2 \times 3 \times a_{\overline{10}|} + v^{12} \times (Da)_{\overline{3}|}$ at 10%

$(Da)_{\overline{10}|} = \dfrac{10 - a_{\overline{10}|}}{i} = \dfrac{10 - 6.1446}{0.10} = 38.5543$

$$(Da)_{\overline{3}|} = \frac{3-2.4869}{0.10} = 5.1310$$

$\Rightarrow$ PV of income = 15.9314 + 15.2345 + 1.6349 = 32.8008

$\Rightarrow$ NPV of project = 32.8008 − 18.2090

= £14.592 million

(ii) Since the outgo precedes the income and the project has a positive NPV, the IRR must be greater than 10% to bring the NPV down to zero.

7 (i) $$\left(1+\frac{i^{(2)}}{2}\right)^2 = 1.05$$

$$\Rightarrow i^{(2)} = 4.939\%$$

$$g(1-t_1) = \frac{0.07}{1.10} \times 0.75 = 0.0477$$

So $i^{(2)} > g(1-t_1) \Rightarrow$ there is a capital gain on the contract

(ii) Since there is a capital gain, the loan is least valuable to the investor if the repayment is made by the borrower at the latest possible date. Hence, we assume redemption occurs 15 years after issue in order to calculate the minimum yield achieved.

(iii) If A is the price per £100 of loan:

$$A = 100 \times 0.07 \times 0.75\, a^{(2)}_{\overline{15}|} + (110 - 0.3(110 - A))v^{15} \text{ at } 5\%$$
$$= 100 \times 0.07 \times 0.75 \times 1.012348 \times 10.3797 + (110 - 0.3(110 - A)) \times 0.48102$$

$$\text{Hence } A = \frac{55.1663 + 37.0385}{1 - 0.3 \times 0.48102} = 107.75$$

8 (i) Let i_1 and i_2 be the interest rates in years 1 and 2, respectively.

Value of a/c after 2 years is a random variable, S, where

$$S = 10{,}000(1+i_1)(1+i_2)$$

$= 10{,}000 S_2$ where $S_2 = (1 + i_1)(1 + i_2)$

Now $E(S_2) = E[(1 + i_1)(1 + i_2)]$

$= E(1 + i_1)E(1 + i_2)$ using independence

$= (1 + E(i_1))\,(1 + E(i_2))$

where $E(i_1) = \frac{1}{3} \times 0.03 + \frac{1}{3} \times 0.04 + \frac{1}{3} \times 0.06$

$= 0.043333$

$E(i_2) = 0.7 \times 0.05 + 0.3 \times 0.04$

$= 0.047$

$\Rightarrow E(S_2) = 1.043333 \times 1.047 = 1.09237$

$\Rightarrow E(S) = 10{,}923.7$

(ii) $\mathrm{Var}(S) = \mathrm{Var}\,(10{,}000\ S_2)$

$= 10^8 \times \left(E(S_2^2) - (E(S_2))^2\right)$

Now

$E(S_2^2) = E[(1 + i_1)^2(1 + i_2)^2]$

$= E[(1+i_1)^2]\,.\,E[(1+i_2)^2]$ — independence

$= E(1 + 2i_1 + i_1^2)\,.\,E(1 + 2i_2 + i_2^2)$

$= (1 + 2E(i_1) + E(i_1^2))\,(1 + 2E(i_2) + E(i_2^2))$

where

$E(i_1^2) = \frac{1}{3} \times (0.03)^2 + \frac{1}{3} \times (0.04)^2 + \frac{1}{3} \times (0.06)^2$

$= 0.002033$

$$E(i_2^2) = 0.7 \times (0.05)^2 + 0.3 \times (0.04)^2$$

$$= 0.002230$$

$$\Rightarrow E(S_2^2) = (1 + 2 \times 0.043333 + 0.002033) \times (1 + 2 \times 0.047 + 0.00223)$$

$$= 1.193465$$

$$\Rightarrow \mathrm{Var}(S_2) = E(S_2^2) - (E(S_2))^2$$

$$= 1.193465 - 1.09237^2$$

$$= 0.00019278$$

$$\Rightarrow \mathrm{Var}(S) = 19{,}278$$

9 (i) $t \le 8$

$$V(t) = e^{-\int_0^t (0.03+0.01s)ds}$$

$$= e^{-\left[0.03s + \frac{0.01s^2}{2}\right]_0^t}$$

$$= e^{-\left[0.03t + \frac{0.01t^2}{2}\right]}$$

$t > 8$

$$V(t) = e^{-\left[\int_0^8 \delta(s)ds + \int_8^t 0.05ds\right]}$$

$$= V(8)\,.\,e^{-0.05(t-8)}$$

$$= e^{-0.56}\,.\,e^{-0.05(t-8)}$$

$$= e^{-(0.16+0.05t)}$$

(ii) (a) $\text{PV} = 500e^{-(0.16+0.05\times15)}$

$$= 500e^{-0.91}$$

$$= 201.26$$

(b) $$500\left(1-\frac{d^{(4)}}{4}\right)^{60} = 201.26$$

$\Rightarrow d^{(4)} = 6.02096\%$ p.a. convertible quarterly

(iii) $$\text{PV} = \int_{10}^{14} e^{-(0.16+0.05t)}\, 10e^{-0.02t}\, dt$$

$$= \int_{10}^{14} 10e^{-0.16-0.07t}\, dt$$

$$= 10e^{-0.16}\left[\frac{e^{-0.07t}}{-0.07}\right]_{10}^{14}$$

$$= 10e^{-0.16}(7.0941 - 5.3616)$$

$$= 14.763$$

10 (i) Working in 000's

PV of liabilities = $3v^5 + 5v^{10} + 7v^{15} + 9v^{20} + 11v^{25}$ at 7%

= 11.57034 i.e. £11,570.34

(ii) The DMT of the liabilities is

$$\frac{15v^5 + 50v^{10} + 105v^{15} + 180v^{20} + 275v^{25}}{11.57034}$$

$$= \frac{171.3530}{11.57034} = 14.8097 \text{ years}$$

(iii) Let A be the nominal value of stock A bought and B be the nominal value of stock B bought.

Then: PV of assets = PV of liabilities

$\Rightarrow A(0.05\, a_{\overline{26}|} + v^{26}) + B(0.04\, a_{\overline{32}|} + v^{32}) = 11{,}570.34$ at 7%

$\Rightarrow A(0.05 \times 11.8258 + 0.17220) + B(0.04 \times 12.6466 + 0.11474) = 11{,}570.34$

$\Rightarrow 0.76349\, A + 0.62060\, B = 11{,}570.34$ (1)

and DMT of assets = DMT of liabilities

$\Rightarrow A(0.05\,(Ia)_{\overline{26}|} + 26v^{26}) + B(0.04\,(Ia)_{\overline{32}|} + 32v^{32}) = 14.8097 \times 11{,}570.34$

Now

$$(Ia)_{\overline{26}|} = \frac{\ddot{a}_{\overline{26}|} - 26v^{26}}{i} \quad \text{at } 7\%$$

$$= \frac{1.07 \times 11.8258 - 26 \times 0.17220}{0.07}$$

$$= 116.8058$$

and $$(Ia)_{\overline{32}|} = \frac{1.07 \times 12.6466 - 32 \times 0.11474}{0.07}$$

$$= 140.8597$$

Hence

$$A(0.05 \times 116.8058 + 26 \times 0.17220)$$

$$+ B(0.04 \times 140.8597 + 32 \times 0.11474) = 171{,}353$$

$\Rightarrow 10.3175A + 9.3061B = 171{,}353 \qquad (2)$

Then $(2) - \dfrac{10.3175}{0.76349}\,(1)$

$\Rightarrow 0.9196B = 14{,}996.03$

Hence $B = £16{,}307$ and $A = £1{,}900$

11 (i) Let monthly repayment be X. Working in quarters:

$$10{,}000 = 3Xa^{(3)}_{\overline{60}|} + 30\left(v^{20}a^{(3)}_{\overline{40}|} + v^{40}a^{(3)}_{\overline{20}|}\right) \text{ at } 2\%$$

Now $i^{(3)} = 3\left((1.02)^{\frac{1}{3}} - 1\right)$

$$= 0.0198681$$

Hence

$$10{,}000 = 3X\frac{i}{i^{(3)}}a_{\overline{60}|} + 30\frac{i}{i^{(3)}}\left(v^{20}a_{\overline{40}|} + v^{40}a_{\overline{20}|}\right) \text{ at } 2\%$$

$$= 3X \times \frac{0.02}{0.0198681} \times 34.7609$$

$$+\ 30 \times \frac{0.02}{0.0198681} \times (0.67297 \times 27.3555 + 0.45289 \times 16.3514)$$

$$= 104.9750X + 779.5859$$

$\Rightarrow X = £87.83$ per month

(ii) Interest in 1st month

$$= 10{,}000 \times \left((1.02)^{\frac{1}{3}} - 1\right)$$

$$= £66.23$$

$\Rightarrow$ Capital repaid $= 87.83 - 66.23$

$= £21.60$

(iii) O/S loan on 1 April 2002 is

$$3 \times (87.83 + 10)\ a^{(3)}_{\overline{28}|} + 30v^8 a^{(3)}_{\overline{20}|} \text{ at } 2\%$$

$$= 293.49 \times \frac{i}{i^{(3)}}a_{\overline{28}|} + 30\frac{i}{i^{(3)}} . v^8 . a_{\overline{20}|} \text{ at } 2\%$$

$$= 293.49 \times \frac{0.02}{0.0198681} \times 21.2813$$

$$+ 30 \times \frac{0.02}{0.0198681} \times 0.85349 \times 16.3514$$

$$= 6{,}287.31 + 421.45$$

$$= 6{,}708.76$$

(iv) Let annual amount be £Y

$$6{,}708.76 = Ya_{\overline{4|}} \text{ at } i\%$$

where $1 + i = (1.02)^4$

$$\Rightarrow i = 8.243\%$$

Hence $6{,}708.76 = Y \times \dfrac{1 - v^4}{i}$ at 8.243%

$$\Rightarrow Y = \frac{6{,}708.76}{3.2943} = £2{,}036.48$$