EXAMINATIONS

April 2000

Subject 102 — Financial Mathematics

EXAMINERS' REPORT

1 Let $f_{t,n}$ = n-year forward rate at time t

$y_{0,k}$ = k-year spot rate of interest at time $t = 0$

Then $(1 + y_{0,3})^3 = (1 + y_{0,1})\,(1 + f_{1,2})^2$

$$\Rightarrow (1 + f_{1,2})^2 = \frac{(1.055)^3}{1.045} = 1.123676$$

$$\Rightarrow f_{1,2} = 6.004\% \text{ per annum}$$

2 An agreement to exchange, on fixed terms, particular cash flow in one currency for a cash flow in another.

3 (i) $$\left(1+\frac{i^{(2)}}{2}\right)^2 = \left(1+\frac{i^{(12)}}{12}\right)^{12}$$

$$\Rightarrow 1 + \frac{i^{(2)}}{2} = \left(1+\frac{0.07}{12}\right)^6$$

$$\Rightarrow i^{(2)} = 7.103\% \text{ per annum}$$

(ii) $$\left(1-\frac{d^{(12)}}{12}\right)^{12} = \left(1+\frac{i^{(12)}}{12}\right)^{-12}$$

$$\Rightarrow 1 - \frac{d^{(12)}}{12} = \left(1+\frac{0.07}{12}\right)^{-1}$$

$$\Rightarrow d^{(12)} = 6.959\%$$

4 Debentures are part of the loan capital of the company. The term "loan capital" usually refers to long term borrowings rather than short term. The issuing company provides some form of security to holders of the debenture. This is usually in the form of a floating charge against the assets of the company.

Unsecured loan stocks have no explicit assets backing them and holders rank alongside other unsecured creditors. Yields will be higher than on the debentures to reflect the higher risk of default.

5 TWRR is i% such that

$$(1+i)^3 = \frac{212}{180} * \frac{230}{237} * \frac{295}{248} * \frac{309}{311}$$

$$= 1.35086$$

$$\Rightarrow i = 10.544\% \text{ p.a.}$$

6 (i) $$\ddot{s}^{(12)}_{\overline{5.5}|} = \frac{(1+i)^{5.5} - 1}{d^{(12)}} \quad \text{at } 13\% \text{ p.a.}$$

where $\left(1 - \frac{d^{(12)}}{12}\right)^{12} = v = \frac{1}{1.13}$

$\Rightarrow d^{(12)} = 0.1216$

$$\Rightarrow \ddot{s}^{(12)}_{\overline{5.5}|} = \frac{(1.13)^{5.5} - 1}{0.1216} \qquad = 7.8827$$

(ii) The answer to (i) represents the accumulation after 5½ years of the following annuity:

An annuity of $\frac{1}{12}$ unit payable monthly in advance for $5\frac{1}{2}$ years at an effective interest rate of 13% per annum.

7 100 shares provide a first dividend of £8.

$$\text{Present value} = 8(v + 1.08v^2 + 1.08 \times 1.07 \times v^3$$
$$+ 1.08 \times 1.07 \times 1.05v^4$$
$$+ 1.08 \times 1.07 \times 1.05^2 v^5$$
$$+ \ldots) \qquad \text{at } 7\% \text{ p.a.}$$

$$= 8(0.93458 + 0.94331)$$

$$+ \frac{8 \times 1.08 \times 1.07}{(1.07)^3}\left[\frac{1}{1 - \frac{1.05}{1.07}}\right]$$

$$= 15.023 + 403.738$$

$$= £418.76$$

8 $$\int_0^6 100\exp\left\{\int_t^6 0.06\,ds + \int_6^{12}(0.05 + 0.0002s^2)\,ds\right\}dt$$

$$= \int_0^6 100\exp\{0.36 - 0.06t + 0.6 + 0.1152 - 0.3 - 0.0144\}\,dt$$

$$= 213.999 \times \int_0^6 e^{-0.06t}\,dt$$

$$= 213.999\left[\frac{e^{-0.06t}}{-0.06}\right]_0^6$$

$$= 213.999 \times \left(\frac{1 - e^{-0.36}}{0.06}\right)$$

$$= £1,078.28$$

9 (i) The "no arbitrage" assumption means that it is assumed that the investor is unable to make a risk-free trading profit.

(ii) Present value of dividends, I, is

$$I = 2(e^{-0.07\times 0.25} + e^{-0.07\times 0.5})$$

$$= 3.8965$$

$$T - t = \frac{7}{12} = 0.58333$$

Hence, forward price, F, is

$$F = (60 - 3.8965)\,e^{0.07\times 0.58333}$$

$$= £58.44$$

10 (i) $5.91\% > 7\% \times (1 - 0.25)$

$\Rightarrow$ Assume stock is redeemed as late as possible (i.e. 1 April 2010)

For £100 nominal:

P' = Price at 1 April 1991

$$= 0.75 \times 7a^{(2)}_{\overline{19}|} + 100v^{19} \quad \text{at } 6\%$$

$$= 0.75 \times 7 \times 1.014782 \times 11.1581 + 100 \times 0.33051$$

$$= 59.446 + 33.051 = 92.497$$

So price at 1 July 1991, P is

$$P = 92.497 \times (1.06)^{3/12} = 93.854$$

and price of £10,000 nominal is £9,385.40

(ii) $4.94\% < 7\% \times (1 - 0.25)$

$\Rightarrow$ Assume stock is redeemed at earliest possible date
(i.e. 1 April 2004)

For £100 nominal:

$$P = 0.75 \times 7a^{(2)}_{\overline{5}|} + 100v^{5} \quad \text{at } 5\%$$

$$= 5.25 \times 1.012348 \times 4.3295 + 100 \times 0.78353$$

$$= 101.364$$

Hence price of £10,000 nominal is £10,136.40

11 (i) $E(1 + i) = 1.07$

$\text{Var}(1 + i) = 0.04$

$$\Rightarrow 1.07 = \exp\left(\mu + \frac{\sigma^2}{2}\right) \qquad (1)$$

$$0.04 = \exp(2\mu + \sigma^2)\,[\exp(\sigma^2) - 1] \qquad (2)$$

$$\Rightarrow \frac{(2)}{(1)^2} = \frac{0.04}{1.07^2} = \exp(\sigma^2) - 1$$

$$\Rightarrow \quad \sigma^2 \quad = \ln\left(\frac{0.04}{(1.07)^2}+1\right)$$

$$= 0.0343411 \;\; \Rightarrow \sigma = 0.185314$$

$$\Rightarrow \;\; 1.07 \quad = \exp\left(\mu + \frac{0.0343411}{2}\right)$$

$$\therefore \; \mu = \ln(1.07) - \frac{0.0343411}{2}$$

$$= 0.05049$$

(ii) (a)

$\ln(1 + i_t) \sim N(0.05049, 0.0343411)$

$\Rightarrow \;\; \ln(S_{15}) \sim N(15 \times 0.05049, \; 15 \times 0.0343411)$

$= N(0.75735, \; 0.5151165)$

(b) $\Pr(S_{15} > 2.5)$

$\Rightarrow \;\; \Pr((\ln S_{15}) > \ln 2.5 = 0.916291)$

$$= \quad (\Pr(z) > \frac{0.916291 - 0.75735}{\sqrt{0.5151165}} = 0.221454)$$

where $z \sim N(0, 1)$

i.e. $\Pr(z > 0.221454)$

$= \quad 1 - \Phi(0.221454)$

$= \quad 1 - 0.588$

$= \quad 0.412$

12 (i) Work in £ millions

Let Discounted Payback Period from 1.1.2000 be n.

Then,

$$-18-10v^{½}-5v$$

$$-5\times 9\,(v^2+1.05v^3+\ldots+(1.05)^{n-3}\,v^{n-1})$$

$$+5\times 12.1\,(v^3+1.05v^4+\ldots+(1.05)^{n-3}\,v^{n})=0 \quad \text{at } 9\%$$

Hence,

$$18+9.5783+4.5872=(60.5v^3-45v^2)\left(\frac{1-\left(\frac{1.05}{1.09}\right)^{n-2}}{1-\frac{1.05}{1.09}}\right)$$

and $\text{RHS}=8.8415\times 27.25\times\left(1-\left(\frac{1.05}{1.09}\right)^{n-2}\right)$

Hence, $\frac{32.1655}{8.8415\times 27.25}=1-\left(\frac{1.05}{1.09}\right)^{n-2}$

$$\Rightarrow \left(\frac{1.05}{1.09}\right)^{n-2}=0.8665$$

$$\Rightarrow (n-2)\ \log\left(\frac{1.05}{1.09}\right)=\log 0.8665$$

$$\Rightarrow n-2=\frac{-0.1433}{-0.0374}=3.833$$

$$\Rightarrow n=5.833$$

But sales are only made at the end of each calendar year.

$$\Rightarrow \text{DPP}=6 \text{ years}$$

(ii) The DPP would be shorter using an effective rate of interest less than 9% p.a. This is because the income (in the form of car sales) does not commence until a few years have elapsed whereas the bulk of the outgo occurs in the early years. The effect of discounting means that using a lower rate of interest has a greater effect on the value of the income than on the value of the outgo (although both values increase). Hence the DPP becomes shorter.

13 (i) Loan $= 200(Da)_{\overline{20|}} + 2{,}000\, a_{\overline{20|}}$ at 9%

$$= 200\left(\frac{20 - a_{\overline{20|}}}{0.09}\right) + 2{,}000\, a_{\overline{20|}}$$

$$= 200\left(\frac{20 - 9.1285}{0.09}\right) + 2{,}000 \times 9.1285$$

$$= 42{,}415.89$$

(ii) Loan outstanding after 7th payment

$$= 200(Da)_{\overline{13|}} + 2000a_{\overline{13|}} \quad \text{at } 9\%$$

$$= 200\left(\frac{13 - 7.4869}{0.09}\right) + 2{,}000 \times 7.4869$$

$$= 27{,}225.13$$

Hence,

Year	*Loan o/s at start*	*Payment*	*Interest element*	*Capital element*
8	27225.13	4600	2450.26	2149.74
9	25075.39	4400	2256.79	2143.21

(iii) Loan o/s after 9th payment

$$= 25075.39 - 2143.21 = 22932.18$$

Let 10th payment be $X + 2200$

Then

$$22932.18 = 200(Da)_{\overline{11|}} + Xa_{\overline{11|}} \quad \text{at } 7\%$$

$$= 200\left(\frac{11 - a_{\overline{11|}}}{0.07}\right) + Xa_{\overline{11|}}$$

$$= 200\left(\frac{11 - 7.4987}{0.07}\right) + 7.4987X$$

$$= 10003.71 + 7.4987X$$

$\Rightarrow X = 1724.09$

$\Rightarrow$ 10th payment $= 1724.09 + 2200$

$= £3{,}924.09$

14 (i) Let $(Va)_{\overline{n}|} = v + 4v^2 + 9v^3 + \ldots + n^2v^n$

Then $v.(Va)_{\overline{n}|} = v^2 + 4v^3 + \ldots + n^2v^{n+1}$

$\Rightarrow (1-v)\ (Va)_{\overline{n}|} = v + 3v^2 + 5v^3 + \ldots + (2n-1)\ v^n - n^2v^{n+1}$

$= 2v + 4v^2 + 6v^3 + \ldots + 2nv^n - a_{\overline{n}|} - n^2\ v^{n+1}$

$$\Rightarrow (Va)_{\overline{n}|} = \frac{2(Ia)_{\overline{n}|} - a_{\overline{n}|} - n^2v^{n+1}}{1-v}$$

(ii) (a) Work in £000's

PV of liabilities $= 95a_{\overline{20}|} + 5(Ia)_{\overline{20}|}$ at 7%

$$= 95 \times 10.5940 + 5 \times \frac{\ddot{a}_{\overline{20}|} - 20v^{20}}{i}$$

$$= 1006.43 + 5 \times \left(\frac{11.3356 - 20 \times 0.25842}{0.07}\right)$$

$= 1446.94$

Numerator of duration

$$= 95(Ia)_{\overline{20}|} + 5(1.v + 2.2v^2 + \ldots + 20.20v^{20})$$

$$= 95 \times 88.1029 + \frac{5}{1-v}\left\{2(Ia)_{\overline{20}|} - a_{\overline{20}|} - 400v^{21}\right\}$$

$$= 8369.78 + \frac{5}{0.06542}\{2 \times 88.1029 - 10.5940 - 400 \times 0.24151\}$$

$= 8369.78 + 5274.21 = 13{,}643.99$

Hence, duration of liabilities $= \dfrac{13643.99}{1446.94} = 9.43$ years

(b) Let nominal amount of 25-year stock be A
and nominal amount of 12-year stock be B.

Then PV of assets =

$$Av^{25} + B \times 0.08 \times a_{\overline{12}|} + Bv^{12}$$

$$(= \text{PV of liabilities} = 1446.94) \qquad (1)$$

and duration of assets =

$$(25Av^{25} + B \times 0.08 \times (Ia)_{12} + 12\ Bv^{12})\ /\ 1446.94$$

$$(= \text{duration of liabilities} = \frac{13643.99}{1446.94}) \quad (2)$$

From (1),

$$1446.94 = A \times 0.18425 + B \times 0.08 \times 7.9427 + B \times 0.44401$$

$$\Rightarrow \quad 1446.94 = 0.18425A + 1.07943B \qquad (3)$$

From (2),

$$13643.99 = 25 \times 0.18425 \times A + \left[0.08 \times \left(\frac{1.07 \times 7.9427 - 12 \times 0.44401}{0.07}\right) + 12 \times 0.44401\right] \times B$$

$$\Rightarrow \quad 13643.99 = 4.60625A + (3.62351 + 5.32812)\ B$$

$$= 4.60625A + 8.95163B \qquad (4)$$

Now, $\dfrac{4.60625}{0.18425} \times (3)$

$$\Rightarrow \quad 36173.50 = 4.60625A + 26.98575B$$

$$\Rightarrow \quad 22529.51 = 18.03412B$$

$$\Rightarrow \quad B = 1249.27$$

$$A = 534.27$$

Hence, amount invested in each security is:

Stock A: $534.27v^{25} = 98.44$ (= £98,440)

Stock B: $1249.27 \times 0.08 \times a_{\overline{12}|} + 1249.27v^{12}$

$= 1249.27 \times 1.07943 = 1348.50$ (= £1,348,500)