EXAMINATIONS

April 2004

Subject 102 — Financial Mathematics

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

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Chairman of the Board of Examiners

22 June 2004

EXAMINERS' COMMENT

Candidates appeared to be less well prepared than in previous recent diets. In particular, many candidates lost marks on parts of questions which were essentially standard bookwork (for example question 3(ii)). In 'show that' types of questions (such as question 11 part (i)) candidates are required to show detailed steps in deriving the result required in order to obtain full marks.

Please note that differing answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this.

However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

1 Work in 000's.

Let total accumulated investment = X and i_y = investment return for year y

$$E(X) = E\left[4(1+i_{2004})(1+i_{2005})(1+i_{2006}) + 4(1+i_{2005})(1+i_{2006}) + 4(1+i_{2006}) + 105 + 4\right]$$

Due to independence:

$$E(X) = 4\left[E(1+i_{2004})E(1+i_{2005})E(1+i_{2006}) + E(1+i_{2005})E(1+i_{2006}) + E(1+i_{2006})\right] + 109$$

$= 4[1.055 \times 1.06 \times 1.045 + 1.06 \times 1.045 + 1.045] + 109$

= 122.29

Many candidates lost some credit by failing to provide a derivation of the requisite formulae, as asked for in the question. A common mistake was to omit the last coupon payment and/or the redemption proceeds.

2 Accumulation at t = 15

$$= \int_0^8 50 \exp\left\{\int_t^8 0.05\,ds + \int_8^{15}\left(0.04 + 0.0004s^2\right)ds\right\}dt$$

$$= \int_0^8 50 \exp\left\{0.05(8-t) + \left[0.04s + \frac{0.0004s^3}{3}\right]_8^{15}\right\}dt$$

$$= \int_0^8 50 \exp\{0.05(8-t) + 1.05 - 0.388267\}\,dt$$

$$= \int_0^8 50 \exp(1.061733 - 0.05t)\,dt$$

$$= 144.568870 \int_0^8 e^{-0.05t}dt$$

$$= \frac{144.568870}{0.05}\left[1 - e^{-0.40}\right]$$

= 953.23

OR

PV of cashflow at t = 0:

$$\int_0^8 \rho(t)\, \nu(t)\, dt$$

where $\rho(t) = 50, \nu(t) = e^{-\int_0^t 0.05dt} = e^{-0.05t}$

Then PV $= \int_0^8 50e^{-0.05t}dt = -\frac{50}{0.05}\left[e^{-0.05t}\right]_0^8 = \frac{50}{0.05}\left[1-e^{-0.40}\right]$

= 329.67995

and accumulating to t = 15:

$$329.67995 \exp\left(\int_0^8 0.05ds + \int_8^{15}\left(0.04+0.0004s^2\right)ds\right)$$

$$= 329.67995 \exp\left[0.40+1.05-0.388267\right]$$

$$329.67995 \times e^{1.061733}$$

= 953.23

This question was well answered. There are several ways in which this question might be answered – two alternatives are given above. Candidates received credit for any appropriate method.

3 (i) TWRR is i such that

$$\left(1+i\right)^2 = \frac{137}{120} \times \frac{173}{157} \times \frac{205}{221} = 1.166937$$

$\Rightarrow i = 8.025\%$ p.a.

(ii) The strengths of the TWRR are that distortions due to the size and timings of the investment of new money are eliminated when comparing investment managers over the same period. This is not the case for MWRR.

The weakness of the TWRR is that more information is required (i.e. value of fund each time new money is received).

Some candidates failed to provide the annual effective TWRR in part (i). Many candidates failed to gain full marks for part (ii), even though this is straightforward bookwork.

4 Let i_1 = one-year spot rate, and

i_2 = two-year spot rate

$$A = \frac{1}{1+i_1},\ \ B = \frac{1}{(1+i_2)^2}$$

For the 2-year fixed interest stock we have:

$$105.40 = 8A + (8+98)B\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

From the 2-year par yield details we have

$$100 = 4.15A + (4.15+100)B\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Hence, $\frac{8}{4.15} * (2) - (1)$

$\Rightarrow 87.37108 = 94.77108B$

$\Rightarrow B = 0.921917$

and $(2) \Rightarrow 100 = 4.15A + 104.15 \times 0.921917$

$\Rightarrow A = 0.959601$

$\Rightarrow i_1 = 4.2100\%$

$B = 0.921917 \Rightarrow i_2 = 4.1488\%$

This question was badly answered. Many candidates were unable to obtain equations (1) and (2) correctly, or tried to work with the redemption yield.

5 (i) Redington's first condition states that the present value of the assets should equal the present value of the liabilities:

p.v. of assets $= 7.404v^2 + 31.834v^{25}$ at 7%
$= 7.404 * 0.87344 + 31.834 * 0.18425$
= 6.4669 + 5.8654
=12.3323

p.v. of liabilities $= 10v^{10} + 20v^{15}$ at 7%
$= 10 * 0.50835 + 20 * 0.36245$

= 5.0835 + 7.2490
= 12.3325

Allowing for rounding, Redington's first condition applies.

Redington's second condition states that the DMT (discounted mean term) of the assets should equal the DMT of the liabilities.

DMT of assets $= (7.404 * 2 * v^2 + 31.834 * 25 * v^{25}) / 12.3323$ at 7%
= (6.4669*2 + 5.8654*25)/12.3323
= 159.569/12.3323
=12.939

DMT of liabilities $= (10 * 10 * V^{10} + 20 * 15 * V^{15})/12.3325$ at 7%
$= (5.0835 * 10 + 7.249 * 15) / 12.3325$
=159.570/12.3325
=12.939

Allowing for rounding, Redington's 2nd condition applies.

(ii) Profit $= 7.404v^2 + 31.834v^{25} - 10v^{10} - 20v^{15}$ at 7.5%
= 6.407 + 5.220 – 4.852 – 6.759
= 0.016 i.e. a profit of £16,000

(iii) It can be seen that the spread of the assets is greater than the spread of the liabilities. This will mean that Redington's third condition for immunization is also satisfied, and that therefore a profit will occur if there is a small change in the rate of interest. Hence we would have anticipated a profit in (ii).

Parts (i) and (ii) were very well answered, but candidates are asked not to use abbreviations without defining what they mean when stating Redington's conditions. Part (iii) was poorly answered.

6 (i) The "no arbitrage" assumption means that it is assumed that the investor is unable to make a risk-free trading profit.

(ii) Present value of dividends, *I*, is (in pence):

$$I = 20\left[1.01\, e^{\frac{-0.05}{4}} + (1.01)^2 e^{\frac{-0.05}{2}} + \ldots + (1.01)^8 e^{-0.10}\right]$$

$$+20(1.01)^8 e^{-0.10}\left[1.015 e^{\frac{-0.05}{4}} + (1.015)^2 e^{\frac{-0.05}{2}} + (1.015)^3 e^{-\frac{3}{4}x0.05} + (1.015)^4 e^{-0.05}\right]$$

$$= 20 \times 1.01 e^{\frac{-0.05}{4}} \left[\frac{1 - \left(1.01 e^{\frac{-0.05}{4}}\right)^8}{1 - \left(1.01 e^{\frac{-0.05}{4}}\right)} \right]$$

$$+ 20 \times (1.01)^8 e^{-0.1125} \times 1.015 \left[\frac{1 - \left(1.015 e^{\frac{-0.05}{4}}\right)^4}{1 - \left(1.015 e^{\frac{-0.05}{4}}\right)} \right]$$

= 158.177 + 78.854

= 237.031

So the forward price is:

$$(4.50 - 2.37031) e^{0.15} = 2.474$$

i.e. £2.47

Many candidates used a rate of interest rather than a force of interest of 5% per annum

7 (i) In '000's

Time	*Cashflow*	*Description*
1 June 2000	–94.000	Purchase price
1 December 2000	1.530	Coupon
1 June 2001	1.605	Coupon
1 December 2001	1.665	Coupon
1 June 2002	1.695	Coupon
	+113.000	Redemption money

(ii) (a) CGT due is :

$$0.35 \times (113 - 94 \times \tfrac{118}{102}) = 1.489$$

(b) Yield is such that

$$94 = (1.530v^{\frac{1}{2}} + 1.605v^{1} + 1.665v^{1\frac{1}{2}} + 1.695v^{2}) \times 0.75$$

$$+ (113 - 1.489)\, v^{2}$$

Roughly:

$$94 = 4.871v + 111.511\, v^{2}$$

$$\Rightarrow v \simeq \frac{-4.871 + \sqrt{4.871^2 + 4 \times 94 \times 111.511}}{2 \times 111.511}$$

$$\Rightarrow i \simeq 11.5\%$$

$$\left.\begin{array}{r} \text{At } 12\%, RHS \text{ is } 93.122 \\ 11\%, RHS \text{ is } 94.778 \end{array}\right\} \rightarrow i = 11.47\% \text{ p.a.}$$

In part (i) some candidates used a nominal dividend of 3% per half year rather than 1.5%. Also, candidates were expected to provide the dates of the cashflows as well as the amounts for full marks. In part (ii)(a) the indexing for capital gains tax purposes was often done incorrectly.

8 (i) Let $i\,\%$ = money rate of return

Then

$$9900 = 800\ a^{(2)}_{\overline{25|}} + 11{,}000v^{25}$$

$$-200v^{\frac{1}{4}}a_{\overline{25|}} - 0.3\left(11{,}000 - 9{,}900\right)v^{25\frac{1}{4}}$$

$$= 800a^{(2)}_{\overline{25|}} + 11000v^{25} - 200v^{\frac{1}{4}}a_{\overline{25|}} - 330v^{25\frac{1}{4}}$$

1st approximation: $i = \dfrac{600 + \frac{11000-330-990}{25}}{9900}$

= 6.4%

Try 7%: RHS = 800 × 1.017204 × 11.6536

+ 11000 × 0.18425 – 200 $v^{\frac{1}{4}}$ × 11.6536

$-330 \times v^{25\frac{1}{4}}$

$= 9158.61$

Try 6%: RHS = $800 \times 1.014782 \times 12.7834$

$+ 11000 \times 0.23300 - 200v^{\frac{1}{4}} \times 12.7834 - 330\, v^{25\frac{1}{4}}$

$= 10345.41$

$$\Rightarrow i = 0.07 - \frac{9900 - 9158.61}{10345.41 - 9158.61} \times 0.01$$

$= 0.0638$ i.e. 6.38% p.a.

Hence, if $r\%$ = real return

$$1 + r = \frac{1.0638}{1.03} = 1.0328$$

$\Rightarrow$ real return = 3.28% p.a.

(ii) If tax were collected 2 months later (i.e. 1 June rather than 1 April) then investor is deferring paying the tax. Hence, real return would be higher than 3.28% p.a.

Many candidates failed to deal with the occurrence and timing of payments of income tax and capital gains tax correctly.

9 (i) Consider t where:

$120000 = 14000\, a_{\overline{t|}}^{(2)}$ at 7%

$$= 14000 \times \frac{i}{i^{(2)}} \times a_{\overline{t|}}$$

$= 14000 \times 1.017204\, a_{\overline{t|}}$

$\Rightarrow \quad a_{\overline{t|}} = 8.426460$

$$\Rightarrow \quad \frac{1 - \left(\frac{1}{1.07}\right)^t}{0.07} = 8.426460$$

$$\Rightarrow \quad \frac{1}{(1.07)^t} = 0.410148$$

$$\Rightarrow \quad t \log\left(\frac{1}{1.07}\right) = \log 0.410148$$

$$\Rightarrow \quad t = 13.17255$$

$\Rightarrow$ DPP = 13.5 years (as annuity paid every 6 months)

(ii) PV of profit

$$= 14000\left(\underset{7\%}{a^{(2)}_{\overline{13.5|}}} + \underset{7\%}{v^{13.5}}\ \underset{5\%}{a^{(2)}_{\overline{11.5|}}}\right) - 120,000$$

Where $\underset{7\%}{a^{(2)}_{\overline{13.5|}} = \dfrac{1-v^{13.5}}{i^{(2)}}} = \dfrac{1-v^{13.5}}{0.068816} = 8.7020$

and $\underset{5\%}{a^{(2)}_{\overline{11.5|}}} = \dfrac{1-v^{11.5}}{i^{(2)}} = \dfrac{1-v^{11.5}}{0.049390} = 8.6943$

Hence PV of profit

$$= 14,000(8.7020 + 0.401161 \times 8.6943) - 120000$$

$$= 50,657.40$$

Hence, profit after 25 years

$$= 50657.40 \times (1.07)^{13.5}(1.05)^{11.5}$$

$$= 221,309.90$$

Several candidates missed the fact that the DPP must occur at the end of a six-month period (i.e. 13.5 years, not 13.17255 years)

10 (i) We know $(1+i_t) \sim \text{LogNormal}(\mu, \sigma^2)$ where:

$$E(1+i_t) = 1 + j = 1.04$$

$$\text{Var}(1+i_t) = \text{Var}(i_t) = s^2 = 0.02$$

Hence

$$1.04 = \exp\left(\mu + \frac{\sigma^2}{2}\right)$$

$$0.02 = \exp(2\mu + \sigma^2) \cdot \left(\exp(\sigma^2) - 1\right)$$

$$\Rightarrow e^{\sigma^2} - 1 = \frac{0.02}{(1.04)^2}$$

$$\Rightarrow \sigma^2 = 0.018322$$

$$\Rightarrow \mu = \text{Ln}(1.04) - \frac{0.018322}{2} = 0.0300597.$$

Let X be the amount to be invested at time 0.

$$S_5 \sim \text{Log Normal}\left(5\mu, 5\sigma^2\right)$$

We want $P_r\left(X.S_5 \ge 5000\right) = 0.99$

i.e. $P_r\left(S_5 \ge \frac{5000}{X}\right) = 0.99$

So $1 - \Phi\left(\frac{Log\left(\frac{5000}{x}\right) - 5\mu}{\sqrt{5\sigma^2}}\right) = 0.99$

$$\Rightarrow \Phi\left(\frac{Log\left(\frac{5000}{x}\right) - 5\mu}{\sqrt{5\sigma^2}}\right) = 0.01$$

$$\Rightarrow \frac{\text{Log}\left(\frac{5000}{x}\right) - 5\mu}{\sqrt{5\sigma^2}} = -2.3263$$

$$\text{So } \operatorname{Log}\left(\frac{5000}{X}\right) = -2.3263 \times \sqrt{5x0.018322} + 5 \times 0.0300597$$

$$= -0.55381$$

$$\Rightarrow \frac{5000}{X} = 0.574758$$

$$\Rightarrow X = 8,699.31$$

(ii) At first glance it seems odd that the investor needs to invest substantially more than £5,000 to have a 99% chance of having £5,000 in 5 years time.

However, the odd result is explained by the fact that the variance of the interest rate is so high relative to its mean. There is therefore a significant risk that the investment will decrease in value over the next 5 years.

In general well answered, although a common mistake was to omit the negative sign before 2.3263.

11 (i) Loan outstanding on 1/9/98 = present value of annuity payments

$$= 1000\left(v^{\frac{10}{12}} + 1.05v^{\frac{14}{12}} + (1.05)^2 v^{\frac{18}{12}} + \ldots\ldots + (1.05)^{14} v^{\frac{66}{12}} \right)$$

$$= 1000V^{\frac{10}{12}} \left[\frac{1-[v^{\frac{4}{12}}(1.05)]^{15}}{[1-v^{\frac{4}{12}}(1.05)]} \right]$$

= 17691.77 (=17692 to nearest £)

(ii) Loan o/s on 30/6/99 = 17691.77 × $(1.06)^{\frac{10}{12}}$

= 18,572.04

⇒ interest in 1st instalment = 18572.04 – 17691.77

= 880.27

⇒ Capital repaid = 1000 – 880.27 = 119.73

(iii) Capital o/s after 6 repayments = Pv of payments at 1/3/2001

$$= 1000 \times (1.05)^6 \left(v^{\frac{4}{12}} + 1.05 v^{\frac{8}{12}} + \ldots + (1.05)^8 v^{\frac{36}{12}} \right)$$

$$= 1000 (1.05)^6 v^{\frac{4}{12}} \left[\frac{1 - \left[v^{\frac{4}{12}} (1.05) \right]^9}{\left[1 - v^{\frac{4}{12}} (1.05) \right]} \right]$$

= 13341.57

$\Rightarrow$ Interest in 7th payment

= 13341.57 × $\left((1.06)^{\frac{4}{12}} - 1 \right)$

= 261.67

and loan repaid in 7th instalment

$$= (1.05)^6 \times 1000 - 261.67$$

= 1078.43

In general, this question was well answered, though candidates did not always provide sufficient working in their answer to part (i) in showing that the amount of the loan was £17,692, as asked in the question, to gain full credit.