REPORT OF THE BOARD OF EXAMINERS

September 2003

Subject 102 — Financial Mathematics

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

J Curtis
Chairman of the Board of Examiners

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Many candidates had difficulty with questions requiring descriptions of concepts, definitions and explanations in their own words. It is important that candidates understand the subject well enough to express important topics and issues in their own words as well as in mathematical language.

Please note that differing answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this.

However, candidates maybe penalised where excessive rounding has been used or where insufficient working is shown.

1 $$\left(1-\frac{90}{365}\times .06\right)=(1+i)^{-\frac{90}{365}}$$

$$\therefore\ i=\left(1-\frac{90}{365}\times 0.06\right)^{-\frac{365}{90}}-1=6.2313\%$$

2 $$(1+i)^3=\frac{80}{120}\times\frac{200}{110}\times\frac{200}{210}=1.15440$$

$\therefore\ i = 4.9024\%$ per annum

3 (a) Bond issued and payments made by a government.
Coupon and redemption payments linked to an index which reflects inflation (typically lagged inflation).
Payments fixed in relation to this index, therefore bond provides inflation protection.

Credit was given for other relevant points.

(b) Most bonds have payments linked to inflation with a time lag. There is therefore a gap between the reference date for inflation used to calculate the payment from a bond and the date on which a payment is received. If inflation is higher than anticipated between those two dates, the real value of the payments will be reduced.

4 Present value of dividends is:

$$0.3v_{4\%}^{\frac{1}{2}} + 0.3v_{4.5\%} = 0.3 \times (0.980581 + 0.956938)$$

$$= 0.581256$$

Hence forward price, F, is:

$$F = (6 - 0.581256) \times (1.045) = 5.66259$$

5 $(1 + i)$ is lognormally distributed with mean 1.0015 and variance 9×10^{-6}.

$$\Rightarrow 1.0015 = \exp\left[\mu + \frac{\sigma^2}{2}\right]$$

$$9 \times 10^{-6} = \exp[2\mu + \sigma^2][\exp(\sigma^2) - 1]$$

$$\Rightarrow \frac{9 \times 10^{-6}}{1.0015^2} = \exp(\sigma^2) - 1$$

$$\Rightarrow \sigma^2 = 8.9730 \times 10^{-6}$$

$$\Rightarrow \mu = \ln 1.0015 - \frac{8.9730 \times 10^{-6}}{2} = 0.0014944$$

$$\ln(1 + i) \sim N(0.0014944, 8.9730 \times 10^{-6})$$

$$\Pr\left(\frac{\ln(1+i) - 0.0014944}{\sqrt{8.9730 \times 10^{-6}}} \le -1.28155\right) = 0.1$$

$$\Pr(\ln(1 + i) \le -0.0023445) = 0.1$$

$$\therefore \Pr(i \le -0.0023417) = 0.1$$

$$\Rightarrow j = -0.23417\%$$

6 $$125 = \frac{5v^{\frac{1}{4}}}{1.015^{\frac{1}{4}}} + \frac{5 \times 1.04 \times v^{1\frac{1}{4}}}{1.015^{1\frac{1}{4}}} + \frac{5 \times 1.04^2 \times v^{2\frac{1}{4}}}{1.015^{2\frac{1}{4}}} + \ldots$$

$$= \frac{5v^{1/4}}{1.015^{1/4}}\left(1+\frac{1.04v}{1.015}+\frac{1.04^2v^2}{1.015^2}+\ldots\right)$$

$$= 4.98142v^{1/4}\left(\frac{1}{1-1.02463v}\right)$$

$$= \frac{4.98142(1+i)^{3/4}}{1+i-1.02463} = \frac{4.98142(1+i)^{3/4}}{i-0.02463}$$

Try i = 6% RHS = 147.129

Try i = 7% RHS = 115.511

Interpolation gives:

$$0.07-\left(\frac{125-115.511}{147.129-115.511}\right)\times 0.01 = 0.06700$$

i = 6.70% p.a. effective

7 (i) $100{,}000 = 12\times Xa_{\overline{25|}}^{(12)}$ @ 5% p.a.

$$= 12\times X\frac{i}{i^{(12)}}\times a_{\overline{25|}}$$

$$\frac{i}{i^{(12)}} = \frac{0.05}{.048889} = 1.022715$$

$a_{\overline{25|}} = 14.0939$

$$\therefore X = \frac{100{,}000}{12\times 1.022715\times 14.0939} = 578.14 \text{ per month}$$

(ii) (a) $12 \times 578.14 \times 1.022715 \times a_{\overline{13|}} = C_{12}$

$a_{\overline{13|}} = 9.3936$

$\therefore C_{12} = 66{,}650.12$

(b) Interest portion is $\frac{i^{(12)}}{12}\times 66{,}650.12$

$$= \frac{.048889}{12} \times 66{,}650.12 = 271.54$$

Capital portion is 578.14 – 271.54

$= 306.60$

8 (i)

- generally secure when issued by governments with good rating
- low volatility of return relative to other investments
- can provide guaranteed return over period to maturity
- can be risky in "real terms"

(ii) Value of that part of the bond redeemed after five years (per £100 nominal)

$$0.7 \times 4 \times a_{\overline{5|}} + 50v^5 \text{ @ 6\% p.a.}$$

$$0.7 \times 4 \times 4.21236 + 50 \times 0.74726 = 49.1576$$

Value of that part of the bond redeemed after 10 years:

$$0.7 \times 4 \times a_{\overline{10|}} + 50v^{10} \text{ @ 6\% p.a.}$$

$= 0.7 \times 4 \times 7.36009 + 50 \times 0.558395 = 48.5280$

Value of bond is:

$49.1576 + 48.5280 = £97.6856$ per £100 nominal

9 (i) $S_n = (1 + i_1)(1 + i_2) \ldots (1 + i_n)$

$E(i_1) = E(i_2) = \ldots = E(i_n) = j$

$\text{Var}(i_1) = \text{Var}(i_2) = \text{Var}(i_n) = s^2$

$E(S_n) = E[(1 + i_1)(1 + i_2)(1 + i_3) \ldots (1 + i_n)]$

due to independence

$$= E(1 + i_1)\, E(1 + i_2)\, E(1 + i_3) \ldots E(1 + i_n)$$

$$= (1 + j)^n$$

$\text{Var}(S_n) = E[S_n^2] - (E[S_n])^2$

$E(S_n^2) = E[(1 + i_1)^2 (1 + i_2)^2 \ldots (1 + i_n)^2]$

due to independence

$= E[(1 + i_1)^2]\, E[(1 + i_2)^2] \ldots E[(1 + i_n)^2]$

$= \{1 + 2j + E(i^2)\}^n$

$E(i^2) = s^2 + j^2$

$\therefore E(S_n^2) = (1 + 2j + j^2 + s^2)^n$

$\text{Var}(S_n) = (1 + 2j + j^2 + s^2)^n - (1 + j)^{2n}$

(ii) (a) $1.06^8 = 1.59385$

(b) variance is: $(1 + 2 \times 0.06 + 0.06^2 + 0.08^2)^8 - (1.06)^{16}$

$2.65844 - 2.54035 = 0.11809$

standard deviation is 0.34364

10 (i) (a) $e^{\int_0^8 0.04 + 0.001t\,dt}$

$$= e^{\left[0.04t + \frac{0.001t^2}{2}\right]_0^8}$$

$$= \exp\left[0.04 \times 8 + \frac{0.001 \times 64}{2}\right]$$

$$= \exp(0.352) = 1.421909$$

(b) $e^{\int_0^9 0.04 + 0.001t\,dt}$

$$= e^{\left[0.04t + \frac{0.001t^2}{2}\right]_0^9}$$

$$= \exp\left[0.04 \times 9 + \frac{0.001 \times 81}{2}\right]$$

$= \exp(0.4005) = 1.492571$

(c) $\dfrac{1.492571}{1.421909} = 1.049695$

(ii) (a) $(1 + i)^8 = 1.421909 \quad \therefore\ i = 4.4982\%$ p.a. effective

(b) $(1 + i)^9 = 1.492571 \quad \therefore\ i = 4.5505\%$ p.a. effective

(c) 4.9695% p.a. effective

11 (i) (a) Duration of liabilities:

$$\frac{\sum tC_t v^t}{\sum C_t v^t} = \frac{\sum_{t=1}^{20} tv^t + \sum_{t=21}^{40} 0.5tv^t}{\sum_{t=1}^{20} v^t + \sum_{t=21}^{40} 0.5v^t}$$

$$= \frac{(Ia)_{\overline{20|}} + 0.5(Ia)_{\overline{40|}} - 0.5(Ia)_{\overline{20|}}}{a_{\overline{20|}} + 0.5v^{20}a_{\overline{20|}}}$$

$$= \frac{(Ia)_{\overline{40|}} + (Ia)_{\overline{20|}}}{a_{\overline{40|}} + a_{\overline{20|}}} = 11.30266 \text{ years}$$

$$(Ia)_{\overline{n|}} = \frac{\ddot{a}_{\overline{n|}} - nv^n}{i} \qquad (Ia)_{\overline{40|}} = 201.0031 \qquad (Ia)_{\overline{20|}} = 98.7004$$

$$a_{\overline{40|}} = 15.0463 \qquad a_{\overline{20|}} = 11.4699 \text{ at 6\% p.a. effective}$$

(b) Duration of assets $= \dfrac{\sum_{t=1}^{15} 10tv^t + 15 \times 100 \times v^{15}}{\sum_{t=1}^{15} 10v^t + 100v^{15}}$

$$= \frac{10(Ia)_{\overline{15|}} + 1500v^{15}}{10a_{\overline{15|}} + 100v^{15}}$$

$(Ia)_{\overline{15|}} = 67.2668$

$a_{\overline{15|}} = 9.7122$ @ 6% p.a. effective

$v^{15} = 0.417265$

$= 9.3524$ years

(ii) (a) For immunisation, the duration of the assets must equal the duration of the liabilities. It is not possible to purchase assets of a long enough duration.

(b) A loss would be made if interest rates decreased. The present value of both assets and liabilities would rise but because the duration/volatility of the liabilities is greater, the value of the liabilities would rise by more than that of the assets.

12 (i) (a) Bond yields are determined by investor's expectations of future short-term interest rates so that returns from bonds of different terms to maturity are equal to expected returns from a series of short-term investments.

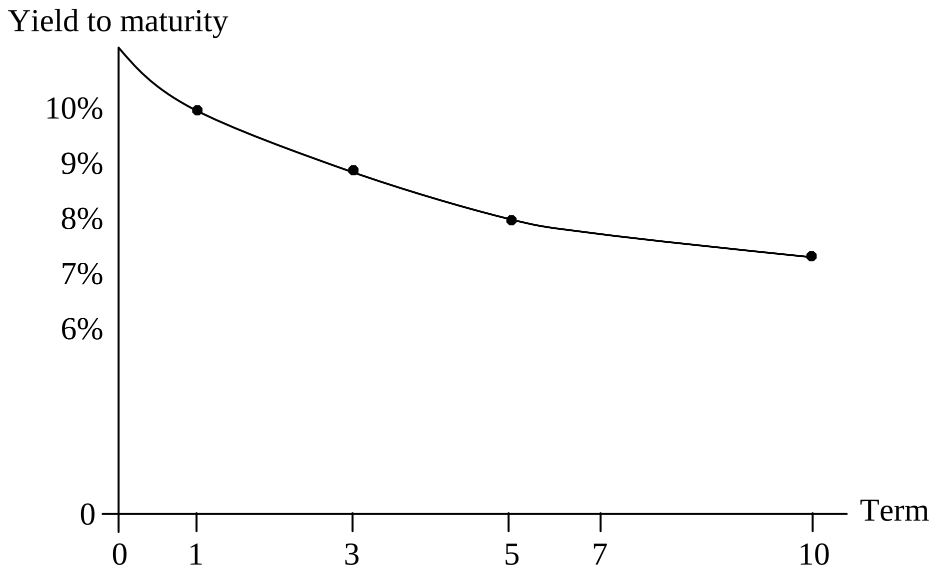
(b) The expectations yield curve is modified under liquidity performance to reflect a preference for shorter-term investments so that long-term investments provide a yield to maturity greater than that implied by expectations of future short-term interest rates.

The expectations yield curve is modified under market segmentation to reflect supply and demand conditions in particular segments of the market.

(ii) (a)

one year:	yield is 10%	$\therefore i_1 = 0.1$
three years:	$(1 + i_3)^3 = 1.1 \times 1.09 \times 1.08$	$\therefore i_3 = 0.08997$
five years:	$(1 + i_5)^5 = 1.1 \times 1.09 \times 1.08 \times 1.07^2$	$\therefore i_5 = 0.08194$
ten years:	$(1 + i_{10})^{10} = 1.1 \times 1.09 \times 1.08 \times 1.07^7$	$\therefore i_{10} = 0.07595$

(b),



(c) The gross redemption yields from coupon paying bonds are, in effect, weighted averages of the earlier zero coupon bond yields. Later terms to redemption yields will be affected by earlier higher zero coupon yields and hence will not decrease as quickly as the zero coupon bond yield curve.

13 (i) (a) The discounted payback period is the point at which net revenues from the project can be used to repay all loans necessary to finance the project outgoings accumulated with interest (or the point at which the net present value or accumulated profit of the project becomes positive for the first time).

(b) The payback period is the point at which total net revenues are greater than total net outgoings.

(ii) The discounted payback period simply shows when a project became profitable in present value terms, not how profitable it is. However, this information can be useful in the decision making process. The payback period ignores interest altogether and is therefore clearly an inferior criterion. It is also possible for the discounted payback period and the payback period to be before the end of the project but for the NPV to be negative.

(iii) Present value of costs (all at 1/1/2004)

Cost of making bid: 0.2

Running costs: $\bar{a}_{\overline{0.25|}} v^7 = v^7 \dfrac{1-e^{-0.09531\times 0.25}}{0.09531} = v^7 \dfrac{1-0.97645}{0.09531}$

@ i = 10%, δ = 0.09531 $\qquad = 0.24709 \times 0.51316$
$\qquad = 0.126797$

Stadium building costs: $\bar{a}_{\overline{5|}} v^2$

$\bar{a}_{\overline{5|}} = \dfrac{i}{\delta} a_{\overline{5|}}$ $\qquad a_{\overline{5|}} = 3.790787$

$i/\delta = 1.049206$

$v^2 = 0.826446$

$\therefore$ stadium building costs = 3.287037

Present value of all revenue (except sale of stadium) at 1/1/2004

Sale of television right = 0.3 × running costs = 0.3 × 0.126797

= 0.038039

Other revenue:

2004 – 2011: $(1+i)^{½}\ 0.1 \times (Ia)_{\overline{8|}}$

$$(Ia)_{\overline{8|}} = \frac{\ddot{a}_{\overline{8|}} - 8v^8}{0.1}$$

$\ddot{a}_{\overline{8|}} = 1.1 \times 5.33493 = 5.86842$

$v^8 = 0.466507$

$$(Ia)_{\overline{8|}} = \frac{5.86842 - 8 \times 0.466507}{0.1} = 21.36364$$

$\therefore$ value is 1.04881 × 0.1 × 21.36364 = 2.240640

Revenue 2012 –2015 = $(1+i)^{½} \times v^9(0.8 + 0.6v + 0.4v^2 + 0.2v^3)$

$v^9 = 0.424098$
$v = 0.909091$
$v^2 = 0.826446$
$v^3 = 0.751315$

Revenue 2012 – 2015 = 1.04881 × 0.424098(0.8 + 0.6 × 0.909091
+ 0.4 × 0.826446 + 0.2 × 0.751315)

= 0.812333

Required equation of value is:

$$0.2 + 0.126797 + 3.287037 = 0.038039 + 2.240640 + 0.812333 + Xv^{11}$$

$\therefore Xv^{11} = 0.522822$

$v^{11} = 0.350494$

$\therefore X = 1.49167$

= £149.167m