EXAMINATIONS

September 2004

Subject 102 — Financial Mathematics

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

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Chairman of the Board of Examiners

23 November 2004

The questions requiring verbal reasoning (such as Q9(ii)and (iii)) ended not to be well answered with many candidates producing vague or ‘woolly’ words which did not demonstrate that they understood the relevant points.

Where candidates use non-standard notation or abbreviations in their answers, these should be defined where they are first introduced.

Please note that differing answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this.

However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

1 First investor achieved higher rate of return if:

$$\left(\frac{93.9}{91}\right)^2 > \frac{100}{93.9} \text{ or if } \frac{100}{93.9} < 1.064752$$

The inequality does not hold. Therefore second investor achieved a higher rate of return.

2 (i) $F_{t,1}$ is such that $1 + f_{t,1} = e^{F_{t,1}}$

$\therefore\ F_{t,1} = \ln 1.04 = 0.039221$

(ii) The instantaneous forward rate is the forward force of interest applying in the time $t \to t + \Delta t$ as $\Delta t \to 0$ (or, alternatively) is F_t where $F_t = \lim\limits_{r \to 0} F_{t,r}$ and where $F_{t,r}$ is the force of interest equivalent to the forward rate of interest applicable over the time period $t \to (t + r)$.

3 Duration $= \dfrac{10\int_0^{20} tv^t dt}{10\int_0^{20} v^t dt}$

$$= \frac{(\bar{I}\bar{a})_{\overline{20|}}}{\bar{a}_{\overline{20|}}} = \frac{\left(\bar{a}_{\overline{20|}} - 20v^{20}\right)\big/\delta}{\bar{a}_{\overline{20|}}}$$

$\bar{a}_{\overline{20|}} = \dfrac{i}{\delta} a_{\overline{20|}} = 1.019869 \times 13.5903 \qquad \delta = 0.039221 \qquad v^{20} = 0.45639$

$= 13.86033$

$$\therefore \text{ Duration } = \frac{(13.86033 - 20 \times 0.45639)/0.039221}{13.86033}$$

$= 8.70565$

$= 8.71$ years

4 Need first to calculate the repayment. Let annual rate of payment = X.

$$X\ddot{a}_{\overline{2}|}^{(12)} = 15{,}000 \qquad \ddot{a}_{\overline{2}|}^{(12)} = \frac{i}{d^{(12)}} a_{\overline{2}|}$$

$$i = 0.1236 \qquad d^{(12)} = 0.115974 \qquad v^2 = 0.792094$$

$$X\left(\frac{1-0.792094}{0.115974}\right) = 15{,}000$$

$\therefore\ X = 8367.29$

Total charge for credit $= 2 \times 8367.29 - 15{,}000$

$= 1734.58$

Flat rate of interest $= \dfrac{1734.58}{2\times 15{,}000} = 5.782\%$

5 (a) The investor is immunised if:

(1) The present value of the assets is equal to the present value of the liabilities.

(2) The volatilities of the asset and liability cash flow series are equal and

(3) The convexity of the asset cash flow series is greater than the convexity of the liability cash flow series.

(3) can also be expressed in terms of the spread of the times of payment around the discount mean term; (1), (2) or (3) can also be expressed in terms of appropriately defined notation.

(b) Duration $= (1 + i) \times$ volatility

(c) Duration $= \dfrac{D\Sigma tv^t}{D\Sigma v^t}$ where D is the coupon rate

$$= \frac{(Ia)_{\overline{\infty}|}}{a_{\overline{\infty}|}} = \frac{(1+i)a_{\overline{\infty}|}/i}{a_{\overline{\infty}|}} = \frac{(1+i)/i^2}{1/i} = (1+i)\frac{1}{i}$$

$$\text{Volatility} = \frac{\text{Duration}}{(1+i)} = \frac{1}{i}$$

6 (a) NPVs of both projects are equal where:

$$1.7v^8 = v^8 + 0.321v^9 + 0.229v^{10} + 0.245v^{11}$$

i.e. when:

$$0.7 = 0.321v + 0.229v^2 + 0.245v^3$$

Try $i = 8\%$

$$\text{RHS} = 0.321 \times 0.92593 + 0.229 \times 0.85734 + 0.245 \times 0.79383$$

$$= 0.68804$$

Try $i = 7\%$

$$\text{RHS} = 0.321 \times 0.93458 + 0.229 \times 0.87344 + 0.245 \times 0.81630$$

$$= 0.7$$

$\therefore\ i = 7\%$

(b) General reasoning:

B is "longer" project. Therefore NPV rises more quickly as i falls

$\therefore$ if $i^* < i'$ NPV will be higher than from project A.

Alternatively, could use illustrative calculation at any rate of interest $i^* < i'$ e.g. $i^* = 0$.

NPV of project B $= 1 + 0.321 + 0.229 + 0.245$
$= 1.795$

NPV of project A = 1.7

7 (i) Relevant points include:

- Shares are issued by commercial undertakings and other bodies
- Shareholders receive dividends paid out of profits after all other obligations have been met
- The return is made up of dividends received and any increase (or less any decrease) in the market price of the shares
- Higher risk than corporate bonds issued by the same company
- Lowest ranking form of finance issued by companies
- No legal obligation to pay dividends
- Generally low initial yield but dividends expected to grow over time
- No fixed redemption time

- Voting rights are attached
- Marketability varies according to size of company

Credit was given for other relevant points.

(ii) Let current price = P; next expected dividend = d

$$P = \frac{dv_{5\%}^{½}}{1.02^{½}} + \frac{d(1.03)v_{5\%}^{1½}}{1.02^{1½}} + \frac{d(1.03)^2 v_{5\%}^{2½}}{1.02^{2½}} + \ldots$$

$g.p.$ with $C.R.$ $\frac{v(1.03)}{1.02}$ first term $\frac{dv_{5\%}^{½}}{1.02^{½}}$

$$\therefore P = \frac{dv_{5\%}^{½}}{1.02^{½}}\left(\frac{1}{1-\frac{v(1.03)}{1.02}}\right)$$

$$\therefore\ P = d \times 0.966285\left(\frac{1}{1-0.96172}\right) = 25.24255d$$

$$\therefore \frac{d}{p} = 0.03962 \text{ or } 3.962\%$$

8 (a) $(1 + i_t) \sim LN(\mu, \sigma^2)$

$\ln(1 + i_t) \sim N(\mu, \sigma^2)$

$\ln[(1 + i_t)^{10}] = \ln(1 + i_t) + \ln(1 + i_t) + \ldots + \ln(1 + i_t) \sim N(10\mu, 10\sigma^2)$

$(1 + i)^{10} \sim LN(10\mu, 10\sigma^2)$

$$E(1 + i_t) = \exp\left(\mu + \frac{\sigma^2}{2}\right) = 1.06$$

$\mathrm{Var}(1 + i_t) = \exp(2\mu + \sigma^2)\,[\exp(\sigma^2) - 1] = 0.08^2$

$$\frac{0.08^2}{1.06^2} = \exp(\sigma^2) - 1 \qquad \therefore \sigma^2 = 0.0056798$$

$$\therefore \exp\left(\mu + \frac{0.0056798}{2}\right) = 1.06 \quad \therefore \mu = \ln 1.06 - \frac{0.0056798}{2}$$

$$= 0.055429$$

$\therefore \ln(S_{10}) \sim N(0.55429, 0.056798)$

$S_{10} \sim LN(0.55429, 0.056798)$

$$E(S_{10}) = \exp\left(0.55429 + \frac{0.056798}{2}\right) = 1.790848 = £1.7908\text{m}$$

These calculations can be done more simply in alternative ways in which case the above calculations will be needed for (b). If student just uses $(1 + j)^{10}$ where j = expected value of i, give two marks and allocate 6 marks for (b). Formula should be proved.

(b) Require $\Pr(S_{10}) < 1.61172$

$= \Pr(\ln S_{10}) < \ln 1.61172$ where

$\ln(S_{10}) \sim N(0.55429, 0.056798)$

$$= \Pr(Z) = \frac{\ln(S_{10}) - 0.55429}{\sqrt{0.056798}} < \frac{\ln 1.61172 - 0.55429}{\sqrt{0.056798}} = -0.32304$$

$= \Pr(Z) < 1 - \Phi(0.32304)$ where $Z \sim N(0, 1)$

$= 1 - 0.627 = 0.373$

9 (i) (a) One-year spot rate is i such that $(1 + i) = \frac{100}{97}$ $\therefore$ $i = 3.0928\%$

Two-year spot rate is i such that $(1 + i)^2 = \frac{100}{93}$ $\therefore$ $i = 3.6952\%$

Three-year spot rate is i such that $(1 + i)^3 = \frac{100}{88}$ $\therefore$ $i = 4.3532\%$

Four-year spot rate is i such that $(1 + i)^4 = \frac{100}{83}$ $\therefore$ $i = 4.7684\%$

(b) Price of coupon paying bond = P

$$P = 4\times\left(\frac{97}{100}+\frac{93}{100}+\frac{88}{100}+\frac{83}{100}\right)+100\times\frac{83}{100}$$

$\therefore\ P = 97.44$

gry is such that

$$97.44 = 4a_{\overline{4|}} + 100v^4$$

For $i = 4\%$ RHS = 100

$i = 5\%$ RHS = 96.454

Using interpolation $i = -\dfrac{97.44-96.454}{100-96.454}\times 0.01 + 0.05$

$= 4.72\%$

(ii) The coupon paying bond could be thought of as being made up of four zero coupon bonds redeemable at times 1, 2, 3, 4. The earlier ones obtain a lower rate of return than the four-year spot rate and the rate of return from the coupon paying bond is a weighted average of the four spot rates.

(iii) According to the liquidity preference theory short-term bonds should provide a lower rate of return than long-term bonds because investors value liquidity and the lower capital value risk. The yield curve therefore has an upward slope when expectations of future interest rates are constant — like the yield curve in (i).

10 (i) Monthly instalment is $\dfrac{X}{12}$ where:

$$X\ddot{a}^{(12)}_{\overline{25|}6\%} = 100{,}000 \qquad \frac{i}{d^{(12)}} = 1.032211;\ a_{\overline{25|}} = 12.7834$$

$$\therefore\ X = \frac{100{,}000}{1.032211\times 12.7834} = 7578.533 \text{ and } \frac{X}{12} = 631.544$$

(ii) After 2 years loan outstanding is:

$$X\ddot{a}^{(12)}_{\overline{23|}6\%} = 1.032211\times 12.3034\times 7578.533$$

$$= 96245.132$$

Interest required $= \dfrac{d^{(12)}}{12} \times 96245.132$

$= \dfrac{0.058128}{12} \times 96245.132 = 466.21$

Capital repaid $= 631.54 - 466.21 = 165.33$

(iii) (a) Loan outstanding is: $X\ddot{a}^{(12)}_{\overline{15}|6\%}$

$7578.533 \times 1.032211 \times 9.7122$

$= 75{,}975.094$

Revised instalment is $\dfrac{Y}{12}$ such that:

$Y\ddot{a}^{(12)}_{\overline{15}|} = 75{,}975.094 \qquad a_{\overline{15}|2\%} = 12.8493$

$\dfrac{i}{d^{(12)}} = 1.010801$

$\therefore\ Y = \dfrac{75{,}975.094}{1.010801 \times 12.8493} = 5849.599$

$\dfrac{Y}{12} = 487.4666$

(b) Annual rate of difference in instalment

$= 7{,}578.533 - 5{,}849.599 = 1{,}728.934$

Accumulation of difference is:

$1{,}728.934\ddot{a}^{(12)}_{\overline{15}|2\%} \qquad \dfrac{i}{d^{(12)}} = 1.010801;\ a_{\overline{15}|2\%} = 12.8493$

$\therefore$ accumulation is $1{,}728.934 \times 1.010801 \times 12.8493$

$= 22{,}455.54$

which represents the "profit" to the borrower from exercising the option to repay to loan.

Full marks were given where candidates calculated the value at some other time.

11 (i) Let present value = V

$$V = 4a^{(2)}_{\overline{20}|} + 110v^{20}_{6\%} - 0.25(110 - V)v^{20} - 0.25 \times 4a_{\overline{20}|}$$

at 6%

$$a_{\overline{20}|} = 11.4699 \qquad v^{20} = 0.31180$$

$$\frac{i}{i^{(2)}} = 1.014782$$

$$V = 4 \times 1.014782 \times 11.4699 + 110 \times 0.31180$$
$$- 0.25(110 - V)\, 0.31180 - 0.25 \times 4 \times 11.4699$$

$$V = 46.55779 + 34.298 - 8.5745 + 0.07795V$$
$$- 11.4699$$

$$\therefore\ V(1 - 0.07795) = 60.8114 \qquad \therefore\ V = 65.9524$$

(ii) Easier to work in half years with a rate of return of 2.9563%

Require $\Sigma t C_t v^t$ for the numerator.

For the gross payments, this expression is:

$$2\sum_{t=1}^{40} tv^t + 40 \times 110v^{40} \quad \text{at } i = 2.9563\% \qquad (1)$$

For the income tax payments, this is

$$0.25 \times 4(2v^2 + 4v^4 + ... + 40v^{40}) \qquad (2)$$

to be subtracted from the numerator

(1) is $2(Ia)_{\overline{40}|} + 40 \times 110v^{40} \qquad (Ia)_{\overline{40}|} = \dfrac{\ddot{a}_{\overline{40}|} - 40v^{40}}{0.029563}$

$$a_{\overline{40}|} = 23.27893 \qquad v^{40} = 0.311805$$

$$(Ia)_{\overline{40}|} = \frac{1.029563 \times 23.27893 - 40 \times 0.311805}{0.029563}$$

$$= 388.828$$

$\therefore$ (1) becomes $2 \times 388.828 + 40 \times 110 \times 0.311805$

$= 2149.598$

Evaluation of (2)

$$2v^2 + 4v^4 + \ldots + 40v^{40} = A \qquad (3)$$

multiplying by $(1 + i)^2$ to give

$$2 + 4v^2 + \ldots + 40v^{38} \qquad (4)$$

subtract (3) from (4) to give

$$(1 + i)^2A - A = 2 + 2v^2 + 2v^4 + \ldots + 2v^{38} - 40v^{40}$$

$$= 2(1 + v^2 + v^4 + \ldots + v^{38}) - 40v^{40}$$

$$= 2\left(\frac{1-v^{40}}{1-v^2}\right) - 40v^{40} \qquad v^2 = 0.943396$$

$$= 2\left(\frac{1-0.311805}{1-0.943396}\right) - 40 \times 0.311805 = 11.84393$$

$$A = 11.84393/[(1 + i)^2 - 1] = 11.84393/0.06$$

$$= 197.39883$$

Duration is $\dfrac{2149.598 - 197.39883}{46.55779 + 34.298 - 11.4699}$

$= 28.1354$ half years

or 14.068 years

12 (i) (a) $A(5, 10) = e^{\int_5^{10} 0.05 dt} = \exp[0.05t]_5^{10} = \exp[0.25]$

$= 1.284025$

$A(10, 14) = e^{\int_{10}^{14} 0.08+0.003t\, dt} = \exp\left[0.08t + \frac{0.003t^2}{2}\right]_{10}^{14}$

$= \exp[1.414 - 0.95] = 1.590423$

$A(14, 15) = e^{\int_{14}^{15} 0.08+0.003t\, dt} = \exp\left[0.08t + \frac{0.003t^2}{2}\right]_{14}^{15}$

$= \exp[1.5375 - 1.414] = 1.13145$

$100 \times A(5, 15) = 100 \times 1.284025 \times 1.590423 \times 1.13145$

$= 231.0583$

(b) $100 \times 1.284025 \times 1.590423 = 204.2143$

(c) $100 \times 1.13145 = 113.145$

These marks should be allocated differently if the work is distributed differently between parts (a), (b) and (c).

(d) $100e^{10\delta} = 231.0583$

$\therefore\ 10\delta = \ln 2.310583 \quad \therefore \delta = 0.083750$

(ii) Require $\int_0^5 \rho(t)v(t)dt$ where

$\rho(t) = 100e^{0.01t}$ and $v(t) = e^{-0.05t}$

$\therefore$ require $100\int_0^5 e^{0.01t}e^{-0.05t}dt = 100\int_0^5 e^{-0.04}dt$

$= 100\left[\frac{e^{-0.04t}}{-0.04}\right]_0^5 = 100(-20.4683 + 25) = 453.17$

(iii) Present value of dividend is: $7e^{-(0.5\times0.05)} = 6.82717$

Hence forward price, F is:

$$F = (300 - 6.82717)\, e^{\delta t} \quad \text{where } \delta = 0.05 \text{ and } t = 1$$

$\therefore\ F = (300 - 6.82717)e^{0.05} = 308.2041$

END OF EXAMINERS' REPORT