# REPORT OF THE BOARD OF EXAMINERS ON THE EXAMINATIONS HELD IN

April 2002

## Subject 102 — Financial Mathematics

### Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

K Forman
Chairman of the Board of Examiners

11 June 2002

# EXAMINERS' COMMENT

*Please note that differing answers may be obtained depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this.*

*However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.*

**1** $f_t = 80(1.0125) = 81\text{p}$

**2** The bond would have an (eight month) indexation lag. This means that if future inflation is higher than the assumption, the investor would not receive inflation compensation for the whole of the time between the purchase date and the date on which a payment would be received.

**3** The purchase of a long-dated fixed interest swap to pay floating and receive fixed would enable the insurance company to receive a series of fixed interest rate payments appropriate to immunise the liabilities. The commitment to pay floating would be met by the returns from the cash investment made by the insurance company.

**4** Equation of value is:

$$99(1+i)^2 = 6.5(1+i) + 6.6 + 101$$

$$\therefore\ 99(1+i)^2 - 6.5(1+i) - 107.6 = 0$$

$$\text{use solution} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore\ \text{solution} = 1.075875$$

$$\therefore\ i = 7.5875\%$$

$\therefore$ Rate of return p.a. convertible half yearly is 15.175%

**5** (i)

$$A(0,8) = e^{\int_0^8 0.005t + 0.0001t^2\,dt}$$

$$= e^{\left[\frac{0.005t^2}{2} + \frac{0.0001t^3}{3}\right]_0^8}$$

$$= e^{0.16 + 0.01707}$$

$$= 1.193711$$

$\therefore$ accumulation of £100 is £119.37

(ii) $i$ is such that $(1+i)^8 = 1.193711$

$\therefore\ i = 2.238\%$

**6** (i) Since $i > (1-t_1)g$ [i.e. $0.08 > 0.75 \times 0.06$] there is a capital gain.

*Candidates who merely assumed that a capital gain was made without justifying the assumption were penalised a mark.*

Let $P$ = price per £100 nominal, then

$$P = (6a_{\overline{10}|} \times 0.75 + 100v^{10} - 0.25(100 - P)v^{10}) \times (1.08)^{1/4}$$

$$a_{\overline{10}|} = 6.71008 \qquad v^{10} = 0.463193$$

$$\therefore\ P = 30.78195 + 47.21912 - 11.80478 + 0.11805P$$

$$\therefore\ P \times 0.88195 = 66.19629$$

$$\therefore\ P = 75.0567 = £75.06$$

(ii) Must satisfy equation $(1 + \xi) \times (1.03) = 1.08$

$$\therefore\ \xi = 4.854\%$$

**7** (i) (a)

$$P = \frac{5}{1.04} + \frac{5}{1.04 \times 1.045} + \frac{105}{1.04 \times 1.045 \times 1.048}$$

$$= 101.597$$

(b) $P = 5a_{\overline{3}|} + 100v^3$ at gross redemption yield

If $i = 5\%$, $P = 100$

if $i = 4.5\%$ $\quad a_{\overline{3}|} = 2.74896;\ v^3 = 0.876297$

$P = 101.375$

if $i = 4\%$ $\quad a_{\overline{3}|} = 2.77509;\ v^3 = 0.889000$

$P = 102.775$

$$\therefore\ i = 0.04 + \left(\frac{102.775 - 101.597}{102.775 - 101.375}\right) \times 0.005$$

$$= 4.421\%$$

(ii) The gross redemption yield is, in effect, a weighted average of the forward rates where the weights depend on the cash flows. If the coupon rate were higher, the cash flows would be weighted towards the earlier times when lower forward rates pertain. Therefore, the gross redemption yield would decrease.

**8** (i) $(Ia)_{\overline{n|}} = v + 2v^2 + 3v^3 + \ldots + nv^n$

$$(1+i)(Ia)_{\overline{n|}} = 1 + 2v + 3v^2 + \ldots + nv^{n-1}$$

$$\therefore\ i(Ia)_{\overline{n|}} = 1 + v + v^2 + \ldots + v^{n-1} - nv^n$$

$$i(Ia)_{\overline{n|}} = \ddot{a}_{\overline{n|}} - nv^n$$

$$\therefore\ (Ia)_{\overline{n|}} = \frac{\ddot{a}_{\overline{n|}} - nv^n}{i}$$

(ii) Purchase price of the bond would be:

$$P = 5a_{\overline{20|}} + 100v^{20} \text{ @ 3\% per half year}$$

$$a_{\overline{20|}} = 14.87747 \qquad v^{20} = 0.553676$$

$$\therefore\ P = 129.755$$

Numerator of DMT is:

$$\sum_{t=1}^{20} tC_t v^t = 5\sum_{t=1}^{20} tv^t + 20 \times 100v^{20}$$

$$= 5(Ia)_{\overline{20|}} + 2{,}000v^{20}$$

$$(Ia)_{\overline{20|}} = \frac{\ddot{a}_{\overline{20|}} - 20v^{20}}{0.03} = \frac{1.03 \times 14.87747 - 20 \times 0.553676}{0.03}$$

$$= 141.6758$$

$\therefore$ numerator is $5 \times 141.6758 + 1{,}107.352 = 1{,}815.731$

$\therefore$ DMT in half years is:

$$\frac{1{,}815.731}{129.755} = 13.994$$

$\therefore$ DMT in years is: $6.997 \approx 7$ years.

(iii) A lower coupon bond would have the timing of the payments more weighted towards the redemption time thus giving a longer DMT.

**9** (i) (a) $V_C = 6v^6 + 6 \times 1.1v^7 + 6 \times 1.1^2v^8 + \ldots + 6 \times 1.1^6v^{12} + 6 \times 1.1^6 \times 1.03v^{13} + 6 \times 1.1^6 1.03^2v^{14} + \ldots$

$$\therefore\ V = 6v^6\left[\frac{1-1.1^7v^7}{1-1.1\times v}\right] + 6\times1.1^6\times1.03v^{13}\times\left(\frac{1}{1-1.03\times v}\right)$$

$v = 0.943396 \quad v^6 = 0.704961 \quad v^7 = 0.665057 \quad v^{13} = 0.468839$

$1.1^6 = 1.771561 \quad 1.1^7 = 1.948717$

$$\therefore\ V = 6\times0.704961\left[\frac{1-1.948717\times0.665057}{1-1.1\times0.943396}\right]$$

$$+\ 6\times1.771561\times1.03\times0.468839\times\frac{1}{1-0.971698}$$

$= 33.17939 + 181.36400 = 214.543$

(b) $V_B = 4v + 4(1.005)v^2 + 4(1.005)^2v^3 + \ldots$

$$= 4v\times\left(\frac{1}{1-1.005\times v}\right) = 72.727$$

(ii) Revised value of Boring plc is the same with $v = \dfrac{1}{1.07}$

$\therefore\ V = 61.538$

$$\therefore\ \%\text{ change in value is: }\left(\frac{61.538-72.727}{72.727}\right)\times100\%$$

$= -15.38\%$

Revised value of Cyber plc is:

$$V = 6 \times 0.666342\left[\frac{1-1.948717\times 0.622750}{1-1.1\times 0.934579}\right]$$

$$+\ 6 \times 1.1^6 \times 1.03 \times 0.414964 \times \frac{1}{1-1.03\times 0.934579}$$

$$= 30.45408 + 121.52720 = 151.981$$

$\therefore$ % change in value is: $\left(\dfrac{151.981-214.543}{214.543}\right)\times 100\%$

$$= -29.16\%$$

(iii) This arises because Cyber's cash flows are much "later". Therefore their duration and interest rate volatility are higher.

**10** (i) Discounted payback period only indicates when a project would come into profitability (in PV terms). It does not indicate how profitable the project is.

(ii) Working in £m

PV of payments from project A is:

$$\frac{3.5}{1.04^{10}} = 2.364475$$

$\therefore$ NPV = 2.364475 − 1 = 1.364475 = £1,364,475

PV of payments from project B is:

$$0.08\bar{a}_{\overline{1}|} + 0.09v\bar{a}_{\overline{1}|} + \ldots + 0.17v^9\bar{a}_{\overline{1}|}$$

$$= 0.07\bar{a}_{\overline{10}|} + 0.01(I\bar{a})_{\overline{10}|}$$

$$= \frac{0.04}{0.039221}\left(a_{\overline{10}|}\times 0.07 + 0.01\times\frac{\ddot{a}_{\overline{10}|}-10v^{10}}{0.04}\right)$$

$$v^{10} = 0.675564 \quad a_{\overline{10}|} = 8.110896$$

$\therefore$ PV $= 1.019862 \times (0.567763 + 0.419923)$

$$= 1.007303$$

$\therefore$ NPV = 1.007303 − 1 = 0.007303 = £7,303

(iii) Clearly DPP of project A is 10 as there are no cash flows before time 10 but project is profitable.

Clearly DPP of project B must be less than 10 as project is profitable at time 10 and income is received continuously.

(iv) Project A is better as it has a higher NPV, although the DPP is later because the cash flows are weighted towards the end.

**11** (i) Let the accumulation of 1 unit over 10 years = $S_{10}$

$$E(S_{10}) = E[(1+i_1)(1+i_2)\ldots(1+i_{10})]$$

Assuming $i_t$'s are independent, this gives:

$$E(S_{10}) = E(1+i_1)E(1+i_2) \therefore E(1+i_{10})$$

$$= 1.07 \times 1.07 \ldots 1.07 = 1.07^{10} = 1.96715$$

$$E(S_{10}^2) = E[(1+i_1)^2(1+i_2)^2 \ldots (1+i_{10})^2]$$

Assuming $i_t$'s are independent, this gives:

$$E(S_{10}^2) = E(1+i_1)^2 E(1+i_2)^2 \ldots E(1+i_{10})^2$$

$$E(1+i_t)^2 = E(1+2i_t+i_t^2)$$

$$= 1 + 2 \times 0.07 + 0.07^2 + 0.09^2$$

$$= 1.153$$

$\therefore\ E(S_{10}^2) = 1.153^{10}$ and $\text{Var}(S_{10}) = 1.153^{10} - 1.07^{20}$

$$= 0.282657$$

S.D.$(S_{10}) = 0.53165$

$\Rightarrow$ Accumulation of £1,000 = £1,967.15 and standard deviation = £531.65

(ii) $1 + i_t \sim LN(\mu, \sigma^2)$

$\ln(1+i_t) \sim N(\mu, \sigma^2)$

$\ln((1+i_t))^{10} = \ln(1+i_t) + \ldots + \ln(1+i_t) \sim N(10\mu, 10\sigma^2)$

$\therefore\ (1+i_t)^{10} \sim LN(10\mu,\ 10\sigma^2)$ let $10\mu = \mu'$ and $10\sigma^2 = \sigma'^2$

$$E(1+i_t) = \exp\left(\mu + \frac{\sigma^2}{2}\right) = 1.07$$

$$\text{Var}(1+i_t) = \exp(2\mu+\sigma^2)\,[\exp(\sigma^2)-1] = 0.09^2$$

$$\therefore\ \frac{0.09^2}{1.07^2} = \exp(\sigma^2) - 1 \quad \therefore\ \sigma^2 = 0.007050$$

$$\therefore\ \exp\left(\mu + \frac{0.00705}{2}\right) = 1.07 \quad \therefore\ \mu = \ln 1.07 - \frac{0.007050}{2}$$

$$= 0.064134$$

$\therefore\ \ln(S_{10}) \sim N(0.64134,\ 0.0705)$

we require probability $S_{10} < 0.983575$

$$= \text{probability that } \frac{\ln(S_{10}) - 0.64134}{\sqrt{0.0705}} < \frac{\ln 0.983575 - 0.64134}{\sqrt{0.0705}}$$

$= \Pr(Z < -2.4778)$ where $Z \sim N(0,1)$

$= 0.00661$

(iii) Require $\Pr(1{,}200(S_{10}) < 1{,}400)$

$= \Pr(S_{10} < 1.16667)$

$$= \Pr\left(\frac{\ln(S_{10}) - 0.64134}{\sqrt{0.0705}} < \frac{\ln 1.16667 - 0.64134}{\sqrt{0.0705}}\right)$$

$= \Pr(Z < -1.8349)$ where $Z \sim N(0,1)$

$= 0.03326$