EXAMINATIONS

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Subject 102 — Financial Mathematics

EXAMINERS' REPORT

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1 $$\left(1 - d \times \frac{91}{365}\right) = (1+i)^{-\frac{91}{365}} \text{ where } i = 0.05$$

$$\therefore \quad 1 - d \times \frac{91}{365} = 0.98791$$

$$\therefore \quad d = 0.048495$$

2 $$V_0 = \frac{d_1}{1+r} + \frac{d_1(1+g)}{(1+r)^2} + \frac{d_1(1+g)^2}{(1+r)^3} + \ldots$$

$$g.p. \quad n \to \infty \quad a = \frac{d_1}{1+r}; \quad R = \frac{1+g}{1+r} \quad S_\infty = \frac{a}{1-R}$$

$$\therefore \quad V_0 = \frac{d_1/(1+r)}{1 - \frac{1+g}{1+r}} = \frac{d_1}{(1+r)-(1+g)} = \frac{d_1}{r-g}$$

3 Let forward price be f_{40}.

$$f_{40} = 100(1.03)^{\frac{40}{182.5}} - 5 \times (1.03)^{\frac{20}{182.5}}$$

$$= 100.64997 - 5.016223$$

$$= 95.63375$$

4 $$A = 1{,}000 S^{(2)}_{\overline{12|}5\%}\,(1.015)^{32} + \frac{500}{S_{\overline{2|}1½\%}} \,.\, S_{\overline{32|}1½\%}$$

$$A = 1{,}000 \times \frac{i}{i^{(2)}} \times S_{\overline{12|}5\%}\,(1.015)^{32} + \frac{500}{S_{\overline{2|}1½\%}} \,.\, S_{\overline{32|}1½\%}$$

$$\frac{i}{i^{(2)}} \text{ @ } 5\% = 1.012351 \quad (1.015)^{32} = 1.61032 \quad S_{\overline{32|}1½\%} = 40.68829$$

$$S_{\overline{12|}5\%} = 15.9171 \quad S_{\overline{2|}1½\%} = 2.015$$

$$\therefore \quad A = 25{,}948.201 + 10{,}096.34988 = £36{,}044.551$$

5 Calculate purchase price at next coupon payment

$P \;=\; 8a_{\overline{7}|}^{(2)} + 100v^7$ @ 6%

$= \; 8 \times \dfrac{0.06}{0.059126} \times 5.5824 + 100 \times 0.665057$

$= 45.31935 + 66.5057 = 111.82505$

$\therefore$ Price ex-div. 8 days earlier is: $111.82505(1.06)^{-\frac{8}{365}}$

$= 111.6823$

6 $\ln(1 + i_t) \sim N(0.06, 0.0009)$

Interquartile range for $\ln(1 + i_t)$ can be found to give us interquartile range for $100(1 + i_t)$

$$\frac{\ln(1+i_t) - 0.06}{\sqrt{0.0009}} \sim N(0,1)$$

Interquartile range for $N(0, 1)$ is $-0.674 + 0.674$

$\therefore$ range for $\ln(1 + i_t)$ is 0.03978, 0.08022 = 0.04044

$\therefore$ range for $100(1 + i_t)$ is 104.058, 108.353 = 4.295

7 (i) $v(1) = 1.08^{-1} = 0.925926$

$v(2) = 1.08^{-1} \times 1.07^{-1} = 0.86535$

$v(3) = 1.08^{-1} \times 1.07^{-1} \times 1.06^{-1} = 0.81637$

$v(4) = 1.08^{-1} \times 1.07^{-1} \times 1.06^{-1} \times 1.05^{-1} = 0.77749$

$P \;= 5 \times (0.925926 + 0.86535 + 0.81637 + 0.77749) + 100 \times 0.77749$

$= 94.6747$

Find i such that:

$$94.6747 = 5a_{\overline{4}|} + 100v^4$$

Try $i = 6.5\%$ RHS = 17.12899 + 77.7323 = 94.8613

Try $i = 7\%$ RHS = 16.9361 + 76.2895 = 93.2256

$$\therefore \quad i = 0.065 + \left(\frac{94.8613 - 94.6747}{94.8613 - 93.2256}\right) \times 0.005 = 6.5570\%$$

(ii) The forward rates are falling with term. The gross redemption yield is a weighted average of those falling forward rates. It is therefore higher than the four year forward rate.

8 The internal rate of return is the interest rate that solves the equation of value, i.e. the interest rate at which the present value of the cost = present value of the income.

Initial cost = £1,000,000

Rental outgo is £10,000 at start of every quarter for 10 years and then £12,000 at start of every quarter for following ten years.

Let i = internal rate of return, then value of outgo

$$= 1{,}000{,}000 + 40{,}000\ddot{a}^{(4)}_{\overline{10}|} + 48{,}000v^{10}\,\ddot{a}^{(4)}_{\overline{10}|} \text{ at } i\%$$

Value of revenue

$$= 100{,}000\bar{a}_{\overline{1}|} + 200{,}000\bar{a}_{\overline{1}|}\,(v + 1.03v^2 + \ldots + 1.03^2v^3 + \ldots + 1.03^{18}\,v^{19})$$

$$= 100{,}000\bar{a}_{\overline{1}|} + 200{,}000\bar{a}_{\overline{1}|}\,v(1 + 1.03v + \ldots + 1.03^{18}\,v^{18})$$

$$= 100{,}000\bar{a}^{i}_{\overline{1}|} + 200{,}000\bar{a}^{i}_{\overline{1}|}\,v_i\,\ddot{a}^{j}_{\overline{19}|} \text{ where } j = \frac{1+i}{1.03} - 1$$

We require the value of $i\%$ for which

$$0 = 100{,}000\bar{a}^{i}_{\overline{1}|} + 200{,}000\bar{a}^{i}_{\overline{1}|}\,v_i\,\ddot{a}^{j}_{\overline{19}|} - 1{,}000{,}000 - 40{,}000\ddot{a}^{(4)_i}_{\overline{10}|} - 48{,}000v_i^{10}\,\ddot{a}^{(4)_i}_{\overline{10}|}$$

For initial estimate, $1{,}000 + 44 \times 20v^{10} = 200(1.03)^9 \times 20v^{10}$

$1{,}000 = (5{,}219 - 880)\,v^{10} = 4{,}339\,v^{10}$

$v^{10} = 1{,}000 / 4{,}339 = 0.23047$

$v = 0.8635$

$\Rightarrow \quad i = 15.8\%$

Try $i = 15\%$

Value of cost $= 1{,}000{,}000 + 40{,}000 \times 5.48106 + 48{,}000 \times 1.35483$
$= 1{,}284{,}274$

Value of income $= 100{,}000 \times 0.93326 + 200{,}000 \times 0.93326 \times 0.86957$
$\times\, 8.402564 = 1{,}457{,}121$

PV cost – PV income $= -172{,}847$

Try $i = 20\%$

Value of cost $= 1{,}000{,}000 + 40{,}000 \times 4.704595 + 48{,}000 \times 0.759818$
$= 1{,}224{,}655$

Value of income $= 100{,}000 \times 0.914136 + 200{,}000 \times 0.914136$
$\times\, 0.833333 \times 6.671391 = 1{,}107{,}840$

PV cost – PV income $= 116{,}815$

By interpolation, $i = 15 + \dfrac{172{,}847}{116{,}815 + 172{,}847} \times 5 = 17.98\%$

At 18%

PV cost $= 1{,}000{,}000 + 40{,}000 \times 4.989216 + 48{,}000 \times 0.953262$
$= 1{,}245{,}325$

PV income $= 1{,}000{,}000 \times 0.921626 + 200{,}000 \times 0.921626$
$\times\, 0.847458 \times 7.272457 = 1{,}228{,}177$

So PV cost – PV income $= 17{,}148$

By interpolation, $i = 15 + \dfrac{172{,}847}{17{,}148 + 172{,}847} \times 3 = 17.73\%$

So IRR $\bumpeq$ 17.7%

9 (i) $Da_{\overline{n}|} = nv + (n-1)v^2 + (n-2)v^3 + \ldots + 2v^{n-1} + v^n$

$(1+i)\,Da_{\overline{n}|} = n + (n-1)v + (n-2)v^2 + \ldots + 2v^{n-2} + v^{n-1}$

$\therefore \quad i(Da)_{\overline{n}|} = n - v - v^2 - v^3 - \ldots - v^n$

$\therefore \quad Da_{\overline{n}|} = \dfrac{n - a_{\overline{n}|}}{i}$

(ii) (a) $20v + 19v^2 + \ldots + 11v^{10}$

$= 10v + 9v^2 + \ldots + v^{10} + 10v + 10v^2 + \ldots + 10v^{10}$

$= Da_{\overline{10}|} + 10a_{\overline{10}|}$ @ 6%

$= \dfrac{10 - a_{\overline{10}|}}{0.06} + 10 \times a_{\overline{10}|} = \dfrac{10 - 7.3601}{0.06} + 10 \times 7.3601$

$= 117.5993$

(b) Interest component

$= 0.06 \times 117.5993 = 7.055958$

Capital component

$= 20 - 7.055958 = 12.94404$

(c) Capital o/s at end of 8th year $= 11v^2 + 12v$

Capital o/s at end of 7th year $= 11v^3 + 12v^2 + 13v$

$\therefore$ capital repayment $= 11v^3 + v^2 + v$

$= 11 \times 0.839619 + 0.889996 + 0.943396$

$= 11.069201$

10 (i) $P = 5a_{\overline{20|}} + 100v^{20}$

$= 5 \times 11.4699 + 100 \times 0.311805 = 88.53$

(ii) (a) $P' = 0.7 \times 5a_{\overline{10|}} + 100v^{10} - (100 - P') \times 0.3v^{10}$

$= 0.7 \times 5a_{\overline{10|}} + 70v^{10} + 0.3P'v^{10}$

$\therefore \quad P' = \dfrac{3.5a_{\overline{10|}} + 70v^{10}}{1 - 0.3v^{10}}$

$a_{\overline{10|}} = 7.18883 \qquad v^{10} = 0.53273$

$\therefore \quad P' = \dfrac{62.45201}{1 - 0.159819} = 74.33161$

(b) $88.53 = 5a_{\overline{10|}} + 74.33161v^{10}$

Estimate of yield: $\dfrac{5}{88.53} - \dfrac{(88.53 - 74.33161)/10}{88.53}$

$= 4.04401\%$

Return must be higher. Try 4.5%

$a_{\overline{10|}} = 7.91272 \qquad v^{10} = 0.64393$

Price would be: 87.4279

Try 4%

$a_{\overline{10|}} = 8.110896 \qquad v^{10} = 0.67556$

Price would be: 90.76994

$$i = -\frac{88.53 - 87.4279}{90.76994 - 87.4279} \times 0.005 + 0.045 = 0.04335$$

11 (i) $A(0, 10) = e^{\int_0^{10} 0.05\,dt} = e^{[0.05t]_0^{10}} = e^{0.5} = 1.64872$

$$A(10, 20) = e^{\int_{10}^{20} 0.006t\,dt}$$

$$= e^{\left[\frac{0.006t^2}{2}\right]_{10}^{20}} = e^{[1.2-0.3]} = e^{0.9} = 2.45960$$

$$A(20, 25) = e^{\int_{20}^{25} 0.003t + 0.0002t^2\,dt}$$

$$= e^{\left[\frac{0.003t^2}{2} + \frac{0.0002t^3}{3}\right]_{20}^{25}}$$

$$= e^{[0.9375+1.04166-0.6-0.5333]}$$

$$= e^{0.845836} = 2.32992$$

Required PV

$$= \frac{1}{A(0,10)\ A(10,20)\ A(20,25)} = \frac{1}{1.64872 \times 2.45960 \times 2.32992}$$

$$= \frac{1}{9.4483} = 0.10584$$

(ii) $A(19, 20) = e^{\int_{19}^{20} 0.006t\,dt} = e^{\left[\frac{0.006t^2}{2}\right]_{19}^{20}}$

$$= e^{1.2-1.083}$$

$$= 1.12412$$

$$\therefore \quad i = 12.412\%$$

(iii) $v(t) = e^{-\int_0^t 0.05ds} = e^{-0.05t} \qquad \rho(t) = e^{-0.03t}$

$\therefore$ we require $\int_0^5 e^{-0.05t}\ e^{-0.03t}\ dt = \int_0^5 e^{-0.08t}\ dt$

$$= \left[\frac{-e^{-0.08t}}{0.08}\right]_0^5 = -8.37900 + 12.5 = 4.121$$

12 (i) (a) DMT $= \dfrac{\sum_{r=1}^{n} t_r \, Ct_r \, e^{-\delta t_r}}{\sum_{r=1}^{n} Ct_r \, e^{-\delta t_r}}$

$$V_\delta = -\frac{\partial PV}{\partial \delta} \cdot \frac{1}{PV}$$

$$PV = \sum_{r=1}^{n} Ct_r \, e^{-\delta t_r}$$

$$\frac{\partial PV}{\partial \delta} = -\sum_{r=1}^{n} t_r \, Ct_r \, e^{-\delta t_r}$$

$$\therefore \; -\frac{\partial PV}{\partial \delta} \cdot \frac{1}{PV} = \frac{\sum_{r=1}^{n} t_r \, Ct_r \, e^{-\delta t_r}}{\sum_{r=1}^{n} Ct_r \, e^{-\delta t_r}}$$

$$= \text{DMT}$$

(b) By chain rule $\dfrac{\partial PV}{\partial i} = \dfrac{\partial PV}{\partial \delta} \cdot \dfrac{\partial \delta}{\partial i}$

$\therefore$ We need to find $\dfrac{\partial PV}{\partial i}$ and multiply DMT by $\dfrac{\partial \delta}{\partial i}$

$\delta = \ln(1 + i) \quad \therefore \; \dfrac{\partial \delta}{\partial i} = (1 + i)^{-1}$

$\therefore \; V_i = \dfrac{V_\delta}{1+i}$ or $\dfrac{\text{DMT}}{1+i}$

(ii) PV of liabilities $= 1{,}000{,}000 \, a_{\overline{10|}} + 1{,}500{,}000 v^{10} \, a_{\overline{10|}}$ at 5%

$= 1{,}000{,}000 \times 7.721735 + 1{,}500{,}000 \times 0.613913 \times 7.721735$

$= 14{,}832{,}448$

Numerator for
DMT of liabilities $= 1{,}000{,}000 \; Ia_{\overline{10|}} + 1{,}500{,}000 \; [(Ia)_{\overline{20|}} - (Ia)_{\overline{10|}}]$

$$= 1{,}000{,}000 \; \frac{\ddot{a}_{\overline{10|}} - 10v^{10}}{.05} + 1{,}500{,}000 \left[\frac{\ddot{a}_{\overline{20|}} - 20v^{20}}{.05} - \frac{\ddot{a}_{\overline{10|}} - 10v^{10}}{.05} \right]$$

$$= 1{,}000{,}000 \times 39.37378 + 1{,}500{,}000 \times [110.9506 - 39.37378]$$

$$= 146{,}739{,}045$$

PV of assets:

Let nominal amount of bond purchased be X.
Let coupon rate of bond be g%.

Then

$$\text{PV of assets} = 10{,}000{,}000v^{10} + \frac{g}{100} Xa_{\overline{19|}} + Xv^{19} \text{ at } 5\%$$

$$= 10{,}000{,}000 \times 0.6139133 + \frac{g}{100} X\, 12.08532 + X\, 0.395734$$

$$= 6{,}139{,}133 + \frac{g}{100} X\, 12.08532 + X\, 0.395734$$

Numerator for

$$\text{DMT of assets} = 100{,}000{,}000v^{10} + \frac{g}{100} X(Ia)_{\overline{19|}} + 19Xv^{19}$$

$$= 61{,}391{,}325 + \frac{g}{100} X\, 103.41283 + X\, 7.518946$$

Equating values of assets and liabilities and DMTs of assets and liabilities (or equivalently the numerators of the DMTs of the assets and liabilities) gives

$$14{,}832{,}448 = 6{,}139{,}133 + \frac{g}{100} X\, 12.08532 + X\, 0.395734$$

$$146{,}739{,}045 = 61{,}391{,}325 + \frac{g}{100} X\, 103.41283 + X\, 7.518946$$

$$\Rightarrow \quad 12.08532 \frac{g}{100} X + 0.395734\, X = 8{,}693{,}315 \qquad \text{A}$$

$$103.41283 \frac{g}{100} X + 7.518946\, X = 85{,}347{,}720 \qquad \text{B}$$

Multiplying A by $\dfrac{103.41283}{12.08532}$ and subtracting from B gives

$$4.13269\ X = 10{,}959{,}925$$

so $\quad X = 2{,}652{,}008$

Substituting in A gives

$$\frac{g}{100} = \frac{8{,}693{,}315 - 0.395734 \times 2{,}652{,}008}{12.08532 \times 2{,}652{,}008}$$

$$\Rightarrow \qquad \frac{g}{100} = 0.2385$$

so coupon $= 23.85\%$

(iii) The third condition is that the convexity of the assets should be greater than the convexity of the liabilities. (This can be written as the spread of the asset terms around the discounted mean term is greater than the spread of the liability terms around the discounted mean term.)