Faculty of Actuaries **Institute of Actuaries**

EXAMINATIONS

April 2001

Subject 102 — Financial Mathematics

EXAMINERS' REPORT

1 A certificate of deposit is a certificate stating that some money has been deposited. They are issued by banks and building societies. Terms to maturity are usually in the range 28 days to 6 months. Interest is payable on maturity.

The degree of security and marketability will depend on the issuing bank. There is an active secondary market in certificates of deposit.

2

$$(1.11)^3 = \frac{460}{400} \times \frac{500}{460+50} \times \frac{650}{500+40} \times \frac{X}{650+60}$$

$$\Rightarrow X = (1.11)^3 \times 710 \times \frac{400}{460} \times \frac{510}{500} \times \frac{540}{650}$$

$$= 715.50$$

3 (i)

$$\frac{1}{\left(1+\frac{i^{(2)}}{2}\right)^2} = \left(1-\frac{0.08}{4}\right)^4$$

$$\Rightarrow i^{(2)} = \left(\sqrt{\frac{1}{(0.98)^4}} - 1\right)2 = 0.08247$$

$\Rightarrow$ 8.247% p.a. convertible half-yearly

(ii)

$$\left(1-\frac{d^{(12)}}{12}\right)^{12} = \left(1-\frac{0.08}{4}\right)^4$$

$$d^{(12)} = \left(1-(0.98)^{\frac{4}{12}}\right) \times 12$$

$$= 0.080539$$

$\Rightarrow$ 8.0539% p.a. convertible monthly

4 (i) A forward contract is an agreement made between two parties under which one agrees to buy from the other a specified amount of an asset at a specified price on a specified future date.

The investor agreeing to sell the asset is said to hold a "short forward position" in the asset, and the buyer is said to hold a "long forward position".

(ii) Forward price $= 150e^{(0.05-0.03)\times\frac{3}{12}} - 30e^{0.05\times\frac{1}{12}}$

$= £120.63$

5 (i)

$$P = \frac{d_1}{(1+i)^{\frac{4}{12}}} + \frac{d_1(1+g)^{\frac{1}{2}}}{(1+i)^{\frac{10}{12}}} + \frac{d_1(1+g)}{(1+i)^{\frac{16}{12}}} + \frac{d_1(1+g)^{\frac{3}{2}}}{(1+i)^{\frac{22}{12}}} + \ldots$$

$$= d_1\, v^{\frac{4}{12}}\left[1 + \left(\frac{1+g}{1+i}\right)^{\frac{1}{2}} + \left(\frac{1+g}{1+i}\right)^{1} + \left(\frac{1+g}{1+i}\right)^{\frac{3}{2}} + \ldots\right]$$

$$= d_1\, v^{\frac{4}{12}}\left[\frac{1}{1-\left(\frac{1+g}{1+i}\right)^{\frac{1}{2}}}\right]$$

$$= \frac{d_1\, v^{\frac{4}{12}}\,(1+i)^{\frac{6}{12}}}{(1+i)^{\frac{1}{2}}-(1+g)^{\frac{1}{2}}} = \frac{d_1\,(1+i)^{\frac{2}{12}}}{(1+i)^{\frac{1}{2}}-(1+g)^{\frac{1}{2}}}$$

(ii) We need to modify the above to:

$$\text{Price} = \frac{d_1\,(1+i)^{\frac{4}{12}}}{(1+i)^{\frac{1}{2}}-(1+g)^{\frac{1}{2}}}$$

Hence

$$18 = \frac{0.5\,(1+i)^{\frac{4}{12}}}{(1+i)^{\frac{1}{2}}-(1.04)^{\frac{1}{2}}}$$

$i = 10\%$, RHS = 17.79

$i = 9\%$, RHS = 21.24

$\Rightarrow \quad i = 10\%$ p.a. to the nearest 1%

6 (i) $A(t) = \exp\left(\int_0^t \delta(r)dr\right)$

For $0 \le t < 8$, $\int_0^t \delta(r)dr = \int_0^t (0.04 + 0.01r)\, dr$

$$= [0.04r + 0.005r^2]_0^t$$

$$= 0.04t + 0.005t^2$$

$\Rightarrow$ for $0 \le t < 8$, $A(t) = \exp(0.04t + 0.005t^2)$

For $t \ge 8$,

$$A(t) = A(8)\,.\,\exp\left(\int_8^t 0.07dr\right)$$

$$= A(8)\,.\,\exp(0.07\,(t-8))$$

$$= \exp(0.64)\,.\,\exp(0.07t - 0.56)$$

$$= \exp(0.07t + 0.08)$$

(ii) Present values $= \dfrac{100}{A(10)}$

$$= \frac{100}{\exp(0.78)}$$

$$= 45.84$$

7 First calculate the effective annual interest rate:

$$1 + i = (1.03)^2 = 1.0609$$

$\Rightarrow$ $i = 6.09\%$ p.a.

PV of initial investments (working in £ millions):

$$2 + 1.5\, v_{6.09\%}^{\frac{7}{12}} = 3.4492\text{m}$$

PV of net income:

$$v\left(0.2\bar{a}_{\overline{10|}} + 0.1(I\bar{a})_{\overline{10|}}\right)$$

$$= \quad v\left(0.2\bar{a}_{\overline{10|}} + 0.1\times\frac{\ddot{a}_{\overline{10|}} - 10v^{10}}{\delta}\right)$$

$$= \quad v\left[0.2\times\left(\frac{1-v^{10}}{\ln 1.0609}\right) + 0.1\times\left(\frac{1-v^{10}}{d\,.\ln 1.0609} - \frac{10v^{10}}{\ln 1.0609}\right)\right] \text{ @ } 6.09\%$$

$$= \quad v[0.2 \times 7.5498 + 0.1 \times (131.5197 - 93.6567)]$$

$$= \quad 4.9922\text{m}$$

PV of Sale proceeds $= 3v^{11}$ @ 6.09%

$= 1.5657\text{m}$

$\Rightarrow$ NPV of project $= 4.9922 + 1.5657 - 3.4492$

$= £3.1087\text{m}$

8 (i) We can find forward rates $f_{1,1}$ and $f_{2,1}$ using the spot rates y_1, y_2 and y_3:

$(1 + y_2)^2 = (1 + y_1)(1 + f_{1,1})$ and

$(1 + y_3)^3 = (1 + y_2)^2 (1 + f_{2,1})$

$\Rightarrow \quad 1.042^2 = (1.041)(1 + f_{1,1})$

$\Rightarrow \quad f_{1,1} = 4.30\%$

and $\quad (1.043)^3 = (1.042)^2 (1 + f_{2,1})$

$\Rightarrow \quad f_{2,1} = 4.50\%$

(ii) (a) Price per £100 nominal

$$= \quad 3\left(v_{4.1\%} + v_{4.2\%}^2 + v_{4.3\%}^3\right) + 110v_{4.3\%}^3$$

$$= \quad 3(0.96061 + 0.92101 + 0.88135) + 110 \times 0.88135$$

$$= \quad 105.24$$

(b) Let yc_2 = 2-year par yield

$$1 = yc_2\left(v_{4.1\%} + v_{4.2\%}^2\right) + v_{4.2\%}^2$$

$\Rightarrow$ $1 = yc_2\,(0.96061 + 0.92101) + 0.92101$

$\Rightarrow$ $yc_2 = 0.04198$ i.e. 4.198% p.a.

9 (i) Let j denote the mean yield, then

$$1 + j = \exp\left(\mu + \frac{\sigma^2}{2}\right) = 1.0757305$$

$\Rightarrow$ $j = 0.0757305$

We require

$$20{,}000\ E(X_{10}) + 150{,}000\ E(S_{10})$$

$$= 20{,}000 s_{\overline{10|}} + 150{,}000\,(1+j)^{10} \text{ at rate } j\%$$

$$= 20{,}000\left(\frac{1.0757305^{10} - 1}{0.0757305}\right) + 150{,}000 \times (1.0757305)^{10}$$

$$= 20{,}000 \times 14.1961 + 311{,}261.98$$

$$= 595{,}183.99$$

where X_{10} represents the accumulation after 10 years of £1 p.a. paid in arrears for 10 years

and S_{10} represents the accumulation after 10 years of £1 paid now.

(ii) We require $\Pr(Z\,.\,S_{10} \geq 600{,}000) = 0.99$

where Z = single amount paid now

$$\Rightarrow \quad \Pr\left(S_{10} \geq \frac{600{,}000}{Z}\right) = 0.99$$

Now $\dfrac{\log S_{10} - 10\mu}{\sigma\sqrt{10}} \sim N(0,1)$

So we want $\Phi\left(\dfrac{\log\left(\dfrac{600{,}000}{Z}\right)-10\mu}{\sigma\sqrt{10}}\right) = 0.01$

So, from tables, $\dfrac{\log\left(\dfrac{600{,}000}{Z}\right)-10\mu}{\sigma\sqrt{10}} = -2.326$

So $\dfrac{600{,}000}{Z} = \exp\left(-2.326\,\sigma\sqrt{10}+10\mu\right)$

$= 1.139112$

$\Rightarrow \quad Z \quad = 526{,}726.25$

10 (i) DMT $= \dfrac{1.v+2v^2+\ldots+10v^{10}}{v+v^2+\ldots+v^{10}}$

$= \dfrac{(Ia)_{\overline{10|}}}{a_{\overline{10|}}} = \dfrac{\ddot{a}_{\overline{10|}}-10v^{10}}{i}\Big/ a_{\overline{10|}}$

$= \dfrac{7.0236\times 1.07-10\times 0.50835}{0.07\times 7.0236}$

$= 4.946$ as required

(ii) We will consider three conditions necessary for immunisation

(1) $V_A = V_L$

$V_A = a_{\overline{10|}} + Xv^n$ at 7%

$= 7.0236 + Xv^n$

$V_L = 7v^5 + 8v^8$ at 7%

$= 9.64698$

$\Rightarrow Xv^n = 2.62338$ (a)

(2) $V_A' = V_L'$ where $V_A' = \dfrac{dV_A}{d\delta}$ and $V_L' = \dfrac{dV_L}{d\delta}$

$V_A' = (Ia)_{\overline{10|}} + n\,.\,Xv^n = 4.946 \times 7.0236 + n\,.\,Xv^n$

$= 34.7393 + n\,.\,Xv^n$

$V_L' = 5 \times 7v^5 + 8 \times 8v^8$

$= 62.20310$

$\Rightarrow\; n\,.\,Xv^n = 27.46380$ (b)

(a) and (b) $\Rightarrow n = 10.46886$

$\Rightarrow X = 2.62338 \times (1.07)^{10.46886}$

$= 5.32692$

(3) $V_A'' > V_L''$

$$V_A'' = \sum_{t=1}^{10} t^2\,.\,v^t + n^2\,.\,Xv^n$$

$= 228.451 + (10.46886)^2 \times 5.32692 \times v^{10.46886}$

$= 515.966$

$V_L'' = 5^2 \times 7v^5 + 8^2 \times 8v^8$

$= 422.761$

$\Rightarrow$ Condition (3) satisfied

Thus, $X = 5.32692m$ and $n = 10.46886$ years will achieve immunisation.

11 (i) (a) Let R = total annual repayment

$$R = \frac{80{,}000}{a_{\overline{25|}}^{(12)}} \text{ at } 8\%$$

$$= \frac{80{,}000}{1.036157 \times 10.6748} = £7{,}232.77 \text{ p.a.}$$

$\Rightarrow$ Monthly instalment = £602.73 per month

Interest in 1st instalment $= 80{,}000 \times \left(1.08^{\frac{1}{12}} - 1\right)$

$= 514.72$

$\Rightarrow$ Capital in 1st instalment $= 602.73 - 514.72$

$= £88.01$

(b) Capital outstanding after 19 years = 7,232.77 $a_{\overline{6}|}^{(12)}$ at 8%

$= 7{,}232.77 \times 1.036157 \times 4.6229$

$= 34{,}645.33$

Total amount of instalments paid in last 6 years

$= 6 \times 7232.77 = 43{,}396.62$

Hence, total interest paid in last 6 years =

$43{,}396.62 - 34{,}645.33 = £8{,}751.29$

(c) Capital outstanding = $602.73 v^{\frac{1}{12}} = 598.88$

$\Rightarrow$ Interest $= 602.73 - 598.88 = £3.85$

(ii) If repayments are made less frequently than monthly, the total annual repayment increases since the borrower makes interest payments less frequently.

The amount of capital outstanding after 19 years will be unaltered. Therefore, the total interest repaid during the last 6 years (being the difference between the total payments and total capital repaid in the last 6 years) will increase.

12 (i) The coupon will be:

$$\frac{0.03 \times 100}{2} \times \left(\frac{\text{Index}_{\text{July 1999}}}{\text{Index}_{\text{July 1995}}}\right)$$

$$= \quad 1.5 \times \frac{126.7}{110.5} = £1.72 \text{ per } £100 \text{ nominal}$$

(ii) T*here are many ways candidates may layout their solution.*

Measure time, t, in half years from 16 September 1999 and let i be the real yield per half year.

Set $(1 + r) = (1.04)^{½}$ and month 0 = September 1999

Then $Q(t) = Q(0)\ (1 + r)^t$ is the estimated value of the index at time t, where

$$Q(0) = 127.4$$

The first interest payment at time 1 is 1.72 and the value of the index will be

$$Q(1) = Q(0)\ .\ (1 + r)$$

For $t \geq 2$, the investor's t^{th} interest payment will be received in month $6t$, and will be of amount

$$1.5 \times \frac{(1+r)^{\frac{(6t-8)}{6}}}{110.5} \times Q(0)$$

$$= \quad 1.5\ Q(0)\ \frac{(1+r)^{\left(t-\frac{4}{3}\right)}}{110.5}$$

This payment will be received at time t, when the value of the index will be $Q(t)$.

Redemption proceeds will be paid at time 5 with the final coupon payment.

The redemption proceeds will be

$$\frac{100Q(0)\ (1+r)^{\left(5-\frac{4}{3}\right)}}{110.5}$$

and the value of the index will be $Q(0)\ (1 + r)^5$

Thus the real yield equation is:

$$111.0 = \frac{1.72}{(1+r)}\,.\,v + \sum_{t=2}^{5} \frac{1.5Q(0)\,(1+r)^{\left(t-\frac{4}{3}\right)}}{110.5\times(1+r)^t}\,v^t$$

$$+\frac{100Q(0)\,(1+r)^{\left(5-\frac{4}{3}\right)}}{110.5\times(1+r)^5}\,v^5 \qquad (*)$$

i.e. $$111.0 = \frac{1.72}{(1+r)}\,.\,v + 1.5\times\frac{127.4}{110.5}\times(1+r)^{\frac{-4}{3}}\sum_{t=2}^{5} v^t$$

$$+100\times\frac{127.4}{110.5}\,(1+r)^{\frac{-4}{3}}\,v^5$$

$\Rightarrow$ $$111.0 = 1.6866v + 1.6848\ (a_{\overline{5}|} - v)$$

$$+\ 112.3186v^5$$

$\Rightarrow$ $$111.0 = 0.0018v + 1.6848\,a_{\overline{5}|} + 112.3186v^5$$

At 2%, RHS = 109.67

At 1½%, RHS = 112.32

Linear interpolation: $i \simeq 0.020 - \left(\frac{111-109.67}{112.32-109.67}\times 0.005\right)$

$= 0.01749$

$\Rightarrow$ yield for year is 3.53% p.a.

(iii) From equation (*), if the retail price index had been greater than 110.5, the right hand side would be less than 111.0 with i = 1.749% per half year. Hence, the real yield, i, would need to be less than 1.749% per half year for the right hand side to equal 111.0.

$\Rightarrow$ Real yield decreases