Faculty of Actuaries | Institute of Actuaries

EXAMINATIONS

September 2000

Subject 102 — Financial Mathematics

EXAMINERS' REPORT

1 One party agrees to pay a floating rate and receive a fixed interest rate and the other party agrees to pay a fixed interest rate and receive a floating interest rate. The fixed payments are at a constant rate for an agreed term and the floating payments will be linked to the level of a short-term interest rate.

2 $$\left(1-\frac{90}{365}\times 0.05\right) = \left(1+\frac{i^{(2)}}{2}\right)^{-\frac{90}{182.5}}$$

$$\Rightarrow \left[\left(1-\frac{90}{365}\times 0.05\right)^{-\frac{182.5}{90}} -1\right]\times 2 = i^{(2)}$$

$$\Rightarrow i^{(2)} = 5.0949\%$$

3 Difference between holding the asset and holding the forward is that in the latter case, the purchase price can be invested at the risk free rate,

hence

$$f_{91} = 1.2 \times (1.05)^{\frac{91}{365}} = 1.2147$$

4 A preference share pays a dividend which is generally fixed. An ordinary share pays a dividend out of residual profits which is at the discretion of the company. A preference share dividend is a prior charge so that, in general, no ordinary share dividend can be paid if a preference share dividend is outstanding. A preference share holder may have no voting rights and is likely to get prior ranking in a winding up.

5 $$96 = 4a_{\overline{20|}} + 100v^{20}$$

@ i = 4% RHS = 100

@ i = 4.5% $a_{\overline{20|}}$ = 13.00794 v^{20} = 0.414643

RHS = 93.4961

Interpolation gives $i = 0.04 + 0.005 \times \left(\frac{100-96}{100-93.4961}\right)$

$= 4.3\%$

The answer may be stated as a semi-annual yield.

6 (a) $E(i) = 0.3 \times 7 + 0.5 \times 8 + 0.2 \times 10 = 8.1\%$

Single premium is X such that $X(1.081)^{10} = 10{,}000$

Hence $X = £4{,}589.26$

(b) Expected profit is

$4{,}589.26[0.3 \times (1.07)^{10} + 0.5 \times (1.08)^{10} + 0.2(1.1)^{10}] - 10{,}000$

$= £42.94$

7 $$250 = 1.03^{\frac{1}{4}}\left(\frac{10v^{\frac{9}{12}}}{1.03} + \frac{10(1.05)v^{1\frac{9}{12}}}{1.03^2} + \frac{10(1.05)^2 v^{2\frac{9}{12}}}{1.03^3} + \ldots\right)$$

$$\Rightarrow 250 = \left(\frac{10v^{\frac{9}{12}} \times (1.03)^{-1}}{1 - \left(\frac{1.05}{1.03}\right)v}\right) \times 1.03^{\frac{1}{4}}$$

$$\Rightarrow 250 = 254.854v + 9.78075v^{\frac{9}{12}}$$

Real return should be approximately 6% (= dividend yield + dividend growth − inflation)

Try $i = 6\%$, $v = 0.943396$

RHS = 249.791

For $i = 5.5\%$, $v = 0.947867$, RHS = 250.9636

Interpolation gives $\left(\frac{250 - 249.791}{250.9636 - 249.791}\right) \times (0.947867 - 0.943396)$

$+ 0.943396 = 0.944193$

$\Rightarrow i = 5.91\%$

8 $(1 + i)$ is lognormally distributed with mean 1.001 and variance 4×10^{-6}

$$\Rightarrow \quad 1.001 = \exp\left[\mu + \frac{\sigma^2}{2}\right]$$

$$4 \times 10^{-6} = \exp[2\mu + \sigma^2] \times [\exp(\sigma^2) - 1]$$

$$\Rightarrow \quad \frac{4 \times 10^{-6}}{1.001^2} = \exp(\sigma^2) - 1$$

$$\Rightarrow \quad \sigma^2 = 3.992 \times 10^{-6}$$

$$\Rightarrow \quad \mu = 0.0009975$$

$$\Rightarrow \quad \ln(1 + i) \sim N(9.9975 \times 10^{-4}, 3.992 \times 10^{-6})$$

$$\Rightarrow \quad \Pr\left(\frac{\ln(1+i) - 9.975 \times 10^{-4}}{\sqrt{3.992 \times 10^{-6}}} \le -1.645\right) = 0.05$$

$$\Rightarrow \quad \Pr(\ln(1 + i) \le -2.28921 \times 10^{-3}) = 0.05$$

$$\Rightarrow \quad \Pr(i \le -0.002287) = 0.05$$

$$\Rightarrow \quad j = -0.2287\%$$

9 (i) $(Ia)_{\overline{n}|} = v + 2v^2 + 3v^3 + \ldots + nv^n$

$$(1 + i)\,(Ia)_{\overline{n}|} = 1 + 2v + 3v^2 + \ldots + nv^{n-1}$$

$$(1 + i)\,(Ia)_{\overline{n}|} - (Ia)_{\overline{n}|} = i(Ia)_{\overline{n}|}$$

$$\Rightarrow \quad i(Ia)_{\overline{n}|} = 1 + v + v^2 + \ldots + v^{n-1} - nv^n$$

$$= \ddot{a}_{\overline{n}|} - nv^n$$

$$\Rightarrow \quad (Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i}$$

(ii) (a) $10v + 12v^2 + 14v^3 + \ldots + 48v^{20}$

$$= 2v + 4v^2 + 6v^3 + \ldots + 40v^{20}$$

$$+ 8v + 8v^2 + 8v^3 + \ldots + 8v^{20}$$

$= 2(Ia)_{\overline{20|}} + 8a_{\overline{20|}}$ @ 3%

$= \quad 2\left(\dfrac{\ddot{a}_{20} - 20v^{20}}{0.03}\right) + 8 \times a_{\overline{20|}}$ @ 3%

$a_{\overline{20|}} = 14.8775 \qquad v^{20} = 0.55368$

$\ddot{a}_{\overline{20|}} = 15.3238$

$= \quad 283.3467 + 119.0200 = 402.3667$

(b) 16th payment is 40

Capital o/s is $2(Ia)a_{\overline{5|}} + 38a_{\overline{5|}}$

$= \quad 2 \times \dfrac{\ddot{a}_{\overline{5|}} - 5v^5}{0.03} + 38a_{\overline{5|}}$

$\ddot{a}_{\overline{5|}} = 4.71710 \qquad v^5 = 0.86261 \qquad a_{\overline{5|}} = 4.57971$

$\Rightarrow$ capital o/s $= 26.93667 + 174.02898$

$= 200.96565$

(c) Interest component is $0.03 \times 200.96565 = 6.03$

Hence capital component is $40 - 6.03 = 33.97$

10 (i) $A(0, 5) = e^{\int_0^5 0.04dt} = \exp[0.04t]_0^5$

$= \exp[0.04 \times 5]$

$= 1.2214$

$A(5, 10) = e^{\int_5^{10} 0.01t^2 - 0.01t\, dt}$

$= \exp\left[\dfrac{0.01t^3}{3} - \dfrac{0.01t^2}{2}\right]_5^{10}$

$\Rightarrow$ present value = 0.06447

(ii) A(9, 10) $= e^{\int_9^{10} 0.01t^2 - 0.01t\, dt}$

$$= \exp\left[\frac{0.01t^3}{3} - \frac{0.01t^2}{2}\right]_9^{10}$$

$\Rightarrow i = 1.2441 = 124.41\%$

(iii) (a) $e^{-\int_0^t \delta(s)ds} = e^{-\int_0^t 0.04 ds} = e^{-0.04t}$

(b) $\int_0^5 \rho(t)\, v(t)\, dt = \int_0^5 e^{0.04t} \times e^{-0.04t}\ dt$

$$= \int_0^5 e^0\ dt$$

$= 5$

11 (i) (a) Clearly $i_1 = 6\%$

$p_2 = 6(1.06)^{-1} + 106(1 + i_2)^{-2}$

$p_2 = 6a_{\overline{2}|} + 100v^2$ @ 6.3%

$$\Rightarrow (1 + i_2)^{-2} = \frac{6a_{\overline{2}|6.3\%} + 100v^2_{6.3\%} - 6(1.06)^{-1}}{106}$$

$a_{\overline{2}|6.3\%} = 1.82571$ $\quad 1.06^{-1} = 0.94340$

$v^2_{6.3\%} = 0.88498$

$\Rightarrow (1 + i_2)^{-2} = 0.884829 \quad i_2 = 6.30907\%$

$p_3 = 6(1.06)^{-1} + 6(1.0630907)^{-2} + 106(1 + i_3)^{-3}$

$p_3 = 6a_{\overline{3}|6.6\%} + 100v^3_{6.6\%}$

$$\Rightarrow \ (1+i_3)^{-3} = \frac{6a_{\overline{3}|6.6\%} + 100v^3_{6.6\%} - 6(1.06)^{-1} - 6(1.0630907)^{-2}}{106}$$

$a_{\overline{3}|6.6\%} = 2.643614$

$v^3_{6.6} = 0.82552$

$1.06^{-1} = 0.94340$

$1.0630907^{-2} = 0.884829$

$\Rightarrow \ (1+i_3)^{-3} = 0.824947 \Rightarrow i_3 = 6.62476\%$

(b) $f_{0,1} = 6\%$

$(1+f_{1,1})\,(1.06) = (1.0630907)^2 \Rightarrow f_{1,1} = 6.61904\%$

$(1+f_{2,1})\,(1.0630907)^2 = (1.0662476)^3$

$\Rightarrow f_{2,1} = 7.25896\%$

(ii) The accumulation factors related to spot rates are geometric averages of accumulation factors related to the forward rate for the same year as well as for all those relating to all previous years. Therefore, if forward rates are increasing, spot rates, being an average of a spot rate for the given year and for previous years, will increase more slowly.

12 (i) PV of outgoings $= 180{,}000\ \ddot{a}_{\overline{3}|7\%}$

$= 180{,}000 \times 2.80802$

$= 505{,}443.2701$

PV of incoming payments $= 25{,}000 \displaystyle\int_0^{25} (1.06)^t\,(1.07)^{-t}\,dt$

This is equivalent to present value of annuity with $v^t = \left(\dfrac{1.06}{1.07}\right)^t$

$= 0.990654^t$ and $i = 0.943396\%$

$\equiv\ 25{,}000\bar{a}_{\overline{25}|}$ @ $i = 0.943396\%$

$= \ 25{,}000\left(\dfrac{1-0.990654^{25}}{\delta}\right) \qquad \delta = 0.00938974$

= 557,061.0639

$\Rightarrow$ NPV = £51617.79

(ii) Need to find t such that

$$505{,}443.2701 = 25{,}000\left(\frac{1-0.990654^t}{0.00938974}\right)$$

$\Rightarrow$ $0.189839 = 1 - 0.990654^t$

$\Rightarrow$ $0.990654^t = 0.810161$

$\Rightarrow$ $t = 22.42$ years (NB time is not an integer)

(iii) We require g such that

$$505{,}443.2701 = 25{,}000 \int_0^{25} (1+g)^t \ (1.07)^{-t}\, dt$$

$$\Rightarrow\ 505{,}443.2701 = 25{,}000 \left(\frac{1-v^{25}}{\delta}\right)$$

where v and δ are calculated such that $v = \dfrac{1+g}{1.07}$

$$\Rightarrow\ 20.217731 = \frac{1-v^{25}}{\delta}$$

Let i' be such that $\dfrac{1}{1+i'} = v$

Let $i' = 1.5\%$ $\quad \delta = 0.0148886$

$v^{25} = 0.689206$

RHS = 20.87463

Let $i' = 2\%$ $\quad \delta = 0.019803$

$v^{25} = 0.609531$

RHS = 19.7180

Interpolation gives $0.015 + \left(\dfrac{20.87463 - 20.21773}{20.87463 - 19.7180}\right) \times 0.005$

$i = 0.01784$

$\Rightarrow\ v = 0.98247$

$\Rightarrow\ g = 5.12458\%$

13 (i) (a) $100{,}000\ a_{\overline{5}|}$ @ 5% $\qquad a_{\overline{5}|} = 4.32948$

$\Rightarrow$ PV = 432,947.667

(b) $\displaystyle\sum_{t=1}^{5} t \times 100{,}000 \times v^t = 100{,}000\ (Ia)_{\overline{5}|}$ @ 5%

$$(Ia)_{\overline{5}|} = \frac{\ddot{a}_{\overline{5}|} - 5v^5}{i} = 12.5664 \text{ @ } 5\%$$

numerator = 1,256,640

$\Rightarrow$ Duration = 1,256,640 / 432,947.667

= 2.9

(ii) PV of zero coupon bond holdings is:

$$Av^5 + Bv = 432{,}947.667 \qquad (1)$$

Duration numerator is:

$$5Av^5 + Bv = 1{,}256{,}640 \qquad (2)$$

$\Rightarrow\quad 5Av^5 - Av^5 = 1{,}256{,}640 - 432{,}947.667$

$\Rightarrow\quad A = \dfrac{823{,}692.333}{4v^5} \qquad v^5 = 0.783526$

$\Rightarrow\quad A = 262{,}815.89$

Using equation (1),

$$B = \frac{432{,}947.667 - 262{,}815.89v^5}{v}$$

$v = 0.952381\ \Rightarrow\ B = 238{,}375.80$

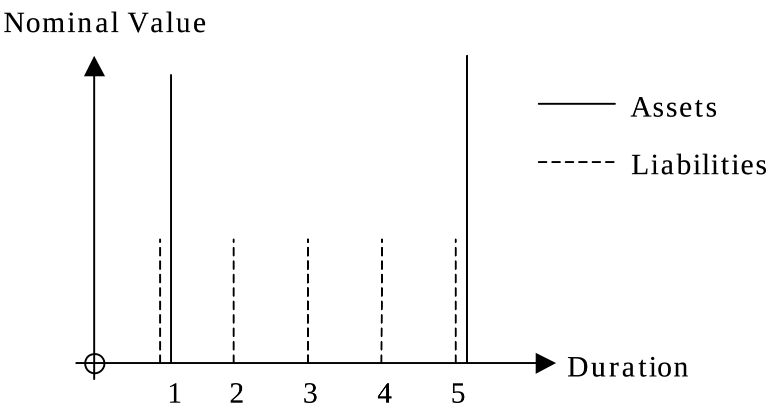
(iii) **Convexity of assets**

$25Av^5 + Bv = 25 \times 262815.89 \times (1.05)^{-5} + 238375.80\ (1.05)^{-1}$

$= 5375102.741$

$$\text{Convexity} = \frac{(1.05)^{-2} \times 5375102.741 + (1.05)^{-2} \times 1256640}{432947.667}$$

$$= 13.89356 \qquad 13.89 \text{ years}^2.$$



Assets appear to be more "spread out" than liabilities, so likely that immunisation against small changes in market rate (in Redington sense) has been achieved.

Check — Convexity of liabilities

$100000(v + 4v^2 + 9v^2 + 16v^3 + 25v^5) = 4510643.101$

$$\text{Convexity} = \frac{(1.05)^{-2}\left\{4510643.101 + 1256640\right\}}{432947.667}$$

$$= 12.0825 \qquad 12.08 \text{ years}^2.$$

So the surmise is true.