REPORT OF THE BOARD OF EXAMINERS

September 2003

Subject 106 — Actuarial Mathematics 2

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

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Chairman of the Board of Examiners

11 November 2003

EXAMINATIONS

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In numerical questions, candidates were not unduly penalised for errors in earlier parts of each question which affected their answers to the rest of the question.

1 Loadings: commission
expenses
profit
other contingencies

Adjust basic values for:

unusual experience
large claims
trends in claims
changes in risk, cover, cost of reinsurance

2 Given an event, distribution of number of claims has

mean, $E[N] = 2.5$
variance, $V[N] = 3.75$

(using formulae in tables)

The mean and variance of the number of claims annually are

$$120 \times 0.02 \times 2.5 = 0.05 \times 120 = 6$$

and $120 \times (0.02 \times 3.75 + 0.02 \times 0.98 \times 2.5^2) = 120 \times 0.1975 = 23.7$.

In Question 2, a common mistake was to find the mean and variance of the wrong type of negative binomial.

3 (i) Let X be a typical claim size, so X has a Pareto distribution with parameters $\alpha = 3$ and $\lambda = 1$, and has mean 0.5 and second moment 1.

The number N of claims by $t = 2$ has a Poisson distribution with mean 40.

Let S be the total amount claimed by $t = 2$.

Then

$$E(S) = E(X)E(N) = 20,$$

and

$$V(S) = E(N)E(X^2) = 40.$$

(ii) The premium income by $t = 2$ is $(1 + \theta) \times 40 \times 0.5 = 26$, so probability of ruin at $t = 2$ is

$$P(S > 10 + 26) = P\left(\frac{S-20}{\sqrt{40}} > \frac{36-20}{\sqrt{40}}\right)$$

$$= P\left(\frac{S-20}{\sqrt{40}} > 2.53\right)$$

$$\approx 1 - \Phi(2.53)$$

$$= 0.0057$$

Question 3 was well answered.

4 (i) Let $M(r)$ be the moment generating function of the gamma distribution, so $M(r) = (\alpha/(\alpha - r))^2$, for $r < \alpha$.

The adjustment coefficient R solves

$$M(r) = 1 + (1 + 0.5)\frac{2r}{\alpha},$$

which is

$$\left(\frac{\alpha}{\alpha - r}\right)^2 = 1 + \frac{3r}{\alpha}.$$

We solve

$$\alpha^3 = \alpha(\alpha - r)^2 + 3r(\alpha - r)^2,$$

which simplifies to

$$r(3r^2 - 5\alpha r + \alpha^2) = 0.$$

Solving the quadratic, we find roots

$$r = \frac{5\alpha \pm \sqrt{25\alpha^2 - 12\alpha^2}}{6}$$

giving $r = \alpha(5 \pm \sqrt{13})/6 = 1.434\alpha,\ 0.232\alpha$.

Since R must be in $(0, \alpha)$, we have $R = 0.232\alpha$.

(ii) (a) By Lundberg's inequality, the probability of ruin $\psi(u)$ with initial capital u satisfies $\psi(u) \le e^{-0.232\alpha u}$ for all $u > 0$.

(b) This upper bound decreases as α increases.

This makes intuitive sense since if α increases, the claim sizes are on average smaller and we expect a smaller probability of ruin.

5 (i) $P(X > M) = e^{-cM^2}$

$$e^{-1{,}000^2 c} = 0.01$$

$$c = -\frac{\log 0.01}{1{,}000^2} = 0.000004605$$

(ii) $P(X > M) = e^{-\lambda M}$

$$e^{-1{,}000\lambda} = 0.01$$

$$\lambda = -\frac{\log 0.01}{1{,}000} = 0.004605$$

(iii) (a) $f_1(x)$: $e^{-cM^2} = 0.001$

$$M^2 = -\frac{\log 0.001}{c} = 1{,}500{,}000$$

$\therefore M = 1{,}225$

$f_2(x)$: $e^{-\lambda M} = 0.001$

$$M = -\frac{\log 0.001}{\lambda} = 1{,}500$$

(b) The Weibull distribution has a lower amount for the 1 in 1,000 claim. It has a lighter tail.

6 Column totals:

4,700	4,063	2,101	
6,714	5,803	4,941	2,209

Chain ladder development factors:

1.235	1.216	1.051

For 2001, $f = 1.57867$.

Initial estimate of ultimate is 3,599.04.

Estimate of outstanding:

$$3{,}599.04 \times \left(1 - \frac{1}{1.57867}\right) = 1{,}319.25\,.$$

Bornhuetter-Ferguson estimate of ultimate is 3,333.25.

Paid to date: 561.

Estimate of outstanding is <u>2,772.25</u>.

Question 6 was well answered.

7 (i) For a Beta distribution, $\mu = \alpha / (\alpha + \beta) = 0.2$ and

$\sigma^2 = \alpha\beta / [(\alpha + \beta)^2.(\alpha + \beta + 1)] = 0.25^2$

From μ we can see that $\beta = 5\alpha - \alpha = 4\alpha$

From μ and σ^2: $0.25^2/0.20^2 = \beta / [\alpha.(\alpha + \beta + 1)]$

hence: $1.5625 = 4\alpha / (5\alpha^2 + \alpha) = 4 / (5\alpha + 1)$

Rearranging gives: $\alpha = 0.312$ and $\beta = 1.248$

(ii) Likelihood is Binomial where $n = 50$ and $x = 12$

Posterior $\propto$ prior $\times$ likelihood

$\propto \theta^{\alpha-1}.(1-\theta)^{\beta-1}.\ \theta^{x}.(1-\theta)^{n-x}$

$\propto \theta^{-0.688}.(1-\theta)^{0.248}.\ \theta^{12}.(1-\theta)^{38}$

$\propto \theta^{11.312}.(1-\theta)^{38.248}$

= Beta (12.312, 39.248)

Posterior mean = 12.312/51.560

= 0.239

(iii) Posterior $\propto \theta^{\alpha-1}.(1-\theta)^{\beta-1}.\ \theta^{x}.(1-\theta)^{n-x}$

$\propto \theta^{\alpha+x-1}.(1-\theta)^{\beta+n-x-1}$

= Beta $(\alpha + x, \beta + n - x)$

Posterior mean $= (\alpha + x) / (\alpha + x + \beta + n - x)$

$= \alpha / (\alpha + \beta + n) + x / (\alpha + \beta + n)$

$= [(\alpha + \beta) / (\alpha + \beta + n)].[\alpha / (\alpha + \beta)] + [n / (\alpha + \beta + n)].[x / n]$

$= Z.(x / n) + (1 - Z).\mu$

where $Z = n / (\alpha + \beta + n)$

(iv) (a) $Z = 50/51.56 = 0.970$

Z represents the weight we place on the sample data.

(b) (1) If the standard deviation of the prior increases, we place less weight on the collateral data and more on the sample data, therefore Z would increase.

(2) If the sample size increases, we place more weight on the sample data, therefore Z would increase.

(c) The limiting value of Z as σ and/or n tend to infinity, is 1.

This means that we place all weight on the sample data and ignore the collateral information

Question 7(iv)(a), many candidates used the posterior values of α and β in the formulae for Z, instead of the prior values found in 7(i).

8 (i) If Y has a Poisson distribution with mean μ, then

$$f(y, \mu) = e^{-\mu}\mu^y / y! = \exp\left(\frac{y \log \mu - \mu}{1} - \log y!\right),$$

which is of exponential family form.

The link function is $g(\mu) = \log(\mu)$.

The linear predictor is $\eta = \alpha_i$.

So this is a generalised linear model.

(ii) The likelihood is

$$\prod_{i=1}^{3}\prod_{j=1}^{m}\frac{e^{-\mu_{ij}}\mu_{ij}^{y_{ij}}}{y_{ij}!},$$

so the log-likelihood is

$$\sum_{i=1}^{3}\sum_{j=1}^{m}(-\mu_{ij} + y_{ij}\log(\mu_{ij}) - \log(y_{ij}!))$$

i.e., in terms of α_i's, writing y_{i+} for the sum of the observations in the ith group, the log-likelihood is

$$l(\alpha_1, \alpha_2, \alpha_3) = -\sum_{i=1}^{3} me^{\alpha_i} + \sum_{i=1}^{3} y_{i+}\alpha_i + \text{constant}.$$

Differentiating,

$$\frac{\partial l}{\partial \alpha_i} = -me^{\alpha_i} + y_{i+},$$

so the maximum likelihood estimator of α_i is

$$\hat{\alpha}_i = \log(y_{i+}/\mathrm{m}).$$

(iii) *Comparing models 2 and 3*:

There are 60 observations altogether.

Model 3 has one parameter estimate, and so has degrees of freedom 59.

Model 2 has degrees of freedom 58.

The drop in deviance in going from model 3 to model 2 is 72.53 − 61.64 = 10.89.

The corresponding drop in degrees of freedom is 59 −58 = 1.

So to test for a significant improvement, compare 10.89 to a χ_1^2.

The upper 5% point of χ_1^2 is 3.841, the upper 1% point is 6.635, this is a significant improvement. We prefer model 2 to model 3.

Comparing models 2 and 1:

Model 1 has degrees of freedom 57.

The drop in deviance is 61.64 − 60.40 = 1.24, and this should be compared to χ_1^2.

It is not significant; do not prefer model 1 to model 2.

(iv) *Interpretation of models*:

Model 3 says that there is no difference in the average number of claims for the three age groups.

Model 2 says that there is no difference in the average number of claims between age groups 1 and 2, but that the third age group may be different.

Model 1 gives the possibility of different average number of claims for each age group.

In Question (i), showing the exponential family form was well done, but some candidates did not state the linear predictor correctly. Question 8(ii) was poorly done, with common mistakes being to omit the product over I for the likelihood, and algebraic mistakes in *deriving the log likelihood in terms of α_i's from the likelihood. In Question 8(iii), full marks were also given if candidates used the method given in Core Reading of comparing the drop in deviance with twice the change in the number of parameters. Question (iii) was well done. However, some candidates did not state how they arrived at their conclusions about whether models were significant improvements or not.*

9 (i) mean and standard deviation = 500

therefore $\lambda = 0.002$

The expected amount per claim ceded to the reinsurer is:

$$\int_{2,500}^{\infty} (x - 2,500)\lambda e^{-\lambda x}.dx$$

if $z = x - 2500$

this gives $\int_0^{\infty} z\lambda e^{-\lambda(z+M)}.dz$

$$= e^{-\lambda M} \int_0^{\infty} z\lambda e^{-\lambda}.dz$$

the expression in the integral is the mean of an exponential distribution, parameter λ, therefore:

$$= e^{-\lambda M}(1/\lambda)$$

$= 500e^{-5}$ i.e. 3.3690 per claim

For the whole portfolio:

= 3.3690 × 0.30 × 200
= £202.14

(ii) The expected amount ceded implies that the reinsurer has applied a loading for expenses and profit of:

= 300/202.14 – 1

= 48.4%

The insurer would pay a premium that is significantly greater than the expected value of the corresponding risk ceded.

This may be acceptable depending on the insurer's attitude to risk

(iii) $\exp(\mu + \tfrac{1}{2}\sigma^2) = 500 \quad \exp(2\mu + \sigma^2).[\exp(\sigma^2) - 1] = 500^2$

$\exp(\sigma^2) - 1 = 500^2/500^2$

$\sigma^2 = \ln(1 + 1) = 0.69315$

$\mu = \ln 500 - 0.34657 = 5.86803$

The expected amount per claim ceded to the reinsurer is:

$$\int_{2,500}^{\infty} xf(x).dx - M.P(X > M)$$

$$\int_{2,500}^{\infty} xf(x).dx = e^{\mu+\frac{1}{2}\sigma^2}\left(1-\Phi\frac{\ln 2,500-\mu-\sigma^2}{\sigma}\right)$$

$= 500[1 - \Phi(1.517)]$

$= 32.32$

and $2,500.P[X > 2,500] = 2,500.[1 - \Phi((\ln 2,500 - \mu)/\sigma)]$

$= 2,500.[1 - \Phi(2.349)]$

$= 23.54$

therefore expected amount ceded per claim:

$= 32.32 - 23.54$

$= 8.78$ per claim

For the whole portfolio:

$= 8.78 \times 0.30 \times 200$

$= £527$

(iv) The expected amount ceded is greater than the reinsurance premium by 527/300 = 175.6%.

The reason for the increase is that the lognormal distribution has a much heavier tail than the exponential distribution.

The insurer's profit is expected to increase if *C* agrees to take out the reinsurance policy.

The insurer should take out the policy irrespective of its attitude to risk.

10 (i) $\lambda / (\alpha - 1) = 1{,}500{,}000 \qquad \alpha\lambda^2 / [(\alpha - 1)^2(\alpha - 2)] = (2{,}500{,}000)^2$

$(2{,}500{,}000)^2 / (1{,}500{,}000)^2 = \alpha / (\alpha - 2) = 2.77778$

therefore $\alpha = 3.125$

so $\lambda = 3{,}187{,}500$

Expected loss without reinsurance = £1,500,000

Expected amount ceded to reinsurer is:

$$\int_{2,000,000}^{\infty} (x - 2{,}000{,}000) f(x).dx$$

$$\int_{2,000,000}^{\infty} x \frac{3.125.\lambda^{3.125}}{(\lambda + x)^{4.125}}.dx = \left[-x.(\lambda/(\lambda + x))^{3.125}\right]_{2,000,000}^{\infty}$$

$$+\int_{2,000,000}^{\infty} x \frac{\lambda^{3.125}}{(\lambda + x)^{3.125}}.dx$$

$$= 2{,}000{,}000.(3{,}187{,}500/5{,}187{,}500)^{3.125} + \left[-\frac{\lambda^{3.125}}{2.125.(\lambda + x)^{2.125}}\right]_{2,000,000}^{\infty}$$

$= 2{,}000{,}000 \times (51 / 83)^{3.125} + 3{,}187{,}500^{3.125} / (2.125 \times 5{,}187{,}500^{2.125})$

$= 436{,}584 + 532{,}889$

$= 969{,}473$

$$2{,}000{,}000.\int_{2,000,000}^{\infty} f(x).dx = 2{,}000{,}000.(3{,}187{,}500/5{,}187{,}500)^{3.125}$$

$= 436{,}584$

Therefore amount ceded to reinsurer is £532,889.

Expected amount payable with reinsurance = 1,500,000 – 532,889 = 967,211

(ii) Annual expected profits

D_0 Reinsurance premium £0
Net total claims: Θ_0 = £0
Θ_1 = £1,500,000
Θ_2 = £1,500,000 × 2 = £3,000,000
Θ_3 = £1,500,000 × 3 = £4,500,000

D_1 Reinsurance premium £500,000
Net total claims: Θ_0 = £0
Θ_1 = £967,111
Θ_2 = £967,111+ £1,500,000 = 2,467,111
Θ_3 = £967,111 + 2 × £1,500,000 = 3,967,111

D_2 Reinsurance premium £1,000,000
Net total claims: Θ_0 = £0
Θ_1 = £967,111
Θ_2 = £967,111 × 2 = £1,934,222
Θ_3 = £967,111 × 2 + £1,500,000 = £3,434,222

Hence decision matrix

	Θ_0 '000	Θ_1 '000	Θ_2 '000	Θ_3 '000
D_0	0	1,500	3,000	4,500
D_1	500	1,467	2,967	4,467
D_2	1,000	1,967	2,934	4,434

(iii) (a) Minimax = minimise maximum loss.

D_0 4,500
D_1 4,467
D_2 **4,434** ← answer is D_2

(b) The insurer would expect to minimise the maximum loss by taking out the policy offering the most reinsurance, i.e. D_2.

(iv) $P(\text{0 claims}) = e^{-0.90} = 0.407$
$P(\text{1 claim}) = 0.90e^{-0.90} = 0.365$
$P(\text{2 claims}) = 0.90^2 e^{-0.90}/2 = 0.165$
$P(\text{>2 claims}) = 1 - 0.407 - 0.365 - 0.165 = 0.063$

Expected loss

$D_0 = (0.407\times0) + (0.365\times1{,}500) + (0.165\times3{,}000) + (0.063\times4{,}500) = \mathbf{1{,}326}$
$D_1 = (0.407\times500) + (0.365\times1{,}467) + (0.165\times2{,}967) + (0.063\times4{,}467) = 1{,}509$
$D_2 = (0.407\times1{,}000) + (0.365\times1{,}967) + (0.165\times2{,}934) + (0.063\times4{,}434) = 1{,}888$

Therefore the answer is D_0 (no reinsurance).

In Question 10(i), many candidates made errors in evaluating the first integral.