Faculty of Actuaries **Institute of Actuaries**

EXAMINATIONS

14 September 2000 (am)

Subject 106 — Actuarial Mathematics 2

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Write your surname in full, the initials of your other names and your Candidate's Number on the front of the answer booklet.*

2. *Mark allocations are shown in brackets.*

3. *Attempt all 9 questions, beginning your answer to each question on a separate sheet.*

Graph paper is not required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet and this question paper.

In addition to this paper you should have available, Actuarial Tables and an electronic calculator.

1 An actuarial student is considering his post-examination holiday plans. His travel agent tells him that there will definitely be vacancies in either Hotel A or Hotel B but not at both. He must book his flight now to take advantage of current huge discounts on flight prices. Flights to airport 1 cost £110 and flights to airport 2 cost £160. Accommodation at Hotel A costs £890 and accommodation at Hotel B costs £490. If he chooses to fly to airport 1, a taxi to Hotel A will cost £10 and a taxi to Hotel B will cost £70. If he chooses to fly to airport 2, a taxi to Hotel A will cost £65 and a taxi to Hotel B will cost £5. The student believes there is a 90% chance of the vacancies being at Hotel B. To which airport should he book a flight in order to minimise the expected cost? [4]

2 In the context of Generalised Linear Models, consider the exponential distribution with density function $f(x)$, where

$$f(x) = \frac{1}{\mu} e^{-x/\mu} \qquad (x > 0).$$

(i) Show that $f(x)$ can be written in the form of an exponential family of distributions. [1]

(ii) Show that the canonical parameter, θ, is given by $\theta = -\frac{1}{\mu}$. [1]

(iii) Determine the variance function and the dispersion parameter. [3]
[Total 5]

3 The table below shows annual aggregate claim statistics for 3 risks over 4 years. Annual aggregate claims for risk i, in year j are denoted by X_{ij}.

Risk i	$\bar{X}_i = \frac{1}{4}\sum_{j=1}^{4} X_{ij}$	$s_i^2 = \frac{1}{3}\sum_{j=1}^{4}(X_{ij} - \bar{X}_i)^2$
1	2,517	4,121,280
2	7,814	7,299,175
3	2,920	3,814,001

(i) Calculate the value of the credibility factor for Empirical Bayes Model 1. [3]

(ii) Using the numbers calculated in (i) to illustrate your answer, describe the way in which the data affect the value of the credibility factor. [4]
[Total 7]

4 In a large portfolio of non-life policies involving a new product, let θ denote the proportion of policies on which claims are made in the first year. The value of θ is unknown and is assumed to have a beta prior distribution with parameters α and β and mean μ_0.

(i) If a random sample of n such policies gives rise to x claims in the first year, shown that the posterior mean of θ is given by

$$w_n \mu_0 + (1 - w_n) \frac{x}{n}$$

expressing the weight w_n as a function of α, β and n. [3]

(ii) Two alternative assessments A and B, of the prior probability density function of θ are made as follows:

$$f_A(\theta) = 3(1-\theta)^2 \qquad 0 \le \theta \le 1,$$

$$f_B(\theta) = 4\theta^3 \qquad 0 \le \theta \le 1.$$

81 claims subsequently arise during the year from 1,000 randomly selected policies.

(a) Sketch the two prior densities and comment briefly on the nature of these two sets of prior beliefs.

(b) Determine the posterior Bayes estimate of θ for each prior assessment based on the squared error loss function, and comment briefly on these posterior estimates.

[5]
[Total 8]

5 An insurance company has a portfolio of policies with a per-risk excess of loss reinsurance arrangement with a deductible of $M(>0)$. Claims made to the direct insurer, denoted by X, have a Pareto distribution with cumulative distribution function

$$F(x;\alpha) = 1 - \left(\frac{200}{200+x}\right)^{\alpha}.$$

There were a total of n claims from the portfolio. Of these, l were for amounts less then the deductible. The claims less than the deductible are $\{x_i : i = 1, 2, \ldots, l\}$. The value of the statistic $\sum_{i=1}^{l} \log(200 + x_i) = y$ is given.

(i) Show that the maximum likelihood estimate of α, $\hat{\alpha}$, is given by

$$\hat{\alpha} = \frac{l}{(n-l)\log(200+M) - n\log 200 + y}$$ [5]

(ii) From last year's experience we have the following information:

$M = 600, \quad n = 500, \quad l = 400, \quad y = 2209.269$

(a) Use the expression derived in (i) to verify that the maximum likelihood estimate of α based on these figures is $\hat{\alpha} = 1.75$.

(b) Assuming that $\alpha = 1.75$, estimate the average amounts paid by the insurer and the reinsurer on a claim made during the year. [6]

[Total 11]

6 The tables below show the cumulative cost of incurred claims and the number of claims reported each year for a certain cohort of insurance policies. The claims are assumed to be fully run-off at the end of development year 2.

Cumulative cost of incurred claims:

		Development	
Accident Year	*0*	*1*	*2*
0	288	634	893
1	465	980	
2	773		

The numbers of claims reported in each year are:

		Development	
Accident Year	*0*	*1*	*2*
0	110	85	55
1	167	113	
2	285		

Given that the total amount paid in claims to date, relating to accident years 0, 1 and 2, is £2,750, calculate the outstanding claims reserve using the average cost per claim method. [11]

7 (i) Define the adjustment coefficient for a compound Poisson process, R, and describe how it can be used to assess reinsurance arrangements on the basis of security. [4]

(ii) Claims occur on a portfolio of insurance business according to a Poisson process. Individual claims are a fixed amount of £500 and the insurance company uses a premium loading of 0.15.

A reinsurer offers excess of loss reinsurance for which it uses a premium loading of 0.2, and the insurance company is considering purchasing this reinsurance, with a retention of m ($0 \le m \le 500$).

(a) Derive an equation for the adjustment coefficient, R, in terms of m.

(b) By differentiating this equation with respect to m, show that R satisfies the following differential equation.

$$1.2R + (1.2m - 25)\frac{dR}{dm} = \left(R + m\frac{dR}{dm}\right)e^{mR}$$

[Hint: It should be noted that R is a function of m.]

(c) Determine the optimal value of m by setting $\frac{dR}{dm} = 0$ and substituting for R in the adjustment coefficient equation derived in (a).

(d) Compare the values of R and the expected profit with no reinsurance with the values with the optimal value of m, and comment.

[Note: It is not necessary to calculate the value of R when there is no reinsurance.] [13]

[Total 17]

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8 An insurance company operates a no claims discount (NCD) system with three categories: 0%, 30% and 60% discount. The rules for moving between these categories are as follows:

If a policyholder makes no claims in a year, they move to the next higher discount level (or stay at the highest level) next year.

If a policyholder makes one or more claims, they move to the 0% level next year (or stay there).

Suppose that the probability that a policyholder does not make a claim (irrespective of discount level) is p, and that the premium in the 0% discount level is c.

(i) Write down the transition matrix. [2]

(ii) Derive the steady state distribution of policyholders in each discount level, in terms of p. [5]

(iii) (a) Calculate the average premium paid in the steady state, A, in terms of p and c.

(b) By considering the value of A when $p = 0.9$ ($A_{0.9}$), and $p = 0.8$ ($A_{0.8}$), comment on the effectiveness of the NCD system.

(c) The company is considering changing the NCD system so that the discount levels are 0%, $100d$% and $200d$%. (The present NCD system has $d = 0.3$.) Determine the value of d such that $A_{0.8} = 1.5A_{0.9}$, and comment on this. [10]

[Total 17]

9 The claim amount in the coming year from a specific type of risk is modelled as the random variable

$$X = IB$$

where I is an indicator random variable and B is a random variable representing the amount of the claim given that the claim occurs. Let μ and σ^2 denote the mean and variance of B respectively. The indicator random variable is such that $P(I = 1) = q$, where $I = 1$ indicates that a claim takes place in the year, otherwise $P(I = 0) = 1 - q$. For this type of risk, at most one claim can be made in the year.

(i) (a) By deriving an expression for the moment generating function of X in terms of the moment generating function of B, or otherwise, show that the mean and variance of X are given by

$$E[X] = q\mu \,, \qquad \mathrm{Var}(X) = q\sigma^2 + q(1-q)\,\mu^2$$

(b) Suppose that $B = a + bZ$ (£000s; $a, b > 0$) where Z has a Poisson distribution with parameter λ, so that B takes the values a, $a + b$, $a + 2b$, ... Determine expressions for the mean, variance and probability distribution of X in terms of a, b, λ and q. [9]

(ii) Consider further two types of the above risk:

Risk Type	*Claim Amount in Coming Year*	*Parameters*
A	X	$a = 2$, $b = 1$, $\lambda = 1.5$, $q = 0.1$
B	Y	$a = 3$, $b = 2$, $\lambda = 1.0$, $q = 0.2$

(a) Given that the two types of risk are independent, calculate the probability that $X + Y$ is greater than 5 (£000), without resorting to any approximations.

(b) A portfolio consists of 100 independent type A risks and 50 independent type B risks. Calculate the probability that the average claim amount per risk in the coming year is greater than 2 (£000), by using an appropriate approximation. [11]

[Total 20]