Faculty of Actuaries **Institute of Actuaries**

EXAMINATIONS

September 2000

Subject 106 — Actuarial Mathematics 2

EXAMINERS' REPORT

1 θ_1 = There are vacancies at A
θ_2 = There are vacancies at B
α_1 = Student chooses to fly to airport 1
α_2 = Student chooses to fly to airport 2

Then

	α_1	α_2
θ_1	1010	1115
θ_2	670	655

Expected costs:

α_1 = 10%. 1010 + 90%. 670 = 704
α_2 = 10%. 1115 + 90%. 655 = 701

and so he should choose airport 2.

2 (i) $f(x) = \frac{1}{\mu} e^{-\frac{x}{\mu}} = \exp\left[-\frac{x}{\mu} - \log\mu\right]$

which is in the form of an exponential family

(ii) The canonical parameter is $\theta = -\frac{1}{\mu}$

(iii) $b(\theta) = \log \mu = \log\left(-\frac{1}{\theta}\right) = -\log(-\theta)$

$$b'(\theta) = -\frac{1}{-\theta} \times (-1) = -\frac{1}{\theta}$$

$$b''(\theta) = \frac{1}{\theta^2} = \mu^2$$

Hence $V(\mu) = \mu^2$

The dispersion parameter is 1.

3 (i) $E[s^2(\theta)]$: 5078152
$V[m(\theta)]$: 7425771

$$z = \frac{n}{n + \frac{E[s^2(\theta)]}{V[m(\theta)]}} = \frac{4}{4 + \frac{5078152}{7425771}} = 0.854$$

(ii) z depends on n, $E[s^2(\theta)]$ and $V[m(\theta)]$.

$E[s^2(\theta)]$ measures the variation of the data within risks and $V[m(\theta)]$ measures the variation between the risks. It can be seen that the variation between risks is relatively large, which means that more weight is given to the individual risks (z is close to 1). If n were larger, z would also be larger because there would be more information on each individual risk.

4 (i) Given $p(\theta) \propto \theta^{\alpha-1}(1-\theta)^{\beta-1}$ with $\mu_0 = E(\theta) = \dfrac{\alpha}{\alpha+\beta}$

We observe $p(x \mid \theta) \propto \theta^{x}(1-\theta)^{n-x}$

and we want $p(\theta \mid x) \propto p(x \mid \theta)\, p(\theta) \propto \theta^{\alpha+x-1}(1-\theta)^{\beta+n-x-1}$

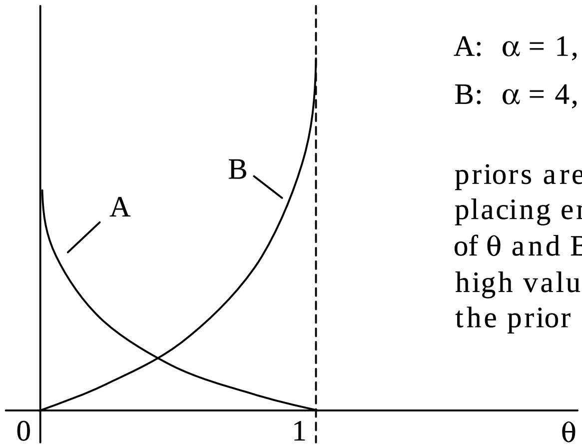
so that
$$\mu = E(\theta \mid x) = \frac{\alpha+x}{\alpha+x+\beta+n-x} = \frac{\alpha+x}{\alpha+\beta+n} = \frac{\alpha}{\alpha+\beta+n} + \frac{x}{\alpha+\beta+n}$$

$$= \frac{\alpha+\beta}{\alpha+\beta+n}\,\frac{\alpha}{\alpha+\beta} + \frac{n}{\alpha+\beta+n}\,\frac{x}{n}$$

$$= w_n\,\mu_0 + (1-w_n)\,\frac{x}{n}$$

where
$$w_n = \frac{\alpha+\beta}{\alpha+\beta+n}$$

(ii) (a)



A: $\alpha = 1, \beta = 3, \quad E(\theta) = \frac{1}{4}$

B: $\alpha = 4, \beta = 1, \quad E(\theta) = \frac{4}{5}$

priors are "opposites" with A placing emphasis on a low value of θ and B placing emphasis on a high value of θ. Note for example the prior means.

(b) With a squared error loss function, look at the posterior means.

Get

$$A: \ \alpha = 1, \beta = 3 \Rightarrow w_n = \frac{4}{4+n} = 0.003984$$

$$E(\theta \mid x) = 0.003984 \times \tfrac{1}{4} + 0.996016 \times \frac{81}{1000} = 0.0817$$

$$B: \ \alpha = 4, \beta = 1 \Rightarrow w_n = \frac{5}{5+n} = 0.004975$$

$$E(\theta \mid x) = 0.004975 \times \tfrac{4}{5} + 0.995025 \times \frac{81}{1000} = 0.0846$$

Results are nearly identical.

The sample is large, so prior beliefs are dominated by the information from the data.

5 (i) $f(x) \ = F'(x) = \dfrac{\alpha 200^{\alpha}}{(200+x)^{\alpha+1}}$

The likelihood, $L(\alpha)$ is given by:

$$L(\alpha) \ = \prod_{i=1}^{l} \frac{\alpha 200^{\alpha}}{(200+x_i)^{\alpha+1}} \times \left(\frac{200}{200+m}\right)^{\alpha(n-l)}$$

$$= \frac{\alpha^{l} 200^{\alpha n}}{(200+m)^{\alpha(n-l)}} \prod_{i=1}^{l} (200+x_i)^{-\alpha-1}$$

$$l(\alpha) = \log L(\alpha) = l \log \alpha + \alpha n \log 200 - \alpha(n-l)\log(200+M) - (\alpha+1)\sum_{i=1}^{l}\log(200+x_i)$$

$$\frac{\partial l}{\partial \alpha} = \frac{l}{\alpha} + n\log 200 - (n-l)\log(200+M) - \sum_{i=1}^{l}\log(200+x_i)$$

$$\frac{\partial^2 l}{\partial \alpha^2} = -\frac{l}{\alpha^2}$$ which is negative and so this is a maximum

$\frac{\partial l}{\partial \alpha} = 0$ when $\alpha = \hat{\alpha}$, where

$$\hat{\alpha} = \frac{l}{(n-l)\log(200+M) - n\log(200) + \sum_{i=1}^{l}\log(200+x_i)}$$

(ii) (a) $$\hat{\alpha} = \frac{400}{100\log 800 - 500\log 200 + 2209.269} = \frac{400}{228.57}$$

$$= 1.75$$

(b) We require $E(Y) = \int_0^M xf(x) + MP(X > M)$

$$= E(X) - \int_M^\infty (x-M)\,f(x)\,dx$$

$$E(X) = \frac{\lambda}{\alpha-1} = \frac{200}{0.75} = 266.67$$

Also $$\int_M^\infty (x-M)\,f(x)\,dx = \int_0^\infty zf(z+M)\,dz$$

$$= \frac{200^\alpha}{800^\alpha}\int_0^\infty \frac{z\alpha\, 800^\alpha}{(800+z)^{\alpha+1}}\,dz$$

$$= \left(\frac{1}{4}\right)^\alpha \frac{\lambda}{\alpha-1} = \left(\frac{1}{4}\right)^\alpha \frac{800}{\alpha-1} = \frac{800}{11.3137\times 0.75}$$

$$= 94.28$$

Thus $E(Y) = 266.67 - 94.28 = \underline{172.39}$.

6 *Note that it is acceptable to use either the grossing-up method or the chain-ladder method to calculate the projected number of claims.*

Cumulative number of claims reported:

110	195	250
167	280	
285		

Average incurred cost per claim:

			ultimate
2.618	3.251	3.572	**3.572**
73.29%	**91.01%**	**100%**	
2.784	3.500		**3.846**
72.39%	**91.01%**		
2.712			**3.723**
72.84%			

Projected ultimate average cost per claim in bold above.

Calculate projected numbers of claims:

			ultimate
110	195	250	250
44.00%	78.00%	100%	
167	280		359
46.52%	78.00%		
285			630
45.26%			

Projected total claims:

3.572	×	250	=	893
3.846	×	359	=	1381
3.723	×	630	=	2345
Total				4619
less claims paid				–2750
o/s claims reserve				**1869**

7 (i) R is defined as the smallest positive solution to:

$$\lambda + cR = \lambda M_X(R)$$

If there is no positive solution, then $R = 0$.

where c is the rate of premium income received by the insurer
λ is the claim frequency
$M_X(R)$ is the MGF of the individual claim size distribution

Security can be assumed by the probability of ruin $\psi(u)$. By Lundberg's inequality, $\psi(u) \le e^{-Ru}$, we see that a large value of R is good for security.

(ii) (a) $c = 500 \times 1.15\lambda$

$$X_I = \begin{cases} X & X \le m \\ m & X > m \end{cases}$$

$$X_R = \begin{cases} 0 & X \le m \\ X - m & X > m \end{cases}$$

Hence $X_I = m$ and $X_R = 500 - m$

$\therefore\ c_R = 1.2 \times (500 - m)\,\lambda$

and $M_{X_I}(R) = e^{mR}$

The adjustment coefficient equation is

$$\lambda + (500 \times 1.15\lambda - 1.2 \times (500 - m)\,\lambda)\,R = \lambda e^{mR}$$

i.e. $1 + (575 - 600 + 1.2\text{m})\,R = e^{mR}$

i.e. $1 + (1.2\text{m} - 25)\,R = e^{mR}$

(b) $1 + (1.2\text{m} - 25)\,R(m) = e^{mR(m)}$

Differentiate wrt m:

$$1.2R(m) + (1.2m - 25)\,\frac{dR(m)}{dm} = \left(R(m) + \frac{m\,dR(m)}{dm}\right)e^{mR(m)}$$

(c) Put $\dfrac{dR(m)}{dm} = 0$:

$$1.2R(m) = R(m)\, e^{mR(m)}$$

Hence $1.2 = e^{mR(m)}$

$$\therefore\ R(m) = \log(1.2)\,/\,m$$

Substituting into the adjustment coefficient equation:

$$1 + (1.2m - 25)\,\frac{\log 1.2}{m} = e^{\log 1.2} = 1.2$$

$$\therefore\ 1 + 1.2\log 1.2 - \frac{25\log 1.2}{m} = 1.2$$

$$\therefore\ m = \frac{25\log 1.2}{1.2\log 1.2 - 0.2} = £242.6 \simeq £243$$

(d) When there is no reinsurance, the value of R will be lower than when m = £243. The expected profit is

No reinsurance: $0.15 \times 500 \times \lambda = 75\lambda$

With reinsurance: $500 \times 1.15\lambda - 1.2 \times 257.4\lambda - 242.6\lambda$

$= 23.5\lambda$

So security is higher, but expected profit is lower when m = £243. It is a trade-off between these two factors.

8 (i) The transition matrix is

$$\begin{pmatrix} 1-p & p & 0 \\ 1-p & 0 & p \\ 1-p & 0 & p \end{pmatrix}$$

(ii) $\pi p = \pi$

$$(\pi_0\ \pi_1\ \pi_2)\begin{pmatrix} 1-p & p & 0 \\ 1-p & 0 & p \\ 1-p & 0 & p \end{pmatrix} = (\pi_0\ \pi_1\ \pi_2)$$

$(1 - p)\ (\pi_0 + \pi_1 + \pi_2) = \pi_0$

$p\pi_0 = \pi_1$
$p(\pi_1 + \pi_2) = \pi_2$

Since $(\pi_0 + \pi_1 + \pi_2) = 1, \quad \pi_0 = 1 - p$

$\therefore\ \pi_1 = p(1 - p)$

and $p\pi_1 = (1 - p)\ \pi_2$

$$\therefore \pi_2 = \frac{p}{1-p}\ \pi_1 = p^2$$

The steady state distribution is

$(1 - p,\ p(1 - p),\ p^2)$

(iii) (a) $A \quad = (\ 1 - p)\ c + p\ (1 - p) \times 0.7c + p^2 \times 0.4c$

$= c(1 - p + 0.7p - 0.7p^2 + 0.4p^2)$

$= c(1 - 0.3p - 0.3p^2)$

$= c(1 - 0.3p\ (1 + p))$

(b) $A_{0.9} = 0.487c$

$A_{0.8} = 0.568c$

Although P(claim) is twice as much in $A_{0.8}$ as in $A_{0.9}$, the average premium is only slightly higher. The NCD system is not effective.

(c) $A \quad = (1 - p)\ c + p\ (1 - p) \times (1 - d)\ c + p^2\ (1 - 2d)\ c$

$= c\ (1 - p + p - pd - p^2 + p^2\ d + p^2 - 2p^2\ d)$

$= c\ (1 - pd - p^2\ d)$

$= c\ (1 - p\ (1 + p)\ d)$

$c\ (1 - 0.8\ (1.8)\ d) = 1.5c\ (1 - 0.9\ (1.9)\ d)$

$1 - 1.44d = 1.5 - 2.565d$

$1.125d \quad = 0.5$

$d \quad = 0.444$

To get a premium 50% higher, the discount categories have to be 44% and 89%. Clearly this is not practical. It might be more appropriate to have the premium 100% higher, but this would make the discount categories even more unrealistic.

9 (i) (a) $M_X(t) = E(e^{tX}) = E\{E(e^{tX} \mid I)\}$

$$= E(e^{tX} \mid I = 0)\, P(I = 0) + E((e^{tX} \mid I = 1))\, P(I = 1)$$

$$= 1 \,.\, (1 - q) + M_B(t) \,.\, q$$

Then

$$M'_X(t) = qM'_B(t) \Rightarrow E(X) = qE(B) = q\mu$$

and

$$M''_X(t) = qM''_B(t) \Rightarrow E(X^2) = qE(B^2)$$

so that

$$E(X) = q\mu$$

$$\text{Var}(X) = E(X^2) - \{E(X)\}^2$$

$$= q\{\text{Var}(B) + \{E(B)^2\} - q^2\mu^2$$

$$= q\{\sigma^2 + \mu^2\} - q^2\mu^2 = q\sigma^2 + q(1 - q)\,\mu^2$$

(b) $B = a + bZ$ with $Z \sim \text{Poi}(\lambda)$ $\quad \therefore\ E(Z) = \text{Var}(Z) = \lambda$

so $\mu = E(B) = a + b\lambda$ $\quad \therefore\ E(X) = q(a + b\lambda)$

$\sigma^2 = \text{Var}(B) = b^2\lambda$ $\quad \therefore\ \text{Var}(X) = qb^2\lambda + q(1 - q)\,(a + b\lambda)^2$

Also

$$M_B(t) = E(e^{tB}) = E(e^{t(a+bZ)}) = e^{at}\, M_Z(bt) = e^{at}\ e^{\lambda(e^{bt}-1)}$$

Then

$$M_X(t) = qe^{at}\ e^{\lambda(e^{bt}-1)} + 1 - q$$

$$= qe^{-\lambda}\ e^{at}\ e^{\lambda e^{bt}} + 1 - q$$

$$= qe^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i\ e^{(a+ib)t}}{i!} + 1 - q$$

X takes values 0, a, $a + b$, $a + 2b$, ...

so selecting coefficients of e^{xt} gives

$$P(X = 0) = 1 - q$$

$$P(X = a + b(x-1)) = \frac{qe^{-\lambda}\ \lambda^{x-1}}{(x-1)!}\ ;\ x = 1, 2, 3, \ldots$$

(ii) (a) X takes values 0, 2, 3, 4, 5, ...

Y takes values 0, 3, 5, 7, ...

We can compute $P(X + Y \le 5)$ as a convolution:

x	$p_X(x)$	y	$p_Y(y)$
0	0.9	0	0.8
2	0.022313	3	0.073576
3	0.033470	5	0.073576
4	0.025102	7	....
5	0.012551		
6	....		

$X + Y$	*sample points* *(x, y)*	*probability*
0	(0, 0)	0.720000
2	(2, 0)	0.017850
3	(0, 3); (3, 0)	0.092994
4	(4, 0)	0.020082
5	(0, 5); (2, 3); (5, 0)	0.077901
		0.928827

$\therefore\ P(X + Y > 5) = \underline{0.071173}$

(b) $E(X) = 0.1\ (2 + 1.5) = 0.35;$

$V(X) = 0.1 \times 1 \times 1.5 + 0.1 \times 0.9 \times 3.5^2 = 0.15 + 1.1025 = 1.2525$

$E(Y) = 0.2\ (3 + 2) = 1$

$V(Y) = 0.2 \times 4 \times 1 + 0.2 \times 0.8 \times 5^2 = 0.8 + 4 = 4.8$

$$\bar{c} = \frac{1}{150}\left(\sum_{i=1}^{100} X_i + \sum_{j=1}^{50} Y_j\right)$$

$$E(\bar{c}) = \frac{1}{150}(100\ E(X) + 50\ E(Y)) = 0.5667$$

$$V(\bar{c}) = \frac{1}{150^2}\ (100\ \mathrm{Var}(X) + 50\ \mathrm{Var}\ (Y)) = 0.016233$$

Under the normal modelling distribution, we require

$$1 - \Phi\left(\frac{2 - 0.567}{0.12741}\right) = 1 - \Phi\ (11.25) = 0$$