Faculty of Actuaries **Institute of Actuaries**

EXAMINATIONS

September 2002

Subject 106 — Actuarial Mathematics 2

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

K G Forman
Chairman of the Board of Examiners
12 November 2002

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EXAMINERS' COMMENTS

In numerical questions, candidates were not unduly penalised for errors in earlier parts of each question which affected their answers to the rest of the question.

There are various alternative methods for Question 7(iii), all of which gained full credit if done correctly. A common error in Question 7 was to make inflation adjustments to cumulative (rather than incremental) figures.

Question 8 was poorly done. Many candidates omitted the conditioning in Question 8 (ii) and (iii).

The examiners were disappointed that many candidates found difficulty in reproducing the standard bookwork in Question 9 (ii).

1 Writing x for $(x_1, \ldots, x_{20})$, we have

$$f(x \mid \lambda) = \lambda^{20} \exp\left(-\lambda \sum_{i=1}^{20} x_i \right).$$

The prior distribution is gamma(2,2).

The posterior density of λ is

$$\begin{aligned} f(\lambda \mid x) &\propto f(x \mid \lambda) f(\lambda) \\ &\propto \lambda^{20} e^{-24\lambda} \times \lambda e^{-2\lambda} \\ &= \lambda^{21} e^{-26\lambda}. \end{aligned}$$

This means that the posterior distribution of λ is a gamma (22,26).

2 (i) The normal density can be written

$$\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2} (y-\mu)^2 \right) = \exp\left(\frac{y\mu - \mu^2/2}{\sigma^2} - ½\left(\frac{y^2}{\sigma^2} + \log(2\pi\sigma^2) \right) \right).$$

This is of exponential family form

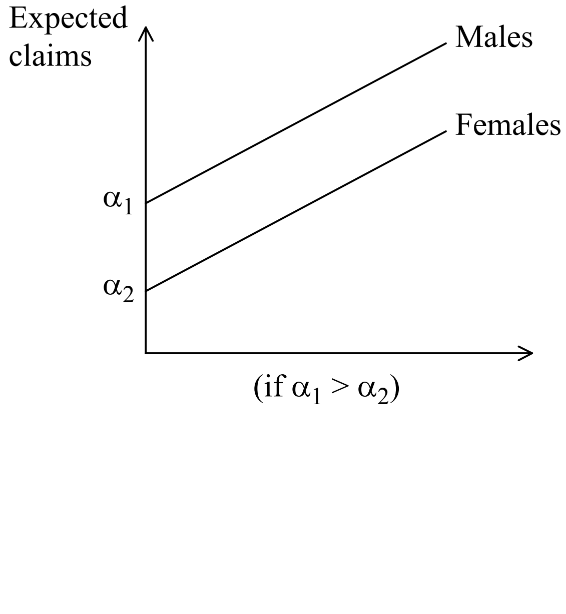
$$\exp\left(\frac{y\theta - b(\theta)}{a(\phi)} + c(y, \phi) \right),$$

with natural parameter $\theta = \mu$ and scale parameter $\phi = \sigma^2$.

The link function is the canonical link function $g(\mu) = \mu$ and the linear predictor is $\alpha_i + \beta x_{ij}$.

So this is a generalised linear model.

(ii)



(if $\alpha_1 > \alpha_2$)

3 (i) The total claim amount has moment generating function

$$M_T(t) = E(e^{tT}) = E(E(e^{tT} \mid N)) = E(M(t)^N) = \exp(\lambda(M(t) - 1)).$$

(ii)

(a) The total amount claimed from all 210 risks is the sum of three independent random variables S_1, S_2 and S_3, where S_i is the total amount claimed from risks of type i, $i = 1, 2, 3$. Using (i),

$$M_S(t) = M_{S_1}(t)M_{S_2}(t)M_{S_3}(t)$$

$$= \left(\exp\left(\frac{1}{1-400t} - 1\right)\right)^{40} \left(\exp\left(2\left(\frac{1}{1-500t} - 1\right)\right)\right)^{120} \left(\exp\left(2.5\left(\frac{1}{1-600t} - 1\right)\right)\right)^{50}$$

$$= \exp\left(40\left(\frac{1}{1-400t} - 1\right) + 2 \times 120\left(\frac{1}{1-500t} - 1\right) + 2.5 \times 50\left(\frac{1}{1-600t} - 1\right)\right)$$

$$= \exp\left(405\left(\frac{40}{405}\frac{1}{(1-400t)} + \frac{240}{405}\frac{1}{(1-500t)} + \frac{125}{405}\frac{1}{(1-600t)} - 1\right)\right)$$

(b) This can be recognised as the moment generating function of a compound Poisson distribution with Poisson parameter $\lambda = 405$. The claim severity distribution has moment generating function

$$M(t) = \frac{40}{405}\frac{1}{(1-400t)} + \frac{240}{405}\frac{1}{(1-500t)} + \frac{125}{405}\frac{1}{(1-600t)},$$

and this corresponds to density function

$$f(x) = \frac{40}{405}\frac{1}{400}e^{-x/400} + \frac{240}{405}\frac{1}{500}e^{-x/500} + \frac{125}{405}\frac{1}{600}e^{-x/600}.$$

[Note: the fractions can be simplified, and this is acceptable.]

4

	1/3	*1/3*	*1/3*			
	θ_1	θ_2	θ_3	*min*	*max*	*Expected profit*
d_1	25	19	7	7	25	**17**
d_2	10	30	8	**8**	30	16
d_3	0	2	34	0	**34**	12

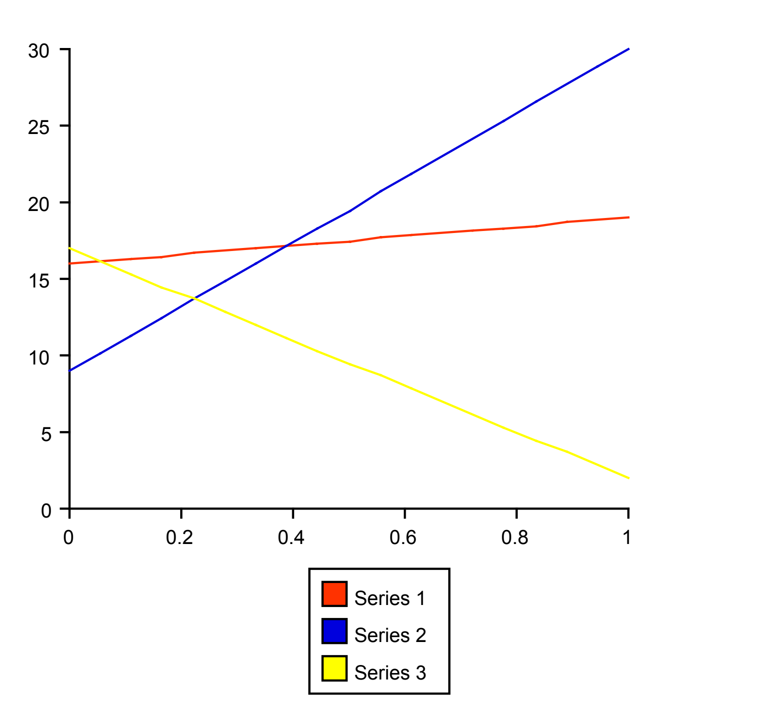
(i) minimax means minimise the maximum loss, which is the same as maximise the minimum profit, so the minimax solution is d_2 (sell hot food).

(ii) The trader's partner would choose d_3 (sell umbrellas).

(iii) The Bayes criterion solution is d_1 (sell ice-cream).

(iv) If $p(\theta_2) = p$ then $p(\theta_1) = p(\theta_3) = 1/2 \times (1 - p)$ and the Bayes risk

for d_1 is $25/2 \times (1 - p) + 19p + 7/2 \times (1 - p) = 16 + 3p$
for d_2 is $10/2 \times (1 - p) + 30p + 8/2 \times (1 - p) = 9 + 21p$
for d_3 is $0/2 \times (1 - p) + 2p + 34/2 \times (1 - p) = 17 - 15p$



and the maximum values are:

d_3 for $0 < p < 1/18$
d_1 for $1/18 < p < 7/18$
d_2 for $7/18 < p < 1$

so for $p > 1/2$ the Bayes criterion solution is d_2.

5 (i) Upper bound: The adjustment coefficient $R > 0$ satisfies

$$\lambda M(R) - \lambda - (1 + \theta)\, \lambda m_1 R = 0,$$

i.e.

$$M(R) - 1 - (1 + \theta) m_1 R = 0,$$

where $M(r)$ is the moment generating function of X_1. Using $M(R) = \sum_{k=1}^{M} p_k e^{Rk}$ and expanding the exponential gives

$$\begin{aligned} 1 + (1 + \theta) m_1 R &= \sum_{k=1}^{M} e^{Rk} p_k \\ &> \sum_{k=1}^{M} \left(1 + Rk + \frac{R^2 k^2}{2} \right) p_k \\ &= 1 + R m_1 + \frac{R^2}{2} m_2 . \end{aligned}$$

Then we have

$$((1 + \theta) - 1) m_1 \;\; R > \frac{R^2}{2} m_2$$

$$R < \frac{2\theta m_1}{m_2}.$$

Lower bound: We use the inequality

$$e^{Rx} \leq \frac{x}{M} e^{RM} + 1 - \frac{x}{M},\;\; 0 \leq x \leq M.$$

Then R satisfies

$$\begin{aligned} 1 + (1 + \theta) m_1 R &= \sum_{k=1}^{M} e^{Rk} p_k \\ &\leq \sum_{k=1}^{M} \left(\frac{k}{M} e^{RM} + 1 - \frac{k}{M} \right) p_k \\ &= \frac{e^{RM} m_1}{M} + 1 - \frac{m_1}{M}, \end{aligned}$$

so

$$1 + \theta \le \frac{1}{RM}(e^{RM} - 1) < e^{RM}.$$

Then

$$R > \frac{1}{M}\log(1 + \theta).$$

(ii) We have $M = 2$ and $p_1 = p_2 = 1/2$, so $m_1 = 1.5$ and $m_2 = 0.5(1 + 4) = 2.5$.

The upper bound for R is

$$\frac{2\theta m_1}{m_2} = \frac{2 \times 0.2 \times 1.5}{2.5} = 0.24.$$

The lower bound is

$$\frac{1}{M}\log(1 + \theta) = \tfrac{1}{2}\log 1.2 = 0.09116.$$

Using Lundberg's inequality and the lower bound above, the probability $\psi(u)$ of ultimate ruin with initial reserve u satisfies

$$\psi(u) \le e^{-Ru} \le e^{-0.09116u}.$$

6 (i) The transition matrix is:

$$\begin{pmatrix} p & 1-p & 0 \\ 0.8p & 0 & 1-0.8p \\ 0 & 0.6p & 1-0.6p \end{pmatrix}$$

(ii) The steady state distribution is the solution to:

$$\begin{aligned} p\pi_0 \quad &+ 0.8p\pi_1 &= \pi_0 \\ (1-p)\pi_0 \quad &+ 0.6p\pi_2 &= \pi_1 \\ (1-0.8p)\pi_1 \quad &+ (1-0.6p)\pi_2 &= \pi_2 \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{aligned}$$

Therefore, expressing π_1 and π_2 in terms of π_0:

$$\pi_1 = \pi_0\,(1-p)\,/\,0.8p$$
$$\pi_2 = \pi_0\,(1-p)\,(1-0.8p)\,/\,0.48p^2$$

And so:

$$\pi_0 + \pi_0\,(1-p)\,/\,0.8p + \pi_0\,(1-p)\,(1-0.8p)\,/\,0.48p^2 = 1$$
$$\pi_0\,\{0.48p^2 + 0.6p(1-p) + (1-p)\,(1-0.8p)\} = 0.48p^2$$

Which gives:

$$\pi_0 = 0.48p^2\,/\,(1-1.2p+0.68p^2)$$
$$\pi_1 = 0.6p(1-p)\,/\,(1-1.2p+0.68p^2)$$
$$\pi_2 = (1-p)\,(1-0.8p)\,/\,(1-1.2p+0.68p^2)$$

(iii)
$$\begin{aligned} A(p, c) &= \pi_0\,c + \pi_1\,c\,(1-0.3) + \pi_2\,c\,(1-0.5) \\ &= c\,\{0.48p^2 + 0.6p(1-p)0.7 + (1-p)\,(1-0.8p)0.5\} \\ &\qquad /\,(1-1.2p+0.68p^2) \\ &= c\,(0.5 - 0.48p + 0.46p^2)\,/\,(1-1.2p+0.68p^2) \end{aligned}$$

(iv)
$$\begin{aligned} A(0.15, c) &= 1.5 * A(0.1, 1{,}000) \\ 0.52478\,c &= 1.5 * 0.51488 * 1{,}000 \\ c &= 1{,}471. \end{aligned}$$

7 (i) Under the basic chain ladder method development factors are:

(4196 + 4715 + 5315) / (2457 + 2648 + 3084) = 14226 / 8189 = 1.7372
(4969 + 5561) / (4196 + 4715) = 10530 / 8911 = 1.1817
5010 / 4969 = 1.0083

The central assumption underlying this method is that, for each accident year, the amount of claims paid in each development year is a constant proportion of the total claims paid from that accident year. Implicit in this is either that the rate of claims inflation is constant or that weighted average past claims inflation will be repeated in the future.

(ii) First need to calculate incremental claims:

	Development year			
Accident year	*0*	*1*	*2*	*3*
1998	2457	1739	773	41
1999	2648	2067	846	
2000	3084	2231		
2001	3341			

Adjust for past inflation to convert to mid-2001 prices:

	Development year			
Accident year	*0*	*1*	*2*	*3*
1998	2861	1983	798	41
1999	3020	2133	846	
2000	3183	2231		
2001	3341			

Calculate inflation-adjusted cumulative payments:

	Development year			
Accident year	*0*	*1*	*2*	*3*
1998	2861	4844	5642	5683
1999	3020	5153	5999	
2000	3183	5414		
2001	3341			

So development factors are 1.7002, 1.1644 and 1.0073.

The assumption underlying the use of this model is that, for each accident year, the amount of claims paid, in real terms, in each development year is a constant proportion of the total claims, in real terms, from that accident year. Separate, explicit assumptions are needed for both past and future claims inflation.

(iii) *Note that this solution follows the method given in the core reading for obtaining the fitted value. This is not necessarily the method that would be used in practice.*

Under the basic chain ladder method "fitted" cumulative payments are:

	Development year			
Accident year	*0*	*1*	*2*	*3*
1998	2457	4268	5044	5086
1999	2648	4600	5436	
2000	3084	5358		
2001	3341			

Analysis of "fitted" versus "actual" incremental payments:

		Development year			
Accident year		*0*	*1*	*2*	*3*
1998	Actual	2457	1739	773	41
	Fitted	2457	1811	776	42
	Error	-	+72	+3	+1
1999	Actual	2648	2067	846	
	Fitted	2648	1952	836	
	Error	-	−115	−10	
2000	Actual	3084	2231		
	Fitted	3084	2274		
	Error	-	+43		
2001	Actual	3341			
	Fitted	3341			
	Error	-			

Under the inflation-adjusted chain ladder method "fitted" cumulative payments before allowing for past-inflation are:

	Development year			
Accident year	*0*	*1*	*2*	*3*
1998	2457	4177	4864	4900
1999	2648	4502	5242	
2000	3084	5243		
2001	3341			

"Fitted" incremental claims before allowing for past inflation:

Accident year	*Development year* 0	*1*	*2*	*3*
1998	2457	1720	687	36
1999	2648	1854	740	
2000	3084	2159		
2001	3341			

Analysis of "fitted" versus "actual" incremental payments after allowing for past inflation:

Accident year		*Development year* 0	*1*	*2*	*3*
1998	Actual	2457	1739	773	41
	Fitted	2457	1756	775	42
	Error	-	+17	+2	+1
1999	Actual	2648	2067	846	
	Fitted	2648	2049	844	
	Error	-	–18	–2	
2000	Actual	3084	2231		
	Fitted	3084	2229		
	Error	-	–2		
2001	Actual	3341			
	Fitted	3341			
	Error	-			

Clearly the inflation-adjusted chain ladder method with the given assumptions about past inflation gives a better fit to the observed data than the basic chain ladder method. However, even under the basic chain ladder method none of the errors are large enough to suggest that this model is inaccurate. There is no guarantee that either model will prove to be a good guide to the future.

8 (i) $$I = \int_a^b \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(\ln x - \mu)^2} dx$$

Put $y = \ln x$, so $dy = \frac{1}{x} dx$ and $dx = e^y dy$

$$I = \int_{\ln a}^{\ln b} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y-\mu)^2} e^y dy$$

$$= \int_{\ln a}^{\ln b} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(y^2 - 2\mu y + \mu^2 - 2y\sigma^2)\right\} dy$$

$$= \int_{\ln a}^{\ln b} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}((y - \mu - \sigma^2)^2 - 2\mu\sigma^2 - \sigma^4)\right\} dy$$

$$= e^{\mu + ½\sigma^2} \int_{\ln a}^{\ln b} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y - \mu - \sigma^2)^2} dy$$

$$= e^{\mu + ½\sigma^2} \left(\Phi\left(\frac{\ln b - \mu - \sigma^2}{\sigma}\right) - \Phi\left(\frac{\ln a - \mu - \sigma^2}{\sigma}\right) \right)$$

(ii)

(a) $e^{\mu + ½\sigma^2} = 264$ $\quad (e^{\mu + ½\sigma^2})^2 (e^{\sigma^2} - 1) = 346^2$

$$e^{\sigma^2} - 1 = \frac{346^2}{264^2}$$

$\sigma^2 = \ln(1 + 1.7177) = 1$

$\mu = \ln 264 - 0.5 = 5.076$

$$X_I = \begin{cases} 0 & X < 100 \\ X - 100 & X > 100 \end{cases}$$

Claims paid by the insurance company are $X - 100 \mid X > 100$.

$$E[X - 100 \mid X > 100] = \frac{\int_{100}^{\infty} x f(x) dx}{P(X > 100)} - 100$$

$$\int_{100}^{\infty} xf(x)dx \qquad = e^{\mu+\frac{1}{2}\sigma^2}\left(1-\Phi\left(\frac{\ln 100-\mu-\sigma^2}{\sigma}\right)\right)$$

$$= 264(1 - \Phi(-1.471))$$

$$= 264 \times 0.9294 = 245.35$$

$$P(X > 100) \qquad = 1-\Phi\left(\frac{\ln 100-\mu}{\sigma}\right) = 1 - \Phi(-0.471)$$

$$= 0.6812$$

$$E[X - 100 \mid X > 100] = \frac{245.35}{0.6812} - 100 \qquad = 260.18$$

(b)

Let $Y = 1.1X$

$E[Y] \quad = 1.1E[X] \quad = 290.4$
$V[Y] \quad = 1.1^2 V[X] \quad = (380.6)^2$

$\therefore \qquad e^{\sigma^2} - 1 = 1.717$
$\qquad \sigma^2 = 1$

$\mu = \ln 290.4 - 0.5 = 5.1714$

$$Y_I = \begin{cases} 0 & Y < 100 \\ Y - 100 & 100 < Y < 1{,}100 \\ 1{,}000 & Y > 1{,}100 \end{cases}$$

Claim amount payable by the insurer is $Y_I \mid Y > 100$

$$E[Y_I \mid Y > 100] \quad = \frac{\int_{100}^{1,100} (y-100) f(y)dy + 1{,}000P(Y > 1{,}100)}{P(Y > 100)}$$

$$= \frac{\int_{100}^{1,100} yf(y)dy - 100P(100 < Y < 1{,}100) + 1{,}000P(Y > 1{,}100)}{P(Y > 100)}$$

$$\int_{100}^{1,100} yf(y)dy = 290.4\left(\Phi\left(\frac{\ln 1,100 - 5.1714 - 1}{1}\right) - \Phi\left(\frac{\ln 100 - 5.1714 - 1}{1}\right)\right)$$

$$= 290.4(\Phi(0.832) - \Phi(-1.566))$$

$$= 290.4(0.7973 - 0.0587)$$

$$= 214.49$$

$$P(Y < 1,100) = \Phi\left(\frac{\ln 1,100 - 5.1714}{1}\right) = \Phi(1.832) = 0.9665$$

$$P(Y < 100) = \Phi\left(\frac{\ln 100 - 5.1714}{1}\right) = \Phi(-0.566) = 0.2857$$

$$\therefore E[Y_I \mid Y > 100] = \frac{214.49 - 100(0.9665 - 0.2857) + 1,000(1 - 0.9665)}{1 - 0.2857}$$

$$= 251.87$$

9 (i) (a) $E(X_j) = E(E(X_j \mid \theta)) = E(m(\theta)) = m$.

(b) $E(X_j X_k) = E(E(X_j X_k \mid \theta)) = E(m(\theta)^2)$ for $j \neq k$

(c) $E(X_j X_k) \neq E(X_j)E(X_k)$ for $j \neq k$

so X_j and X_k are not independent.

(ii) Differentiating with respect to a_0 and setting to zero gives

$$E\left(m(\theta) - a_0 - \sum_{j=1}^{4} a_j X_j\right) = 0,$$

and so

$$a_0 = m\left(1 - \sum_{j=1}^{4} a_j\right).$$

Note that

$$E(X_k m(\theta)) = E(E(X_k m(\theta) \mid \theta)) = E(m(\theta)^2),$$

$$E(X_j X_k) = E(m(\theta)^2), j \neq k,$$

$$\begin{aligned} E(X_k^2) &= E(E(X_k^2 \mid \theta)) \\ &= E(V(X_k \mid \theta) + (E(X_k \mid \theta))^2) \\ &= \frac{1}{P_k} E(s^2(\theta)) + E(m(\theta)^2). \end{aligned}$$

Differentiating with respect to a_k ($k \neq 0$) and setting to zero gives

$$E\left(X_k m(\theta) - a_0 X_k - \sum_{j=1}^{4} a_j X_j X_k \right) = 0.$$

Using the above relationships this becomes

$$E(m(\theta)^2) - a_0 m - \sum_{j=1}^{4} a_j \, E(m(\theta)^2) - \frac{a_k E(s^2(\theta))}{P_k} = 0,$$

which gives, using the expression above for a_0,

$$a_k = \frac{P_k V(m(\theta))\left(1 - \sum_{j=1}^{4} a_j\right)}{E(s^2(\theta))}$$

Sum over k:

$$\sum_{k=1}^{4} a_k = \sum_{k=1}^{4} P_k \left(1 - \sum_{j=1}^{4} a_j \right) \frac{V(m(\theta))}{E(s^2(\theta))},$$

and so

$$\sum_{j=1}^{4} a_j = \frac{\sum_{j=1}^{4} P_j}{\sum_{j=1}^{4} P_j + E(s^2(\theta)) / V(m(\theta))}.$$

Substituting into the expressions for a_0 and a_k, we obtain

$$a_0 = \frac{m E(s^2(\theta)) / V(m(\theta))}{\sum_{j=1}^{4} P_j + E(s^2(\theta)) / V(m(\theta))}$$

and

$$a_k = \frac{P_k}{\sum_{j=1}^{4} P_j + E(s^2(\theta)) / V(m(\theta))}$$

The credibility premium per car is

$$\frac{E(s^2(\theta))/V(m(\theta))}{\sum_{j=1}^4 P_j + E(s^2(\theta))/V(m(\theta))} \times m + \frac{\sum_{j=1}^4 P_j}{\sum_{j=1}^4 P_j + E(s^2(\theta))/V(m(\theta))} \times \bar{X},$$

where $\bar{X} = \sum_{j=1}^4 P_j X_j / \sum_{j=1}^4 P_j$. The credibility factor is

$$Z = \frac{\sum_{j=1}^4 P_j}{\sum_{j=1}^4 P_j + E(s^2(\theta))/V(m(\theta))}.$$

(iii) The credibility premium per car is $Z\bar{X} + (1 - Z)m$. We have

$$\sum P_j = 64$$

$E(s^2(\theta))/V(m(\theta))$ is estimated as 106.32/5.8 = 18.3310, so Z is 64/(64 + 18.3310) = 0.7773.

Further, $\bar{X}$ = 5,100/64 = 79.6875,
so that credibility premium per car is 75.9275.

Hence for the fleet of 16 cars the premium is 16 × 75.9275 = 1,214.84.

If the estimate of $V(m(\theta))$ increases, then the estimate of Z increases and relatively more weight is put on the data from this particular fleet.
This happens because an increase $V(m(\theta))$ means an increase in the variability between fleets and so less emphasis on collateral information.

If Z increases, then $Z \times 79.69 + (1 - Z) \times 62.8$ also increases. The credibility premium moves closer to $\bar{X}$, and, since this is greater than the estimated value of m, this implies an increase in the premium.