Faculty of Actuaries **Institute of Actuaries**

EXAMINATIONS

April 2001

Subject 106 — Actuarial Mathematics 2

EXAMINERS' REPORT

Examiners Comments:

In the numerical questions, students were not unduly penalised for errors in earlier parts of each question which affected their answers to the rest of the question. This mostly occurred in question 7 for students who had not calculated the transition matrix correctly. Marks were deducted for errors in the transition matrix but full credit was given for the later parts of the question where students used the correct method to determine the proportions at each NCD level and their final answer were wrong only due to the original mistake.

In question 5, some students rightly concluded that ignoring fixed costs would not affect the final answer hence their figures calculated differed from the examiners' solution. In these cases, full marks were awarded where the student explicitly stated that fixed costs could be ignored.

1 Cumulative claims:

2,541	3,570	3,787
2,824	3,614	
1,981		

$$\lambda_3 = \frac{3{,}787}{3{,}570} = 1.06078$$

$$\lambda_2 = \frac{7{,}184}{5{,}365} = 1.33905$$

Grossing up factors:

94.27%
70.40%

2 (i) $$f_Y(y;\theta,\varphi) = \binom{n}{ny}\mu^{ny}(1-\mu)^{n-ny}$$

$$= \exp\left\{n\left[y\log\left(\frac{\mu}{1-\mu}\right)+\log(1-\mu)\right]+\log\binom{n}{ny}\right\}$$

This is of the form $\exp\left\{\dfrac{y\theta-b(\theta)}{a(\varphi)}+c(y,\varphi)\right\}$

i.e. <u>exponential family form</u>

(ii) The natural parameter is

$$\theta = \log\left(\frac{\mu}{1-\mu}\right) \qquad \text{so } \mu = \left(\frac{e^\theta}{1+e^\theta}\right)$$

$$\varphi = n, \quad a(\varphi) = \frac{1}{\varphi}$$

$$b(\theta) = -\log(1-\mu) = -\log\left(\frac{1}{1+e^\theta}\right) = \log(1+e^\theta)$$

$$c(y,\varphi) = \log\binom{n}{ny}$$

Canonical link is $g(\mu) = \theta(\mu)$

so $\underline{g(\mu) = \log\left(\frac{\mu}{1-\mu}\right)}$

$V(\mu) = b''(\theta)$

$$b'(\theta) = \frac{e^\theta}{1+e^\theta}$$

$$b''(\theta) = \frac{e^\theta}{(1+e^\theta)^2} = \mu(1-\mu)$$

so $\underline{V(\mu) = \mu(1 - \mu) \text{ is the variance function}}$

3 Let X be a typical claim. Then

$E(X) = ¼ \times 100 + ½ \times 200 + ¼ \times 250 = 187.5$

$E(X^2) = ¼ \times 100^2 + ½ \times 200^2 + ¼ \times 250^2 = 38{,}125$

Let S be the aggregate claims at time 3, then

$E(S) = E(X) \times 3\lambda = 11{,}250$

$\text{Var}(S) = E(X^2) \times 3\lambda = 2{,}287{,}500$

standard deviation of S is 1,512.4483

So S is approximately $N(11{,}250,\ 2{,}287{,}500)$

Want $0.05 \geq P(1{,}000 + (1 + \theta)\, E(X)\ .\ 3\lambda - S < 0)$

$= P(S > 1{,}000 + (1 + \theta) \times 11{,}250)$

$$= P\left(N(0,1) > \frac{1{,}000 + (1+\theta)\times 11{,}250 - 11{,}250}{1512.4483}\right)$$

$$= P\left(N(0,1) > \frac{1{,}000 + 11{,}250\theta}{1512.4483}\right)$$

So $$\frac{1{,}000 + 11{,}250\theta}{1512.4483} > 1.645$$

i.e. $\underline{\theta > 0.1323}$

4 (i) The moment generating function of S is

$$M_S(t) = E(e^{St}) = E[E(e^{St} \mid N)]$$

$$= E[\{E(e^{Xt})\}^N]$$

$$= G_N(M_X(t))$$

(ii) (a) Let N be the number of claims.

$$P(N=n) = \int_0^\infty P(N=n|\lambda)\, 2e^{-2\lambda} d\lambda$$

$$= \int_0^\infty \frac{e^{-\lambda}\lambda^n}{n!} 2e^{-2\lambda} d\lambda$$

$$= \frac{2}{3^{n+1}} \int_0^\infty \frac{3^{n+1}\lambda^n e^{-3\lambda}}{n!} d\lambda$$

$$= \underline{\frac{2}{3^{n+1}}} \qquad n = 0, 1, 2, \ldots$$

(b) N has pgf $\displaystyle\sum_0^\infty s^n \frac{2}{3^{n+1}} = \frac{2}{3}\sum_0^\infty \left(\frac{s}{3}\right)^n = \frac{2}{3-s}$

For a gamma(α,λ), $\left.\begin{array}{l}\text{mean} = \alpha/\lambda = 2 \\ \text{variance} = \alpha/\lambda^2 = 2\end{array}\right\} \Rightarrow \alpha = 2 \quad \lambda = 1$

So $\quad M_X(t) = \left(\dfrac{1}{1-t}\right)^2$

So $\quad M_S(t) = \dfrac{2}{3 - \dfrac{1}{(1-t)^2}} = \dfrac{2(1-t)^2}{3(1-t)^2 - 1} = \underline{\dfrac{2(1-t)^2}{3t^2 - 6t + 2}}$

Note: candidates do not have to simplify the expression $M_S(t)$: any correct expression is satisfactory.

5 (i)

d_1 100 policies premium £85 per annum
d_2 150 policies premium £81 per annum
d_3 200 policies premium £79 per annum

θ_1 Intensity I_1 claim costs £40 per policy per annum
θ_2 Intensity I_2 claim costs £45 per policy per annum
θ_3 Intensity I_3 claim costs £57 per policy per annum
θ_4 Intensity I_4 claim costs £60 per policy per annum

	Figures in £000s		
Strategy	d_1	d_2	d_3
Total premiums	8,500	12,150	15,800
Fixed expenses	1,500	1,500	1,500
Per policy expenses	1,800	2,700	3,600
Premium less expenses	5,200	7,950	10,700

Hence annual profits (£000s)

	θ_1	θ_2	θ_3	θ_4
d_1	1,200	700	–500	–800
d_2	1,950	1,200	–600	–1,050
d_3	2,700	1,700	–700	–1,300

Minimax = minimise maximum loss

d_1 –800 ← choose d_1, set premiums at £85 per annum
d_2 –1,050
d_3 –1,300

(ii) Bayes criterion

$d_1 = 0.1 \times 1{,}200 + 0.4 \times 700 - 0.3 \times 500 - 0.2 \times 800 = 90$
$d_2 = 0.1 \times 1{,}950 + 0.4 \times 1{,}200 - 0.3 \times 600 - 0.2 \times 1{,}050 = 285$
$d_3 = 0.1 \times 2{,}700 + 0.4 \times 1{,}700 - 0.3 \times 700 - 0.2 \times 1{,}300 = 480$ ← choose d_3

Choose d_3, premiums of £79 per annum

6 (i) (a) Likelihood function

$$f(\underset{\sim}{x}|\theta) \propto \prod_{i=1}^{n} \theta e^{-\theta x_i}$$

$$\propto \theta^n e^{-\theta \sum_{i=1}^{n} x_i}$$

As a function of θ, this is in the form of a gamma distribution.

Hence the conjugate prior distribution is

$$f(\theta) = \frac{\lambda^\alpha \theta^{\alpha-1} e^{-\lambda\theta}}{\Gamma(\alpha)}$$

(b) Find values for the parameters α and λ

$E(\theta)$: $\alpha / \lambda = 0.315$

$V(\theta)$: $\alpha / \lambda^2 = 0.251^2$

$\lambda = 0.315 / 0.251^2$

$= 5.000$

$\alpha = 0.315\lambda$

$= 1.575$

Posterior = Prior × Likelihood

$$\propto \lambda^\alpha\, \theta^{\alpha-1}\, e^{-\lambda\theta}.\ \theta^n\ e^{-\theta\Sigma x_i}$$

$$\propto \theta^{n+\alpha-1} e^{-\theta(\lambda+\Sigma x_i)}$$

Hence Posterior

$= \text{gamma}\ (n + \alpha, \lambda + \Sigma x_i)$

$= \text{gamma}\ (11.575, 41.720)$

Estimator under quadratic loss = mean of the Posterior

$= 11.575 / 41.720 = 0.27744$

(ii) Hence Posterior

$$= \text{gamma}\ (n + \alpha, \lambda + \Sigma x_i)$$

$$= \text{gamma}\ (81.575, 298.760)$$

Estimator under quadratic loss = mean of the Posterior

$$= 81.575 / 298.760 = 0.27305$$

(iii) The mean of the prior distribution is 0.315.

The maximum likelihood estimate of θ is $\dfrac{1}{3.672} = 0.272$. As more data is collected, the Bayesian estimate approaches the maximum likelihood estimate.

7 (i)

0% level	P(0 claims)	$= e^{-0.29}$	$= 0.7483$
	P ($\geq$1 claim)	$= 1 - 0.7483$	$= 0.2517$
25% level	P(0 claims)	$= e^{-0.22}$	$= 0.8025$
	P($\geq$ 1 claim)	$= 1 - 0.8025$	$= 0.1975$
50% level	P(0 claims)	$= e^{-0.18}$	$= 0.8353$
	P(1 claim)	$= 0.18e^{-0.18}$	$= 0.1503$
	P($\geq$ 2 claims)	$= 1 - 0.8353 - 0.1503$	$= 0.0144$
75% level	P(0 claims)	$= e^{-0.10}$	$= 0.9048$
	P(1 claim)	$= 0.10e^{-0.10}$	$= 0.0905$
	P($\geq$ 2 claims)	$= 1 - 0.9048 - 0.0905$	$= 0.0047$

Therefore the transition matrix is:

$$\begin{bmatrix} 0.2517 & 0.7483 & 0 & 0 \\ 0.1975 & 0 & 0.8025 & 0 \\ 0.0144 & 0.1503 & 0 & 0.8353 \\ 0.0047 & 0 & 0.0905 & 0.9048 \end{bmatrix}$$

(ii) In a steady state

$$\Pi \begin{bmatrix} 0.2517 & 0.7483 & 0 & 0 \\ 0.1975 & 0 & 0.8025 & 0 \\ 0.0144 & 0.1503 & 0 & 0.8353 \\ 0.0047 & 0 & 0.0905 & 0.9048 \end{bmatrix} = \Pi$$

$$\begin{aligned}
0.2517\,\pi_0 + 0.1975\,\pi_1 + 0.0144\,\pi_2 + 0.0047\pi_3 &= \pi_0 \\
0.7483\,\pi_0 + 0.1503\,\pi_2 &= \pi_1 \\
0.8025\,\pi_1 + 0.0905\,\pi_3 &= \pi_2 \\
0.8353\,\pi_2 + 0.9048\,\pi_3 &= \pi_3
\end{aligned}$$

and $\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$

Hence

$$\pi_3 = 8.7742\,\pi_2$$
$$\pi_2 = 3.8969\,\pi_1$$
$$\pi_1 = 1.8062\,\pi_0$$

$$\pi_0 + 1.8062\,\pi_0 + 7.0386\,\pi_0 + 61.7579\,\pi_0 = 1$$

$\pi_0 = 0.0140$
$\pi_1 = 0.0252$
$\pi_2 = 0.0983$
$\pi_3 = 0.8625$

8 (i) (a)

$$f(\theta) \propto \theta^{\beta-1}(1-\theta)^{\beta-1}$$

$$f(\underline{x}\,|\,\theta) = \prod_{i=1}^{n}\binom{m}{x_i}\theta^{x_i}(1-\theta)^{m-x_i}$$

$$\propto \theta^{\Sigma x_i}(1-\theta)^{nm-\Sigma x_i}$$

so $$f(\theta\,|\,\underline{x}) \propto \theta^{\Sigma x_i+\beta-1}(1-\theta)^{nm+\beta-\Sigma x_i-1}$$

so $$\theta\,|\,\underline{x} \sim \text{Beta}(\Sigma x_i + \beta,\ nm + \beta - \Sigma x_i)$$

(b) $l = \log \text{like}(\theta) = (\Sigma x_i)\log\theta + (nm - \Sigma x_i)\log(1-\theta) + \text{constant}$

$$\frac{\partial l}{\partial\theta} = \frac{\sum x_i}{\theta} - \frac{nm - \sum x_i}{1-\theta} = \frac{nm}{\theta(1-\theta)}\left(\frac{\sum x_i}{nm} - \theta\right)$$

This is zero when $\Sigma x_i (1 - \theta) = (nm - \Sigma x_i)\, \theta$

i.e. $\hat{\theta} = \dfrac{\Sigma x_i}{nm}$

$$\left.\begin{array}{ll} \dfrac{\partial l}{\partial \theta} > 0 \text{ for } & \theta < \dfrac{\Sigma x_i}{nm} \\ \quad < 0 & \theta > \dfrac{\Sigma x_i}{nm} \end{array}\right\} \Rightarrow \max$$

So $\underline{\text{mle} = \dfrac{\Sigma x_i}{nm} = g(\underline{x})}$

(c) Need posterior mean.

$$E[\theta \mid \underline{x}] = \frac{\Sigma x_i + \beta}{nm + 2\beta}$$

$$= \frac{nm}{nm + 2\beta} \frac{\Sigma x_i}{nm} + \frac{2\beta}{nm + 2\beta} \frac{1}{2}$$

which is of the form $Z\, g(\underline{x}) + (1 - Z)\, \mu$

where $g(\underline{x}) = \dfrac{\Sigma x_i}{nm}$, $\mu = ½$

Find μ: $f(\theta) = \theta^{\beta-1} (1 - \theta)^{\beta-1}$

symmetric about $\theta = ½$

mean = ½ $\underline{\mu = \text{prior mean}}$

(d) $Z = \dfrac{nm}{nm + 2\beta} = \dfrac{m}{m + \dfrac{2\beta}{n}} \to 1 \text{ as } n \to \infty$

As more data obtained, put more relative weight on the estimate based in data, rather than the prior estimate.

(ii) $Z = \dfrac{nm}{nm + 2\beta}$ $g(\underline{x})$ $(\Sigma x_i = 12)$ Cred est

$\beta = 1$	0.9677	0.2	0.2097
$\beta = 4$	0.8824	0.2	0.2353

$$\text{Prior variance} = \frac{\beta^2}{(2\beta)^2(2\beta+1)}$$

$$= \frac{1}{4(2\beta+1)}$$

If $\beta = 1$ prior var is 0.0833
$\beta = 4$ prior var is 0.0278

Credibility factor is lower when $\beta = 4$, and prior variance is lower, indicating greater precision in our prior estimate, so give relatively more weight to prior estimate.

9 (i) $\displaystyle\int_a^\infty x f(x)dx = \int_{\log a}^\infty e^y f(y)dy$

$$= \int_{\log a}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-\left(y^2 - 2y\mu + \mu^2 - 2y\sigma^2\right)}{2\sigma^2}} dy$$

$$= e^{\mu + \frac{1}{2}\sigma^2} \int_{\log a}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(y-\mu-\sigma^2)^2}{2\sigma^2}} dy$$

$$= e^{\mu + \frac{1}{2}\sigma^2} \left\{1 - \Phi\left(\frac{\log a - \mu - \sigma^2}{\sigma}\right)\right\}$$

(ii)

(a) Find value of σ and μ

$\exp(\mu + \sigma^2/2) = 9.070$
$\exp 2(\mu + \sigma^2/2) \times (\exp \sigma^2 - 1) = 10.132^2$

$$\rightarrow \exp \sigma^2 = 1 + \frac{10.132^2}{9.070^2} = 2.248$$

$\rightarrow \sigma^2 = 0.80999$ $\sigma = 0.90$ and $\mu = 1.80$

$$P(X > 25) = 1 - \Phi\left(\frac{\log 25 - 1.80}{0.90}\right)$$

$$= 1 - \Phi\,(1.577)$$

$$= 1 - 0.9426$$

$$= 0.0574$$

$$P(X > 30) = 1 - \Phi\left(\frac{\log 30 - 1.80}{0.90}\right)$$

$$= 1 - \Phi\,(1.779)$$

$$= 1 - 0.9624$$

$$= 0.0376$$

(b) Expected amount ceded to reinsurer:

$$\int_M^\infty xf(x)dx - MP(X \geq M)$$

$$= 9.070\left\{1 - \Phi\left(\frac{\log M - 1.80 - 0.90^2}{0.90}\right)\right\} - MP(X > M)$$

(1) $= 9.070\,\{1 - \Phi\,(0.677)\} - 25\,\{1 - \Phi\,(1.576)\}$

$= 9.070\,(1 - 0.7507) - 25 \times 0.0574$

$= 0.82524$ i.e. £825.24

(2) $= 9.070\,\{1 - \Phi\,(0.879)\} - 30\,\{1 - \Phi\,(1.779)\}$

$= 9.070\,(1 - 0.8103) - 30 \times 0.0376$

$= 0.59258$ i.e. £592.58

Expected number of claims = $200 \times 20\% = 40$ claims

	Reinsurance policy	*Amount ceded to reinsurer*	*Premium paid*	*Reduction in insurer's profit*
(1)	40×825.24	= £33,010	£48,500	£15,490
(2)	40×592.58	= £23,703	£38,200	£14,497 ←

Choose policy (2) for best value for money.

(c) Mean = 9.796
Standard Deviation = 10.943

$\sigma = 0.900$ as before
$\mu = 1.877$

(1) $= 9.796 \{1 - \Phi(0.591)\} - 25 \{1 - \Phi (1.491)\}$

$= 9.796 (1 - 0.7227) - 25 \times (1 - 0.9320)$

$= 1.01643$ i.e. £1,016.43

(2) $= 9.796 \{1 - \Phi (0.794)\} - 30 \{1 - \Phi(1.694)\}$

$= 9.796 (1 - 0.7864) - 30 \times (1 - 0.9549)$

$= 0.73943$ i.e. £739.43

Hence:

	Amount ceded	*Premium paid*	*Reduction in profit*
(1)	£40,657	£48,500	£7,843 ←
(2)	£29,577	£38,200	£8,623

Choose policy (1) for best value for money.