EXAMINATIONS

September 2004

Subject 106 — Actuarial Mathematics 2

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

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Chairman of the Board of Examiners

23 November 2004

In numerical questions, candidates were not unduly penalised for errors in earlier parts of the question which affected their answers to the later parts of the question.

1 (i) Write

$$f(y) = \exp\left(2\left(-\frac{y}{\mu} - \log\mu\right) + \log y + \log 4\right).$$

This is of exponential family form

$$f(y) = \exp\left(\frac{y\theta - b(\theta)}{a(\phi)} + c(y,\phi)\right)$$

with natural parameter $\theta = -1/\mu$ (accept $1/\mu$).

(ii) Model I says that there is a possibly different mean preparation time for each type of coffee.

Model II says that cappuccino and espresso have the same mean preparation time, but that the mean preparation time for filter coffee is possibly different.

In Question 1(ii), a common mistake was to say that Model I implies no difference in the mean preparation times for the three types of coffee.

2 (i) Let S be the total amount of claims arising from a single hurricane, and let X be a typical claim. Then

$$E(S) = \lambda E(X) = \lambda/2,$$

and

$$V(S) = \lambda E(X^2) = \lambda(1 + 0.5^2) = 5\lambda/4.$$

(ii) Let T be the total amount of claims arising from the hurricanes in one year. Then $T = S_1 + \ldots + S_N$, where $S_1, S_2, \ldots$ are iid with the same distribution as S in (i), and N has a Poisson distribution with mean μ. Then

$$E(T) = \mu E(S) = \lambda\mu/2,$$

and

$$V(T) = \mu E(S^2) = \mu\left(\frac{5\lambda}{4} + \frac{\lambda^2}{4}\right) = \frac{\lambda\mu(5+\lambda)}{4}.$$

3

	Per Trip	*Annual*
1	30	95
2	60	95
3	90	95
4	120	95
5	150	95
Max:	150	95
E[Cost]:	93	95

Minimax decision is to buy the annual policy.

Bayes decision is to buy policies per trip

Question 3 was done well.

4 (i) The likelihood is

$$f(n_1, n_2 \mid \lambda) = \frac{e^{-\lambda}\lambda^{n_1}}{n_1!} \times \frac{e^{-2\lambda}(2\lambda)^{n_2}}{n_2!}$$

$$\propto e^{-3\lambda}\lambda^{n_1+n_2},$$

and so the loglikelihood is

$$l(\lambda) = -3\lambda + (n_1 + n_2)\log(\lambda) + \text{constant}.$$

Differentiating with respect to λ and setting to zero gives

$$\hat{\lambda} = \frac{n_1 + n_2}{3}.$$

(ii) (a) The prior density is $f(\lambda) = ve^{-v\lambda}$, so that the posterior density of λ given n_1, n_2 is

$$\begin{aligned} f(\lambda \mid n_1, n_2) &\propto f(n_1, n_2 \mid \lambda)\, f(\lambda) \\ &\propto e^{-3\lambda}\lambda^{n_1+n_2}e^{-v\lambda} \\ &= \lambda^{n_1+n_2}e^{-(v+3)\lambda} \end{aligned}$$

The posterior distribution is a gamma with parameters $n_1 + n_2 + 1$ and $v + 3$.

(b) Under quadratic loss, the Bayesian estimate is the posterior mean, i.e. it is

$$\frac{n_1+n_2+1}{v+3} = \frac{3}{v+3}\frac{n_1+n_2}{3}+\frac{v}{v+3}\frac{1}{v},$$

which is of the form of a credibility estimate $Z\hat{\lambda}+(1-Z)\times$ prior mean, where the credibility factor is $Z = 3/(v + 3)$ and the prior mean is $1/v$.

Many candidates misinterpreted Question 4(i) to mean that there are n_1 independent identically distributed random variables having a Poisson distribution with mean λ, and, in addition, n_2 independent identically distributed random variables having a Poisson distribution with mean 2λ, and this error led to the wrong likelihood. However, later marks were awarded for correct use of this wrong likelihood.

5 (i) The transition matrix is

$$\begin{bmatrix} q & 1-q \\ q & 1-q \end{bmatrix}$$

$(\pi_0 + \pi_1)\, q = \pi_0$

$\therefore\ \pi_0 = q$ since $\pi_0 + \pi_1 = 1$

and $\pi_1 = 1 - q$

The steady state distribution is $(q, 1 - q)$

(ii) Average premium is $1000q + 1000d\,(1 - q) = 1000\,(q + d(1 - q))$

$1000(0.2 + 0.8d) = 1.5 \times 1000\,(0.1 + 0.9d)$

$0.2 + 0.8d = 0.15 + 1.35d$

$0.05 = 0.55d$

$d = \dfrac{0.05}{0.55} = \dfrac{1}{11} = 0.091$

(iii) The value of d is extremely small, making this unrealistic. This is when the premium is 50% higher but it should be even higher. Thus, the NCD system is not effective.

Question 5 was done well.

6 (i) Conventional approach uses data from risk itself only. Credibility approach combines this with information from other sources using a credibility premium.

$$Z\overline{X} + (1 - Z)\mu$$

(ii) **Bayes**

Advantage: Not an approximation.

Disadvantage: Have to assume full distribution is correct.

EBCT

Advantage: Can be used when distribution is not known.

Disadvantage: An approximation.
May not take account of tail of distribution (e.g.)

(iii) Model 1:

$X_{ij} \mid \theta_i$ iid

$(\theta_i, X_{ij}), (\theta_k, X_{km})$ are iid $(i \neq k)$

Model 2:

$X_{ij} \mid \theta_i$ are independent (not id)

$(\theta_i, X_{ij}), (\theta_k, X_{km})$ are independent $(i \neq k)$

$\theta_1, \ldots, \theta_N$ iid

Difference is that $X_{ij} \mid \theta_i$ are not identically distributed to allow for different volumes of business.

In Question 6(ii), credit was given for other sensible and correct advantages and disadvantages.

7 (i)

414	460	482	488	500
453	506	526		
496	558			
540				
	1524	1008	488	500
1363	966	482		

Chain-ladder development factors:

	1.11812	1.04348	1.01245	1.02459
Forecasts:				
2000				500
2001			532.5	545.6
2002		582.3	589.5	604.0
2003	603.8	630.0	637.9	653.6

(ii) Average cost per claim:

2000	8.254	9.328	9.504	9.660	9.800
2001	10.627	13.613	13.321		
2002	11.782	14.337			
2003	12.322				

Grossing-up method:

84.22%	95.19%	96.98%	98.57%	
77.37%	99.10%			13.736
79.83%				14.758
				15.312

Average: 80.47% Average: 97.14%

(iii)

	Number of Claims	*Cost per Claim*	*Projected Loss*
2000	500	9.800	4900
2001	545.6	13.736	7494.9
2002	604.0	14.758	8914.1
2003	653.6	15.312	10007.5
			31316.6

Claims paid to date: 19212
Reserve = 12104.6

i.e. £12,104,600.

Question 7 was done well, although it is important to read the question carefully to make sure that the required methods are applied to the relevant figures.

8 (i) (a) The adjustment coefficient R satisfies

$$\lambda M(R) = \lambda + cR$$

(or equivalent).

(b) When claims are exponentially distributed with mean μ, the adjustment coefficient R_{exp} solves

$$\lambda \frac{1}{1-\mu r} = \lambda + cr$$

Hence

$$\lambda = \lambda(1 - \mu r) + cr(1 - \mu r)$$

$$0 = c\mu r^2 - cr + \lambda\mu r$$

$$= c\mu r\left(r - \frac{c-\lambda\mu}{c\mu}\right),$$

and so

$$R = \frac{c-\lambda\mu}{c\mu} = \frac{1}{\mu} - \frac{\lambda}{c}.$$

(c) When the premium loading factor is $\theta = 0.25$ and $\mu = 100$, then $c = 1.25\lambda\mu = 125\lambda$ and so $R_{\text{exp}} = 0.002$.

(d) The probability of ruin with initial capital u satisfies $\psi(u) \le e^{-Ru}$ for all $u > 0$.

(e) If the premium loading factor is θ, then $c = (1 + \theta)\lambda\mu$, so that $R_{\text{exp}} = \theta/((1 + \theta)\mu)$.

Hence if μ is increased then the adjustment coefficient is reduced, leading to an increased bound on $\psi(u)$, which makes sense since if claims are on average larger, then we expect a higher risk of ruin.

(f) If the premium loading factor is increased then the adjustment coefficient is increased, leading to a decreased bound on $\psi(u)$, which makes sense since increasing the premium loading factor means reduced risk.

(ii) (a) R^* solves

$$e^{100r} = 1 + 1.25 \times 100r$$

Search for root of this equation. For example,

R	e^{100r}	$1 + 125r$
0.002	1.22	1.25
0.004	1.49	1.50
0.005	1.65	1.63
0.0045	1.57	1.56

Hence R^* is between 0.004 and 0.0045, so that $R^* = 0.004$ to one significant figure.

(b) Then $R^* > R_{\text{exp}}$, which makes sense since we expect the ruin probability to be greater for the exponential distribution as it has heavier tails than the distribution concentrated on one point (or other sensible comment).

(iii) (a) The insurer's new premium income per unit time, net of reinsurance is

$$\tilde{c} = 100\lambda\,(1.25 - 1.30(1 - \alpha)).$$

The insurer's expected aggregate claim net of reinsurance per unit time is $100\lambda\alpha$ so that we require

$$1.25 - 1.30(1 - \alpha) > \alpha,$$

so that $\alpha > 0.05/0.30 = 0.17$.

(b) The insurer pays a proportion α of the original claim, so that the new claim size distribution is exponential with mean $\tilde{\mu} = 100\alpha$.

Hence we can use the result in (i)(b) for exponentially distributed claims to find that the new adjustment coefficient is

$$\tilde{R} = \frac{1}{100\alpha} - \frac{\lambda}{100\lambda(1.30\alpha - 0.05)} = \frac{1}{80} - \frac{1}{99} = 0.0024.$$

which is larger than R_{exp} resulting in a reduced Lundberg bound.

9 (i) Mean: nqm_1

Variance: $nqm_2 - nq^2 m_1^2$

Skewness: $nqm_3 - 3nq^2 m_2 m_1 + 2nq^3 m_1^3$

where $m_k = E[X^k]$

For the exponential $m_1 = \mu$, $m_2 = 2\mu^2$, $m_3 = 6\mu^3$

Variance $= 2nq\mu^2 - nq^2\mu^2 = nq\mu^2(2 - q)$

Skewness $= 6nq\mu^3 - 6nq^2\mu^3 + 2nq^3\mu^3$

$= 2nq\mu^3(3 - 3q + q^2)$

	Mean	*Var*	*Skewness*
Small	56.507	1379.647	50531.554
Medium	169.691	9302.743	765038.299
Large	46.517	12019.316	4658528.856
	272.715	22701.707	5474098.710

Standard deviation = 150.671

$$\text{Coefficient of skewness} = \frac{\text{Skewness}}{(\text{Variance})^{3/2}} = 1.6$$

(ii) $P(S > (1 + \theta)\, E(S)) = 0.05$

$$P\left(\frac{S - E(S)}{\sqrt{\text{Var}(S)}} > \frac{\theta E(S)}{\sqrt{\text{Var}(S)}}\right) = 0.05$$

$$\theta = 1.645\frac{\sqrt{\text{Var}(S)}}{E(S)} = 1.645 \times \frac{\sqrt{22701.707}}{272.715}$$

$$= 0.909$$

(iii) $P(S > 1.25E(S) + \text{Capital}) = 0.05$

$$\frac{0.25E(S) + \text{Capital}}{\sqrt{\text{Var}(S)}} = 1.645$$

$$\text{Capital} = 1.645\sqrt{\text{Var}(S)} - 0.25E(S)$$

$$= 179.675$$

i.e. £179,675

(iv) The assumption of independence will depend on the proximity of the risks. For example, if they are close and there is the danger of a catastrophic fire caused, for example, by a natural disaster, the assumption may not be valid.

However, it is probably likely to be reasonable.

The normal approximation is unsuitable when looking at the upper tail, as here, especially when the distribution is positively skewed. The consequence is that the capital required and the premium loading will be underestimated.

In Question 9(i), the first part (finding the mean) was done well. Many candidates used wrong formulae for the variances for the three types of buildings, and many candidates omitted the coefficient of skewness.

10 (i)

$$\int_d^{L+d} xf(x)dx = \int_d^{L+d} x\frac{\alpha\lambda^\alpha}{(\lambda+x)^{\alpha+1}}dx$$

$$= \left[-x\frac{\lambda^\alpha}{(\lambda+x)^\alpha}\right]_d^{L+d} + \int_d^{L+d}\frac{\lambda^\alpha}{(\lambda+x)^\alpha}dx$$

$$= \left[-x\frac{\lambda^\alpha}{(\lambda+x)^\alpha}\right]_d^{L+d} + \left[-\frac{1}{\alpha-1}\frac{\lambda^\alpha}{(\lambda+x)^{\alpha-1}}\right]_d^{L+d}$$

$$= \left[-x\frac{\lambda^\alpha}{(\lambda+x)^\alpha} - \frac{\lambda^\alpha}{(\alpha-1)(\lambda+x)^{\alpha-1}}\right]_d^{L+d}$$

$$= \left[\frac{-\lambda^\alpha(x(\alpha-1)+\lambda+x)}{(\alpha-1)(\lambda+x)^\alpha}\right]_d^{L+d}$$

$$= \frac{\lambda^\alpha}{(\alpha-1)}\left[-\frac{\alpha x+\lambda}{(\lambda+x)^\alpha}\right]_d^{L+d}$$

$$= \frac{\lambda^\alpha}{\alpha-1}\left[\frac{\alpha d+\lambda}{(\lambda+d)^\alpha}-\frac{\alpha(L+d)+\lambda}{(\lambda+L+d)^\alpha}\right]$$

(ii) (a) $\frac{\lambda}{\alpha-1} = 3000$ and $\left(\frac{\lambda}{\alpha-1}\right)^2\frac{\alpha}{\alpha-2} = 6000^2$

$\therefore\ \frac{\alpha}{\alpha-2} = \frac{6000^2}{3000^2} = 4$

$\therefore\ \alpha = 4\alpha - 8$

$\therefore\ \alpha = 8/3$ and $\lambda = 3000 \times 5/3 = 5000$

$$X_R = \begin{cases} 0 & X < d \\ X-d & d \le X < d+L \\ L & X > d+L \end{cases}$$

$$E[X_R \mid X > d] = \frac{E[X_R]}{P(X>d)}$$

$$E[X_R] = \int_d^{d+L}(x-d)f(x)dx + LP(X>d+L)$$

$$= \int_d^{d+L} xf(x)dx - dP(d<X<d+L) + LP(X>d+L)$$

$$\int_{8000}^{8000+6000} xf(x)dx = \frac{5000^{8/3}}{5/3}\left[\frac{8/3\times 8000+5000}{(5000+8000)^{8/3}}-\frac{8/3\times 14000+5000}{(5000+14000)^{8/3}}\right]$$

$$= 513.78$$

$$P(8000 < X < 14000) = \left(\frac{\lambda}{\lambda+8000}\right)^\alpha - \left(\frac{\lambda}{\lambda+14000}\right)^\alpha$$

$$= 0.0498$$

$$\therefore\ E[X_R] = 513.78 - 8000 \times 0.0498 + 6000 \times \left(\frac{\lambda}{\lambda + 14000}\right)^{\alpha}$$

$$= 513.78 - 398.37 + 170.63$$

$$= 286.04$$

$$P(X > 8000) = 0.07824$$

$$\therefore\ E[X_R \mid X > 8000] = \frac{286.04}{0.07824} = £3656.14.$$

(b)

$$X'_R = \begin{cases} 0 & 1.1X < d \\ 1.1X - d & d < 1.1X < d + L \\ L & 1.1X > d + L \end{cases}$$

$$= \begin{cases} 0 & X < \dfrac{d}{1.1} \\ 1.1X - d & \dfrac{d}{1.1} < X < \dfrac{d+L}{1.1} \\ L & X > \dfrac{d+L}{1.1} \end{cases}$$

$$E[X'_R] = 1.1\int_{\frac{d}{1.1}}^{\frac{d+L}{1.1}} xf(x)dx - dP\left(\frac{d}{1.1} < X < \frac{d+L}{1.1}\right) + LP\left(X > \frac{d+L}{1.1}\right)$$

$$\int_{\frac{8000}{1.1}}^{\frac{14000}{1.1}} xf(x)dx = \frac{5000^{8/3}}{5/3}\left[\frac{8/3 \times \frac{8000}{1.1} + 5000}{\left(5000 + \frac{8000}{1.1}\right)^{8/3}} - \frac{8/3 \times \frac{14000}{1.1} + 5000}{\left(5000 + \frac{14000}{1.1}\right)^{8/3}}\right]$$

$$= 535.72$$

$$P\left(X > \frac{8000}{1.1}\right) = \left(\frac{\lambda}{\lambda + \frac{8000}{1.1}}\right)^{\alpha} = 0.0912173$$

$$P\left(X > \frac{14000}{1.1}\right) = 0.0342143$$

$$\therefore \; E[X'_R] = 1.1 \times 535.72 - 8000 \times 0.057003 + 6000 \times 0.0342143$$

$$= 338.55$$

$$\therefore \; E[X'_R | X'_R > 8000] = \frac{338.55}{0.0912173} = £3{,}711.51$$

In Question 10(i), there were many slips in the integration, and not all candidates showed how their expression simplified to the one given in the question.

In Question 10(ii)(a), many candidates had an incorrect expression for the amount paid by the reinsurer. Another common source of error was to forget about the conditioning, or to condition on the wrong event (e.g. conditioning on the claim being between £8000 and £14000, instead of correctly conditioning on the claim being bigger than £8000).

END OF EXAMINERS' REPORT