EXAMINATIONS

12 April 2002 (am)

Subject 106 — Actuarial Mathematics 2

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*

2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*

3. *Mark allocations are shown in brackets.*

4. *Attempt all 10 questions, beginning your answer to each question on a separate sheet.*

Graph paper is not required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available Actuarial Tables and your own electronic calculator.

1 For each of m independent risks, there is probability 0.2 that a claim is made in a year and probability 0.8 that no claim is made. Claim sizes are independent with mean 400 and variance 110.

Determine the expected value and the variance of the total amount claimed in one year. [3]

2 The loss function under a decision problem is given by:

	θ_1	θ_2	θ_3
d_1	14	12	13
d_2	13	15	14
d_3	11	15	5

(i) Determine the minimax solution to this problem. [2]

(ii) Given the probability distribution $p(\theta_1) = 0.25$, $p(\theta_2) = 0.25$, $p(\theta_3) = 0.5$ determine the Bayes criterion solution. [2]
[Total 4]

3 Claims arrive according to a Poisson process. Individual claim sizes are independent with density:

$$f(x) = xe^{-x} \qquad (x > 0)$$

and the insurer uses a premium loading factor of θ.

(i) Derive the equation for the adjustment coefficient for this process. [2]

(ii) If $\theta = 0.4$, calculate the adjustment coefficient, and determine an upper bound for the probability of ultimate ruin if the initial surplus is 50. [4]
[Total 6]

4 The loss amount, X, on a certain type of insurance policies, has a Pareto distribution with density function $f(x)$, where:

$$f(x) = \frac{3 \times 400^3}{(400 + x)^4} \qquad (x > 0)$$

A policyholder deductible of £100 is applied to these policies.

(i) Calculate the expected claim size paid by the insurance company. [5]

(ii) Comment on the difference between your answer to (i) and the expected loss amount, $E[X]$. [2]
[Total 7]

5 The delay triangles given below relate to a certain portfolio of insurance policies, for which it may be assumed that the claims are fully run off by the end of development year 2.

The cost of claims incurred each year is given in the table below:

	Development Year		
Accident Year	*0*	*1*	*2*
1999	2,317	1,437	582
2000	3,287	1,792	
2001	4,816		

The cumulative number of reported claims is shown in the table below:

	Development Year		
Accident Year	*0*	*1*	*2*
1999	132	197	207
2000	183	258	
2001	261		

(i) Given that the total claims paid to date is 10,237 calculate the outstanding claims reserve for this cohort using the average cost per claim method. [7]

(ii) State the assumptions that underlie your result. [2]
[Total 9]

6 The no claims discount system operated by an insurance company for their annual motor insurance business has four levels of discount:

Level 1: 0%
Level 2: 25%
Level 3: 45%
Level 4: 60%

If a policyholder does not make a claim under the policy in a particular year then he or she will go up one level (or stay at level 4), whereas if any claims are made he or she will go down by two levels (or remain at, or move to, level 1). The full premium payable at the 0% discount level is 900.

The probability of an accident occurring is assumed to be 0.2 each year for all policyholders and losses are assumed to follow a lognormal distribution with mean 1,188 and standard deviation 495. However, policyholders claim only if the loss is greater than the total additional premiums that would have to be paid over the next three years, assuming that no further accidents occur.

(i) Calculate the smallest loss amount for which a claim will be made for a policyholder at the 0% discount level. [2]

(ii) Complete the transition matrix below by calculating the missing values.

$$\begin{pmatrix} * & * & * & * \\ 0.147 & 0 & 0.853 & 0 \\ 0.120 & 0 & 0 & 0.880 \\ 0 & 0.197 & 0 & 0.803 \end{pmatrix}$$

[5]

(iii) Calculate the proportion of policyholders at each discount level when the system reaches a stable state. [4]

[Total 11]

7 Claims occur on a portfolio of 100 general insurance policies according to a Poisson process. The expected number of claims per annum on each policy is λ, and the claim size distribution has density function $f(x)$, where

$$f(x) = \frac{1}{10{,}000} x e^{-x/100} \qquad (x > 0)$$

The parameter λ is not the same for all policies in the portfolio, but is modelled as a random variable (independent of the claim sizes) with density function $g(\lambda)$, where:

$$g(\lambda) = 100\lambda e^{-10\lambda} \qquad (\lambda > 0)$$

(i) Calculate the mean and variance of annual aggregate claims. [6]

(ii) Given an initial reserve of 2,000, use a normal approximation to the distribution of aggregate claims to find the relative premium loading that should be used in order to be 95% sure that the reserve at the end of the first year is positive. [3]

(iii) If λ were fixed at its mean value, describe the effect that this would have on the value of the relative premium loading. (No further calculations are required.) [2]

[Total 11]

8 An insurance portfolio consists of m groups of individuals. In the ith group there are n_i (> 1) individuals aged x_i. The number of claims from this group of n_i individuals is a binomial random variable Y_i with parameters n_i and θ_i, with $0 \le \theta_i \le 1$ ($i = 1, \ldots, m$). The random variables $Y_1, \ldots, Y_m$ are independent.

(i) Derive the maximum likelihood estimators $\hat{\theta}_1, \ldots, \hat{\theta}_m$ of $\theta_1, \ldots, \theta_m$. [3]

(ii) (a) If

$$\text{logit}(\theta_i) = \log\left(\frac{\theta_i}{1-\theta_i}\right) = \alpha + \beta x_i,$$

show that the log-likelihood is

$$\alpha \sum_{i=1}^{m} y_i + \beta \sum_{i=1}^{m} x_i y_i - \sum_{i=1}^{m} n_i \log(1 + \exp(\alpha + \beta x_i)) + c,$$

where c does not depend on α or β.

(b) Derive, but do not attempt to solve, the equations satisfied by the maximum likelihood estimators $\hat{\alpha}$ and $\hat{\beta}$ of α and β. [5]

(iii) Let $e_i = n_i \tilde{\theta}_i$ where $\tilde{\theta}_i = \exp(\hat{\alpha} + \hat{\beta} x_i)/(1 + \exp(\hat{\alpha} + \hat{\beta} x_i))$.

Derive the scaled deviance for the model in (ii). [3]

(iv) Data are available for six ages, 17, 18, 19, 20, 21 and 22.

Two models are fitted to the data with the following results:

		Scaled Deviance	*Degrees of Freedom*
Model 1	$\text{logit}(\theta_i) = \alpha$	13.33	5
Model 2	$\text{logit}(\theta_i) = \alpha + \beta x_i$	1.67	*

(a) State the degrees of freedom for Model 2.

(b) Comment on the fit of these two models.

(c) In Model 2, the estimate of β is -0.2492, with standard error 0.07217. Interpret these results and explain how the odds $\theta_{i+1}/(1 - \theta_{i+1})$ are related to the odds $\theta_i/(1 - \theta_i)$, when $x_{i+1} = x_i + 1$. [5]

[Total 16]

9 The number of e-mail messages received each day by an actuarial student has a Poisson distribution with mean λ, where from past experience, the prior distribution of λ is exponential with mean μ.

(i) The student has data $x_1, \ldots, x_n$, where x_i is the number of messages arriving on day i, $i = 1, \ldots, n$.

(a) Derive the posterior distribution of λ.

(b) Show that the Bayesian estimate of λ under quadratic loss can be written in the form of a credibility estimate, and state the credibility factor.

(c) If $\mu = 50$ and the student receives a total of 550 messages over 10 days, calculate the Bayesian estimate of λ under quadratic loss. [6]

(ii) 60% of messages require an answering time (in minutes) that is exponentially distributed with mean 1, and the remaining messages require an answering time that has a Pareto distribution with mean 2 and variance 12.

(a) Determine the probability that a randomly chosen message requires more than M minutes answering time.

(b) Using the value of λ estimated in (i)(c), calculate the expected total amount of time the student spends answering all the messages that arrive on one day.

(c) The student decides to reduce the amount of time spent with e-mail, and adopts a strategy of imposing a maximum cut-off time of 1.5 minutes for answering each message. Still using the estimated value of λ, calculate the reduction in the expected amount of time spent answering the e-mail messages from one day. [10]

[Total 16]

10 The last ten claims under a particular class of insurance policy were:

1,330	201	111	2,368	617
309	35	4,685	442	843

(i) Assuming that the claims came from a lognormal distribution with parameters μ and σ, derive the formula for the maximum likelihood estimates of these parameters and estimate the parameters based on the observed data. [5]

(ii) Assuming that the claims come from a Pareto distribution with parameters α and λ, use the method of moments to estimate these parameters. [3]

(iii) Assuming that the claims come from a Weibull distribution with parameters c and γ, use the method of percentiles (based on the 25^{th} and 75^{th} percentiles) to estimate these parameters. [5]

(iv) If the insurance company takes out reinsurance cover with an individual excess of loss of 3,000 estimate the percentage of claims that will involve the reinsurer under each of the three models above. [4]

[Total 17]