EXAMINATIONS

September 2001

Subject 106 — Actuarial Mathematics 2

EXAMINERS' REPORT

Examiners' Comments

In numerical questions, candidates were not unduly penalised for errors in earlier parts of each question which affected their answers to the rest of the question.

Question 7 contained two misprints. The question should say that the *variances* are σ_1^2 and σ_2^2, not the *standard deviations*. Many candidates corrected this error and gave the solutions given here. Others interpreted the question as written, thus obtaining σ_1^4 and σ_2^4 (instead of σ_1^2 and σ_2^2) in their solutions. Both approaches, provided done correctly, were awarded full marks.

Question 11 on generalised linear models was poorly done in general. Candidates should bear in mind that the examiners will continue to set questions on this part of the syllabus.

1 If $f(x) = \dfrac{\alpha\lambda^{\alpha}}{(x+\lambda)^{\alpha+1}} \quad x > 0$

then $\dfrac{\lambda}{\alpha-1} = 350 \qquad \dfrac{\lambda^2}{(\alpha-1)^2}\dfrac{\alpha}{\alpha-2} = 452^2$

so $\dfrac{\alpha}{\alpha-2} = \dfrac{452^2}{350^2} = 1.6678$

and $\alpha = \dfrac{2\times 1.6678}{0.6678} = 4.995$

$\lambda = 350 \times 3.995 = 1{,}398.2$

so $P(X > 1{,}200) = \left(\dfrac{\lambda}{\lambda+1{,}200}\right)^{\alpha} = \underline{0.0453}$

2 $E[Y] = E[E[Y|X]]$

$= E[2X + 400]$

$= 2E[X] + 400$

$= 500$

$V[Y] = V[E[Y|X]] + E[V[Y|X]]$

$= V[2X + 400] + E\left[\dfrac{X^2}{2}\right]$

$= 4V[X] + ½E[X^2]$

$= 4 \times 14^2 + ½(V[X] + (E[X])^2)$

$= 784 + 1{,}348 = 2{,}132$

Hence standard deviation of Y is 46.17.

3 Let X = gross claim amount.

For 5 claims, $X < 500$
For 26 claims, $\Sigma x = 76{,}457 + 26 \times 500 = 89{,}457$
For 11 claims, $X > 5{,}000$

Now:

$$P(X < 500) = 1 - e^{-500\lambda}$$

$$P(X > 5{,}000) = e^{-5{,}000\lambda}$$

Likelihood is:

$$\left(\prod_{i=1}^{26} \lambda e^{-\lambda x_i}\right)[1 - e^{-500\lambda}]^5\ [e^{-5000\lambda}]^{11}$$

log Likelihood is:

$$26 \log \lambda - \lambda \sum_{i=1}^{26} x_i + 5 \log[1 - e^{-500\lambda}] + 11\ [- 5{,}000\lambda]$$

$$= 26 \log \lambda - 89{,}457\lambda + 5 \log[1 - e^{-500\lambda}] - 55{,}000\lambda$$

$$= 26 \log \lambda - 144{,}457\lambda + 5 \log[1 - e^{-500\lambda}]$$

4 (i) The decision matrix is:

	Winnings			
	History	*Literature*	*Sport*	*General Knowledge*
A	0	100	100	100
B	0	0	150	150
C	200	200	0	0

Expected winnings against:

Player A $0.15 \times 0 + 0.25 \times 100 + 0.20 \times 100 + 0.40 \times 100 = 85$
Player B $0.15 \times 0 + 0.25 \times 0 + 0.20 \times 150 + 0.40 \times 150 = 90 \leftarrow$
Player C $0.15 \times 200 + 0.25 \times 200 + 0.20 \times 0 + 0.40 \times 0 = 80$

Choose Player B

(ii) Expected winnings against:

Player A $(0.15 \times 0 + 0.25 \times 100 + 0.40 \times 100) / (1 - 0.20) = 81.25$
Player B $(0.15 \times 0 + 0.25 \times 0 + 0.40 \times 150) / (1 - 0.20) = 75$
Player C $(0.15 \times 200 + 0.25 \times 200 + 0.40 \times 0) / (1 - 0.20) = 100 \leftarrow$

Choose Player C

5 $M_{S_I}(t) = M_N(\log M_{X_I}(t))$

$$M_{X_I}(t) = \int_0^M e^{tx}\,\lambda e^{-\lambda x}\,dx + e^{tM}\,e^{-\lambda M}$$

$$= \left[-\frac{\lambda}{\lambda - t}e^{-(\lambda - t)x}\right]_0^M + e^{-(\lambda-t)M}$$

$$= \frac{\lambda}{\lambda - t} + \left(1 - \frac{\lambda}{\lambda - t}\right)e^{-(\lambda-t)M}$$

$M_N(t) \qquad ((1-p) + pe^t)^n$

$$\therefore\ M_{S_I}(t) = \left[1 - p + p\left(\frac{\lambda}{\lambda - t} + \left(1 - \frac{\lambda}{\lambda - t}\right)e^{-(\lambda-t)M}\right)\right]^n$$

6 (i)

	Claims	Doesn't claim
0%	0 + 190 + 152 + 133+ …	L + 152 + 133 + …
20%	0 + 190 + 152 + 133 + …	L +133 + 133 + …
30%	0 + 190 + 152 + 133 +	L + 133 + 133 +

0%: claims if	$L + 152 + 133 > 190 + 152$	$L > 57$
20%, 30%: claims if	$L + 133 + 133 > 190 + 152$	$L > 76$

where $L \sim$ lognormal (μ, σ^2) is the loss.

$$P(L > 57) = 1 - \Phi\left(\frac{\ln 57 - 4.5}{\sqrt{0.84}}\right) = 1 - \Phi\,(-0.499) = 0.6911$$

$$P(L > 76) = 1 - \Phi\left(\frac{\ln 76 - 4.5}{\sqrt{0.84}}\right) = 1 - \Phi\,(-0.18) = 0.5714$$

so
$$P = \begin{bmatrix} 0.1\times 0.6911 & 1 - 0.1\times 0.6911 & 0 \\ 0.1\times 0.5714 & 0 & 1 - 0.1\times 0.5714 \\ 0.1\times 0.5714 & 0 & 1 - 0.1\times 0.5714 \end{bmatrix}$$

(ii) E(premium at beginning of next year)

$$= 190 \times 0.1 \times 0.6911 + 152 \times (1 - 0.1 \times 0.6911)$$

$$= \underline{154.63}$$

7 (i) (a) Posterior mean:

$$= \frac{\mu\sigma_1^2 + n\sigma_2^2\,\bar{a}}{\sigma_1^2 + n\sigma_2^2}$$

(b)

$$= \frac{\sigma_1^2}{\sigma_1^2 + n\sigma_2^2}\,\mu + \frac{n\sigma_2^2}{\sigma_1^2 + n\sigma_2^2}\,\bar{a}$$

Hence posterior mean can be expressed as:

$$Z\bar{a} + (1 - Z)\,\mu$$

Where: $Z = \dfrac{n\sigma_2^2}{\sigma_1^2 + n\sigma_2^2} = \dfrac{n}{n + \dfrac{\sigma_1^2}{\sigma_2^2}}$

And: $1 - Z = \dfrac{\sigma_1^2 + n\sigma_2^2 - n\sigma_2^2}{\sigma_1^2 + n\sigma_2^2} = \dfrac{\sigma_1^2}{\sigma_1^2 + n\sigma_2^2}$

(ii) $\bar{a}$ = £163.73

And: $Z = \dfrac{15}{15 + \dfrac{210^2}{84^2}} = 0.70588$

Hence premium is (0.29412 × £155 + 0.70588 × £163.73) × 1.30 = £209.51

(iii) As $n \to \infty$, $Z \to 1$ because more weight is placed on the sample data if we have a greater volume of data.

As $\sigma_1^2 \to \infty$, $Z \to 0$ because less weight is placed on the sample data if we believe the variability of the observed data to be higher.

As $\sigma_2^2 \to \infty$, $Z \uparrow 1$ because more weight is placed on the sample data if we believe the variability of the prior estimate to be higher.

8 (i) $f(\underline{x}\,|\,\lambda) \propto e^{-n\lambda}\ \lambda^{\Sigma x_i}$

$f(\lambda) \propto e^{-5\lambda}$

so $f(\lambda\,|\,\underline{x}) \propto \lambda^{\sum_1^n x_i}\ e^{-(n+5)\lambda}$

and $\underline{\lambda|\underline{x} \sim \text{gamma}\left(\sum_1^n x_i + 1, n+5\right)}$

(ii) (a) Under quadratic loss, the Bayesian estimate is the posterior mean

$$\underline{E(\lambda|\underline{x}) = \frac{\sum_1^n x_i + 1}{n+5}}$$

(b) Under all-or-nothing loss, the Bayesian estimate is the posterior mode.

$$f(\lambda\,|\,\underline{x}) \propto \lambda^{\Sigma x_i}\ e^{-(n+5)\lambda}$$

If $\Sigma x_i = 0$, then mode is at $\lambda = 0$

Otherwise, the posterior density has a maximum when

$$(\Sigma x_i)\ \lambda^{\Sigma x_i - 1}\ e^{-(n+5)\lambda} = (n+5)\ \lambda^{\Sigma x_i}\ e^{-(n+5)}$$

i.e. $\underline{\lambda = \dfrac{\Sigma x_i}{n+5}}$

(iii) Under absolute error loss, the Bayesian estimate is the posterior median

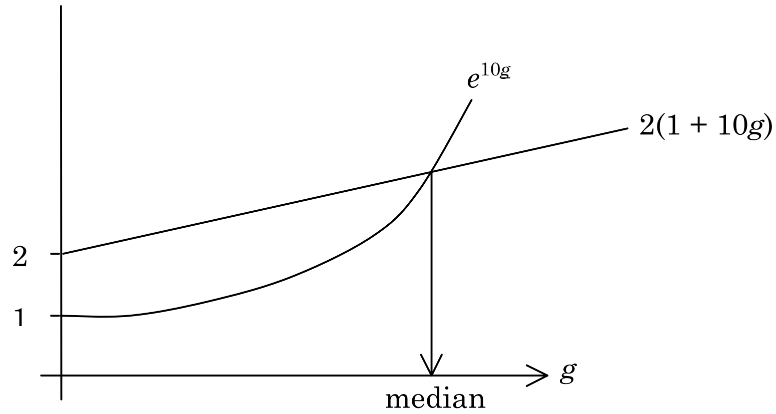
$n = 5, \Sigma x_i = 1$ so $\lambda\,|\,\underline{x} \sim \text{gamma}(2, 10)$

Median given by g where $\int_g^\infty 10^2\ \lambda e^{-10\lambda}\ d\lambda = ½$

i.e. $[-10\lambda e^{-10\lambda}]_g^\infty + \int_g^\infty 10 e^{-10\lambda}\ d\lambda = ½$

i.e. $(10g + 1)\ e^{-10g} = ½$

i.e. $2(10g + 1) = e^{10g}$



g	$2(1 + 10g)$	e^{10g}	root
0.1	4	2.7	> 0.1
0.2	6	7.4	< 0.2
0.15	5	4.5	> 0.15
0.16	5.2	4.95	> 0.16
0.17	5.4	5.47	< 0.17
0.165	5.3	5.21	> 0.165

So root between 0.165 and 0.17. To 2 sig. figs, $\underline{g = 0.17}$

9 Adjust individual claim amounts to mid-1999 prices:

Figures in £000s

	Development year			
Policy year	*0*	*1*	*2*	*3*
1996	2,002	1,507	991	461
1997	2,294	1,684	1,195	
1998	2,543	1,863		
1999	2,935			

Cumulative claim amounts:

Figures in £000s

	Development year			
Policy year	*0*	*1*	*2*	*3*
1996	2,002	3,509	4,500	4,961
1997	2,294	3,978	5,173	
1998	2,543	4,406		
1999	2,935			
Column Sum		**11,893**	**9,673**	**4,961**
Column sum minus last entry	**6,839**	**7,487**	**4,500**	

Development factors for each year:

d_3 = 4,961/4,500 = 1.10244

d_2 = 9,673/7487 = 1.29197

d_1 = 11,893/6,839 = 1.73900

Ultimate claim amount for 1999 policies = 0.77 × 7,731 = 5,953

Hence outstanding amounts arising from 1999 policies:

$$1999,\ 3 = 5{,}953 \times \left(1 - \frac{1}{1.10244}\right) = 553$$

$$1999,\ 2 = 5{,}953 \times \left(\frac{1}{1.10244} - \frac{1}{1.10244 \times 1.29197}\right) = 1{,}220$$

$$1999,\ 1 = 5{,}953 \times \left(\frac{1}{1.10244 \times 1.29197} - \frac{1}{1.10244 \times 1.29197 \times 1.73900}\right) = 1{,}776$$

Adjust for future inflation $= 553 \times 1.05^3 + 1{,}220 \times 1.05^2 + 1{,}776 \times 1.05$

$= £3{,}850{,}000$

10 (i) (a) Let μ be the mean claim size.

Then the adjustment coefficient R is the positive solution of $M(r) = 1 + (1 + \theta)\,\mu r$.

(b) The surplus at time t is $U(t) = u + (1 + \theta)\,\lambda\mu t - S(t)$ where $S(t)$ is the aggregate claims at time t.

The surplus process is $\{U(t): t > 0\}$.

The probability of ruin with initial surplus u is $\psi(u) = P(U(t) < 0$ for some t, $0 < t < \infty)$.

(c) Lundberg's inequality says: $\psi(u) \le \exp\{-Ru\}$.

(ii) For exponential mean μ, the mgf is $M(r) = \dfrac{1}{1-\mu r}$, $r < \dfrac{1}{\mu}$.

So $R > 0$ solves

$$\frac{1}{1-\mu r} = 1 + \frac{5}{4}\mu r$$

$$4 = 4(1 - \mu r) + 5\mu r(1 - \mu r)$$

$$5\mu^2 r^2 - \mu r = 0$$

$R > 0$ so $\underline{R = \dfrac{1}{5\mu}}$ (check: $R < \dfrac{1}{\mu}$)

(iii) (a) Claims ~ ½ × (exponential mean 1) + ½ × (exponential mean ½) (*)

So $\underline{\mu = \tfrac{1}{2} \times 1 + \tfrac{1}{2} \times \tfrac{1}{2} = \tfrac{3}{4}}$

[or evaluate $\displaystyle\int_0^\infty x\, \tfrac{1}{2} e^{-x}(1 + 2e^{-x})\, dx$]

(b) From (*), $M(r) = \tfrac{1}{2} \times \dfrac{1}{1-r} + \tfrac{1}{2} \times \dfrac{2}{2-r}$, $r < 1$

[or evaluate $\displaystyle\int_0^\infty e^{rx}\ \tfrac{1}{2} e^{-x}(1 + 2e^{-x})\, dx$]

$R > 0$ solves

$$½\frac{1}{(1-r)} + \frac{2}{2(2-r)} = 1 + \frac{5}{4} \times \frac{3}{4}r$$

$$8(2 - r) + 16(1 - r) = 16(2 - 3r + r^2) + 15r(2 - 3r + r^2)$$

$$24r = -18r - 29r^2 + 15r^3$$

$R \neq 0 \quad 15r^2 - 29r + 6 = 0$

Solution $\quad r = \dfrac{29 \pm \sqrt{29^2 - 4 \times 6 \times 15}}{30} = 1.6977$ or 0.2356

Need $R < 1$, i.e. R in domain of definition of $M(r)$

So $\underline{R = 0.2356}$

Lundberg's inequality gives $\psi(15) \leq e^{-R\times 15} = \underline{0.0292}$

(c) For exponential mean $\mu = ¾$, we know from (ii) that the adjustment coefficient R^* is $R^* = \dfrac{1}{5\mu} = \dfrac{4}{15} = \underline{0.2667}$ and so Lundberg's Inequality gives $\psi(15) \leq e^{-R^*\times 15} = \underline{0.0183}$.

R^* is larger than the true value, which means that the bound on $\psi(15)$ is smaller than the true bound, and may lead to a false sense of security.

11 (i) (a)

$$f(\underset{\sim}{y}) = \prod_{i=1}^{n} \frac{e^{-\mu_i}\,\mu_i^{y_i}}{y_i\,!}$$

Log likelihood is

$$l(\alpha, \beta) = -\sum_1^n \mu_i + \sum_1^n y_i \log \mu_i - \log(\Pi y_i\,!)$$

$$= -me^{\alpha} - (n-m)\,e^{\alpha+\beta} + \alpha\left(\sum_1^m y_i\right) + (\alpha+\beta)\left(\sum_{m+1}^n y_i\right) - \log(\Pi y_i\,!)$$

$$= \underline{-me^{\alpha} - (n-m)\,e^{\alpha+\beta} + \alpha\left(\sum_1^n y_i\right) + \beta\left(\sum_{m+1}^n y_i\right) - \log(\Pi y_i\,!)}$$

(b)

$$\frac{\partial l}{\partial \alpha} = 0: -me^{\alpha} - (n-m)\,e^{\alpha+\beta} + \sum_{1}^{n} y_i = 0 \qquad (1)$$

$$\frac{\partial l}{\partial \beta} = 0: -(n-m)\,e^{\alpha+\beta} + \sum_{m+1}^{n} y_i = 0 \qquad (2)$$

Substitute (2) in (1): $me^{\alpha} = \sum_{1}^{m} y_i$

$$\text{so} \quad \underline{\hat{\alpha} = \log_e\left(\frac{\sum_1^m y_i}{m}\right)}$$

Substitute $\hat{\alpha}$ in (2): $\hat{\alpha} + \hat{\beta} = \log_e\left(\frac{\sum_{m+1}^n y_i}{n-m}\right)$

$$\text{so} \ \hat{\beta} = \log_e\left(\frac{\sum_{m+1}^n y_i}{n-m}\right) - \log_e\left(\frac{\sum_1^m y_i}{m}\right)$$

(c)

Deviance = 2{log likelihood for full model $- \ l(\hat{\alpha}, \hat{\beta})$}

In the full (or saturated) model, fitted values are $\tilde{\mu}_i = y_i$.

In model (*) the fitted values are $\hat{\mu}_i = \begin{cases} t_1 & i = 1, \ldots, m \\ t_2 & i = m+1, \ldots, n \end{cases}$

where $t_1 = \sum_1^m \frac{y_i}{m}, t_2 = \sum_{m+1}^n \frac{y_i}{(n-m)}$.

So Deviance

$$= 2\left\{-\sum_{i=1}^{n} y_i + \sum_{1}^{n} y_i \log y_i + \sum_{i=1}^{m} t_1 - \sum_{1}^{m} y_i \log t_1 + \sum_{m+1}^{n} t_2 - \sum_{m+1}^{n} y_i \log t_2\right\}$$

$$= \sum_{1}^{m} 2\left\{y_i \log\left(\frac{y_i}{t_1}\right) - (y_i - t_1)\right\} + \sum_{m+1}^{n} 2\left\{y_i \log\left(\frac{y_i}{t_2}\right) - (y_i - t_2)\right\}$$

or equivalent expression

(d)

When $\beta = 0$, $\mu_i = e^{\alpha}$ for all i

maximum likelihood estimate of α is $\tilde{\alpha} = \log\left(\frac{\Sigma y_i}{n}\right)$

with fitted values $\mu_i' = \frac{\Sigma y_i}{n} = \bar{y}$ for all i

The deviance is $2\{\Sigma y_i \log y_i - \Sigma y_i \log \bar{y}\}$

$$= 2\Sigma y_i \log\left(\frac{y_i}{\bar{y}}\right)$$

(ii) (a) $\bar{y} = 6.3$

y_i	*freq*	*Contribution to deviance*
2	2	−9.1792
4	2	−7.2681
5	3	−6.9334
6	3	−1.7564
7	3	4.4251
8	4	15.2891
9	2	12.8403
10	1	9.2407
		16.6581

So Deviance is 16.6581 on 19 df.

(b) Deviance for model (*) is 16.1499 on 18 df.

Both models fit well.

Drop in deviance resulting from including β is 0.5083 on 1 df.

This is not significant when referred to a χ_1^2.

So no evidence to suggest we need β in the model.