EXAMINATIONS

4 April 2003 (am)

Subject 106 — Actuarial Mathematics 2

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*
2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*
3. *Mark allocations are shown in brackets.*
4. *Attempt all 10 questions, beginning your answer to each question on a separate sheet.*

Graph paper is not required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available Actuarial Tables and your own electronic calculator.

1 The lengths of time taken to deal with each of n reports are independent exponentially distributed random variables with mean $1/\lambda$.

Show that the gamma distribution is the conjugate prior for this exponential distribution. [3]

2 For the following main categories of insurance product:

Liability
Property

(i) Describe the cover provided. [2]

(ii) List two examples of insurance cover in each category. [2]

[Total 4]

3 (i) A random variable Y has density of exponential family form:

$$f(y) = \exp\left(\frac{y\theta - b(\theta)}{a(\phi)} + c(y, \phi)\right).$$

State the mean and variance of Y in terms of $b(\theta)$ and its derivatives and $a(\phi)$. [1]

(ii) (a) Show that an exponentially distributed random variable with mean μ has a density that can be written in the above form.

(b) Determine the natural parameter and the variance function.

[3]

[Total 4]

4 The loss function under a decision problem is given by:

	Θ_1	Θ_2	Θ_3
D_1	23	34	16
D_2	30	19	18
D_3	23	27	20
D_4	32	19	19

(i) State which decision can be discounted immediately and explain why. [2]

(ii) Determine the minimax solution to the problem. [2]

(iii) Given the following distribution $P(\Theta_1) = 0.25$, $P(\Theta_2) = 0.15$, $P(\Theta_3) = 0.60$, determine the Bayes criterion solution to the problem. [2]

[Total 6]

5 Claims arrive in a Poisson process rate λ, and $N(t)$ is the number of claims arriving by time t. The claim sizes are independent random variables $X_1, X_2, \ldots$, with mean μ, independent of the arrivals process. The initial surplus is u and the premium loading factor is θ.

(i) (a) Give an expression for the surplus $U(t)$ at time t.

(b) Define the probability of ruin with initial surplus u, $\psi(u)$, and sketch a realisation of the surplus process that shows a ruin event (graph paper not needed).

(c) State the value of $\psi(u)$ when $\theta = 0$.

[4]

(ii) The unit of currency is changed so that one unit of the old currency is worth the same as 2.5 units of the new currency.

Determine a relationship between $\psi(u)$ in (i)(b) and the probability of ruin for the new process. [2]

(iii) Suppose instead that the old unit of currency is used, but the claim arrival rate is doubled. Let $N^*(t)$ be the new number of claims by time t.

(a) Show that $N^*(t)$ has the same distribution as $N(2t)$.

(b) Derive an expression for the surplus $U^*(t)$ for the new process in terms of the surplus process in (i)(a), and determine the probability of ruin for the new process in terms of the probability of ruin in (i)(b).

[3]
[Total 9]

6 The triangle below shows incremental claims for a portfolio of property insurance policies.

Accident Year	*Development Year* 0	1	2	3
1998	4,652	3,079	931	275
1999	6,067	4,555	1,183	
2000	5,822	4,297		
2001	7,934			

(i) Calculate the outstanding claims reserve for this portfolio using the basic chain ladder method. [8]

(ii) State any assumptions used in determining your answer in (i). [2]
[Total 10]

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7 Claims on a portfolio of general insurance policies have a Pareto distribution, with density function $f(x)$, where:

$$f(x) = \frac{\alpha\lambda^{\alpha}}{x^{\alpha+1}} \quad (x > \lambda)$$

Excess of loss reinsurance is arranged with retention M ($M > \lambda$).

(i) (a) Show that $P(X > x) = \left(\frac{\lambda}{x}\right)^{\alpha} \quad (x > \lambda)$.

(b) Derive an expression for the expected amount paid by the reinsurer, on a claim which involves the reinsurer.

[7]

(ii) Last year 10 claims were received, of which 4 involved the reinsurer. The claims which involved the reinsurer are denoted by $\{x_i: i = 1, \ldots, 4\ (x_i > M)\}$.

Write down the likelihood for these data. [4]

[Total 11]

8 The number of claims on a particular risk in a fixed time period has a Poisson distribution with mean λ. There were x_1 and x_2 claims during the first two time periods.

(i) Suppose that λ has prior density $f(\lambda) = 2e^{-2\lambda}$ ($\lambda > 0$).

(a) Determine the Bayesian estimate under quadratic loss for the expected number of claims during the third time period, and show that it is of the form of a credibility estimate.

(b) Show that the corresponding credibility estimate under Model 1 of Empirical Bayes Credibility Theory (EBCT Model 1) is the same as your answer in (i)(a).

[8]

(ii) Now suppose that λ has prior density $f(\lambda) = 1$, $0 < \lambda < 1$.

Derive an expression in the form of the ratio of two integrals for the Bayesian estimate under quadratic loss for the expected number of claims during the third time period (do not evaluate the integrals). [2]

(iii) Assume there were no claims at all during the first two time periods.

(a) Evaluate the expressions in (ii) and in (i)(a).

(b) Derive the corresponding EBCT Model 1 estimate with the prior as in (ii).

(c) Compare it with the Bayesian estimate in (ii), and with the estimate in (i)(a).

[7]
[Total 17]

9 (i) Let N be the number of claims on a risk in one year. Suppose claims $[X_1, X_2, \ldots]$ are independent, identically distributed random variables, independent of N. Let S be the total amount claimed in one year.

(a) Derive $E(S)$ and $V(S)$ in terms of the mean and variance of N and X_1.

(b) Derive an expression for the moment generating function $M_S(t)$ of S in terms of the moment generating functions $M_X(t)$ and $M_N(t)$ of X_1 and N respectively.

(c) If N has a Poisson distribution with mean λ, show that:

$$M_S(t) = \exp(\lambda(M_X(t) - 1)).$$

(d) If N has a binomial distribution with parameters m and q, determine the moment generating function of S in terms of m, q and $M_X(t)$.

[5]

(ii) A portfolio consists of 500 independent risks. For the ith risk, with probability $1 - q_i$ there are no claims in one year, and with probability q_i there is exactly one claim ($0 < q_i < 1$). For all risks, if there is a claim, it has mean μ, variance σ^2 and moment generating function $M(t)$. Let T be the total amount claimed on the whole portfolio in one year.

(a) Determine the mean and variance of T. [4]

The amount claimed in one year on risk i is approximated by a compound Poisson random variable with Poisson parameter q_i and claims with the same mean μ, the same variance σ^2, and the same moment generating function $M(t)$ as above. Let $\tilde{T}$ denote the total amount claimed on the whole portfolio in one year in this approximate model.

(b) Determine the mean and variance of $\tilde{T}$, and compare your answers to those in (ii)(a). [4]

Assume that $q_i = 0.02$ for all i, and if a claim occurs, it is of size μ with probability one.

(c) Derive the moment generating function of T, and show that T has a compound binomial distribution. [2]

(d) Determine the moment generating function of the approximating $\tilde{T}$, and show that $\tilde{T}$ has a compound Poisson distribution. [2]

[Total 17]

10 An insurance company charges an annual premium of £300 and operates a No Claims Discount system as follows:

Level 1	0% discount
Level 2	30% discount
Level 3	60% discount

The rules for moving between levels are as follows:

If the policyholder does not make a claim during the year, they move up one level or are eligible to stay at level 3.

If the policyholder makes 1 claim during the year, they move down one level or stay at level 1.

If the policyholder makes 2 or more claims during the year, they move straight down to, or remain at, level 1.

The insurance company has recently introduced a "protection" system where on reaching, or remaining eligible to remain at, level 3, policyholders are immediately offered the opportunity to "protect" their discount for an additional annual premium of £50. If they make no claims or 1 claim during the year they can remain at level 3. However, if they make 2 or more claims during the year, they move straight down to level 1.

Out of the policyholders who had "protected" their discount level at the beginning of the year and are still eligible to stay at level 3 at the end of the year, 25% choose to "protect" their discount again the following year. From all the other policyholders eligible for level 3 at the end of the year, 10% choose to take the "protection" option.

Policyholders at different levels are found to experience different rates of claiming. The number of claims made per year follows a Poisson distribution with parameter λ as follows:

Level:	1 and 2	3 and "protected"
λ:	0.40	0.25

(i) Derive the transition matrix. [7]

(ii) Calculate the proportions at each of the levels 1, 2, 3 and the "protected" level, when the system reaches a steady state. [10]

(iii) Determine the average premium per policy. [2]

[Total 19]