Faculty of Actuaries | Institute of Actuaries

EXAMINATIONS

April 2000

Subject 106 — Actuarial Mathematics 2

EXAMINERS' REPORT

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(i)

	Annual profits θ_1	θ_2	θ_3	*Expected profit*
d_1	1360	1520	1760	1576
d_2	1407	1541	1742	1587.9 ←
d_3	1250	1350	1500	1385
$p(\theta)$	0.1	0.6	0.3	

decision: choose d_2

(ii) **minimax**

	max
d_1	1760 ←
d_2	1742
d_3	1500

decision: choose d_1

maximin

	min
d_1	1360
d_2	1407←
d_3	1250

decision: choose d_2

2 (i) Premiums in next three years:

	Claim	*No claim*	*Smallest loss for which will claim*
0%	500, 425, 350	425, 350, 250	250
15%	500, 425, 350	350, 250, 250	425
30%	425, 350, 250	250, 250, 250	275
50%	425, 350, 250	250, 250, 250	275

(ii) 0% level: $P(\text{cost} > 250) = e^{-250/1000} = 0.779$

15% level: $P(\text{cost} > 425) = e^{-425/1000} = 0.654$

30% and 50% levels: $P(\text{cost} > 275) = e^{-275/1000} = 0.760$

3 (i) Insurer pays $X - E$ if $X > E$

0 if $X \le E$

Define $Y = X - E \mid X > E$.

$$f_Y(y) = \frac{f_x(y+E)}{P(X>E)} = \frac{\alpha\lambda^\alpha(\lambda+E+y)^{-\alpha-1}}{\int_E^\infty \alpha\lambda^\alpha(\lambda+x)^{-\alpha-1}\,dx}$$

$$= \frac{\alpha\lambda^\alpha(\lambda+E+y)^{-\alpha-1}}{\lambda^\alpha(\lambda+E)^{-\alpha}}$$

$$= \alpha(\lambda+E)^\alpha(\lambda+E+y)^{-\alpha-1}$$

$\rightarrow Y$ is Pareto, parameters α and $\lambda + E$

(ii) $E(Y) = \dfrac{\lambda+E}{\alpha-1}$

(a) $E(Y) = \dfrac{15}{4-1} = 5$

(b) $E(Y) = \dfrac{15+10}{3} = 8\tfrac{1}{3}$

(iii) Small claims are excluded, so the mean claim amount increases.

A misprint occurred in part (ii) of this question. Many candidates spotted this and highlighted it in their answer. However, the misprint was taken into account in marking scripts to ensure that nobody was disadvantaged by it.

4 (i) Let X = No. of people with access to the internet at home in a sample of 50.

$X \sim B(50, \theta)$

$$f(x \mid \theta) = \binom{50}{\theta}\theta^x(1-\theta)^{50-x}$$

$f(\theta) = 1$

$f(\theta \mid x) \propto \theta^x(1-\theta)^{50-x}$

$\theta \mid x \sim \text{beta}(x+1, 50-x+1)$

$x = 29 \Rightarrow \theta \mid x \sim \text{beta}\,(30, 22)$

Quadratic loss function $\rightarrow$ estimate is posterior mean = $\dfrac{30}{52}$.

(ii) $f(x \mid \theta) = \dbinom{50}{\theta} \theta^x (1-\theta)^{50-x}$

$f(\theta) = \theta^3 (1-\theta)^3$

$$f(\theta \mid x) \propto \theta^{x+3} (1-\theta)^{53-x}$$

$x = 29 \Rightarrow f(\theta \mid x) \propto \theta^{32} (1-\theta)^{24}$

"All-or-nothing" loss function $\rightarrow$ estimate is posterior mode.

$$\log f(\theta \mid x) = 32 \log \theta + 24 \log (1-\theta)$$

$$\frac{d}{d\theta} \log f(\theta \mid x) = \frac{32}{\theta} - \frac{24}{(1-\theta)}$$

$$= \frac{32(1-\theta) - 24\theta}{\theta(1-\theta)}$$

Equate to 0 for maximum (mode)

$$32(1-\theta) = 24\theta$$

$$\therefore \theta = \frac{32}{56}$$

5 Cumulative claims:

2587	3678	3929
2053	3351	
3190		

$\lambda_2 = \dfrac{7029}{4640} = 1.5149$, $\lambda_3 = \dfrac{3929}{3678} = 1.0682$

Forecast cumulative claims:

		3579.7
	4832.4	5162.2

Total outstanding claims = 228.7 + 1972.2 = 2200.9

6 (i) (a) $V(m) = m$

(b) $d = 2\int_{\hat{m}}^{y} \frac{y-t}{t}\,dt$

$= 2\left[y \log t - t\right]_{\hat{m}}^{y}$

$= 2\left(y \log\left(\frac{y}{\hat{m}}\right) - \left(y - \hat{m}\right)\right)$

(c) deviance residual sign $(y - \hat{m}) . \sqrt{d}$

Pearson residual $\dfrac{y - \hat{m}}{\sqrt{\hat{m}}}$

(ii) (a) **Formulae:**

$$\log \mu_{ix} = \begin{cases} \alpha_0 \\ \alpha_0 + \alpha_1 x \\ \alpha_0 + \alpha_1 x + \alpha_2 x^2 \\ \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 \\ \alpha_0 + \beta_i + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 \quad (\beta_1 = 0) \end{cases}$$

(b) The term in x^3 is not supported by the change in the deviance. It also renders the parameter estimates $\hat{\alpha}_3$ (and $\hat{\alpha}_2$) <u>non</u>-significant as measured by their respective t–statistics $\left(\frac{\text{estimate}}{\text{s.e.}}\right)$.

Conclusion: Select the formula $\log \mu_{ix} = \alpha_0 + \beta_j + \alpha_1 x + \alpha_2 x^2$

(c) **Additional output:**

- parameter estimates and standard errors for the selected model
- residual plots of various types (e.g. vs fitted values; vs age)

A significant number of candidates did not attempt this question although those that did generally scored fairly well, particularly on part (ii).

7 (i)

$$M_S(t) = E[e^{tS}]$$

$$= E\left[e^{tS_1+tS_2}\right]$$

$$= E\left[e^{tS_1}\right].E\left[e^{tS_2}\right]$$

$$= e^{\lambda_1(M_{X_1}(t)-1)}e^{\lambda_2(M_{X_2}(t)-1)}$$

$$= e^{[\lambda_1 M_{X_1}(t)+\lambda_2 M_{X_2}(t)]-(\lambda_1+\lambda_2)}$$

$$= e^{\lambda[M_Y(t)-1]}$$

where $\lambda = \lambda_1 + \lambda_2$, and $M_Y(t) = \dfrac{\lambda_1 M_{X_1}(t) + \lambda_2 M_{X_2}(t)}{\lambda}$.

Hence S has a compound Poisson distribution, with Poisson parameter $\lambda = \lambda_1 + \lambda_2$.

(ii)

(a)

$$E[S] = 400 \times 0.05 \times 3{,}000 + 150 \times 0.1 \times 3{,}000 + 100 \times 0.1 \times 2000$$

$$= 125{,}000$$

$$V[S] = 400 \times 0.05 \times 3{,}000^2 + 150 \times 0.1 \times 3{,}000^2 + 100 \times 0.1 \times 2000^2$$

$$= 355{,}000{,}000$$

$$E[(S - E(S))^3] = 20 \times 3000^3 + 15 \times 3000^3 + 10 \times 2000^3$$

$$= 1.025 \times 10^{12}$$

$$\rightarrow \text{Coefficient of skewness} = \frac{1.025 \times 10^{12}}{355{,}000{,}000^{3/2}} = 0.153$$

(b)

$$S \sim N(125{,}000,\ 355{,}000{,}000)$$

$$P(S > Y) = P\left[\frac{Y - 125{,}000}{18{,}841} < \frac{S - 125{,}000}{18{,}841}\right] = 0.1$$

$$\Rightarrow Y = 12{,}500 + 18{,}841 \times 1.2816 = 149{,}000$$

(c) Now, $E(S) = 60{,}000$

$V(S) = 180{,}000{,}000$

$S \sim N(60{,}000;\ 180{,}000{,}000)$

$P(S > 100{,}000) = P[N(0,\ 1) > 2.98]$

$= 0.0014$

8 (i) The adjustment coefficient R is the solution of

$$\lambda M_X(r) - \lambda - cr = 0$$

in the positive region for which $M_X(r) = E(e^{rX})$ exists.

Note also, here $c = (1 + \theta)\, E(X)\, \lambda$ so look at

$$M_X(r) - 1 - (1 + \theta)\, E(X)\, r = 0$$

Get

$$M_X(t) = \int_0^\infty \frac{1 + 2x}{3}\ e^{-(1-t)x}\, dx \qquad \text{so require } t < 1$$

$$= \tfrac{1}{3}\int_0^\infty e^{-(1-t)x}\, dx + \tfrac{2}{3}\int_0^\infty xe^{-(1-t)x}\, dx$$

But $y = (1 - t)\, x, \qquad dy = (1 - t)\, dx$

$$M_X(t) = \tfrac{1}{3}\frac{1}{1-t}\,\Gamma(1) + \tfrac{2}{3}\frac{1}{(1-t)^2}\,\Gamma(2)$$

$$\therefore \quad M_X(t) = \tfrac{1}{3}(1-t)^{-1} + \tfrac{2}{3}(1-t)^{-2}$$

$$= \frac{3-t}{3(1-t)^2}\ .$$

Also $M'_X(t) = \tfrac{1}{3}(-1)(-1)(1-t)^{-2} + \tfrac{2}{3}(-2)(-1)(1-t)^{-3}$

$\Rightarrow M'_X(0) = E(X) = \tfrac{5}{3}$

We require the root of

$$\frac{3-r}{3(1-r)^2} - 1 - \frac{11}{8} \cdot \frac{5}{3} r = 0$$

s.t. $0 < r < 1$.

Look at

$$\frac{8(3-r)-(24+55r)(1-r)^2}{24(1-r)^2} = \frac{-r(55r^2-86r+15)}{24(1-r)^2}$$

$$= \frac{-r(11r-15)(5r-1)}{24(1-r)^2}$$

so $\underline{R = 0.2}$ (other roots are 0, $\frac{15}{11} > 1$)

(ii) $U_t = u + ct - S_t$

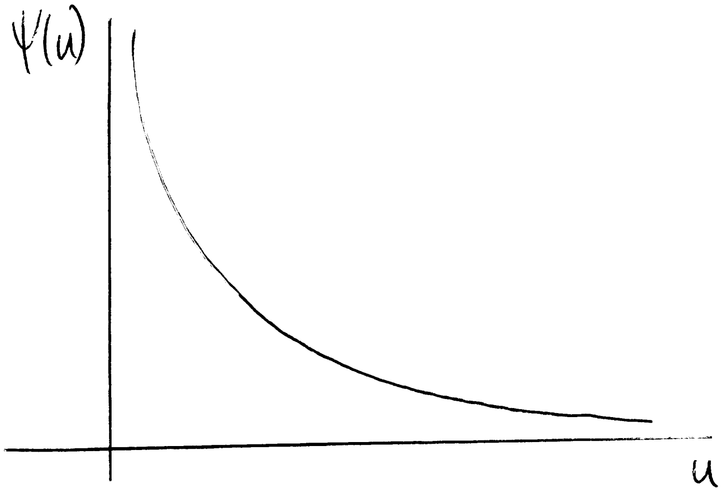
S_t = loss (compound Poisson process)

u = initial capital

c = constant rate

Then $\psi(u) = P(\exists\ t > 0$ s.t. $U_t < 0 \mid U_0 = u)$.

(iii) (a)



(b) Lundberg's upper bound is $e^{-Ru} = e^{-u/5}$.

This is clearly greater than $\psi(u)$ $\forall u$ given the specific nature of $\psi(u)$.

(iv) Look at $X_I = \alpha X$ with $\alpha = 0.84$

$$E(X_I) = \alpha E(X) = 0.84 \times \frac{5}{3} = 1.4 \quad \Rightarrow E(X_R) = E(X) - E(X_I)$$

$$= \frac{5}{3} - 1.4 = 0.2667$$

$\therefore$ the reinsurance premium is $1.5 \times 0.2667 \times \lambda = 0.4\lambda$.

Recall that the insurer's premium before reinsurance was

$$(1 + \theta)\, E(X)\, \lambda = \frac{11}{8} \times \frac{5}{3}\lambda = \frac{55}{24}\lambda$$

We also require

$$M_{X_I}(t) = E(e^{tX_I}) = E(e^{t\alpha X}) = M_X(\alpha t) = \frac{3 - \alpha t}{3(1 - \alpha t)^2} \ .$$

Hence the required equation is

$$\lambda M_{X_I}(r) - \lambda - \left(\frac{55}{24} - 0.4\right)\lambda r = 0$$

or

$$\frac{3 - 0.84r}{3(1 - 0.84r)^2} - 1 - 1.8917r = 0$$

9 (i)

(a) Given

$$f(\theta) \quad \propto \exp -\frac{(\theta - \mu)^2}{2\sigma^2} \propto \exp -\frac{1}{2\sigma^2}\,(\theta^2 - 2\mu\theta)$$

and

$$p(\underline{x} \mid \theta) \propto \prod_{i=1}^{n} \exp -\frac{(x_i - \theta)^2}{2\tau^2} \propto \exp -\frac{1}{2\tau^2} \sum_{i=1}^{n} (\theta^2 - 2x_i\theta)$$

$$\propto \exp -\frac{1}{2\tau^2}(n\theta^2 - 2n\bar{x}\theta) \qquad \left(\sum_{i=1}^{n} x_i = n\bar{x}\right),$$

we want

$$p(\theta \mid \underline{x}) \propto p(\underline{x} \mid \theta)\, p(\theta)$$

$$\propto \exp\left[-\left\{\left(\frac{1}{2\sigma^2}+\frac{n}{2\tau^2}\right)\theta^2-\left(\frac{\mu}{\sigma^2}+\frac{n\bar{x}}{\tau^2}\right)\theta\right\}\right]$$

$$\propto \exp\left[-\frac{\tau^2+n\sigma^2}{2\sigma^2\tau^2}\left\{\theta^2-2\left(\frac{\mu\tau^2+n\bar{x}\sigma^2}{\tau^2+n\sigma^2}\right)\sigma\right\}\right]$$

$$\propto \exp\left[-\frac{\tau^2+n\sigma^2}{2\sigma^2\tau^2}\left(\theta-\frac{\mu\tau^2+n\bar{x}\sigma^2}{\tau^2+n\sigma^2}\right)^2\right]$$

$$\Rightarrow \theta \mid \underline{x} \sim N\left(\frac{\tau^2}{\tau^2+n\sigma^2}\mu+\frac{n\sigma^2}{\tau^2+n\sigma^2}\bar{x},\ \frac{\sigma^2\tau^2}{\tau^2+n\sigma^2}\right)$$

(b) The posterior mean (the point estimator under quadratic loss) is

$$E(\theta \mid \underline{x}) = \frac{\tau^2}{\tau^2+n\sigma^2}\mu+\frac{n\sigma^2}{\tau^2+n\sigma^2}\bar{x}\ .$$

(c) $E(\theta \mid \underline{x}) = (1-Z)\,\mu + Z\bar{x}$

where

$$Z = \frac{n\sigma^2}{\tau^2+n\sigma^2} = \frac{n}{n+\dfrac{\tau^2}{\sigma^2}}$$

is the credibility factor. Hence $E(\theta \mid \underline{x})$ can be expressed in the form of a credibility estimate.

(d) As $n \to \infty$, $\quad Z = \lim_{n\to\infty}\dfrac{\sigma^2}{\sigma^2+\dfrac{\tau^2}{n}} \to 1$ and $E(\theta \mid \underline{x}) \to \bar{x}$

(ii) (a)

		A	B	
Want	$\bar{x}_i$	244	226.8	with $n = 5$.

(1)	A	B	(2)	A	B
τ^2	400	400		400	400
μ	270	260		270	260
σ^2	2500	2500		225	225

For both companies	$Z = \dfrac{\sigma^2}{\sigma^2 + \dfrac{\tau^2}{n}} = 0.969$	$Z = 0.7377$ (0.738)
Credibility premiums	$0.969\bar{x}_i + 0.031\mu$	$0.738\bar{x}_i + 0.262\mu$
A	244.8	250.8
B	227.8	235.5

(b) With larger σ in case (1), prior knowledge is more vague. This is reflected in the lower weighting given to μ relative to $\bar{x}_i$.

(c)

		A	**B**	
Want	$\bar{x}_i$	244	226.8	$\Rightarrow \bar{x} = 235.4$ with $N = 2$, $n = 5$
	s_i^2	314	363.7	

$$\frac{1}{N-1}\sum_i (\bar{x}_i - \bar{x})^2 = 147.92$$

Then $\hat{E}(S^2(\theta)) = \dfrac{1}{N}\sum_i S_i^2 = 338.85$

$$\hat{\mathrm{Var}}(m(\theta)) = \frac{1}{N-1}\sum_i (\bar{x}_i - x)^2 - \frac{1}{n}\frac{1}{N}\sum_i S_i^2 = 147.92 - \frac{338.85}{5} = 80.15$$

and $Z = \dfrac{80.15}{80.15 + \dfrac{338.85}{5}} = \underline{0.5418}$

Credibility premiums are $0.5418\bar{x}_i + 0.4582\bar{x} = 0.5418\bar{x}_i + 107.86$

A	240.1
B	230.7