# EXAMINATIONS

30 September 2004 (am)

## Subject 106 — Actuarial Mathematics 2

*Time allowed: Three hours*

***INSTRUCTIONS TO THE CANDIDATE***

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*

2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*

3. *Mark allocations are shown in brackets.*

4. *Attempt all 10 questions, beginning your answer to each question on a separate sheet.*

***Graph paper is not required for this paper.***

*AT THE END OF THE EXAMINATION*

*Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.*

*In addition to this paper you should have available Actuarial Tables and your own electronic calculator.*

**1** The preparation times for coffee in a high-street coffee shop have density

$$f(y) = \frac{4}{\mu^2} y e^{-2y/\mu}.$$

(i) Show that this can be written in exponential family form, and determine the natural parameter. [2]

(ii) Interpret the two models

$$\text{Model I:} \quad \frac{1}{\mu} = \alpha_i, \quad i = 1, 2, 3$$

$$\text{Model II:} \quad \frac{1}{\mu} = \begin{cases} \alpha & i = 1 \\ \alpha + \beta & i = 2, 3 \end{cases}$$

where $i = 1, 2, 3$ correspond to filter coffee, cappuccino and espresso respectively. [2]

[Total 4]

**2** The number of claims arising from a hurricane in a particular region has a Poisson distribution with mean $\lambda$. The claim severity distribution has mean 0.5 and variance 1.

(i) Determine the mean and variance of the total amount of claims arising from a hurricane. [2]

(ii) The number of hurricanes in this region in one year has a Poisson distribution with mean $\mu$. Determine the mean and variance of the total amount claimed from all the hurricanes in this region in one year. [3]

[Total 5]

**3** A consumer has to decide whether to take out a travel insurance policy which covers him for all trips over the next year, or to purchase a separate policy each time he makes a trip. He is unsure how many trips he will make.

An annual policy would cost £95.

A separate policy for each trip costs £30.

He estimates that the probability distribution of the number of trips he will take over the next year, $X$, is as follows:

| *Number of trips* $x$ | $P(X = x)$ |
|---|---|
| 1 | 0.1 |
| 2 | 0.2 |
| 3 | 0.3 |
| 4 | 0.3 |
| 5 | 0.1 |

Determine the minimax and Bayes decisions. [5]

**4** The number of claims from one group of drivers in a year has a Poisson distribution with mean $\lambda$, and the number of claims from a second group of drivers has a Poisson distribution with mean $2\lambda$. In one year, there are $n_1$ claims from group 1 and $n_2$ claims from group 2.

(i) Derive the maximum likelihood estimator, $\hat{\lambda}$, of $\lambda$. [3]

(ii) Suppose that past experience shows that $\lambda$ has an exponential distribution with mean $1/v$.

(a) Derive the posterior distribution of $\lambda$.

(b) Show that the Bayesian estimate of $\lambda$ under quadratic loss may be written in the form of a credibility estimate combining the prior mean of $\lambda$ with the maximum likelihood estimate $\hat{\lambda}$ in (i). State the credibility factor.

[4]

[Total 7]

**5** A no claims discount system has two levels of discount, such that the premiums paid at each level are:

| *Discount Category* | *Premium (£'s)* |
|---|---|
| 0 | 1,000 |
| 1 | 1,000$d$ |

where $0 \le d \le 1$.

If a policyholder does not claim, they are in discount category 1 next year. Otherwise, they are in discount category 0.

The probability that a policyholder does not claim is $1 - q$ ($0 \le q \le 1$).

(i) Write down the transition matrix and derive the steady state distribution, in terms of $q$. [4]

(ii) Calculate the value of $d$ which should be used, so that policyholders for whom $q = 0.2$ pay an average premium (in the steady state distribution) which is 50% greater than that paid by policyholders for whom $q = 0.1$. [3]

(iii) Comment on the effectiveness of the no claims discount system, using your answer to (ii). [2]

[Total 9]

**6** An insurance company has to estimate the risk premium for the coming year for a certain risk.

(i) Describe how the credibility approach to calculating the risk premium differs from a conventional approach. [2]

(ii) State the advantages and disadvantages of using Bayesian estimation and empirical Bayes credibility theory estimation. [3]

(iii) State the differences between the assumptions in empirical Bayes credibility theory Model 1 and Model 2, and state why Model 2 is more likely to be useful in practice. [4]

[Total 9]

**7** The tables below show the cumulative incurred claims data and number of reported claims, by accident year and development year (in £'000s).

**Cumulative incurred claims**

| *Accident Year* | *Development Year* | | | | |
|---|---|---|---|---|---|
| | *0* | *1* | *2* | *3* | *Ult* |
| 2000 | 3,417 | 4,291 | 4,581 | 4,714 | 4,900 |
| 2001 | 4,814 | 6,888 | 7,007 | | |
| 2002 | 5,844 | 8,000 | | | |
| 2003 | 6,654 | | | | |

**Cumulative number of reported claims**

| *Accident Year* | *Development Year* | | | | |
|---|---|---|---|---|---|
| | *0* | *1* | *2* | *3* | *Ult* |
| 2000 | 414 | 460 | 482 | 488 | 500 |
| 2001 | 453 | 506 | 526 | | |
| 2002 | 496 | 558 | | | |
| 2003 | 540 | | | | |

(i) Estimate the ultimate number of claims, for each accident year, using the chain-ladder technique. [4]

(ii) Estimate the ultimate average incurred cost per claim, for each accident year, using the grossing-up method. [4]

(iii) Using the results from (i) and (ii), calculate the total reserve required, assuming that claims paid to date are £19,212,000. [3]

[Total 11]



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**8** Claims arrive in a Poisson process rate $\lambda$, and the claim severity distribution has mean $\mu$ and moment generating function $M(t)$. The premium income per unit time is $c$ where $c > \lambda\mu$.

(i) (a) Write down an equation satisfied by the adjustment coefficient.

(b) Derive the adjustment coefficient in terms of $\lambda$, $\mu$ and $c$, when the claims are exponentially distributed with mean $\mu$.

(c) Calculate the adjustment coefficient $R_{\text{exp}}$ when $\mu = 100$ and the premium loading factor is 25%.

(d) State Lundberg's inequality for the probability of ruin with initial capital $u$.

(e) Determine and comment on the effect on $R_{\text{exp}}$ if the mean claim size is increased but the premium loading factor remains the same.

(f) Determine and comment on the effect on $R_{\text{exp}}$ if instead the premium loading factor is increased but the mean claim size stays the same.

[7]

(ii) Now suppose that the premium loading factor is 25%.

(a) Determine to one significant figure the adjustment coefficient $R^*$ if each claim is exactly £100.

(b) Compare $R^*$ with $R_{\text{exp}}$ and comment.

[3]

(iii) For the exponential case in (i)(c) with $\mu = 100$, proportional reinsurance is arranged with retention $\alpha$. The insurer's premium income per unit time before paying the reinsurance premium is as in (i)(c) with premium loading factor 25%, and the reinsurer uses a premium loading factor of 30%.

(a) Determine a condition on $\alpha$ that ensures that the insurer's premium income per unit time, net of reinsurance, exceeds the insurer's expected aggregate claims per unit time, net of reinsurance.

(b) Determine the insurer's new adjustment coefficient, taking reinsurance into account, when $\alpha = 0.8$. Compare this value with $R_{\text{exp}}$.

[5]

[Total 15]

**9** A general insurance company has a portfolio of fire insurance policies, which offer cover for just one fire each year.

Within the portfolio, there are three types of buildings for which the average cost of a claim and probability of a claim are given in the table below.

| *Type of building* | *Number of Risks Covered* | *Average Cost of a Claim (£'000s)* | *Probability of a Claim* |
|---|---|---|---|
| Small | 147 | 12.4 | 0.031 |
| Medium | 218 | 27.8 | 0.028 |
| Large | 21 | 130.3 | 0.017 |

It is assumed that the cost of a claim has an exponential distribution, and that all the buildings in the portfolio represent independent risks for this insurance cover.

(i) Show that the mean and standard deviation of annual aggregate claims from this portfolio of insurance policies are £272,715 and £150,671, respectively, and calculate the coefficient of skewness. [8]

(ii) Using a normal distribution to approximate the distribution of annual aggregate claims, calculate the premium loading factor necessary such that the probability that annual aggregate claims exceed premium income is 0.05. [3]

(iii) Market conditions dictate that the insurer can only charge a premium which includes a loading of 25%. Calculate the amount of capital that the insurer must allocate to this line of business in order to ensure that the probability that annual aggregate claims exceed premium income and capital is 0.05 (again using a normal approximation). [2]

(iv) Comment on the assumption of independence and the use of a normal approximation, in relation to your answers to (ii) and (iii). [4]

[Total 17]

**10** (i) The random variable $X$ has a Pareto distribution with parameters $\alpha$ and $\lambda$. Show that, for $L$, $d > 0$:

$$\int_d^{L+d} xf(x)dx = \frac{\lambda^\alpha}{\alpha-1}\left[\frac{\alpha d+\lambda}{(\lambda+d)^\alpha} - \frac{\alpha(L+d)+\lambda}{(\lambda+L+d)^\alpha}\right]$$ [6]

(ii) Claims on a certain class of insurance policy have a Pareto distribution with mean £3,000 and standard deviation £6,000. The insurance company arranges a layer of excess-of-loss reinsurance with a retention level of £8,000. The maximum amount the reinsurer will pay on any individual claim is £6,000.

(a) Calculate the mean claim amount paid by the reinsurer on claims which involve the reinsurer.

(b) Next year the claim amounts on these policies are expected to increase by 10% but the reinsurance treaty will remain unchanged. Calculate the mean claim amount to be paid next year by the reinsurer on claims which involve the reinsurer. [12]

[Total 18]

## END OF PAPER