Faculty of Actuaries **Institute of Actuaries**

REPORT OF THE BOARD OF EXAMINERS

April 2003

Subject 106 — Actuarial Mathematics 2

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

J Curtis
Chairman of the Board of Examiners

3 June 2003

General comments

Most candidates attempted all of the questions although some were clearly short on time towards the end of the paper.

There was a tendency by some candidates to write out definitions from the core reading rather than applying the specifics given in the questions.

1 Let $x_1, \ldots, x_n$ be the observed times taken. Then

$$f(x_1, \ldots, x_n \mid \lambda) = \lambda^n \exp\left(-\lambda \sum_{i=1}^{n} x_i\right).$$

Suppose the prior is a gamma distribution with density

$$f(\lambda) = v^{\alpha}\lambda^{\alpha-1}e^{-v\lambda}, \quad \lambda > 0.$$

Then the posterior satisfies

$$\begin{aligned} f(\lambda \mid x_1, \ldots, x_n) &\propto \lambda^n \exp\left(-\lambda \sum_{i=1}^{n} x_i\right)\lambda^{\alpha-1}e^{-v\lambda} \\ &= \lambda^{n+\alpha-1} e^{-\lambda\left(\sum_{i=1}^{n} x_i + v\right)}, \end{aligned}$$

so the posterior distribution is also a gamma. Thus the gamma is the conjugate prior.

This question was generally well answered. A number of candidates lost marks for using sloppy notation.

2 (i) Liability cover provides indemnity where the insured, owing to some form of negligence, is legally liable to pay compensation to a third party.

Property cover indemnifies the policyholder against loss of or damage to his/her own material property.

(ii) Liability cover; any two of the following

employers' liability
motor third party liability
public liability
product liability

professional indemnity

Property cover; any two of the following

residential building
moveable property
commercial building
land vehicles
marine craft
aircraft

Both parts of this question were poorly answered. In (i) very few mentioned negligence and compensation to a third party. In (ii) many used examples such as 'house insurance' or 'car insurance'.

3 (i) $E(Y) = b'(\theta)$ and $V(Y) = a(\phi)b''(\theta)$.

(ii) (a)
$$f(y) = \frac{1}{\mu}\exp\left(-\frac{y}{\mu}\right)$$
$$= \exp\left(\frac{-(y/\mu) - \log\mu}{1} + 0\right),$$

which is of the exponential family form.

(b) The natural parameter is $\theta = -\mu^{-1}$.

$$b(\theta) = -\log(-\theta) \text{ so } b'(\theta) = -\theta^{-1} \text{ and } b''(\theta) = \theta^{-2} = \mu^2.$$

so the variance function is $V(\mu) = \mu^2$.

Nearly all candidates correctly answered (i). Part (ii) was poorly answered by many candidates with sloppy algebra and answers being given in terms of theta rather than mu.

4 (i) Decision D_4 can be discounted immediately.

D_4 is dominated by D_2 because the loss in each scenario for D_2 is either the same as or less than that for D_4.

(ii) The maximum loss under each of the remaining decisions is:

D_1 34
D_2 30
D_3 **27** ←

Choose D_3 which minimises the maximum loss.

(iii) The expected loss under each decision is:

D_1 $23 \times (0.25) + 34 \times (0.15) + 16 \times (0.60) = \mathbf{20.45} \leftarrow$
D_2 $30 \times (0.25) + 19 \times (0.15) + 18 \times (0.60) = 21.15$
D_3 $23 \times (0.25) + 27 \times (0.15) + 20 \times (0.60) = 21.80$

Therefore choose D_1 which has the smallest expected loss.

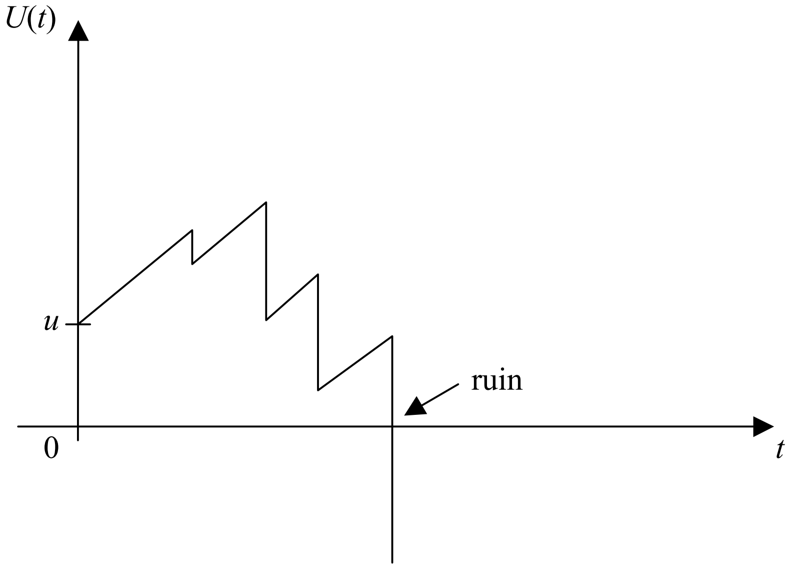
Very well answered by nearly all candidates. A few omitted to show any working to demonstrate that they had correctly determined the answers to (ii) and (iii).

5 (i) (a) The surplus at time t is

$$U(t) = u + (1+\theta)\,\lambda\mu t - \sum_{i=1}^{N(t)} X_i .$$

(b) The probability of ruin is

$$\psi(u) = P(U(t) < 0 \text{ for some } t,\ 0 < t < \infty).$$



(c) When $\theta = 0$, then $\psi(u) = 1$ for all $u > 0$.

(ii) Let $\tilde{U}(t)$ be the new surplus at t, then $\tilde{U}(t) = 2.5U(t)$ and the initial capital is $2.5u$ in the new units.

Then, writing $\tilde{\psi}(\cdot)$ for the new probability of ruin,

$$\begin{aligned}\tilde{\psi}(2.5u) &= P(\tilde{U}(t) < 0 \text{ for some } t,\ 0 < t < \infty)\\ &= P(2.5U(t) < 0 \text{ for some } t,\ 0 < t < \infty)\\ &= \psi(u),\end{aligned}$$

or an equivalent relationship.

(Could give a graphical explanation, or an explanation in words.)

(iii) (a) From properties of Poisson processes, $N(t)$ has a Poisson distribution with mean λt, and $N^*(t)$ has a Poisson distribution with mean $2\lambda t$. So $N^*(t)$ has the same distribution as $N(\alpha t)$ for $\alpha = 2$.

(b) For initial capital u, using ~ to denote "has the same distribution as,"

$$U^*(t) = u + (1+\theta)\, 2\lambda\mu t - \sum_{i=1}^{N^*(t)} X_i$$

$$\sim u + (1+\theta)\lambda\mu(2t) - \sum_{i=1}^{N(2t)} X_i,$$

so $U^*(t) \sim U(2t)$ (or a graphical explanation, or an explanation in words). So

$$\begin{aligned}\psi^*(u) &= P(U^*(t) < 0 \text{ for some } t,\ 0 < t < \infty) \\ &= P(U(2t) < 0 \text{ for some } t,\ 0 < t < \infty) \\ &= \psi(u).\end{aligned}$$

(i) Many candidates used the expression S(t) as the sum of claims without definition rather than sigma Xi. Graphs were often disappointing with many illustrating a recovery from the ruin event.

(ii) This was was well understood although a few candidates failed to explain adequately why there was no change. Either a graphical or written explanation was acceptable.

(iii) Relatively few candidates managed to derive the required expression in (b).

6 (i) Cumulative claims:

4652	7731	8662	8937
6067	10622	11805	
5822	10119		
7934			

Development factors:

R_1 = (10119 + 10622 + 7731) / (5822 + 6067 + 4652)
= 28472 / 16541
= 1.7213

R_2 = (11805 + 8662) / (10622 + 7731)
= 20467 / 18353
= 1.1152

R_3 = 8937 / 8662

= 1.0317

Forecast cumulative claims:

4652	7731	8662	8937
6067	10622	11805	**12179**
5822	10119	**11285**	**11642**
7934	**13657**	**15230**	**15713**

Total outstanding claims

= (12179 + 11642 + 15713) – (11805 + 10119 + 7934)
= 39534 – 29858
= **9676**

(ii) For each accident year the total claim amount in each development year is a constant proportion of the ultimate claim amount for that accident year.

Weighted average past inflation is appropriate estimate of future inflation.

Claims are fully run off by the end of development year 3.

This straightforward question was generally well answered. Many failed to obtain full marks for stating the basic chain ladder assumptions.

7 (i) (a) $P(X > x) = \int_x^\infty \frac{\alpha\lambda^\alpha}{y^{\alpha+1}} dy$

$$= \left[-\frac{\lambda^\alpha}{y^\alpha} \right]_x^\infty = \left(\frac{\lambda}{x} \right)^\alpha$$

(b) $E[X - M | X > M] = \dfrac{\int_M^\infty (x - M) \frac{\alpha\lambda^\alpha}{x^{\alpha+1}} dx}{\left(\frac{\lambda}{M} \right)^\alpha}$

$$\int_M^\infty (x - M) \frac{\alpha\lambda^\alpha}{x^{\alpha+1}} dx = \int_M^\infty \frac{\alpha\lambda^\alpha}{x^\alpha} dx - M \cdot \left(\frac{\lambda}{M} \right)^\alpha$$

$$= \left[-\frac{\alpha\lambda^\alpha}{(\alpha - 1)x^{\alpha-1}} \right]_M^\infty - M\left(\frac{\lambda}{M} \right)^\alpha$$

$$= \frac{\alpha}{\alpha-1}\frac{\lambda^\alpha}{M^{\alpha-1}} - M\left(\frac{\lambda}{M}\right)^\alpha$$

$$= \frac{\lambda^\alpha}{(\alpha-1)M^{\alpha-1}}(\alpha-(\alpha-1)) = \frac{\lambda^\alpha}{(\alpha-1)M^{\alpha-1}}$$

$$\therefore\ E[X-M \mid X>M] = \frac{\lambda^\alpha}{(\alpha-1)M^{\alpha-1}}\frac{M^\alpha}{\lambda^\alpha} = \frac{M}{\alpha-1}$$

(ii) The likelihood is

$$(P(X \le M))^6 \prod_{i=1}^{4} f(x_i)$$

$$= \left(1-\left(\frac{\lambda}{M}\right)^\alpha\right)^6 \prod_{i=1}^{4}\frac{\alpha\lambda^\alpha}{x_i^{\alpha+1}}$$

$$= \left(1-\left(\frac{\lambda}{M}\right)^\alpha\right)^6 \frac{\alpha^4\lambda^{4\alpha}}{\left(\prod_{i=1}^{4} x_i\right)^{\alpha+1}}$$

A variety of successful approaches were adopted by candidates for (i). Some used the Yellow Tables whilst others carried out calculations from first principles. Both approaches were acceptable. Many candidates failed to follow through the algebra in (ii).

Part (ii) was disappointing with many candidates misunderstanding the question. A common error was to express the likelihood as the probability of 4 claims exceeding M multiplied by the product of x1 to x6.

8 (i) (a) Let $k = x_1 + x_2$, so

$$f(x_1, x_2 \mid \lambda) \propto e^{-2\lambda}\lambda^k.$$

The posterior density of λ is

$$f(\lambda \mid x_1, x_2) \propto f(x_1, x_2 \mid \lambda) f(\lambda)$$

$$\propto \lambda^k e^{-4\lambda}, \quad \lambda > 0,$$

i.e. the posterior distribution is a gamma distribution with parameters $k + 1$ and 4.

The Bayesian estimate under quadratic loss is the posterior mean

$$\frac{k+1}{4} = \frac{1}{2}\overline{x} + \frac{1}{2}\frac{1}{2},$$

where $\overline{x} = k/2$ is the mean of x_1 and x_2.

The prior mean of λ is $m = ½$.
The Bayesian estimate is of the form

$$Z\overline{x} + (1 - Z)m,$$

which is of the form of a credibility estimate.

(b) $m(\lambda) = E(X \mid \lambda) = \lambda$, so $m(\lambda)$ has an exponential distribution with mean ½. Hence $V(m(\lambda)) = ¼$.

Further $s^2(\lambda) = V(X \mid \lambda) = \lambda$, so that $E(s^2(\lambda)) = E(\lambda) = ½$.

The EBCT Model 1 credibility factor is

$$Z = \frac{n}{n + \dfrac{E(s^2(\lambda))}{V(m(\lambda))}} = \frac{2}{2 + \dfrac{½}{¼}} = ½,$$

and the corresponding credibility estimate is

$$\frac{1}{2}\overline{x} + \frac{1}{2}m.$$

This is the same as in (i)(a).

(ii) With the alternative prior, the posterior is

$$f(\lambda \mid x_1, x_2) \propto e^{-2\lambda}\lambda^k,\ 0 < \lambda < 1.$$

The Bayesian estimate is the posterior mean

$$\frac{\int_0^1 \lambda^{k+1} e^{-2\lambda} d\lambda}{\int_0^1 \lambda^k e^{-2\lambda} d\lambda}$$

(iii) Now $k = 0$ (so $\overline{x} = 0$).

(a) The numerator in (ii) is then

$$\int_0^1 \lambda e^{-2\lambda} d\lambda = \left[-\frac{\lambda e^{-2\lambda}}{2}\right]_0^1 + \int_0^1 \frac{e^{-2\lambda}}{2} d\lambda$$

$$= -\frac{e^{-2}}{2} + \left[-\frac{e^{-2\lambda}}{4}\right]_0^1$$

$$= \frac{1-3e^{-2}}{4}.$$

The denominator is

$$\int_0^1 e^{-2\lambda} d\lambda = \frac{1-e^{-2}}{2}.$$

Then the expression in (ii) is evaluated as

$$\frac{1-3e^{-2}}{2(1-e^{-2})} = 0.3435.$$

The expression in (i)(a) is the same as that in (i)(b). When $k = 0$ they are

$$\frac{k+1}{4} = 0.25.$$

(b) The prior in (ii) is a uniform distribution on (0, 1), so we still have $E(s^2(\lambda)) = E(\lambda) = ½$. But now $V(m(\lambda)) = V(\lambda) = 1/12$.

Then the credibility factor for EBCT Model 1 is

$$Z = \frac{2}{2 + \frac{½}{1/12}} = 0.25,$$

and the EBCT Model 1 credbility estimate is

$$Z\overline{x} + (1 - Z)m = 0.75 \times 0.5 = 0.375.$$

(c) For the uniform prior in (ii), the EBCT Model 1 estimate is not the same as the Bayesian estimate.

Comparing the EBCT Model 1 credibility estimates for the two priors, they are not the same. Both priors have the same value for $E(s^2(\lambda))$, but the prior in (ii) has a smaller value for $V(m(\lambda))$. As $V(m(\lambda))$ decreases, the credibility factor decreases. In our case, the credibility estimate is $(1 - Z)m$, so this will increase as $V(m(\lambda))$ decreases, and this is what is observed.

(*Or other sensible comments.*)

This question was very poorly answered apart from (i)(a). Few candidates managed (ii) or (iii).

9 (i) (a) $S = X_1 + \ldots X_N$, and so

$$E(S) = E(E(S \mid N)) = E(NE(X_1)) = E(N)E(X_1),$$

and

$$\begin{aligned} V(S) &= E(V(S \mid N)) + V(E(S \mid N)) \\ &= E(NV(X_1)) + V(NE(X_1)) \\ &= E(N)V(X_1) + V(N)(E(X_1))^2. \end{aligned}$$

(b) The moment generating function of S is

$$M_S(t) = E(e^{St}) = E(E(e^{St} \mid N)) = E((M_X(t))^N) = M_N(\log(M_X(t))).$$

(c) Since $M_N(t) = \exp(\lambda(e^t - 1))$, the mgf of S is

$$M_S(t) = \exp(\lambda(M_X(t) - 1)).$$

(d) Here the mgf of N is $(1 - q + qe^t)^m$, so that the mgf of S is

$$M_S(t) = (1 - q + qM_X(t))^m.$$

(ii) (a) Let Y_i be the amount claimed on risk i, so that

$$Y_i = \begin{cases} 0 & \text{with probability } 1 - q_i \\ Z_i & \text{with probability } q_i, \end{cases}$$

where $E(Z_i) = \mu$, $V(Z_i) = \sigma^2$ and the mgf of Z_i is $M(t)$.

Moments of Y_i are

$$E(Y_i) = q_i\mu,$$

and

$$E(Y_i^2) = q_i E(Z_i^2) = q_i(\sigma^2 + \mu^2),$$

so that

$$V(Y_i) = q_i(\sigma^2 + \mu^2) - q_i^2\mu^2 = q_i\sigma^2 + \mu^2 q_i(1 - q_i).$$

The mean of T is

$$E(T) = \sum_{i=1}^{500} E(Y_i) = \mu\sum_{i=1}^{500} q_i.$$

The variance of T is

$$V(T) = \sum_{i=1}^{500}(q_i\sigma^2 + \mu^2 q_i(1-q_i)).$$

(b) Let $\tilde{Y}_i$ have a compound Poisson distribution with Poisson parameter q_i and claim sizes with mean μ, variance σ^2 and mgf $M(t)$. Then from (i)(a)

$$E(\tilde{Y}_i) = q_i\mu,$$

so

$$E(\tilde{T}) = \sum_{i=1}^{500} E(\tilde{Y}_i) = \mu\sum_{i=1}^{500} q_i.$$

Also from (i)(a), the variance of $\tilde{Y}_i$ is

$$V(\tilde{Y}_i) = q_i(\sigma^2 + \mu^2),$$

so

$$V(\tilde{T}) = (\sigma^2 + \mu^2)\sum_{i=1}^{500} q_i.$$

Comparing with the mean and variance of T,

$$E(\tilde{T}) = E(T),$$

and

$$V(\tilde{T}) = \sum_{i=1}^{500}(q_i\sigma^2 + q_i\mu^2)$$

$$= \sum_{i=1}^{500}(q_i\sigma^2 + q_i(1-q_i)\mu^2) + \mu^2\sum_{i=1}^{500} q_i^2,$$

so that $V(\tilde{T}) \geq V(T)$.

(c) Now $q_1 = \ldots = q_{500} = 0.02$ and if claims occur they are equal to μ, so all Y_i's have the same distribution. Write $M_Y(t)$ for the mgf of Y_i. Then

$$M_Y(t) = E(\exp(Y_1 t)) = 0.98 + 0.02e^{-\mu t},$$

so that

$$M_T(t) = (M_Y(t))^{500} = (0.98 + 0.02e^{\mu t})^{500}.$$

From (i)(c) this is the mgf of a compound binomial distribution, with binomial parameters 500 and 0.02 and with claims equal to μ with probability one.

(d) In the approximation for this case, the $\tilde{Y}_i$'s all have the same compound Poisson distribution with mgf

$$M_{\tilde{Y}}(t) = \exp(0.02(e^{\mu t} - 1)),$$

so that

$$M_{\tilde{T}}(t) = (M_{\tilde{Y}}(t))^{500} = \exp(10(e^{\mu t} - 1)).$$

From (i)(b) this is the mgf of a compound Poisson distribution, with Poisson parameter 10 and claims equal to μ with probability one.

Part (i) was well answered although a number of candidates failed to derive E(S) and V(S) in (a).

Part (ii) was not well answered. Only a few candidates distinguished the difference between a summation of qi and 500q. Some credit was given for this approach. A small number of candidates made meaningful comments on the comparison requested in (b). For (c) and (d) most candidates failed to apply the formulae from (i) correctly.

10 (i)

Level 1 & 2	P (0 claims)	$= e^{-0.40}$	$= 0.6703$
	P (> 0 claims)	$= 1 - 0.6703$	$= 0.3297$
Level 3 & P	P (0 claims)	$= e^{-0.25}$	$= 0.7788$
	P (1 claim)	$= 0.25e^{-0.25}$	$= 0.1947$
	P (> 1 claim)	$= 1 - 0.7788 - 0.1947$	$= 0.0265$

Moving to level P
From level 2 = $0.6703 \times 0.10 = 0.0670$
From level 3 = $0.7788 \times 0.10 = 0.0779$
From level P = $(1 - 0.0265) \times 0.25 = 0.2434$

Moving to level 3
From level 2 = $0.6703 \times 0.90 = 0.6033$
From level 3 = $0.7788 \times 0.90 = 0.7009$
From level P = $(1 - 0.0265) \times 0.75 = 0.7301$

Therefore the transition matrix is:

$$\begin{matrix} 0.3297 & 0.6703 & 0 & 0 \\ 0.3297 & 0 & 0.6033 & 0.0670 \\ 0.0265 & 0.1947 & 0.7009 & 0.0779 \\ 0.0265 & 0 & 0.7311 & 0.2434 \end{matrix}$$

(ii) In a steady state

$$\Pi \begin{matrix} 0.3297 & 0.6703 & 0 & 0 \\ 0.3297 & 0 & 0.6033 & 0.0670 \\ 0.0265 & 0.1947 & 0.7009 & 0.0779 \\ 0.0265 & 0 & 0.7311 & 0.2434 \end{matrix} = \Pi$$

$$\begin{aligned} 0.3297\pi_1 + 0.3297\pi_2 + 0.0265\pi_3 + 0.0265\pi_P &= \pi_1 \\ 0.6703\pi_1 + 0.1947\pi_3 &= \pi_2 \\ 0.6033\pi_2 + 0.7009\pi_3 + 0.7311\pi_P &= \pi_3 \\ 0.0670\pi_2 + 0.0779\pi_3 + 0.2434\pi_P &= \pi_P \end{aligned}$$

and $\pi_1 + \pi_2 + \pi_3 + \pi_P = 1$

Hence

$\pi_P = 0.0886\pi_2 + 0.1030\pi_3$
$\pi_3 = 2.0171\pi_2 + 2.4443\pi_P$

$\pi_P = 0.0886\pi_2 + 0.2078\pi_2 + 0.2518\pi_P$
$\pi_3 = 2.0171\pi_2 + 0.2166\pi_2 + 0.2518\pi_3$

$\pi_P = 0.3962\pi_2$
$\pi_2 = 0.3350\pi_3$
$\pi_3 = 4.7776\pi_1$

$\pi_1 + 4.7776\pi_1 + 1.6005\pi_1 + 0.6341\pi_1 = 1$

$\pi_1 = 0.1248$
$\pi_2 = 0.1998$
$\pi_3 = 0.5962$
$\pi_P = 0.0791$

(iii) The premium at each level is:

Level 1 = £300
Level 2 = £210
Level 3 = £120
Level *P* = £120 + £50 = £170

Therefore the average premium is:

0.1248 × £300 + 0.1998 × £210 + 0.5962 × £120 + 0.0791 × £170:

= £164.39

This was generally well done although a number of candidates presented a 3x3 matrix which ignored the Protected Level. Marks for the following part were reduced accordingly.

Some candidates struggled with the algebra in (ii), possibly as a result of time constraints. Credit was awarded even where the candidate had made errors in (i).

*Part (iii) was well answered although a few candidates did not identify the correct premium for the Protected level (£300 * .4 + £50).*