EXAMINATIONS

April 2004

Subject 106 — Actuarial Mathematics 2

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

J Curtis
Chairman of the Board of Examiners

22 June 2004

1 $$P(1.05X < 100) = P\left(X < \frac{100}{1.05}\right)$$

$$= 1 - \left(\frac{\lambda}{\lambda + \frac{100}{1.05}}\right)^{\alpha}$$

$$= 1 - \left(\frac{1000}{1000 + \frac{100}{1.05}}\right)^{3.5}$$

$$= 0.2727$$

This question was well answered.

2

	0.3 θ_1	0.3 θ_2	0.4 θ_3	*Max*	*Expected Loss*
d_1	120	97	131	131	117.5
d_2	132	74	89	132	97.4
d_3	117	141	37	141	92.2

(i) The minimax solution is d_1.

(ii) The Bayes criterion solution is d_3.

Again, this question was well answered. However, some candidates did not show much working or did not explicitly identify the solution. Although credit was given where the answers were clearly correct, some easy marks were lost.

3 (i) Let X_i be the number of claims in month i, so that X_i has a binomial distribution with parameters 5 and θ, $i = 1, \ldots, 12$.

The posterior density is, for $0 < \theta < 1$,

$$f(\theta \mid \mathbf{x}) \propto f(x_1, \ldots, x_{12} \mid \theta)\, f(\theta)$$

$$\propto \theta^{\Sigma_{i=1}^{12} x_i} (1-\theta)^{12\times 5 - \Sigma_{i=1}^{12} x_i}$$

$$= \theta^{10}(1-\theta)^{50}.$$

So the posterior distribution is a beta with parameters $\alpha = 11$ and $\beta = 51$.

(ii) (a) Under quadratic loss, the Bayesian estimate is the posterior mean, i.e. $\alpha/(\alpha + \beta) = 11/62 = 0.1774$.

(b) Under all-or-nothing loss, the Bayesian estimate is the posterior mode. Differentiating the posterior density, the mode solves

$$10\theta^9(1-\theta)^{50} - 50\theta^{10}(1-\theta)^{49} = 0$$

$$10(1-\theta) = 50\theta,$$

and so the Bayesian estimate is $10/60 = 0.1667$.

Most of the candidates who passed the exam did well on this question. Conversely, most of those who did badly failed. Although most candidates understood how to determine the Bayesian estimates in part (ii) many were unable to correctly determine the posterior distribution.

4 (i) Let S_1 and S_2 be the total amount claimed on type 1 and type 2 policies respectively in one year. Then

$$E(S) = E(S_1) + E(S_2) = 1.5 \times 5 + 2 \times 4 = 15.5,$$

and

$$\begin{aligned} V(S) &= V(S_1) + V(S_2) \\ &= 1.5 \times 2 \times 5^2 + 2 \times 2 \times 4^2 \\ &= 75 + 64 \\ &= 139. \end{aligned}$$

(ii) The moment generating function of S is

$$\begin{aligned} M_S(t) &= M_{S_1}(t)M_{S_2}(t) \\ &= \exp\left(1.5\left(\frac{1}{1-5t}-1\right)\right)\exp\left(2\left(\frac{1}{1-4t}-1\right)\right) \\ &= \exp\left(3.5\left(\frac{1.5}{3.5}\frac{1}{1-5t}+\frac{2}{3.5}\frac{1}{1-4t}-1\right)\right). \end{aligned}$$

This is of the form $\exp(\lambda(M(t) - 1))$, i.e. the moment generating function of a compound Poisson distribution, with Poisson parameter 3.5 and claim size density

$$\frac{3}{7}\times\frac{1}{5}e^{-x/5}+\frac{4}{7}\times\frac{1}{4}e^{-x/4},$$

which is a probability density function.

Part (i) was well answered. Most candidates were also able to derive the moment generating function in part (ii) although the number of algebraic errors was relatively high. Very few demonstrated correctly that the amount claimed had a compound Poisson distribution.

5 (i) The adjustment coefficient $R > 0$ satisfies $\dfrac{1}{1-\mu r} = 1 + (1 + \theta)\mu r$, so that

$$1 = 1 - \mu r + (1 + \theta)\mu r\,(1 - \mu r)$$

$$0 = \theta\mu r - (1 + \theta)\mu^2 r^2$$

$$= \mu r(\theta - (1 + \theta)\mu r).$$

Since $R > 0$, we have $R = \dfrac{\theta}{(1+\theta)\mu}$. The Lundberg inequality gives that the probability of ruin $\psi(u)$ satisfies $\psi(u) \le e^{-Ru}$ for all $u > 0$.

(ii) (a) We need $\exp\left(-\dfrac{20\theta}{1+\theta}\right)\le 0.01$, so that

$$\frac{20\theta}{1+\theta}\ge \log_e 100$$

$$\theta \ge \frac{\log_e 100}{20-\log_e 100},$$

and $\theta \ge 0.299$.

(b) This value will decrease if u increases, which makes sense since we expect the probability of ruin to decrease if we increase the initial capital.

Most candidates scored reasonably well on this question. Algebraic manipulation was poor in many cases leading to the wrong answer for (i). The second part was generally well answered although the use of equalities rather than inequalities was careless in most cases.

6 (i) To adjust claims in 2002, we need to inflate from end 2002 to mid 2003 at 2.5% p.a., and from mid 2003 to end 2003 at 3.0% p.a.

i.e. $(1.025)^{½} \times (1.03)^{½}$

Similarly for other years. So inflation adjustment factors are:

$1.025^{½} \times 1.03^{½} = 1.0275$
$1.025^{½} \times 1.025^{½} = 1.025$
$1.03^{½} \times 1.025^{½} = 1.0275$

Inflation adjusted incrementals are:

227.25	100.05	41.1	10
236.97	107.89	45	
220.91	95		
220			

Cumulatives:

227.25	327.30	368.40	378.40
236.97	344.85	389.85	
220.91	315.91		
220			
	988.07	758.26	378.40
685.13	672.16	368.40	
	1.4422	1.1281	1.0271

Forecast cumulatives:

		400.44
	356.38	366.05
317.28	357.92	367.63

Forecast total: 1512.52
Paid to date (allowing for inflation): 1304.16

Reserve: 208.36

(ii) Development pattern is the same each year.

Inflation has been removed from data, so future claims should be inflated.

None of the candidates identified correctly the twist of applying the mid-year interest adjustments to claims payments. A substantial majority correctly worked through the *remainder of the question and scored most of the marks. The question proved difficult to mark as the inflation adjustment factors were not sufficiently different from period to period.*

7 (i) (a) Let $x_1, \ldots, x_5$ be the observed claims. The Bayesian estimate under quadratic loss is the posterior mean, so first find the posterior distribution. The posterior density for θ is

$$f(\theta \mid \mathbf{x}) \propto \exp\left(-\frac{1}{50}\sum_{i=1}^{5}(x_i-\theta)^2 - \frac{1}{72}(\theta-125)^2\right)$$

$$\propto \exp\left(-\frac{1}{2}\left(\left(\frac{5}{25}+\frac{1}{36}\right)\theta^2 - 2\theta\left(\frac{5\overline{x}}{25}+\frac{125}{36}\right)\right)\right)$$

so that the posterior distribution is a normal with mean

$$\frac{\frac{5\overline{x}}{25}+\frac{125}{36}}{\frac{5}{25}+\frac{1}{36}} = \frac{\frac{5}{25}}{\frac{5}{25}+\frac{1}{36}}\overline{x} + \frac{\frac{1}{36}}{\frac{5}{25}+\frac{1}{36}}125.$$

This is of the form $Z\overline{x}+(1-Z)125,$ where 125 is the prior mean for θ.

(b) So it is of the form of a credibility estimate with credibility factor

$$Z = \frac{5}{5+\frac{25}{36}} = 0.8780.$$

If $\overline{x}$ = 122, then the credibility premium is

$$122Z + (1 - Z)125 = 122.37.$$

(c) If the variance of 25 is increased, then the value of Z would decrease, and the credibility estimate would move closer to the prior mean. This makes sense, since increasing this variance means that the claim amounts within each risk are more variable, and so we should put relatively less weight on past data.

(ii) (a) The estimate of $E(m(\theta))$ is $\overline{x}$ = (122 + 164 + 106)/3 = 130.7, so that the credibility premium for risk i is $Zx_i + (1-Z)130.7.$

The estimate for $E(s^2(\theta))$ is (2848 + 1628 + 1887)/12 = 6363/12 = 530.25.

The estimate of var($m(\theta)$) is

$$\frac{1}{2}\sum_{i=1}^{3}(\overline{x}_i - \overline{x})^2 - \frac{1}{60}6363 = 1794.67/2 - 106.05 = 791.29.$$

The estimated credibility factor is

$$\frac{5}{5+\dfrac{530.25}{791.29}} = 0.8818,$$

so that the credibility premium for risk 1 is

$$0.8818 \times 122 + (1 - 0.8818) \times 130.7 = 123.03.$$

(b) This is similar to the value obtained in (i), so the assumptions made in the prior appear not to be inappropriate.

Very few candidates derived the Bayesian estimate as required. Nonetheless most used information from tables to complete the remainder of the question. Marks for part (i) were generally good. Although most scored reasonably on part (ii) a disappointing number failed to calculate var(m(theta)) correctly. Most who made it to (ii)(b) made sensible conclusions.

8 (i) $\int_0^d x^m f(x)dx$

$$= \int_{-\infty}^{\log d} e^{my}\frac{1}{\sigma\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2\right\}dy$$

$$= \int_{-\infty}^{\log d} \frac{1}{\sigma\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{y^2-2\mu y+\mu^2-2m\sigma^2 y}{\sigma^2}\right)\right\}dy$$

$$= \int_{-\infty}^{\log d} \frac{1}{\sigma\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{(y-\mu-m\sigma^2)^2-2m\mu\sigma^2-m^2\sigma^4}{\sigma^2}\right)\right\}dy$$

$$= e^{m\mu+½m^2\sigma^2}\int_{-\infty}^{\log d} \frac{1}{\sigma\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{(y-\mu-m\sigma^2}{\sigma}\right)^2\right\}dy$$

$$= e^{m\mu+½m^2\sigma^2}\Phi\left(\frac{\log d-\mu-m\sigma^2}{\sigma}\right)$$

(ii) $e^{\mu+½\sigma^2} = 800$

$(e^{\mu+½\sigma^2})^2(e^{\sigma^2}-1) = 1200^2$

$e^{\sigma^2}-1 = \dfrac{1200^2}{800^2}$

$\sigma^2 = 1.178655$

$\mu = 6.095284$

(iii) $E[X_I^{(\text{Prop})}] = (1-k)\,E[X]$

$\therefore\ k = 0.3$

$$E[X_I^{(XL)}] = \int_0^d xf(x)dx + dP(X>d)$$

$$= e^{\mu+½\sigma^2}\Phi\left(\frac{\log d-\mu-\sigma^2}{\sigma}\right)+d\left[1-\Phi\left(\frac{\log d-\mu}{\sigma}\right)\right]$$

$$= 800\Phi\left(\frac{\log d-\mu-\sigma^2}{\sigma}\right)+d\left[1-\Phi\left(\frac{\log d-\mu}{\sigma}\right)\right]$$

Hence $800\Phi\left(\dfrac{\log d-7.27394}{1.08566}\right)+d\left[1-\Phi\left(\dfrac{\log d-6.095284}{1.08566}\right)\right] = 0.7\times 800$

$= 560$

when d = 1189.4, the left hand side is

$800\Phi(-0.1775) + 1189.4(1-\Phi(0.9081))$

$= 800\times 0.42956 + 1189.4\times 0.1819 = 560$

(iv) $\text{Var}[X_I^{(\text{Prop})}] = \text{Var}[(1-k)X] = 0.7^2\,\text{Var}[X]$

$= 705600$

$\text{Var}[X_I^{(XL)}] = E\left[X_I^{(XL)^2}\right] - (E(X_I^{(XL)}))^2$

$E\left[X_I^{(XL)^2}\right] = \int_0^d x^2 f(x)dx + d^2 P(X > d)$

$= e^{2\mu+2\sigma^2}\Phi\left(\dfrac{\log d - \mu - 2\sigma^2}{\sigma}\right) + d^2 \times 0.1819$

$= 2080000\Phi(-1.2632) + 1189.4^2 \times 0.1819$

$= 2080000 \times 0.10326 + 257328.9$

$= 472130$

Hence $\text{Var}[X_I^{(XL)}] = 472130 - 560^2 = 158530$

(v) The mean is the same and hence the cost will be the same, if the loadings are the same for proportional and excess-of-loss reinsurance.

The variance is lower for excess-of-loss, which is preferable for the insurer. This is due to the heavy tail of the lognormal, which is not removed by the proportional reinsurance.

Most candidates scored well on part (i) although notation was rather sloppy. Very few identified that the lower bound should be -infinity once the appropriate substitution had been made. Most of the candidates also scored reasonably well on part (ii). However, the later parts were only done well by a small number of candidates with most scoring few marks.

9 (i) (a) The likelihood is

$$\prod_{i=1}^{3}\prod_{j=1}^{4}\frac{e^{-\mu_{ij}}\mu_{ij}^{y_{ij}}}{y_{ij}!},$$

so that the log-likelihood is

$$\sum_{i=1}^{3}\sum_{j=1}^{4}(-\mu_{ij} + y_{ij}\log(\mu_{ij}) - \log(y_{ij}!)).$$

Differentiating with respect to μ_{ij} and setting to zero gives $\hat{\mu}_{ij} = y_{ij}$.

(b) When $\log(\mu_{ij}) = \mu$, this becomes

$$l(\mu) = -12e^{\mu} + \mu\sum_{i=1}^{3}\sum_{j=1}^{4} y_{ij} - \sum_{i=1}^{3}\sum_{j=1}^{4}\log(y_{ij}!).$$

Differentiating with respect to μ, we get

$$\partial l/\partial\mu = -12e^{\mu} + \sum_{i=1}^{3}\sum_{j=1}^{4} y_{ij},$$

so that the maximum likelihood estimator of μ is

$$\hat{\mu} = \log\left(\sum_{i=1}^{3}\sum_{j=1}^{4} y_{ij}/12\right) = \log(\bar{y}).$$

(c) The scaled deviance is 2 × (maximum loglikelihood for saturated model – maximum loglikelihood for the current model). The saturated model has parameters μ_{ij} with maximum likelihood estimators $\hat{\mu}_{ij} = y_{ij}$. The scaled deviance is

$$2\left(-\sum_{i,j} y_{ij} + \sum_{i,j} y_{ij}\log(y_{ij}) + 12\bar{y} - \log(\bar{y})\sum_{i,j} y_{ij}\right) = 2\sum_{i,j} y_{ij}\log(y_{ij}/\bar{y}).$$

(ii) (a) Model 1 says that there is no difference in the mean number of accidents over quarters and years.

Model 2 says that there is no seasonal difference in mean number of accidents over quarters, but there may be a difference over years.

Model 3 says that there is a difference over both years and quarters, and that there is no interaction between years and quarters.

The effects on the mean are multiplicative, with e.g. the same factor for the first quarter for all three years.

(b) The models 1, 2, 3 are nested. The drop in deviance from model 1 to model 2 is 266.35 –202.19 = 64.16 on 11 – 9 = 2 degrees of freedom. Since 64.16 is significant when compared to a χ^2 distribution on 2 degrees of freedom, model 2 is a significant improvement over model 1.

The drop in deviance from model 2 to model 3 is 191.51 on 3 df. Again this is significant (compared to a χ^2 on 3df), so model 3 is a significant improvement over model 2.

Hence model 3, with dependence on both year and quarter, is the preferred model out of these three models.

(iii) This model says that there is dependence on both year and quarter. In terms of the log(mean) the dependence on quarter allows a different value for each quarter, but the dependence on year is given by a linear increase of 0.34 per year. This translates into multiplying the mean by $e^{0.34}$ per year.

This was very poorly done. Many candidates identified the appropriate log-likelihood functions in (i) but few differentiated correctly to determine the maximum likelihood estimates. Similarly the scaled deviance in (i)(c) caused problems for most.

Part (ii) was answered better but a number of candidates failed to identify sufficiently clearly that they had interpreted the models correctly or how they arrived at the recommended model.

Only the best candidates were able to interpret the model correctly in (iii).

10 (i) Premiums are 2500, 2000, 1250

0% level

Claim:	2500	2000
No claim:	2000	1250

Difference: 1250

Policyholder will not claim if cost is less than £1,250

20% level

Claim:	2500	2000
No claim:	1250	1250

Difference: 2000

Policyholder will not claim if cost is less than £2,000

50% level

Not drunk

Claim:	2000	1250
No claim:	1250	1250

Difference: 750

Drunk

Claim:	2500	2000
No claim:	1250	1250

Difference: 2000

Policyholder will not claim if cost is less than £750 if it was not as a result of drunken behaviour, and £2,000 if it was.

(ii) **0% level**

$$P(X > 1250) = 1 - \Phi\left(\frac{\log 1250 - 6.5}{\sqrt{3.5}}\right) = 1 - \Phi(0.3372)$$

$$= 0.3680$$

20% level

$$P(X > 2000) = 1 - \Phi\left(\frac{\log 2000 - 6.5}{\sqrt{3.5}}\right) = 1 - \Phi(0.5885)$$

$$= 0.2781$$

50% level

$$P(X > 750) = 1 - \Phi\left(\frac{\log 750 - 6.5}{\sqrt{3.5}}\right) = 1 - \Phi(0.0642)$$

$$= 0.4744$$

$$P(\text{Claim}) = 0.75 \times 0.4744 + 0.25 \times 0.2781$$

$$= 0.42525$$

(iii) $P(\text{Accident}) = 0.2$

Transition matrix is

$$\begin{bmatrix} 0.2\times 0.3680 & 1-0.2\times 0.3680 & 0 \\ 0.2\times 0.2781 & 0 & 1-0.2\times 0.2781 \\ 0.2\times 0.2781\times 0.25 & 0.2\times 0.4744\times 0.75 & 1-0.2\times 0.42525 \end{bmatrix}$$

i.e. $$\begin{bmatrix} 0.0736 & 0.9264 & 0 \\ 0.05562 & 0 & 0.94438 \\ 0.013905 & 0.07116 & 0.914935 \end{bmatrix}$$

(iv) Solve $\pi P = \pi$

$$0.0736\pi_0 + 0.05562\pi_1 + 0.013905\pi_2 = \pi_0$$
$$0.9264\pi_0 + 0.07116\pi_2 = \pi_1$$
$$0.94438\pi_1 + 0.914935\pi_2 = \pi_2$$

$$\therefore\ \pi_2 = \frac{0.94438}{0.085065}\pi_1 = 11.102\pi_1$$

$$\therefore\ \pi_0 = \frac{(1-0.07116\times 11.102)}{0.9264}\pi_1$$

$$= 0.2267\pi_1$$

$\pi_0 + \pi_1 + \pi_2 = 1$

$0.2267\pi_1 + \pi_1 + 11.102\pi_1 = 1$

$\therefore\ 12.3287\pi_1 = 1$

$\therefore\ \pi_1 = 0.0811$

$\therefore\ \pi_0 = 0.0184$

and $\pi_2 = 0.9005$

$(0.0184, 0.0811, 0.9005)$

This question tended to split those who passed and those who failed. Stronger candidates identified the information required for (iii) and worked methodically through the whole question relatively easily. Many failed to calculate the probabilities correctly in (ii) or the *transition matrix in (iii). Credit was given to those who followed through sensible answers from earlier parts.*

END OF EXAMINERS' REPORT