EXAMINATIONS

11 September 2001 (am)

Subject 106 — Actuarial Mathematics 2

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. Write your surname in full, the initials of your other names and your Candidate's Number on the front of the answer booklet.

2. Mark allocations are shown in brackets.

3. Attempt all 11 questions, beginning your answer to each question on a separate sheet.

Graph paper is not required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet and this question paper.

In addition to this paper you should have available Actuarial Tables and an electronic calculator.

1 The distribution of X, which represents the claim severity from a portfolio of non-life insurance policies, has a Pareto distribution with mean £350 and standard deviation £452.

If the insurer arranges excess of loss reinsurance, with retention £1,200, calculate the probability that a claim will involve the reinsurer. [3]

2 On a portfolio of insurance policies, the claim size, Y is assumed to depend on the age of the policyholder, X. Suppose that the conditional mean and variance of Y are

$$E[Y|X = x] = 2x + 400$$

and $$V[Y|X = x] = \frac{x^2}{2}.$$

The distribution of X over the portfolio is assumed to be normal with mean 50 and standard deviation 14.

Calculate the unconditional mean and standard deviation of Y. [5]

3 A specialist motor insurer writes policies with individual excesses of £500 per claim. The insurer has taken out a reinsurance policy whereby the insurer pays out a maximum of £4,500 in respect of each individual claim, the rest being paid by the reinsurer. The individual claims, gross of reinsurance and the excess, are believed to follow an exponential distribution with parameter λ.

Over the last year, the insurer has gathered the following data:

- There were 5 claims which were not processed because the loss was less than the excess.
- There were 11 claims where the insurer paid out £4,500 and the reinsurer the remainder.
- There were 26 other claims in respect of which the insurer paid out a total of £76,457.

Derive the loglikelihood function of λ. [6]

4 An actuarial student enters a quiz evening. All the questions will be picked by the organiser, from one of four subjects: History, Literature, Sport or General Knowledge. Under the rules of the quiz, entrants choose one opponent to play against for a cash prize, determined in advance according to the level of knowledge of each competitor.

The student can choose from three opponents, Person A, Person B or Person C. He knows that he is better than A at Literature, Sport and General Knowledge but not History; he is better than B at Sport and General Knowledge only and he is better than C at History and Literature only.

If the student wins against A he will receive £100, if he wins against B he receives £150 and if he wins against C he receives £200. It is assumed that the better student always wins.

From his experience of past quiz nights, the student believes that History has a 15% chance of being the chosen subject, Literature a 25% chance, Sport a 20% chance and General Knowledge 40%.

(i) Determine which opponent the student should choose to play against to maximise his expected winnings. [4]

(ii) If the student decides instead that Sport will definitely not be picked but the other three subjects have the same chances of being picked as before **relative to each other**, determine which opponent he should choose. [2]
[Total 6]

5 Consider a portfolio of insurance policies, on which the number of claims has a binomial distribution, with parameters n and p. The claim size distribution is assumed to be exponential with mean $\frac{1}{\lambda}$. Claims are assumed to be independent random variables and to be independent of the number of claims.

The insurer arranges excess of loss reinsurance, with retention M.

Calculate the moment generating function of S_I, where S_I is aggregate annual claims paid by the insurer (net of reinsurance). [6]

6 In a No Claims Discount system for motor insurance, there are three discount levels, 0%, 20% and 30%. The full annual premium is £190.

If no claims are made during a year, the policyholder moves up to the next higher level of discount, or stays at 30%. If one or more claims are made, the policyholder moves down to 0% discount, or stays at 0%. The probability that a policyholder has one accident in a year is 0.1, and the probability of more than one accident is negligible. In the event of an accident, the natural logorithm of the loss has a $N(\mu, \sigma^2)$ distribution, with $\mu = 4.5$ and $\sigma^2 = 0.84$. The policyholder will only make a claim when an accident occurs if the loss is greater than the total extra premiums that would have to be paid over the infinite time horizon, assuming no further accidents occur.

(i) Determine the transition matrix for the Markov chain $X_1, X_2, \ldots$ where X_n is the discount level at the beginning of year n. [6]

(ii) For a policyholder who pays the full premium in the current year, calculate the expected premium at the beginning of the next year. [1]
[Total 7]

7 Claim amounts under a particular insurance portfolio are believed to follow a Normal distribution with standard deviation σ_1^2 and an unknown mean θ. The insurer observes a sample of n policies which have given rise to a claim for which the mean amount is $\bar{a}$. The prior distribution of θ is assumed to be Normal with mean μ and standard deviation σ_2^2.

(i) (a) State the posterior mean for θ.

(b) Show that the posterior mean of θ can be expressed as a weighted average of the prior mean and the sample mean, and derive an expression for the weight placed on the sample mean. [3]

(ii) An insurer believes that individual claim amounts follow a Normal distribution with an unknown mean and a standard deviation of £210. Prior information suggests that the mean should be assumed to follow a Normal distribution with mean = £155 and standard deviation = £84. Over the previous year, the insurer collected data from 15 claims where the total amount paid out was £2,456.

Calculate the credibility factor placed on the sample mean and hence, given that the insurer wishes to add a 30% loading for profit and expenses, calculate the premium the insurer should charge. [2]

(iii) Determine the limiting value of the credibility factor as each of n, σ_1^2 and σ_2^2 increases and briefly describe how it is affected by the insurer's assumptions and the observed data. [3]

[Total 8]

8 The number of claims per month are independent Poisson random variables with mean λ, and the prior distribution for λ is exponential with mean 0.2.

(i) Determine the posterior distribution for λ given the observed values $x_1, \ldots, x_n$ of the number of claims in each of n months. [2]

(ii) Determine the Bayesian estimate of λ

(a) under quadratic loss
(b) under "all-or-nothing" loss [3]

(iii) If $n = 5$ and $\sum_1^5 x_i = 1$, calculate to 2 significant figures the Bayesian estimate of λ under absolute error loss. [4]

[Total 9]

9 Claims paid to date on a motor insurance account are as follows:

Figures in £000s

	Development year			
Policy year	*0*	*1*	*2*	*3*
1996	1,708	1,366	942	461
1997	2,079	1,601	1,195	
1998	2,417	1,863		
1999	2,935			

Inflation for the 12-month period to the middle of each year was as follows:

1997	6.2%
1998	4.9%
1999	5.2%

You are given the following further information:

- Annual premiums written in 1999 were £7,731,000.
- Future inflation from mid-1999 is estimated to be 5.0% per annum.
- The ultimate loss ratio (based on mid-1999 prices) has been estimated at 77%.
- Claims are assumed to be fully run-off by the end of development year 3.

Estimate the outstanding claims **arising from policies written in 1999 only** taking explicit account of the inflation statistics, using the Bornhuetter-Ferguson method. [15]

10 Claims for a particular risk arrive in a Poisson process rate λ. The claim sizes are independent and identically distributed with density $f(x)$ and are independent of the claims arrival process. Assume there is a constant γ $(0 < \gamma < \infty)$ such that $\lim_{r \to \gamma} M(r) = \infty$ where $M(r)$ is the moment generating function of a claim.

Premiums are received continuously at constant rate with premium loading factor $\theta > 0$.

(i) (a) Define the adjustment coefficient, R.

(b) Define the surplus process and the probability $\psi(u)$ of ruin with initial surplus $u > 0$.

(c) Write down Lundberg's inequality. [3]

(ii) Derive the adjustment coefficient if $f(x) = \frac{1}{\mu} e^{-\frac{x}{\mu}}$, $x > 0$, and $\theta = 0.25$. [3]

(iii) Consider the case where $f(x) = \frac{1}{2} e^{-x}(1 + 2e^{-x})$, $x > 0$, and $\theta = 0.25$.

(a) Calculate the expected claim size μ.

(b) Calculate the corresponding adjustment coefficient, and determine an upper bound for $\psi(15)$.

(c) Compare your answers to (iii)(b) with those obtained if the claim sizes are mistakenly assumed to be exponentially distributed with mean μ, and comment briefly. [11]

[Total 17]

11 Observations $Y_1, \ldots, Y_n$ are independent Poisson random variables with $E(Y_i) = \mu_i$ where

$$\log \mu_i = \begin{cases} \alpha & i = 1, \ldots, m \\ \alpha + \beta & i = m+1, \ldots, n. \end{cases} \qquad (*)$$

(i) (a) Show that the loglikelihood can be written as

$$-me^{\alpha} - (n-m)\, e^{\alpha+\beta} + \alpha \sum_{i=1}^{n} y_i + \beta \sum_{i=m+1}^{n} y_i - \log\left(\prod_{i=1}^{n} y_i\,! \right)$$

(b) Derive the maximum likelihood estimates of α and β.

(c) Derive an expression for the deviance for this model.

(d) Derive an expression for the deviance for the model where $\beta = 0$. [13]

(ii) Suppose $n = 20$, $m = 10$ and the observations are

8, 6, 4, 7, 5, 8, 2, 8, 2, 9, 10, 7, 6, 4, 5, 7, 9, 6, 8, 5.

(a) Complete the following table, and hence calculate the deviance for the model in (i)(d).

y_i	*Frequency*	*Contribution to deviance*
2	2	−9.1792
4	2	−7.2681
5	3	−6.9334
6	3	−1.7564
7	3	4.4251
8	4	15.2891
9	2	12.8403
10	?	?

(b) Given that the deviance for the model in (i)(c) is 16.1499, comment on the fit of (*) and of the model with $\beta = 0$. [5]

[Total 18]