

Project L3

Instructions to CA2 candidate

- (i) Read the attached document, which describes the background to this project. You will be given an additional piece of information later on in the day.
- (ii) Use the values of l_x for the mortality table AM92 Ult, supplied in a spreadsheet on your PC, in a new spreadsheet.
- (iii) Scenario 1: Use the values of l_x to generate the relevant annuity and assurance factors needed to calculate the specified theoretical premium rates. You should assume that premiums are payable annually in advance, and benefits are paid at the end of the year of death.
- (iv) Include a demonstration of how you have checked the accuracy of these calculations.
- (v) Calculate the theoretical premium rates for each type of product and show them in a suitable graph.
- (vi) Scenario 2: Calculate the theoretical whole of life premium rates assuming mortality rates are double those in Scenario 1. Include a demonstration of how you have checked these calculations for reasonableness and accuracy. Present the premium rate results in a suitable graph.
- (vii) Use the information obtained in Scenarios 1 and 2 to complete the calculations required for Scenario 3, which will be given to you later in the day.
- (viii) Prepare around five or so pages of summary using bullet points and visual aids, capturing the main features and results. You should cover the following:
 - A summary of your approach to each scenario and any assumptions you used.
 - Scenario 1 – theoretical premium rates and comments on the shape of the rates.
 - Scenario 2 – impact of impaired mortality assumption and comments on the shape of the rates.
 - Scenario 3 – specified premium rates and comments on the shape of the rates.
 - Suggested next steps.

Background for CA2 candidate – Project L3

You are an actuarial student working in a life insurance company. One of the marketing actuaries is interested in enhancing the company's existing range of protection business, particularly for the higher age end of the market.

He has asked you to help him get an idea of what might be possible by calculating theoretical premium rates for different types of product at different ages. These premium rates should ignore expenses, commissions, profit margins, lapses, tax and supervisory reserving requirements.

Scenario 1

The actuary is interested in how the theoretical premium rates for a whole of life product and a ten year term assurance vary by age, and also how the rates differ between the two product types.

He would like the calculations to be performed for ages 50 to 90, at an interest rate of 4.25% p.a. and assuming mortality in line with AM92 Ult.

Scenario 2

He then asks you to recalculate the whole of life premium rates assuming that the lives are in relatively poor health. The impaired mortality rates are assumed to be double those used for the basic calculations in Scenario 1.

He is particularly interested in how this changes the rates calculated under Scenario 1.

A further piece of information will be provided later on in the day.

You may wish to note that, usually, the simplest approach to calculating basic assurance and annuity factors is to make use of recursion formulae, such as

$$A_x = vq_x + v(1-q_x)A_{x+1}$$

Other values may be obtained by similar recursion techniques, or it should be possible to use the basic results to obtain other values, such as

$$A_x^{1:10} = A_x - A_{x+10} * {}_{10}p_x * v^{10}$$

Additional information for CA2 candidate – Project L3

To be provided later on in the day

Scenario 3

The marketing actuary has suggested that the company should sell a whole of life product with an increased cover option, that policyholders can choose whether or not to exercise but only at a specified date. He would like you to calculate the theoretical, additional premium that should be paid for the proposed option.

The product has a basic sum assured of £10,000. At the tenth anniversary, the policyholder can opt to take out a further whole of life policy with a sum assured up to £10,000 at the company's "standard premium rates" and without further evidence of health.

Benefits are paid at the end of the year of death and premiums are payable annually in advance for the whole of life, other than the additional option premium which is payable by all policyholders annually in advance until the option date or earlier death.

Mortality up to the option date should be assumed to be AM92 Ult for all policyholders. Following the option date, this assumption remains unchanged for those who do not take the option. However, those who do take the option are expected to be in relatively poor health. Their mortality rates are expected to be double those under AM92 Ult. However, for its "standard premium rates", the company assumes that all policyholders experience AM92 Ult.

Interest should be assumed to be 4.25% per annum, it is expected that 30% of policyholders will take the option, and expenses, lapses, tax, profit margins and supervisory reserving requirements can be ignored.

The marketing actuary would like to know the size of the option premium for entry ages 50 to 80 on the above basis.

Hint:

The option premium can be calculated by equating the present value of all premiums to the present value of all benefits.

The benefits are the basic sum assured and, if the option is taken, the additional sum assured.

The premiums are the option premium, the basic premium payable (at "standard premium rates") for the basic sum assured, and, if the option is taken, the extra "standard rate" premium payable for the additional sum assured.

UNIT 1 — SIMPLE ASSURANCES AND ANNUITIES

Syllabus objective

- (i) Define simple assurance and annuity contracts, and develop formulae for the means and variances of the present values of the payments under these contracts, assuming constant deterministic interest.

1. Define the following terms:

- whole life assurance
- term assurance
- pure endowment
- endowment assurance
- critical illness assurance
- whole life level annuity
- temporary level annuity
- premium
- benefit

including assurance and annuity contracts where the benefits are deferred.

2. Define the following probabilities: ${}_n|m q_x$, ${}_n|q_x$ and their select equivalents

${}_n|m q_{[x]+r}$, ${}_n|q_{[x]+r}$.

3. Obtain expressions in the form of sums for the mean and variance of the present value of benefit payments under each contract above, in terms of the curtate random future lifetime, assuming that death benefits are payable at the end of the year of death and that annuities are paid annually in advance or in arrear, and, where appropriate, simplify these expressions into a form suitable for evaluation by table look-up or other means.
4. Obtain expressions in the form of integrals for the mean and variance of the present value of benefit payments under each contract above, in terms of the random future lifetime, assuming that death benefits are payable at the moment of death and that annuities are paid continuously, and, where appropriate, simplify these expressions into a form suitable for evaluation by table look-up or other means.
5. Extend the techniques of 3. and 4. above to deal with the possibility that premiums are payable more frequently than annually and that benefits may be payable annually or more frequently than annually.

6. Define the symbols A_x , $A_{x:\overline{n}|}$, $A_{x:\overline{n}|}^1$, $A_{x:\overline{n}|}^{\overline{1}}$, a_x , $a_{x:\overline{n}|}$, ${}_m|a_{x:\overline{n}|}$, \ddot{a}_x , $\ddot{a}_{x:\overline{n}|}$, ${}_m|\ddot{a}_{x:\overline{n}|}$ and their select and continuous equivalents. Extend the annuity factors to allow for the possibility that payments are more frequent than annual but less frequent than continuous.
7. Derive relations between annuities payable in advance and in arrear, and between temporary, deferred and whole life annuities.
8. Derive the relations $A_x = 1 - d\ddot{a}_x$, $A_{x:\overline{n}|} = 1 - d\ddot{a}_{x:\overline{n}|}$, and their select and continuous equivalents.
9. Define the expected accumulation of the benefits in 1., and obtain expressions for them corresponding to the expected present values in 3., 4., and 5. (note: expected values only).

1 Life insurance contracts

- 1.1** Life insurance **contracts** (also called **policies**) are made between a life insurance company and one or more persons called the **policyholders**.

The policyholder(s) will agree to pay an amount or a series of amounts to the life insurance company, called **premiums**, and in return the life insurance company agrees to pay an amount or amounts called the **benefit(s)**, to the policyholder(s) on the occurrence of a specified event.

In this Subject we first consider contracts with a single policyholder and then later show how to extend the theory to two policyholders.

- 1.2** The benefits payable under simple life insurance contracts are of two main types.

- (a) The benefit may be payable on or following the death of the policyholder.
- (b) The benefit(s) may be payable provided the life **survives** for a given term.

An example of the second type of contract is an **annuity**, under which amounts are payable at regular intervals as long as the policyholder is still alive.

More generally, the theory of this Subject may be applied to “near-life” contingencies — such as state of health of a policyholder — or to “non-life” contingencies — such as the cost of replacing a machine at the time of failure. The theory assumes only that the payment is of known amount. Subject ST3, General Insurance Specialist Technical examines the valuation of non-life contingencies where the future payment is of unknown amount.

1.3

The simplest life insurance contract is the **whole life assurance**. The benefit under such a contract is an amount, called the **sum assured**, which will be paid on the policyholder's death.

A **term assurance** contract is a contract to pay a sum assured on or after death, provided death occurs during a specified period, called the **term** of the contract.

A **pure endowment** contract provides a sum assured at the end of a fixed term, provided the policyholder is then alive.

An **endowment assurance** is a combination of (i) a term assurance and (ii) a pure endowment assurance. That is, a sum assured is payable **either** on death during the term **or** on survival to the end of the term. The sums assured payable on death or survival need not be the same, although they often are.

A **critical illness assurance** is a contract to pay a benefit if and when the policyholder is diagnosed as suffering from a particular disease, where the disease is specifically listed in the wording of the policy. Typically a number of diseases are listed in a policy, and the benefit may be available on a whole of life or a term basis.

A **life annuity contract** provides payments of amounts, which might be level or variable, at stated times, provided a life is still then alive.

Here we consider three varieties of life annuity contract.

- (1) Annuities under which payments are made for the whole of life, with level payments, called a **whole life level annuity** or, more usually, an **immediate annuity**
- (2) Annuities under which level payments are made only during a limited term, called a **temporary level annuity** or, more usually, just **temporary annuity**
- (3) Annuities under which the start of payment is deferred for a given term, called a **deferred annuity**

Further, we consider the possibilities that payments are made in advance or in arrear. For the moment we consider only contracts under which **level** payments are made at **yearly** intervals.

2 **Formulae for the present value of benefit payments — means and variances**

2.1

Much actuarial work is concerned with finding a fair price for a life insurance contract. In such calculations we must consider

- (a) the time value of money, and

- (b) the uncertainty attached to payments to be made in the future, depending on the death or survival of a given life

Therefore this Subject requires us to bring together the topics covered in Subject CT1, Financial Mathematics, in particular compound interest, and the topics covered in Subject CT4, Models, in particular the unknown future lifetime and its associated probabilities.

2.2

In this Subject we will usually assume that money can be invested or borrowed at some given rate of interest. We will always assume that the rate of interest is **known**, that is, deterministic. The mathematics of finance includes several useful stochastic models of the behaviour of interest rates, but we will not use them. We will not always assume that the rate of interest is **constant**, however. When the rate of interest **is** constant, we denote the effective compound rate of interest per annum by i and define $v = (1 + i)^{-1}$, and we will use these without further comment.

We will also assume knowledge of the basic probabilities introduced in Subject CT4, Models, namely ${}_t p_x$ and ${}_t q_x$ and their degenerate quantities, when $t = 1$, p_x and q_x . Further, we assume knowledge of the key survival model formulae, the fundamental ones being that:

$${}_t q_x = \int_0^t {}_s p_x \mu_{x+s} ds \quad \text{and} \quad {}_t p_x = \exp \left\{ - \int_0^t \mu_{x+s} ds \right\}.$$

2.3

Using the two basic building blocks described above in 2.1, and the assumptions made in 2.2, we will develop formulae for the means and variances of the present value of contingent benefits.

In Unit 2 we will consider ways of assigning probability values to the unknown future lifetime, so as to evaluate the formulae. Two different assumptions are typically used in practice when determining the probability values. The first is to assume that the underlying mortality depends on age only. The second is to assume that the underlying mortality depends on age plus duration since some specific event. For example, the assumption in the second case can allow for the likely lower level of mortality which might result from a medical test having to have been passed before the insurer agrees to issue a life insurance contract.

The first assumption is described as assuming **ultimate mortality**, the second as assuming **select mortality**. We will return later to discussing select mortality but assume for the moment that ultimate mortality applies.

2.4 Whole life assurance contracts

We begin by looking at the simplest assurance contract, the whole life assurance, which pays the sum assured on the policyholder's death. For the moment we ignore the premiums which the policyholder might pay.

We will use this simple contract to introduce important concepts, in particular the **expected present value** (“EPV”) of a payment contingent on an uncertain future event. We will then apply these concepts to other types of life insurance contracts.

2.5 In mathematics of finance, the present value at time 0 of a payment of 1 to be made at time t is v^t . Suppose, however, that the time of payment is not certain but is a random variable, say H . Then the present value of the payment is v^H , **which is also a random variable**. A whole life assurance benefit is a payment of this type.

2.6 For the moment we will introduce two conventions, which can be relaxed later on.

Convention 1 We suppose that we are considering a benefit payable to a life who is currently aged x , where x is an integer.

Convention 2 We suppose that the sum assured is payable, not **on** death, but at the end of the year of death (based on policy year).

These will not always hold in practice, of course, but they simplify the application of life table functions.

2.7 Under these conventions we see that the whole life sum assured will be paid at time $K_x + 1$, where K_x denotes the curtate random future lifetime of a life currently aged x .

Let the sum assured be S , then the present value of the benefit is Sv^{K_x+1} , a random variable. Obvious questions are, what are the expected value and the variance of Sv^{K_x+1} ?

2.8 By definition of ${}_t p_x$ and ${}_t q_x$ it follows that

$$P[K_x = k] = {}_k p_x \cdot {}_k q_{x+k} = {}_k | q_x \quad (k = 0, 1, 2, \dots)$$

We define a new probability ${}_n | q_x$:

$${}_n | q_x = {}_n p_x \cdot {}_n q_{x+n} \quad (n = 0, 1, 2, \dots)$$

Therefore

$$E[v^{K_x+1}] = \sum_{k=0}^{\infty} v^{k+1} {}_k p_x \cdot {}_k q_{x+k} = \sum_{k=0}^{\infty} v^{k+1} {}_k | q_x.$$

$E[v^{\mathbf{K}_x+1}]$ is the **expected present value** (“EPV”) of a sum assured of 1, payable at the end of the year of death. Such functions play a central rôle in life insurance mathematics and are included in the standard actuarial notation. We define

$$A_x = E[v^{\mathbf{K}_x+1}] = \sum_{k=0}^{\infty} v^{k+1} {}_k|q_x$$

[Note that, for brevity, we have written the sum as $\sum_{k=0}^{\infty}$ instead of $\sum_{k=0}^{\omega-x-1}$. This should cause no confusion, since ${}_k p_x = 0$ for $k \geq \omega - x$.]

If the sum assured is S , then the EPV of the benefit is $S \cdot A_x$. Values of A_x at various rates of interest are tabulated in (for example) the AM92 tables, which can be found in “Formulae and Tables for Examinations”.

Example 1

Find A_{40} (AM92 at 6%)

Solution

0.12313

2.9 Turning now to the variance of $v^{\mathbf{K}_x+1}$, we have

$$\text{Var}[v^{\mathbf{K}_x+1}] = \sum_{k=0}^{\infty} (v^{k+1})^2 {}_k|q_x - (A_x)^2$$

But since $(v^{k+1})^2 = (v^2)^{k+1}$, the first term is just 2A_x where the “2” prefix denotes an EPV calculated at a rate of interest $(1+i)^2 - 1$.

So provided we can calculate EPVs at any rates of interest, it is easy to find the variance of a whole life benefit.

Note that $\text{Var}[Sv^{\mathbf{K}_x+1}] = S^2 \text{Var}[v^{\mathbf{K}_x+1}]$.

Values of 2A_x are tabulated at various rates of interest in (for example) AM92 “Formulae and Tables for Examinations”.

2.10 Term assurance contracts

Consider such a contract, which is to pay a sum assured at the end of the year of death of a life aged x , provided this occurs during the next n years. We assume that n is an integer.

2.11 Let \mathbf{F} denote the present value of this payment. \mathbf{F} is a random variable.

$$\begin{aligned} E[\mathbf{F}] &= \sum_{k=0}^{n-1} v^{k+1} {}_k|q_x + 0 \times {}_n p_x \\ &= \sum_{k=0}^{n-1} v^{k+1} {}_k|q_x \end{aligned}$$

In actuarial notation, we define

$$A_{x:\overline{n}|}^1 = E[\mathbf{F}] = \sum_{k=0}^{n-1} v^{k+1} {}_k|q_x$$

2.12 Along the same lines as for the whole life assurance,

$$\text{Var}[\mathbf{F}] = {}^2A_{x:\overline{n}|}^1 - (A_{x:\overline{n}|}^1)^2$$

where the “2” prefix means that the EPV is calculated at rate of interest $(1+i)^2 - 1$.

2.13 Pure endowment contracts

Consider a pure endowment contract to pay a sum assured of 1 after n years, provided a life aged x is still alive. We assume that n is an integer.

2.14 Let \mathbf{G} denote the present value of the payment.

$$E[\mathbf{G}] = v^n {}_n p_x + 0 \times {}_n q_x.$$

In actuarial notation, we define

$$A_{x:\overline{n}|}^{\frac{1}{0}} = E[\mathbf{G}] = v^n {}_n p_x$$

to be the EPV of a pure endowment benefit of 1, payable after n years to a life aged x .

2.15 Following the same lines as before,

$$\text{Var}[\mathbf{G}] = {}^2A_{x:\overline{n}|} - (A_{x:\overline{n}|})^2$$

where the “2” prefix denotes an EPV calculated at a rate of interest of $(1+i)^2 - 1$.

2.16 Endowment assurance contracts

Consider an endowment assurance contract to pay a sum assured of 1 to a life now aged x at the end of the year of death, if death occurs during the next n years, or after n years if the life is then alive. We suppose that n is an integer.

2.17 Let \mathbf{H} be the present value of this payment. In terms of the present values already defined, $\mathbf{H} = \mathbf{F} + \mathbf{G}$. Therefore

$$\begin{aligned} E[\mathbf{H}] &= E[\mathbf{F}] + E[\mathbf{G}] \\ &= \sum_{k=0}^{n-1} v^{k+1} {}_k/qx + v^n {}_npx \\ &= \sum_{k=0}^{n-2} v^{k+1} {}_k/qx + v^n {}_n\tilde{l}px \end{aligned}$$

The last expression holds because payment at time n is certain if the life survives to age $x + n - 1$. In actuarial notation we define

$$\begin{aligned} A_{x:\overline{n}|} &= E[\mathbf{H}] \\ &= E[\mathbf{F}] + E[\mathbf{G}] \\ &= A_{x:\overline{n}|}^1 + A_{x:\overline{n}|}^{}. \end{aligned}$$

2.18 Note that \mathbf{F} and \mathbf{G} are not independent random variables (one must be zero and the other non-zero).

Therefore $\text{Var}[\mathbf{H}] \neq \text{Var}[\mathbf{F}] + \text{Var}[\mathbf{G}]$. We must find $\text{Var}[\mathbf{H}]$ from first principles. As before, we find that

$$\text{Var}[\mathbf{H}] = {}^2A_{x:\overline{n}|} - (A_{x:\overline{n}|})^2$$

where the “2” prefix denotes an EPV calculated at rate of interest $(1+i)^2 - 1$.

2.19 Critical illness assurance contracts

Consider such a contract, which is to pay a sum assured at the end of the year of diagnosis, from a specified list of diseases, of a healthy life aged x , provided this occurs during the next n years. We assume that n is an integer.

To value this benefit, we proceed much as we did for a term assurance. The EPV of an n -year term assurance is

$$\sum_{k=0}^{n-1} v^{k+1} \times {}_k p_x \times q_{x+k}.$$

For a critical illness assurance, we replace ${}_k p_x$ with ${}_k(ap)_x$, the probability that (x) is still alive at time k and has not yet been diagnosed with a critical illness, and we replace q_{x+k} with $(aq)_{x+k}^i$, the probability that a life aged $x+k$ is diagnosed with a critical illness in the coming year.

The probabilities used in this evaluation are examples of dependent decrements, which we will study in more detail in a later unit.

2.20 Immediate annuity

An **immediate** annuity is one under which the first payment is made within the first year. For the purposes of this section we will assume payments are made in arrear. Consider an annuity contract to pay 1 at the end of each future year, provided a life now aged x is then alive.

If the life dies between ages $x+k$ and $x+k+1$ ($k=0, \dots, \omega-x-1$) which is to say, $\mathbf{K}_x = k$, the present value at time 0 of the annuity payments which are made is $a_{\overline{k}|}$. (We define $a_{\overline{0}|} = 0$.) Therefore the present value at time 0 of the annuity payments is $a_{\overline{\mathbf{K}_x}|}$.

Since we know the distribution of \mathbf{K}_x , we can compute moments of $a_{\overline{\mathbf{K}_x}|}$.

2.21 The expectation of $a_{\overline{\mathbf{K}_x}|}$ defines the actuarial value a_x . So

$$a_x = E[a_{\overline{\mathbf{K}_x}|}] = \sum_{k=0}^{\infty} a_{\overline{k}|} {}_k p_x q_{x+k}.$$

We can write this in a form which is easier to calculate

$$\begin{aligned}
 a_x &= \sum_{k=0}^{\infty} a_{\overline{k}|} q_x = \sum_{k=0}^{\infty} \left(\sum_{j=1}^k v^j \right) q_x \\
 &= \sum_{j=0}^{\infty} \left(\sum_{k=j+1}^{\infty} q_x \right) v^{j+1} \\
 &= \sum_{j=0}^{\infty} {}_{j+1}p_x v^{j+1} \\
 &= \sum_{j=1}^{\infty} {}_j p_x v^j
 \end{aligned}$$

We will defer discussion of $\text{Var}[a_{\overline{\mathbf{K}_x}|}]$ until later.

2.22 Annuity-due

An **annuity-due** is one under which payments are made in advance. Consider an annuity contract to pay 1 at the start of each future year, provided a life now aged x is then alive. By similar reasoning to that above, we see that the present value of these payments is

$\ddot{a}_{\overline{\mathbf{K}_x+1}|}$. In actuarial notation we denote $E[\ddot{a}_{\overline{\mathbf{K}_x+1}|}]$ by \ddot{a}_x .

2.23 We can again write down \ddot{a}_x in a form which is simple to calculate.

$$\begin{aligned}
 \ddot{a}_x = E[\ddot{a}_{\overline{\mathbf{K}_x+1}|}] &= \sum_{k=0}^{\infty} \left(\sum_{j=0}^k v^j \right) {}_k p_x \\
 &= \sum_{j=0}^{\infty} \left(\sum_{k=j}^{\infty} {}_k p_x \right) v^j \\
 &= \sum_{j=0}^{\infty} {}_j p_x v^j
 \end{aligned}$$

2.24 Temporary annuity

A **temporary annuity** differs from a whole life annuity in that the payments are limited to a specified term.

Consider a temporary annuity contract to pay 1 at the end of each of the next n years, provided a life now aged x is then alive.

2.25 The present value of this benefit is $a_{\overline{\text{MIN}[\mathbf{K}_x, n]}}$. In actuarial notation, $E[a_{\overline{\text{MIN}[\mathbf{K}_x, n]}}]$ is denoted $a_{x:\overline{n}|}$. To calculate $a_{x:\overline{n}|}$, we use the following:

$$\begin{aligned} a_{x:\overline{n}|} &= E[a_{\overline{\text{MIN}[\mathbf{K}_x, n]}}] = \sum_{k=1}^{n-1} a_{\overline{k}|} q_x + a_{\overline{n}|} {}_n p_x \\ &= \sum_{k=1}^{n-1} \left(\sum_{j=0}^{k-1} v^{j+1} \right) q_x + \left(\sum_{j=0}^{n-1} v^{j+1} \right) {}_n p_x \end{aligned}$$

Note that ${}_n p_x = {}_n | q_x + {}_{n+1} | q_x + \dots$, so

$$a_{x:\overline{n}|} = \sum_{j=0}^{n-1} {}_{j+1} p_x v^{j+1} = \sum_{j=1}^n {}_j p_x v^j$$

2.26 Temporary annuity-due

A **temporary annuity-due** has payments that are made in advance and are limited to a specified term.

Consider a temporary annuity-due contract to pay 1 at the start of each of the next n years, provided a life now aged x is then alive.

2.27 The present value of the benefit is $\ddot{a}_{\overline{\text{MIN}[\mathbf{K}_x+1, n]}}$. In actuarial notation, $E[\ddot{a}_{\overline{\text{MIN}[\mathbf{K}_x+1, n]}}]$ is denoted $\ddot{a}_{x:n}$. Then

$$\begin{aligned}\ddot{a}_{x:n} &= E[\ddot{a}_{\overline{\text{MIN}[\mathbf{K}_x+1, n]}}] = \sum_{k=0}^{n-1} \ddot{a}_{\overline{k+1}|} {}_k|q_x + \ddot{a}_{\overline{n}|} {}_n p_x \\ &= \sum_{k=0}^{n-1} \left(\sum_{j=0}^k v^j \right) {}_k|q_x + \left(\sum_{j=0}^{n-1} v^j \right) {}_n p_x \\ &= \sum_{j=0}^{n-1} \left(\sum_{k=j}^{n-1} {}_k|q_x + {}_n p_x \right) v^j \\ &= \sum_{j=0}^{n-1} {}_j p_x v^j\end{aligned}$$

2.28 Comments

We have introduced the basic functions of life insurance mathematics — EPVs of simple assurance and annuity contracts. The next step is to explore useful relationships among these EPVs, and then we can apply the same ideas to other types of life insurance contracts.

2.29 A selection of the above functions are tabulated in, for example, *Formulae and Tables for Examinations*. Unit 2 will discuss ways of calculating a greater range of functions.

Example 2

- Find (i) \ddot{a}_{30} (AM92 at 4%)
(ii) \ddot{a}_{75} (PMA92C20 at 4%)

Solution

- (i) 21.834
(ii) 9.456

2.30

The formulae we have derived for EPVs can be interpreted in a simple way which is often useful in practice. Consider, for example

$$A_x = \sum_{k=0}^{\infty} v^{k+1} {}_k|q_x \quad \text{or} \quad \ddot{a}_x = \sum_{k=0}^{\infty} v^k {}_kp_x$$

Each term of these sums can be interpreted as

- An amount payable at time k
- × the probability that a payment will be made at time k
- × a discount factor for k years.

The first term in each case is just 1, but it should be easy to see that this interpretation can be applied to **any** benefit, level or not, payable on death or survival. This makes it easy to write down formulae for EPVs. For example, consider an annuity-due, under which an amount k will be payable at the start of the k^{th} year provided a life aged x is then alive (an increasing annuity-due). With this interpretation of EPVs, we can write down the EPV of this benefit (which is denoted $(I\ddot{a})_x$).

$$(I\ddot{a})_x = \sum_{k=0}^{\infty} (k+1) v^k {}_kp_x.$$

Increasing benefits will be covered in Unit 4.

3 Relationships among EPVs of simple life insurance benefits

3.1

The following relationships are easy to prove.

$$\ddot{a}_x = 1 + a_x$$

$$\ddot{a}_{x:\overline{n}|} = 1 + a_{x:\overline{n-1}|}$$

$$a_x = vp_x \ddot{a}_{x+1}$$

$$a_{x:\overline{n}|} = vp_x \ddot{a}_{x+1:\overline{n}|}$$

Example 3

Find a_{65} (PFA92C20 at 4%)

Solution

$$a_{65} = \ddot{a}_{65} - 1 = 13.871$$

3.2

There is a simple and very useful relationship between the EPVs of certain assurance contracts and the EPVs of annuities-due.

$$\begin{aligned}\ddot{a}_x &= E[\ddot{a}_{\overline{K_x+1}|}] = E\left[\frac{1-v^{K_x+1}}{d}\right] \\ &= \frac{1-E[v^{K_x+1}]}{d} \\ &= \frac{1-A_x}{d}\end{aligned}$$

$$\text{Hence } A_x = 1-d\ddot{a}_x$$

3.3

Along similar lines, we find that

$$A_{x:\overline{n}|} = 1-d\ddot{a}_{x:\overline{n}|}$$

and, as we shall see, similar relationships hold for all of the **whole life** and **endowment** contracts which we consider.

4 Annuity variances**4.1**

These relationships provide the easiest approach to finding the variances of the present values of annuity benefits. We use an annuity-due as an example.

$$\begin{aligned}\text{Var}[\ddot{a}_{\overline{K_x+1}|}] &= \text{Var}\left[\frac{1-v^{K_x+1}}{d}\right] \\ &= \frac{1}{d^2} \text{Var}[v^{K_x+1}] \\ &= \frac{1}{d^2} ({}^2A_x - (A_x)^2)\end{aligned}$$

where the “2” superscript denotes an assurance function calculated at a rate of interest of $(1+i)^2 - 1$.

Similarly, for a temporary annuity-due

$$\text{Var}[\ddot{a}_{\overline{\text{MIN}[\mathbf{K}_x+1, n]}}] = \frac{1}{d^2} ({}^2A_{x:\overline{n}} - (A_{x:\overline{n}})^2)$$

4.2

For a whole life annuity payable annually in arrears, we obtain

$$\begin{aligned} \text{Var}[a_{\overline{\mathbf{K}_x}}] &= \text{Var}[\ddot{a}_{\overline{\mathbf{K}_x+1}} - 1] = \text{Var}[\ddot{a}_{\overline{\mathbf{K}_x+1}}] \\ &= \frac{1}{d^2} ({}^2A_x - (A_x)^2) \end{aligned}$$

and for a temporary annuity payable annually in arrears we have

$$\begin{aligned} \text{Var}[a_{\overline{\text{MIN}[\mathbf{K}_x, n]}}] &= \text{Var}[\ddot{a}_{\overline{\text{MIN}[\mathbf{K}_x+1, n+1]}} - 1] \\ &= \text{Var}[\ddot{a}_{\overline{\text{MIN}[\mathbf{K}_x+1, n+1]}}] \\ &= \frac{1}{d^2} ({}^2A_{x:n+1} - (A_{x:n+1})^2) \end{aligned}$$

It is an excellent test of understanding to see why this last result is correct and

$$\text{Var}[a_{\overline{\text{MIN}[\mathbf{K}_x, n]}}] = \text{Var}[vp_x \ddot{a}_{\overline{\text{MIN}[\mathbf{K}_{x+1}+1, n]}}] = \frac{v^2 p_x^2}{d^2} ({}^2A_{x+1:n} - (A_{x+1:n})^2) \text{ is}$$

not correct, although

$$E[a_{\overline{\text{MIN}[\mathbf{K}_x, n]}}] \mu a_{x:\overline{n}} = vp_x \ddot{a}_{x+1:\overline{n}} = E[vp_x \ddot{a}_{\overline{\text{MIN}[\mathbf{K}_{x+1}+1, n]}}]$$

is correct.

5 Deferred annuities and assurances

5.1 Deferred annuities are annuities under which payment does not begin immediately but is deferred for one or more years.

5.2 Consider, for example, an annuity of 1 per annum payable annually in arrears to a life now aged x , deferred for n years. Payment will be at ages $x + n + 1, x + n + 2, \dots$ provided that the life survives to these ages, instead of at ages $x + 1, x + 2, \dots$. To write down the EPV of this annuity, let \mathbf{X} represent the (random) present value of the annuity. Then, by considering the distribution of \mathbf{X} , we have that

$$\begin{aligned}
 E[\mathbf{X}] &= \sum_{k=0}^n 0 \times P[\mathbf{K}_x = k] + \sum_{k=n+1}^{\infty} {}_n|a_{\overline{k-n}|} P[\mathbf{K}_x = k] \\
 &= \sum_{k=0}^n a_{\overline{k}|} P[\mathbf{K}_x = k] + a_{\overline{n}|} P[\mathbf{K}_x > n] - \sum_{k=0}^n a_{\overline{k}|} P[\mathbf{K}_x = k] - a_{\overline{n}|} P[\mathbf{K}_x > n] \\
 &\quad + \sum_{k=n+1}^{\infty} {}_n|a_{\overline{k-n}|} P[\mathbf{K}_x = k] \\
 &= \sum_{k=0}^{\infty} a_{\overline{k}|} P[\mathbf{K}_x = k] - \sum_{k=0}^n a_{\overline{k}|} P[\mathbf{K}_x = k] - a_{\overline{n}|} P[\mathbf{K}_x > n] \\
 &= a_x - a_{x:\overline{n}|}
 \end{aligned}$$

In actuarial notation, the EPV of this deferred annuity is denoted ${}_n|a_x$, so

$${}_n|a_x = a_x - a_{x:\overline{n}|}$$

Similarly, expressions can be derived for ${}_m|a_{x:\overline{n}|}$, the present value of an n -year temporary annuity deferred for m years (assuming survival to that point).

5.3 An alternative way to evaluate ${}_n|a_x$ follows from

$$\begin{aligned} {}_n|a_x &= a_x - a_{x:\overline{n}|} = \sum_{k=1}^{\infty} v^k {}_k p_x - \sum_{k=1}^n v^k {}_k p_x \\ &= \sum_{k=n+1}^{\infty} v^k {}_k p_x \\ &= v^n {}_n p_x \sum_{k=1}^{\infty} v^k {}_k p_{x+n} \\ &= v^n {}_n p_x a_{x+n} \end{aligned}$$

5.4 The factor $v^n {}_n p_x$ is important and useful in developing EPVs. It plays the role of the pure interest discount factor v^n , where now the payment or present value being discounted depends on the survival of a life aged x .

5.5 Deferred annuities-due can be defined similarly, with the corresponding formulae such as

$${}_n|\ddot{a}_x = \ddot{a}_x - \ddot{a}_{x:\overline{n}|} = v^n {}_n p_x \ddot{a}_{x+n}.$$

Note that $a_x = {}_1|\ddot{a}_x$.

5.6 Although not as common as deferred annuities, deferred assurance benefits can be defined in a similar way. For example, a whole life assurance with sum assured 1, payable to a life aged x but deferred n years is a contract to pay a death benefit of 1 provided death occurs after age $x + n$.

If the benefit is payable at the end of the year of death (if at all), the EPV of this assurance is denoted ${}_n|A_x$. It is easily shown that

$$\begin{aligned} {}_n|A_x &= A_x - A_{x:\overline{n}|}^1 \\ &= v^n {}_n p_x A_{x+n}. \end{aligned}$$

Note the appearance once more of the “discount factor” $v^n {}_n p_x$.

5.7 One useful feature of deferred assurances is that it is easier to find their variances directly than is the case for annuities. For example, let \mathbf{X} be the present value of a whole life assurance and \mathbf{Y} the present value of a temporary assurance with term n years, both for a

sum assured of 1 payable at the end of the year of death of a life aged x . Then $E[\mathbf{X}] = A_x$ and $E[\mathbf{Y}] = A_{x:\overline{n}|}^1$ and

$$E[\mathbf{X} - \mathbf{Y}] = {}_n|A_x = A_x - A_{x:\overline{n}|}^1.$$

Moreover,

$$\text{Var}[\mathbf{X} - \mathbf{Y}] = \text{Var}[\mathbf{X}] + \text{Var}[\mathbf{Y}] - 2 \text{Cov}[\mathbf{X}, \mathbf{Y}]$$

and it can be shown by considering the distributions of \mathbf{X} and \mathbf{Y} , that

$$\text{Cov}[\mathbf{X}, \mathbf{Y}] = {}^2A_{x:\overline{n}|}^1 - A_x A_{x:\overline{n}|}^1$$

where the “2” superscript has its usual meaning. Hence

$$\begin{aligned} \text{Var}[\mathbf{X} - \mathbf{Y}] &= {}^2A_x - (A_x)^2 + {}^2A_{x:\overline{n}|}^1 - (A_{x:\overline{n}|}^1)^2 - 2({}^2A_{x:\overline{n}|}^1 - A_x A_{x:\overline{n}|}^1) \\ &= {}^2A_x - (A_x - A_{x:\overline{n}|}^1)^2 - {}^2A_{x:\overline{n}|}^1 \\ &= {}^2A_x - ({}_n|A_x)^2 - {}^2A_{x:\overline{n}|}^1 \\ &= {}_n|{}^2A_x - ({}_n|A_x)^2 \end{aligned}$$

This can be used to find the variance of the corresponding deferred annuity-due.

6 Continuous annuities and assurances payable at the moment of death

6.1 So far we have assumed that assurance death benefits have been paid at the end of the year of death, and we have concentrated on annuities payable annually. In practice, assurance death benefits are paid a short time after death, as soon as the validity of the claim can be verified. Assuming a delay until the end of the year of death is therefore not a prudent approximation, but assuming that there is no delay and that the sum assured is paid immediately on death is a prudent approximation.

6.2 Related to such assurance benefits are annuities under which payment is made in a continuous stream instead of at discrete intervals. Of course this does not happen in practice, but such an assumption is reasonable if payments are very frequent, say weekly or daily. Later in this Unit we will consider annuities with a payment frequency between continuous and annual.

- 6.3** Consider, therefore, a whole life assurance with sum assured 1, payable immediately on the death of a life aged x . The present value of this benefit is v^{T_x} , and since the density function of T_x is ${}_t p_x \mu_{x+t}$, the EPV of the benefit, denoted \bar{A}_x , is

$$\bar{A}_x = E[v^{T_x}] = \int_0^\infty v^t {}_t p_x \mu_{x+t} dt$$

and its variance is easily seen to be

$$\text{Var}[v^{T_x}] = {}^2\bar{A}_x - (\bar{A}_x)^2$$

where the “2” superscript means an EPV calculated at rate of interest $(1+i)^2 - 1$.

- 6.4** The standard actuarial notation for the EPVs of assurances with a death benefit payable immediately on death, or of annuities payable continuously, is a bar added above the symbol for the EPV of an assurance with a death benefit payable at the end of the year of death, or an immediate annuity with annual payments, respectively.

- 6.5** Term assurance contracts with a death benefit payable immediately on death can be defined in a similar way, with the obvious notation for their EPVs and deferred assurance benefits likewise. We leave the reader to supply the obvious definitions and proofs of the following.

$$\bar{A}_x = \bar{A}_{x:\overline{n}|}^1 + {}_n\bar{A}_x$$

$$\bar{A}_{x:\overline{n}|} = \bar{A}_{x:\overline{n}|}^1 + A_{x:\overline{n}|}^1$$

$${}_n\bar{A}_x = v^n {}_n p_x \bar{A}_{x+n}$$

Note in the second of these that it is only **death** benefits which are affected by the changed time of payment. **Survival** benefits such as a pure endowment are not affected.

- 6.6** It is convenient to be able to estimate \bar{A}_x , $\bar{A}_{x:n}$ and so on in terms of commonly tabulated functions. One simple approximation is **claims acceleration**. Of deaths occurring between ages $x + k$ and $x + k + 1$, say, ($k = 0, 1, 2, \dots$) roughly speaking the average age at death will be $x + k + \frac{1}{2}$. Under this assumption claims are paid on average 6 months before the end of the year of death. Therefore we obtain the approximate EPVs

$$\bar{A}_x \cong (1+i)^{\frac{1}{2}} A_x$$

$$\bar{A}_{x:n}^1 \cong (1+i)^{\frac{1}{2}} A_{x:n}^1$$

$$\bar{A}_{x:n} \cong (1+i)^{\frac{1}{2}} A_{x:n}^1 + A_{x:n}^1$$

Note again that in the case of the endowment assurance only the death benefit is affected by the claims acceleration.

- 6.7** A second approximation is obtained by considering a whole life or term assurance to be a sum of deferred term assurances, each for a term of one year. Then, taking the whole life case as an example,

$$\begin{aligned} \bar{A}_x &= {}_0|\bar{A}_{x:1}^1 + {}_1|\bar{A}_{x:1}^1 + {}_2|\bar{A}_{x:1}^1 + \dots \\ &= \bar{A}_{x:1}^1 + {}_1p_x \bar{A}_{x+1:1}^1 + {}_2p_x \bar{A}_{x+2:1}^1 + \dots \end{aligned}$$

$$\text{Now } \bar{A}_{x+k:1}^1 = \int_0^1 v^t {}_t p_{x+k} \mu_{x+k+t} dt$$

and if we use assume that deaths are uniformly distributed between integer ages such that ${}_t p_{x+k} \mu_{x+k+t} = q_{x+k} \quad (0 \leq t < 1)$

then

$$\bar{A}_{x+k:1}^1 \cong q_{x+k} \int_0^1 v^t dt = q_{x+k} \frac{iv}{\delta}$$

Hence

$$\begin{aligned}\bar{A}_x &\cong \frac{i}{\delta} \left(vq_x + v^2 p_x q_{x+1} + v^3 {}_2p_x q_{x+2} + \dots \right) \\ &= \frac{i}{\delta} A_x\end{aligned}$$

Similarly,

$$\bar{A}_{x:\overline{n}|}^1 \cong \frac{i}{\delta} A_{x:\overline{n}|}^1.$$

6.8 Consider an immediate annuity of 1 per annum payable continuously during the lifetime of a life now aged x . The present value of this annuity is $\bar{a}_{\overline{T_x}|}$, and its EPV, denoted \bar{a}_x , is

$$\bar{a}_x = E[\bar{a}_{\overline{T_x}|}] = \int_0^\infty \bar{a}_{\overline{t}|} {}_t p_x \mu_{x+t} dt.$$

Note that $\bar{a}_{\overline{t}|} = \int_0^t e^{-\delta s} ds$ so that $\frac{d}{dt} \bar{a}_{\overline{t}|} = e^{-\delta t} = v^t$, and then integrate by parts.

$$\begin{aligned}\bar{a}_x &= - \left[\bar{a}_{\overline{t}|} {}_t p_x \right]_0^\infty + \int_0^\infty v^t {}_t p_x dt \\ &= \int_0^\infty v^t {}_t p_x dt.\end{aligned}$$

6.9 Temporary and deferred continuous annuities can be defined, and their EPVs calculated, in a similar way. Using the obvious notation, for example,

$$\begin{aligned}\bar{a}_{x:\overline{n}|} &= E \left[\bar{a}_{\overline{\min(T_x, n)}|} \right] = \int_0^n \bar{a}_{\overline{t}|} {}_t p_x \mu_{x+t} dt + \bar{a}_{\overline{n}|} {}_n p_x = \int_0^n v^t {}_t p_x dt. \\ \bar{a}_x &= \bar{a}_{x:\overline{n}|} + {}_n \bar{a}_x.\end{aligned}$$

6.10 To evaluate these annuities, use the approximation

$$\bar{a}_x \cong \ddot{a}_x - \frac{1}{2}$$

and for temporary annuities

$$\bar{a}_{x:\overline{n}|} \cong \ddot{a}_{x:\overline{n}|} - \frac{1}{2}(1 - v^n {}_n p_x).$$

- 6.11** There is an important relationship between level annuities payable continuously and assurance contracts with death benefits payable immediately on death. For whole life benefits

$$\bar{a}_x = E[\bar{a}_{\overline{T_x}|}] = E\left[\frac{1-v^{T_x}}{\delta}\right] = \frac{1}{\delta}(1-\bar{A}_x).$$

Hence $\bar{A}_x = 1 - \delta\bar{a}_x$.

For temporary benefits,

$$\bar{a}_{x:n|} = E\left[\bar{a}_{\overline{\min[T_x, n]}|}\right] = E\left[\frac{1-v^{\min[T_x, n]}}{\delta}\right] = \frac{1}{\delta}(1-\bar{A}_{x:n|}).$$

Hence $\bar{A}_{x:n|} = 1 - \delta\bar{a}_{x:n|}$.

- 6.12** We can use these relationships to express the variances of annuities payable continuously. For example,

$$\begin{aligned}\text{Var}[\bar{a}_{\overline{T_x}|}] &= \text{Var}\left[\frac{1-v^{T_x}}{\delta}\right] \\ &= \frac{1}{\delta^2} \text{Var}[v^{T_x}] \\ &= \frac{1}{\delta^2} \left({}^2\bar{A}_x - (\bar{A}_x)^2 \right)\end{aligned}$$

where the “2” superscript indicates an EPV calculated at rate of interest $(1+i)^2 - 1$.

7 Expected present values of annuities payable m times each year

We now consider the question of how annuities, with payments made more than once each year but less than continuously, may be evaluated.

We define the expected present value of an immediate annuity of 1 per annum, payable m times each year to a life aged x , as $a_x^{(m)}$.

This comprises payments, each of $\frac{1}{m}$, at ages

$$x + \frac{1}{m}, x + \frac{2}{m}, x + \frac{3}{m}, \text{ and so on.}$$

The expected present value may be written

$$a_x^{(m)} = \frac{1}{m} \sum_{t=1}^{\infty} \frac{v^{\frac{t}{m}} \ell_{x+\frac{t}{m}}}{\ell_x}.$$

If a mathematical formula for ℓ_x is known, this expression may be evaluated directly. More often, an approximation will be needed to evaluate the expression.

An expression for $\ddot{a}_x^{(m)}$ can be valued as a series of deferred annuities with annual payments of $\frac{1}{m}$ and deferred period of $\frac{t}{m}$; $t = 0, 1, 2, m-1$

$$\sum_{t=0}^{t=m-1} \frac{1}{m} {}_{\frac{t}{m}}\ddot{a}_x$$

Using the approximation that a sum of £1 payable a proportion k ($0 < k < 1$) through the year is equivalent to $\£(1-k)$ paid at the start of the year and $\£k$ at the end of the year, we can write

$${}_{\frac{t}{m}}\ddot{a}_x \cong \ddot{a}_x - \frac{t}{m}$$

and so the expected present value of the m thly annuity is approximately

$$\begin{aligned} & \sum_{t=0}^{t=m-1} \frac{1}{m} \left(\ddot{a}_x - \frac{t}{m} \right) \\ &= m \cdot \frac{1}{m} \ddot{a}_x - \frac{1}{m} \cdot \frac{1}{2} \frac{(m-1)m}{m} \end{aligned}$$

That is

$$\ddot{a}_x^{(m)} \cong \ddot{a}_x - \frac{(m-1)}{2m}$$

The corresponding expression for $a_x^{(m)}$ then follows from the relationship

$$\ddot{a}_x^{(m)} = \frac{1}{m} + a_x^{(m)}, \text{ that is,}$$

$$a_x^{(m)} \cong a_x + \frac{m-1}{2m}$$

(Note that, letting $m \rightarrow \infty$, we obtain the expression $\bar{a}_x \cong \ddot{a}_x - \frac{1}{2}$, as referred to in Section 6.10 above.)

These approximations may be used to develop expressions for temporary and deferred annuities.

8 Retrospective accumulations

8.1 In mathematics of finance there are two common viewpoints from which a stream of cashflows may be considered.

- (1) Prospectively, leading to the calculation of present values.
- (2) Retrospectively, leading to the calculation of accumulations.

In this section we discuss the latter approach allowing for the presence of mortality.

8.2 The basic idea is that we consider a **group** of lives, who are regarded as identical and stochastically independent as far as mortality is concerned. At age x , each life transacts an identical life insurance contract. Under these contracts, payments will be made (the direction of the payments is immaterial), depending on the experience of the members of the group. We imagine these payments being accumulated in a fund at rate of interest i . After n years, we divide this fund equally among the surviving members of the group. (If the fund is negative we imagine charging the survivors in equal shares). The question is, what is the **expected share of the fund per survivor**?

8.3 We illustrate with the simplest example of a pure endowment contract. Suppose we begin with N people in the group. After n years, a payment of 1 is made to each survivor. The number of survivors is, we suppose, a random variable. If there are N^1 survivors ($N^1 > 0$) then the fund is N^1 and the share of the fund per survivor is simply 1. We deal with the awkward possibility that $N^1 = 0$ by supposing N to be large enough that the probability that $N^1 = 0$ vanishes.

8.4 Let us now formalise the above. Suppose that there are \mathbf{N}^1 survivors at age $x + n$ out of N “starters” at age x , and that the accumulated fund at age $x + n$ is $\mathbf{F}(N)$. The retrospective accumulation of the benefit under consideration is defined to be

$$\lim_{N \rightarrow \infty} \frac{\mathbf{F}(N)}{\mathbf{N}^1}$$

Clearly \mathbf{N}^1 and $\mathbf{F}(N)$ are random variables. The process of taking the limit eliminates the probability that $\mathbf{N}^1 = 0$, but also since

$$\lim_{N \rightarrow \infty} \frac{\mathbf{F}(N)}{N} = E[\mathbf{F}(1)]$$

$$\lim_{N \rightarrow \infty} \frac{\mathbf{N}^1}{N} = {}_n p_x$$

(by the law of large numbers) the limit is equal to

$$\frac{E[\mathbf{F}(1)]}{{}_n p_x}.$$

8.5 For an example less trivial than the pure endowment, consider a term assurance with term n years. We need only consider a single life and calculate $E[\mathbf{F}(1)]$. It is easy to see that $\mathbf{F}(1)$ has the following distribution:

$$\mathbf{F}(1) = (1+i)^{n-(k+1)} \quad \text{if } \mathbf{K}_x = k \quad (k = 0, 1, \dots, n-1)$$

$$\mathbf{F}(1) = 0 \quad \text{if } \mathbf{K}_x \geq n$$

$$\text{so} \quad E[\mathbf{F}(1)] = \sum_{k=0}^{n-1} (1+i)^{n-(k+1)} {}_k|q_x$$

$$= (1+i)^n A_{x:n}^1.$$

Hence the accumulation of the term assurance benefit is

$$\frac{(1+i)^n A_{x:n}^1}{{}_n p_x}.$$

8.6 Consider an annuity-due with a term of n years. $\mathbf{F}(1)$ has the following distribution

$$\mathbf{F}(1) = (1+i)^{n-(k+1)} \ddot{s}_{\overline{k+1}|} \quad \text{if } \mathbf{K}_x = k \ (k = 0, 1, \dots, n-1)$$

$$\mathbf{F}(1) = \ddot{s}_{\overline{n}|} \quad \text{if } \mathbf{K}_x \geq n$$

Hence

$$\begin{aligned} E[\mathbf{F}(1)] &= \sum_{k=0}^{n-1} (1+i)^{n-(k+1)} \ddot{s}_{\overline{k+1}|} {}_k|q_x + \ddot{s}_{\overline{n}|} {}_n p_x \\ &= (1+i)^n \left(\sum_{k=0}^{n-1} \ddot{a}_{\overline{k+1}|} {}_k|q_x + \ddot{a}_{\overline{n}|} {}_n p_x \right) \\ &= (1+i)^n \ddot{a}_{\overline{x:n}|}. \end{aligned}$$

Therefore the accumulation of the annuity-due is

$$\frac{(1+i)^n \ddot{a}_{\overline{x:n}|}}{{}_n p_x}$$

8.7 To each annuity EPV there corresponds an accumulation denoted by an “ s ” symbol instead of an “ a ” symbol. Thus in the example above we **define**

$$\ddot{s}_{\overline{x:n}|} = \frac{(1+i)^n \ddot{a}_{\overline{x:n}|}}{{}_n p_x}$$

and we define symbols for the accumulation of other annuities similarly. There is no actuarial notation for the accumulation of assurance benefits.

END

UNIT 2 — THE EVALUATION OF ASSURANCES AND ANNUITIES

Syllabus objective

- (ii) Describe practical methods of evaluating expected values and variances of the simple contracts defined in objective (i).
1. Describe the life table functions l_x and d_x and their select equivalents $l_{[x]+r}$ and $d_{[x]+r}$.
 2. Express the following life table probabilities in terms of the functions in 1.: ${}_nP_x$, ${}_nq_x$, ${}_n|m q_x$ and their select equivalents ${}_nP_{[x]+r}$, ${}_nq_{[x]+r}$, ${}_n|m q_{[x]+r}$.
 3. Express the expected values and variances in objective (i) 3. in terms of the functions in 1. and 2.
 4. Evaluate the expected values and variances in objective (i) 3. by table look-up, where appropriate, including the use of the relationships in objectives (i) 7. and 8.
 5. Derive approximations for, and hence evaluate, the expected values and variances in objective (i) 4. in terms of those in objective (i) 3.
 6. Evaluate the expected accumulations in objective (i) 9.
 7. Describe practical alternatives to the life table which can be used to obtain the evaluations in 4., 5., and 6.

1 Introduction

1.1 In Unit 1, equations were established for the expected values and variances of benefits under simple contracts.

The evaluation of these expressions depends upon

- the assumed value of the constant deterministic interest rate; and
- the assumed probability distribution of the unknown future lifetime (or similar contingency)

The interest rate can be specified as a particular number. This Unit will describe practical ways of assigning numerical probabilities of death.

We will first look at probabilities continuing to use our ultimate mortality assumption from Unit 1. Then we will see how the same formulae of Unit 1 may be applied with the probabilities replaced with the equivalents which stem from the select mortality assumption mentioned in Section 2.3 of Unit 1.

- 1.2** The most practical means of evaluation is to assume that mortality follows a probability distribution defined by tracing the number of lives alive at each age in a population, and summarising the results in a “life table”.

2 The life table

We first introduce the life table making the ultimate mortality assumption.

- 2.1** In actuarial applications, we carry out many calculations using the probabilities ${}_t p_x$ and ${}_t q_x$, so it is useful to have some way of tabulating these functions, say for integer values of t and x . However, this would result in very large tables. The life table is a device for calculating all such probabilities from a smaller table, one whose entries depend on age only. The key to the definition of a life table is the relationship

$${}_{t+s} p_x = {}_t p_x {}_s p_{x+t} = {}_s p_x {}_t p_{x+s}.$$

- 2.2** Choose a starting age, which will be the lowest age in the table. We denote this lowest age α . This choice of α will often depend on the data which were available. For example, in studies of pensioners' mortality it is unusual to observe anyone younger than (say) 50, so 50 might be a suitable choice for α in a life table which is to represent pensioners' mortality.

- 2.3** Choose an arbitrary positive number and denote it l_α . We call l_α the **radix** of the life table. It is convenient to interpret l_α as being the number of lives starting out at age α in a homogeneous population, but the mathematics does not depend on this interpretation.

- 2.4** For $\alpha \leq x \leq \omega$, **define** the function l_x by

$$l_x = l_\alpha \times {}_{x-\alpha} p_\alpha$$

We assume that the probabilities ${}_{x-\alpha} p_\alpha$ are all known. Obviously, $l_\omega = 0$.

- 2.5** Now we see that, for $\alpha \leq x \leq \omega$ and for $t \geq 0$,

$${}_t p_x = \frac{{}_{t+x-\alpha} p_\alpha}{{}_{x-\alpha} p_\alpha} = \frac{l_{x+t}}{l_\alpha} \times \frac{l_\alpha}{l_x} = \frac{l_{x+t}}{l_x}$$

Hence, if we know the function l_x for $\alpha \leq x \leq \omega$, we can find any probability ${}_t p_x$ or ${}_t q_x$.

- 2.6** The function l_x is called the **life table**. It depends on age only, so it is more easily tabulated than the probabilities which we need in calculations, although the importance of this has diminished with the widespread use of computers.
- 2.7** If we interpret l_α to be the number of lives known to be alive at age α (in which case it has to be an integer) then we can interpret l_x ($x > \alpha$) as the **expected** number of those lives who survive to age x .
- 2.8** A life table is sometimes given a deterministic interpretation. That is, l_α is interpreted as above, and l_x ($x > \alpha$) is interpreted as the number of lives who will survive to age x , as if this were a fixed quantity. Then the symbol ${}_t p_x = l_{x+t}/l_x$ is taken to be the proportion of the l_x lives alive at age x who survive to age $x + t$. This is the so-called “deterministic model of mortality”. It is not such a fruitful approach as the stochastic model which we have outlined, and we will not use it. In particular, while it is useful in computing quantities like premium rates, it is of no use when we must analyse mortality data.
- 2.9** We now introduce a further life table function d_x . For $\alpha \leq x \leq \omega - 1$, define

$$d_x = l_x - l_{x+1}$$

We interpret d_x as the expected number of lives who die between age x and age $x + 1$, out of the l_α lives alive at age α . Note that

$$q_x = 1 - p_x = \frac{l_x}{l_x} - \frac{l_{x+1}}{l_x} = \frac{d_x}{l_x}.$$

It is trivial to see that

$$d_x + d_{x+1} + \dots + d_{x+n-1} = l_x - l_{x+n}$$

and that (if x and ω are integers)

$$d_x + d_{x+1} + \dots + d_{\omega-1} = l_x.$$

2.10 It is usual to tabulate values of l_x and d_x at integer ages, and often other functions such as μ_x , p_x or q_x as well. For an example, see the English Life Table No. 15 (Males) in the “Formulae and Tables for Examinations”. The following is an extract from that table.

x	l_x	d_x	q_x	μ_x
0	100 000	814	0.008 14	
1	99 186	62	0.000 62	0.000 80
2	99 124	38	0.000 38	0.000 43
3	99 086	30	0.000 30	0.000 33
4	99 056	24	0.000 24	0.000 27
5	99 032	22	0.000 22	0.000 23

(No value given for μ_0 because of the difficulty of calculating a reasonable estimate from observed data.) It is easy to check the relationships

$${}_t p_x = \frac{l_{x+t}}{l_x}$$

$$d_x = l_x - l_{x+1}$$

$$q_x = \frac{d_x}{l_x}.$$

2.11 Notice that $\mu_x > q_x$ at all ages in this part of the table. At some higher ages it is found that $\mu_x < q_x$. In fact, since

$$q_x = \int_0^1 {}_t p_x \mu_{x+t} dt$$

we see that if ${}_t p_x \mu_{x+t}$ is increasing for $0 \leq t \leq 1$,

$$q_x = \int_0^1 {}_t p_x \mu_{x+t} dt > {}_0 p_x \mu_{x+0} = \mu_x$$

while if ${}_t p_x \mu_{x+t}$ is decreasing for $0 \leq t \leq 1$

$$q_x = \int_0^1 {}_t p_x \mu_{x+t} dt < {}_0 p_x \mu_{x+0} = \mu_x$$

It is therefore of interest to note the behaviour of the function ${}_t p_x \mu_{x+t}$ for $0 \leq t < \omega - x$ (recall that this function is the density of \mathbf{T}_x). Figure 1 shows ${}_t p_0 \mu_t$ (i.e. the density of $\mathbf{T} = \mathbf{T}_0$) for the English Life Table No. 15 (Males). It has the following features, which are typical of life tables based on human mortality in modern times.

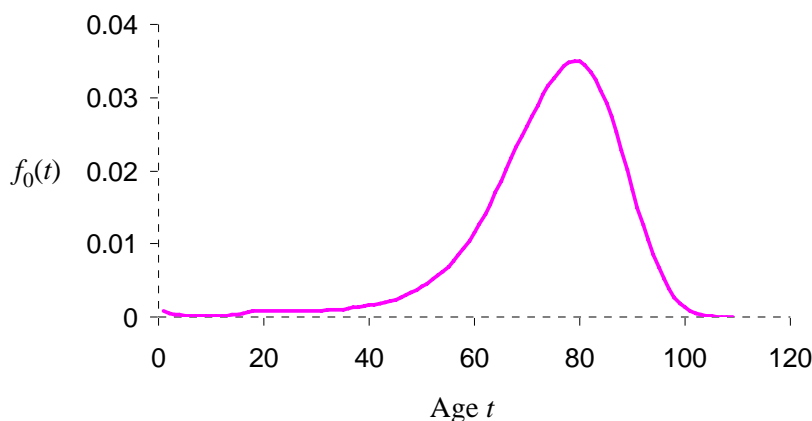


Figure 1

$$f_0(t) = {}_t p_0 \mu_t \text{ (ELT15 (Males) Mortality Table)}$$

- (1) Mortality just after birth (“infant mortality”) is very high.
- (2) Mortality falls during the first few years of life.
- (3) There is a distinct “hump” in the function at ages around 18–25. This is often attributed to a rise in accidental deaths during young adulthood, and is called the “accident hump”.
- (4) From middle age onwards there is a steep increase in mortality, reaching a peak at about age 80.
- (5) The probability of death at higher ages falls again (even though q_x continues to increase) since the probabilities of surviving to these ages are small.

2.12

We will now introduce some more actuarial notation for probabilities of death, and give formulae for them in terms of the life table l_x . Define

$${}_n|m q_x = P[n < \mathbf{T}_x \leq n + m]$$

In words, ${}_n|_m q_x$ is the probability that a life age x will survive for n years but die during the subsequent m years.

It is easy to see that

$${}_n|_m q_x = \frac{l_{x+n} - l_{x+n+m}}{l_x}$$

or alternatively that

$${}_n|_m q_x = {}_n p_x \times {}_m q_{x+n}$$

2.13 An important special case for actuarial calculations is $m = 1$, since we often use probabilities of death over one year of age. By convention, we drop the “ m ” and write

$${}_n|_1 q_x = {}_n| q_x$$

It is easy to see that

$${}_n| q_x = {}_n p_x \cdot q_{x+n} = \frac{l_{x+n} - l_{x+n+1}}{l_x} = \frac{d_{x+n}}{l_x}.$$

2.14 Recall the definition of the curtate future lifetime, \mathbf{K}_x . We now see that the probability function of \mathbf{K}_x can be written

$$P[\mathbf{K}_x = k] = {}_k| q_x.$$

Example 1

In a certain population, the force of mortality equals 0.025 at all ages.

Calculate:

- (i) the probability that a new-born baby will survive to age 5
- (ii) the probability that a life aged exactly 10 will die before age 12
- (iii) the probability that a life aged exactly 5 will die between ages 10 and 12
- (iv) the complete expectation of life of a new-born baby
- (v) the curtate expectation of life of a new-born baby

(Complete and curtate expectations of life, at age x , were defined in the Subject CT4, Models.)

Solution

$$(i) \quad {}_5p_0 = \exp\left(-\int_0^5 0.025 dt\right)$$

$$= e^{-0.125}$$

$$= 0.88250$$

$$(ii) \quad {}_2q_{10} = 1 - {}_2p_{10}$$

$$= 1 - \exp\left(-\int_0^2 0.025 dt\right)$$

$$= 1 - e^{-0.05}$$

$$= 0.04877$$

$$(iii) \quad {}_{5|2}q_5 = {}_5p_5 \cdot {}_2q_{10}$$

$$= e^{-0.125} \times 0.04877$$

$$= 0.88250 \times 0.04877$$

$$= 0.04304$$

$$(iv) \quad \ddot{e}_0 = \int_0^{\infty} {}_t p_0 dt$$

$$= \int_0^{\infty} e^{-0.025t} dt$$

$$= -\frac{1}{0.025} \left[e^{-0.025t} \right]_0^{\infty}$$

$$= \frac{1}{0.025}$$

$$= 40$$

$$\begin{aligned}
 \text{(v)} \quad e_0 &= \sum_{k=1}^{\infty} {}_k p_0 \\
 &= \sum_{k=1}^{\infty} e^{-0.025k} \\
 &= \frac{e^{-0.025}}{1 - e^{-0.025}} = 39.5
 \end{aligned}$$

3 Life table functions at non-integer ages

3.1 Life table functions such as l_x , p_x or μ_x are usually tabulated at integer ages only, but sometimes we need to compute probabilities involving non-integer ages or durations, such as ${}_{2.5}p_{37.5}$. We can do so using approximate methods. We will show two methods.

3.2 In both cases, we suppose that we split up the required probability so that we need only approximate over single years of age. For example, we would write ${}_3p_{55.5}$ as ${}_{0.5}p_{55.5} \times {}_2p_{56} \times {}_{0.5}p_{58}$. The middle factor can be found from the life table. To approximate the other two factors we need only consider single years of age.

3.3 The first method is based on the assumption that, for integer x and $0 \leq t \leq 1$, the function ${}_t p_x \mu_{x+t}$ is a constant. Since this is the density of the time to death from age x , it is seen that this assumption is equivalent to a uniform distribution of the time to death, conditional on death falling between these two ages. Hence it is called the **Uniform Distribution of Deaths** (or UDD) assumption.

3.4 Since ${}_s q_x = \int_0^s {}_t p_x \mu_{x+t} dt$, by putting $s = 1$ we must have ${}_t p_x \mu_{x+t} = q_x$ ($0 \leq t \leq 1$) and therefore

$${}_s q_x = \int_0^s q_x dt = s \cdot q_x.$$

This is sometimes taken as the definition of the UDD assumption. Since q_x can be found from the life table, we can use this to approximate any ${}_s q_x$ or ${}_s p_x$ ($0 \leq s \leq 1$).

3.5

Note that we must have an integer age x in the above formula, so (in our example) we can now estimate ${}_{0.5}p_{58}$ but not ${}_{0.5}p_{55.5}$. It is easy to show that for $0 \leq s < t \leq 1$,

$${}_{t-s}q_{x+s} = \frac{(t-s)q_x}{1-s \cdot q_x}$$

[Hint: ${}_t p_x = {}_s p_x \times {}_{t-s} p_{x+s}$.] This result, with $s = 0.5$, $t = 1$, can be used to estimate probabilities of the form ${}_{0.5}p_{55.5}$.

3.6

The second method of approximation is based on the assumption of a **constant force of mortality**. That is, for integer x and $0 \leq t < 1$, we suppose that

$$\mu_{x+t} = \mu = \text{constant.}$$

Then the formula

$${}_t p_x = \exp \left\{ - \int_0^t \mu_{x+s} ds \right\} = e^{-t\mu}$$

can be used to find the required probabilities. We first have to find μ . Note that $p_x = e^{-\mu}$ so $\mu = -\log p_x$, which we can find from the life table. Next, note that matters are rather simpler than under the UDD assumption since for $0 \leq s < t < 1$ we have

$${}_{t-s} p_{x+s} = \exp \left\{ - \int_s^t \mu_{x+r} dr \right\} = e^{-(t-s)\mu}$$

Hence we can easily calculate any required probability.

4 The general pattern of mortality

4.1 In this part we illustrate the general features of life table functions using ELT15 Males.

4.2 Figure 2 shows the ultimate rates of mortality q_x . The main feature is the rapid, in fact nearly exponential, increase in mortality beyond middle age.

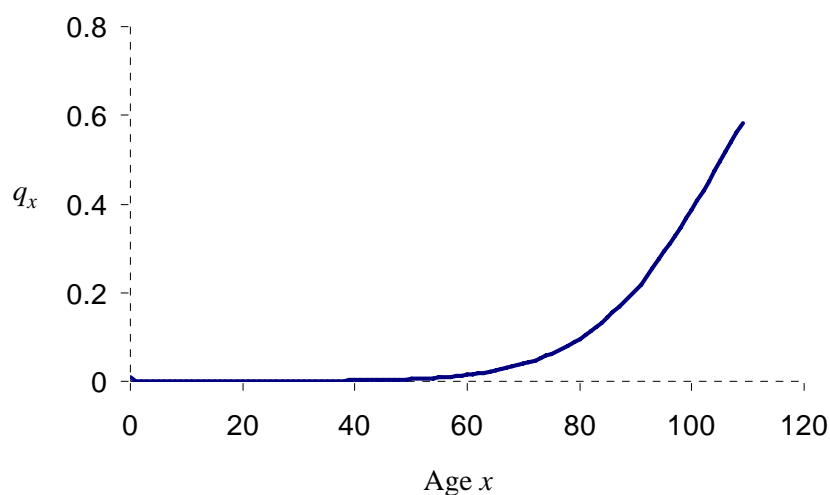


Figure 2

q_x (ELT15 (Males) Mortality Table)

The rates of mortality at older ages are so much larger than those at younger ages that Figure 2 fails to show any detail at younger ages.

The mortality table is tabulated to age 109. In practice, the value of q_x will rise towards 1 by around age 120.

4.3 Figure 3 shows q_x on a vertical base 10 logarithmic scale.

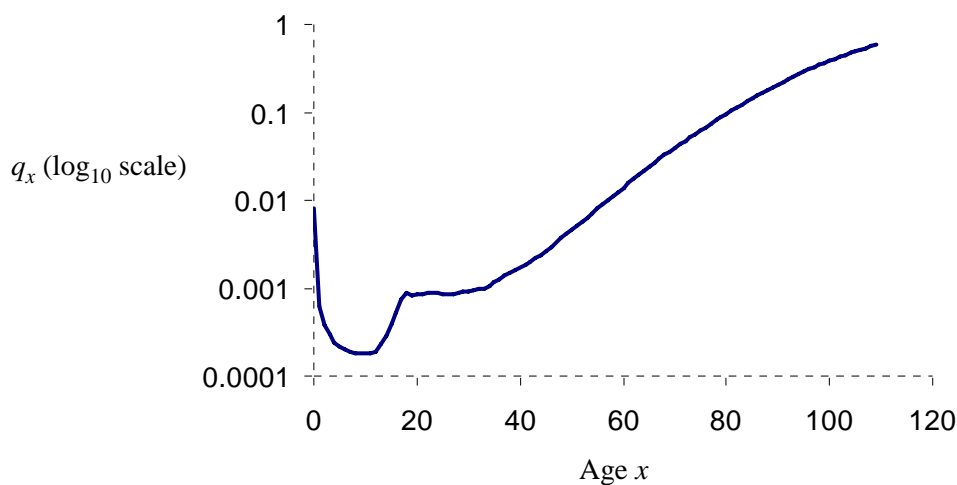


Figure 3

q_x on \log_{10} scale (ELT15 (Males) Mortality Table)

The main features are high infant mortality, an “accident hump” at ages around 20, and the nearly exponential increase at older ages.

In practice, the graph would effectively reach 1 by age 120.

4.4 Figure 4 shows l_x . The main feature is the very slight fall until late middle age, followed by a steep plunge.

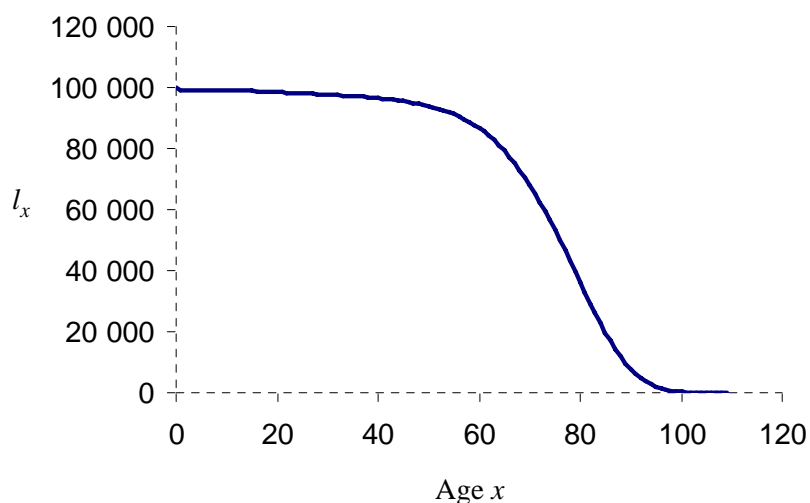


Figure 4

l_x (ELT15 (Males) Mortality Table)

4.5 Figure 5 shows d_x . Although the scale is different, this looks the same as Figure 1. The similarity is explained by the relationship

$$q_x = \frac{d_x}{l_x} = \int_0^1 {}_t p_x \mu_{x+t} dt$$

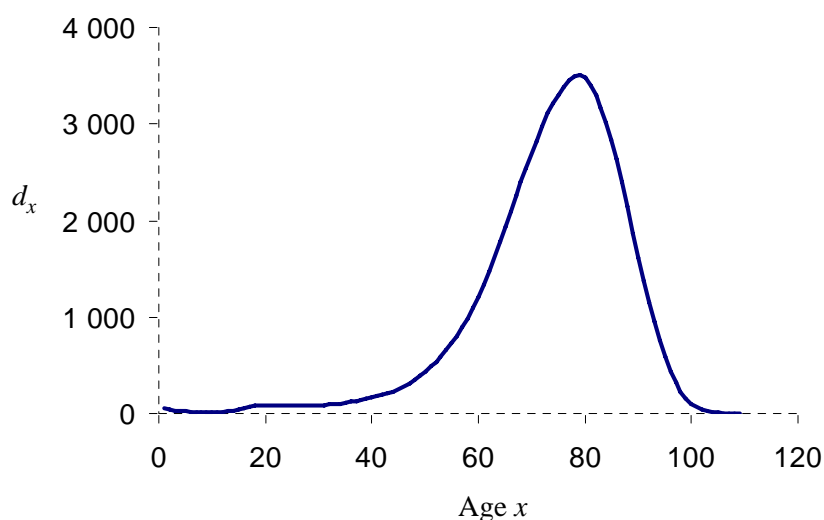


Figure 5

d_x (ELT15 (Males) Mortality Table)

5 Using the life table to evaluate means and variances

5.1 All of the formulae developed in Unit 1 (such as

$$A_x = \sum_{k=0}^{\infty} v^{k+1} {}_k|q_x$$

$$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^k {}_k p_x$$

can (obviously) be evaluated by direct calculation given only the life table probabilities, and with the widespread use of computers this is often the simplest method.

5.2 To save the time involved in the direct calculation using the underlying probabilities, the gold “Formulae and Tables for Examinations” book tabulates a selection of annuity and assurance functions on certain interest rates (including higher interest rates for variance calculations) and various mortality tables.

For functions dependent upon age as well as term, tabulations are restricted in order to save space. For example, in AM92, $\ddot{a}_{x:n|}$ is tabulated for $x + n = 60$ and for $x + n = 65$.

Other values can be calculated using formulae like

$$\ddot{a}_{x:n|} = \ddot{a}_x - v^n {}_n p_x \ddot{a}_{x+n}.$$

Assurance functions may be calculated directly, or by calculation of a corresponding annuity value and then use of a formula like those laid out in Sections 3.2 and 3.3 of Unit 1.

- 5.3** When dealing with annuities paid continuously or with death benefits payable immediately on death, it is straightforward to calculate functions as above and use the relevant approximations for the type of insurance concerned. For example, for whole life contracts, the formulae would be

$$\bar{a}_x \cong \ddot{a}_x - 1/2$$

$$\bar{A}_x \cong (1+i)^{1/2} A_x$$

- 5.4** Retrospective accumulations can be calculated along similar lines, so for example

$$\begin{aligned} \ddot{s}_{x:n|} &= \frac{(1+i)^n l_x (1 + v p_x + \dots + v^{n-1} {}_{n-1} p_x)}{l_{x+n}} \\ &= \frac{l_x (1+i)^n + l_{x+1} (1+i)^{n-1} + \dots + l_{x+n-1} (1+i)}{l_{x+n}}. \end{aligned}$$

6 Evaluating means and variances without use of the life table

- 6.1** The main alternative to using a life table is to postulate a formula and parameter values for the probability ${}_t p_x$ in the equations of Unit 1 and then evaluate the expressions directly.
- 6.2** Equivalently a formula for ${}_t q_x$ or μ_{x+t} could be postulated.
- 6.3** The difficulty of adopting this approach is that the postulation would need to be valid across the whole age range for which the formulae might be applied. As can be seen from the discussion in Section 4 above, the shape of human mortality may make a simple postulation difficult. Simple formulae may, however, be more appropriate and expedient for non-life contingencies.

7 Select mortality

7.1 Introduction

So far, we have made an assumption of ultimate mortality, that is, that mortality varies by age only. As discussed further in Unit 9 of this Subject, many other factors other than just age might affect observed mortality rates. In practice, therefore, the evaluation of assurance and annuity benefits is often modified to allow for other factors, than just age, which affect the survival probabilities.

Many factors can be allowed for by segregating the population, the assumption being that an age pattern of mortality can be discerned in the sub-population. For example, a population may well be segregated by sex and then sex-specific mortality rates (and mortality tables) can be used directly by using the techniques so far described.

Where, however, the pattern of mortality is assumed to depend not just on age, then slightly more complicated survival probabilities are employed. The most important, in the case of human mortality, is where the mortality rates depend upon duration as well as age, called **select rates**.

7.2 Mortality rates which depend on both age and duration

Select rates are usually studied by modelling the force of mortality μ as a function of the age at joining the population and the duration since joining the population. The usual notation is

$[x]+r$ age at date of transition

$[x]$ age at date of joining population

r duration from date of joining the population until date of transition and the transition intensity is written $\mu_{[x],r}$

$l_{[x]+r}$ number of lives alive at duration r having joined the population at age $[x]$ based on some assumed radix

In effect, a model showing how μ varies with r is constructed for each value of $[x]$. Instead of the single life age-specific life table described above we have a series of life tables, one for each value of $[x]$.

7.2.1 Displaying select rates

Once select rates have been estimated, it is conventional to display estimated rates for each age at entry into the population, $[x]$, by age attained at the date of transition i.e. $[x] + r$. This can be done in an array

$$\begin{array}{cccc}
 \dots\dots\dots & & & \\
 \mu_{[x]} & \mu_{[x-1]+1} & \mu_{[x-2]+2} & \mu_{[x-3]+3} \dots\dots\dots \\
 \mu_{[x+1]} & \mu_{[x]+1} & \mu_{[x-1]+2} & \mu_{[x-2]+3} \dots\dots\dots \\
 \mu_{[x+2]} & \mu_{[x+1]+1} & \mu_{[x]+2} & \mu_{[x-1]+3} \dots\dots\dots \\
 \dots\dots\dots & & &
 \end{array}$$

Each diagonal (\backslash) of the array represents a model of how rates vary with duration since joining the population for a particular age at the date of joining i.e. each is a set of life table mortality rates. The rates displayed on the rows of the array are rates for lives who have a common age attained at the time of transition, but different ages at the date of joining the population. If the rates did not depend on the duration since the date of joining the population, then apart from sampling error the rates on each row would be equal. Usually it is the case that rates depend on duration until duration s , and after s they are independent of duration. This phenomena is termed Temporary Initial Selection (see Unit 9) and s is called the length of the select period. In any investigation s is determined empirically by considering the statistical significance of the differences in transition rates along each row and the substantive impact of the different possible values of s .

When a value of s has been determined, then the estimates of the rates for duration $\geq s$ are pooled to obtain a common estimated value which is used in all the life tables in which it is needed. The array can be written

$$\begin{array}{cccc}
 \dots\dots\dots & & & \\
 \mu_{[x]} & \mu_{[x-1]+1} & \mu_{[x-2]+2} \dots\dots\dots & \mu_{[x-s+1]+s-1} \quad \mu_x \\
 \mu_{[x+1]} & \mu_{[x]+1} & \mu_{[x-1]+2} \dots\dots\dots & \mu_{[x-s+2]+s-1} \quad \mu_{x+1} \\
 \mu_{[x+2]} & \mu_{[x+1]+1} & \mu_{[x]+2} \dots\dots\dots & \mu_{[x-s+3]+s-1} \quad \mu_{x+2} \\
 \dots\dots\dots & & &
 \end{array}$$

and the pooled estimate at age attained $x + s$, for example, is given by

$$\mu_{x+s}^{\circ} = \frac{\theta_{[x],s} + \theta_{[x-1],s+1} + \theta_{[x-2],s+2} \dots\dots}{E_{[x],s}^c + E_{[x-1],s+1}^c + E_{[x-2],s+2}^c \dots\dots}$$

The right hand column of the above array represents a set of rates which are common to all the constituent life tables in the model of rates by age and duration. It is called an ultimate table.

The initial select probabilities can be displayed in a similar way.

7.2.2 Constructing select and ultimate life tables

The first step in constructing life tables is to refine the crude estimated rates (μ or q) into a smooth set of rates which statistically represent the true underlying mortality rates. This refinement process, called “graduation”, is dealt with in Subject CT4, Models and we now assume that we have a set of graduated mortality rates.

Using the graduated values of the initial probabilities displayed in the array

[illegible]

a table representing the select and ultimate experience can be constructed.

This is achieved by firstly constructing the ultimate life table based on the final column of the array and the formulae described in part 2 above:

- choose the starting age of the table, k
- choose an arbitrary radix for the table, ℓ_k
- recursively calculate the values of ℓ_x using

$$\ell_{r+1} = \ell_r (1 - q_r)$$

Beginning with the appropriate ultimate value in the final column the select life table functions for each row of the array are determined. This is achieved by “working backwards” up each diagonal using

$$\ell_{[x]+t} = \frac{\ell_{[x]+t+1}}{1 - q_{[x]+t}}$$

for $t = s - 1, \dots, 1, 0$

and noting in the first iteration that $\ell_{[x]+s-1+1} = \ell_{x+s}$

7.2.3 Using tabulated select life table functions

Some probabilities are particularly useful for life contingencies calculations. We have already defined

$${}_{n|m}q_x = \frac{\ell_{x+n} - \ell_{x+n+m}}{\ell_x}$$

$${}_nq_x = \frac{\ell_{x+n} - \ell_{x+n+1}}{\ell_x}$$

representing the m year and 1 year transition probabilities when the event of transition is deferred for n years.

Similar probabilities can be defined for each select mortality table

$${}_{n|m}q_{[x]+r} = \frac{\ell_{[x]+r+n} - \ell_{[x]+r+n+m}}{\ell_{[x]+r}}$$

$${}_nq_{[x]+r} = \frac{\ell_{[x]+r+n} - \ell_{[x]+r+n+1}}{\ell_{[x]+r}}$$

with the special case of $n = 0$ and $m = n$ being of particular interest

$${}_nq_{[x]+r} = \frac{\ell_{[x]+r} - \ell_{[x]+r+n}}{\ell_{[x]+r}}$$

and the complement of this n year transition probability, the n year survival probability is

$${}_nP_{[x]+r} = \frac{\ell_{[x]+r+n}}{\ell_{[x]+r}}.$$

The above probabilities may also be expressed in terms of the number of deaths in the select mortality table by defining

$$d_{[x]+r} = l_{[x]+r} - l_{[x]+r+1}.$$

7.2.4 Evaluating means and variances using select mortality

Corresponding to the assurances and annuities defined in Unit 1 are select equivalents defined as before but assumed to be issued to a select life denoted $[x]$ rather than x .

So, for example, $A_{[x]} = \sum_{k=0}^{k=\infty} v^{k+1} {}_kq_{[x]}$ can be used to calculate the EPV of benefits of a whole life assurance issued to a select life aged $[x]$ at entry.

Similarly, $\ddot{a}_{[x]} = \sum_{k=0}^{k=\infty} {}_kp_{[x]} \cdot v^k$ can be used to calculate the EPV of benefits of a whole life annuity-due, with level annual payments, issued to a select life aged $[x]$ at entry.

The variance formulae established in Unit 1 also apply replacing x with $[x]$, as do the relationships in Section 3.2 of Unit 1, namely that:

$$A_{[x]} = 1 - d\ddot{a}_{[x]}$$

and

$$A_{[x]:\overline{n}|} = 1 - d\ddot{a}_{[x]:\overline{n}|}$$

E N D

UNIT 3 — NET PREMIUMS AND RESERVES

Syllabus objective

- (iii) Describe and calculate, using ultimate or select mortality, net premiums and net premium reserves of simple insurance contracts.
1. Define the net random future loss under an insurance contract, and state the principle of equivalence.
 2. Define and calculate net premiums for the insurance contract benefits in objective (i) 1. Premiums and annuities may be payable annually, more frequently than annually, or continuously. Benefits may be payable at the end of the year of death, immediately on death, annually, more frequently than annually, or continuously.
 3. State why an insurance company will set up reserves.
 4. Describe prospective and retrospective reserves.
 5. Define and evaluate prospective and retrospective net premium reserves in respect of the contracts in objective (i) 1., with premiums as in (iii) 2.
 6. Show that prospective and retrospective reserves are equal when calculated on the same basis.
 7. Derive recursive relationships between net premium reserves at annual intervals, for contracts with death benefits paid at the end of the year of death, and annual premiums.
 8. Derive Thiele's differential equation, satisfied by net premium reserves for contracts with death benefits paid at the moment of death, and premiums payable continuously.
 9. Define and calculate, for a single policy or a portfolio of policies (as appropriate):
 - death strain at risk
 - expected death strain
 - actual death strain
 - mortality profit

1 Equations of value

1.1 Definition

In Subject CT1, Financial Mathematics payments were generally treated as *certain* to be paid. The equation of value where payments are certain has already been introduced there. In most actuarial contexts some or all of the cash flows in a contract are uncertain, depending on the death or survival (or possibly the state of health) of a life.

We therefore extend the concept of the equation of value to deal with this uncertainty, by equating *expected present values* of uncertain cashflows. The equation of expected present values for a contract, usually referred to as the equation of value, is

$\begin{aligned} &\text{The expected present value of the income} \\ &= \text{The expected present value of the outgo} \end{aligned}$

Alternatively, this is referred to as the **principle of equivalence**.

1.2 The income to a life insurer comes from the payments made by policyholders, called the **premiums**. The outgo arises from benefits paid to policyholders and expenses of the insurer. Given a suitable set of assumptions, which we call the **basis**, we may use the equation of value to calculate the premium or premiums which a policyholder must make in return for a given benefit. We may also calculate the amount of benefit payable for a given premium.

1.3 We may also use the equation of value when a policyholder wishes to adjust the terms of a contract after it is effected — for example by changing the contract term.

1.4 In the remainder of this Unit we will concentrate on net premiums and the associated net premium reserves (i.e. we will ignore actual premiums and the associated gross premium reserves). Although the relevance of the net premium valuation in the UK has diminished in recent years, primarily on account of regulatory changes, it continues to play a significant role in territories other than the UK and therefore remains an important part of this Subject. Gross premium valuations, on which much greater emphasis is now placed in the UK, are covered in Unit 5.

2 The basis

2.1 The basis for applying the equation of value for a life insurance contract will specify the mortality and interest rates to be assumed.

2.2 Usually the assumptions will *not* be the best estimates we can find of the individual basis elements, but will be more cautious than the best estimates. For example, if we expect to earn a rate of interest of 8% p.a. on the invested premiums, we may calculate the premiums assuming we earn only 6% p.a. As we are assuming that we earn less interest than we really expect, then the premiums calculated will be higher than they would need to be if the expected rate of 8% were actually earned. Some reasons for this element of caution in the basis are:

1. To allow a contingency margin, to ensure a high probability that the premiums plus interest income meet the cost of benefits allowing for random variation. In other words to ensure a high probability of making a profit.
2. To allow for uncertainty in the estimates themselves.

3 Premiums

3.1 The premium payment arrangement will commonly be one of the following:

- A **single premium** contract, under which benefits are paid for by a single lump sum premium paid at the time the contract is effected. This payment is certain, so that the left hand side of the equation of value is the certain payment, not the expected value of a payment.
- An **annual premium** contract, under which benefits are paid for by a regular annual payment of a level amount, the first premium being due at the time the contract is effected. Premiums continue to be paid in advance until the end of some agreed maximum premium term, often the same as the contract term, or until the life dies if this is sooner. Therefore, there would not usually be a premium paid at the end of the contract term.
- A **true m thly premium** contract, under which benefits are paid for by m level payments made every $1/m$ years. As in the annual premium case, premiums continue to be paid in advance until the end of some agreed maximum premium term or until the life dies if this is sooner. Again, there would not usually be a premium paid at the end of the contract term. Often the premium is paid monthly (that is, $m = 12$). For some types of contract, weekly premiums are possible.

3.2 **Premiums are always paid in advance**, so the first payment is always due at the time the policy is effected.

3.3 Given a basis specifying mortality and interest to be assumed, and given details of the benefits to be purchased, we can use the equation of value to calculate the premium payable.

4 The net premium

4.1 Definition

The **net premium** is the amount of premium required to meet the expected cost of the assurance or annuity benefits under a contract, given mortality and interest assumptions. The net premium is also sometimes referred to as the pure premium or the risk premium.

4.2 The net premium for a contract, given suitable mortality and interest assumptions, is found from the equation of expected present value:

$$\begin{aligned} &\text{The expected present value of the net premium income} \\ &= \text{The expected present value of the outgo on benefits} \end{aligned}$$

4.3 Notation

We have some standard notation for the net premiums of common life insurance contracts, related to the notation for the expected present value of the assurance benefits which the net premium is to pay for:

4.3.1 $P_{x:n|}$ is the net premium payable annually in advance throughout the duration of the contract for an endowment assurance issued to a life aged x with term n years, under which the sum assured is 1, payable at the end of the year of death or at maturity, where ultimate mortality is assumed.

From the net premium definition:

$$P_{x:n|} = \frac{A_{x:n|}}{\ddot{a}_{x:n|}}$$

If the death benefit is instead payable immediately on death then the net premium payable annually in advance is denoted by

$$P(\bar{A}_{x:n|})$$

and

$$P(\bar{A}_{x:n|}) = \frac{\bar{A}_{x:n|}}{\ddot{a}_{x:n|}}$$

If the premium is instead payable m -thly per annum in advance, this would be denoted by placing a superscript (m) above the P and a functions. So, assuming again that the death benefit is payable at the end of the year of death, for example, we would have

$$P_{x:n}^{(m)} = \frac{A_{x:n}}{\ddot{a}_{x:n}^{(m)}}$$

If the premium is payable continuously then, assuming again that the death benefit is payable at the end of the year of death, for example, we would have

$$\bar{P}_{x:n}$$

and

$$\bar{P}_{x:n} = \frac{A_{x:n}}{\bar{a}_{x:n}}$$

The final main variation would be to assume select, rather than ultimate, mortality. Then the above expressions apply but with x replaced by $[x]$.

$$P_{[x]:n}$$

is the net premium payable annually in advance throughout the duration of the contract for an endowment assurance issued to a life aged x with term n years, under which the sum assured is 1, payable at the end of the year of death or at maturity, where select mortality is assumed.

$$P_{[x]:n} = \frac{A_{[x]:n}}{\ddot{a}_{[x]:n}}$$

The above notation is flexible enough for the relevant symbols to be unambiguously combined in different ways when the benefits and premiums have alternative combinations of timings to those shown above. A full description of the approach to this notation can be found, for example, in the “International Notation Section” of the “Formulae and Tables for Examinations”.

Without loss of generality, we now show the corresponding notation for other simple insurance contracts assuming ultimate mortality, annual premiums in advance, and death benefits payable at the end of the year of death. We leave it to the reader to provide the alternative combinations in a similar manner to those outlined above.

- 4.3.2** $P_{x:n}^1$ is the net premium payable annually in advance throughout the duration of the contract for a term assurance issued to a life aged x with term n years, under which the sum assured is 1, payable at the end of the year of death, and where ultimate mortality is assumed.

From the net premium definition:

$$P_{x:n}^1 = \frac{A_{x:n}^1}{\ddot{a}_{x:n}}$$

- 4.3.3** P_x is the net premium payable annually in advance throughout the duration of the contract for a whole life assurance issued to a life aged x , under which the sum assured is 1, payable at the end of the year of death, and where ultimate mortality is assumed.

From the net premium definition:

$$P_x = \frac{A_x}{\ddot{a}_x}$$

- 4.3.4** ${}_tP_x$ is the net premium payable annually in advance for a maximum of t years (or until earlier death) for a whole life assurance issued to a life aged x , under which the sum assured is 1, payable at the end of the year of death, and where ultimate mortality is assumed.

From the net premium definition:

$${}_tP_x = \frac{A_x}{\ddot{a}_{x:t}}$$

- 4.3.5** There is no agreed notation for premiums payable on a critical illness assurance.