INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

April 2019 Examinations

Subject CM2A – Financial Mathematics and Loss Reserving

Introduction

The Examiners' Report is written by the Chief Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and using past papers as a revision aid and also those who have previously failed the subject.

The Examiners are charged by Council with examining the published syllabus. The Examiners have access to the Core Reading, which is designed to interpret the syllabus, and will generally base questions around it but are not required to examine the content of Core Reading specifically or exclusively.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report; other valid approaches are given appropriate credit. For essay-style questions, particularly the open-ended questions in the later subjects, the report may contain more points than the Examiners will expect from a solution that scores full marks.

The report is written based on the legislative and regulatory context pertaining to the date that the examination was set. Candidates should take into account the possibility that circumstances may have changed if using these reports for revision.

Mike Hammer
Chair of the Board of Examiners
July 2019

A. General comments on the *aims of this subject and how it is marked*

1. The aim of Financial Mathematics and Loss Reserving subject is to develop the necessary skills to construct asset liability models, value financial derivatives and calculate reserves for insurance or guarantees. These skills are also required to communicate with other financial professionals and to critically evaluate modern financial theories.

2. The marking approach is flexible in the sense that different answers to those shown in the solution can earn marks if they are relevant and appropriate. Marks for the methodology are also awarded including marks for using the right method even if an error in an earlier part of the question prevents the final answer from being correct. The marking focusses on rewarding students' understanding of the concepts, including their ability to articulate algebra and arguments clearly.

B. General comments on *student performance in this diet of the examination*

1. Students who scored strongly were those who were able to set out their thinking or algebra clearly and explain every step. A number of candidates knew roughly what was required in some questions, for example deriving formulae for the accumulation of premiums, but were not able to set out all the steps fully. There were a few questions where many students gave a 'standard' answer without thinking through the specifics of the question, for example considering the impact of an unexpected dividend on derivative prices.

2. Students performed relatively well on knowledge-based questions, although many missed the opportunity to be awarded full marks. The questions that required more thought tended to differentiate the better students.

C. Pass Mark

The Pass Mark for this exam was 60.

Solutions

Q1

(i) Weak Form EMH [½]
The market price of an investment incorporates all information contained in the price history of that investment. [½]
Knowledge of a stock's price history cannot produce excess performance as this information is already incorporated in the market price. [½]
This form, if true, means that technical analysis (or chartism) techniques (i.e. analysing charts of prices and spotting patterns) will not produce excess performance. [½]

Semi-Strong Form EMH [½]
The market price of an investment incorporates all publicly available information. [½]
Knowledge of any public information cannot produce excess performance, as this information is already incorporated in the market price. [½]
This form, if true, means that fundamental analysis techniques (i.e. analysing accounting statements and other pieces of financial information) will not produce excess performance. [½]

Strong Form EMH [½]
The market price of an investment incorporates all information, both publicly available and that available only to insiders. [½]
Knowledge available only to insiders cannot produce excess performance as this information is already incorporated in market prices. [½]
[Max 4]

(ii) (a) Some of the effects found by studies can be classified as overreaction to events, for example:

The market appears to overreact to past performance [½]
Past winners tend to be future losers and vice versa. [½]

Certain accounting ratios appear to have predictive powers [½]
e.g. companies with high earnings to price, cashflow to price and book value to market value (generally poor past performers) tend to have high future returns. [½]
Again, this is an example of the market apparently overreacting to past growth. [½]

(b) There are also well-documented examples of under-reaction to events:

Firms coming to the market [½]
Evidence from a number of major financial markets including the UK and the US appears to support the idea that stocks coming to the market by Initial Public Offerings and Seasoned Equity Offerings have poor subsequent long-term performance. [½]

Shiller's analysis

Shiller found strong evidence that the observed level of volatility in S&P 500 stock index contradicted the EMH as such volatility was not in line with the subsequent fluctuations in the dividends. [½]
Also, if markets are efficient, broad movements in the perfect foresight price should be correlated with moves in the actual price as both react to the same news. [½]

<u>Stock prices continuing to respond to earnings announcements up to a year after their announcement</u> [½]
This is an example of under-reaction to information which is slowly corrected. [½]

<u>Abnormal excess returns for both the parent and subsidiary firms following a de-merger</u> [½]
This is another example of the market being slow to recognise the benefits of an event. [½]

<u>Abnormal negative returns following mergers (agreed takeovers leading to the poorest subsequent returns)</u> [½]
The market appears to overestimate the benefits from mergers… [½]
…and the stock price slowly reacts as the optimistic view is proved to be wrong. [½]
[Max 6]

The majority of students scored full marks on part (i), which was a standard knowledge based question..

Part (ii) was also knowledge based and students who had learned it scored well, though some did not give enough detail for a six mark question.

Q2

(i) The first order stochastic dominance theorem states that:
Assuming an investor prefers more to less, [½]
A will dominate *B* (i.e. the investor will prefer portfolio A to portfolio B) [½]
if:
$F_A(x) \leq F_B(x)$, for all x, and [½]
$F_A(x) < F_B(x)$ *for some value of x.* [½]
where $F_Y(x)$ represents the cumulative probability distribution function of returns on portfolio Y. [½]
[Max 2]

(ii) The second order stochastic dominance theorem applies when the investor is risk averse… [½]
… as well as preferring more to less. [½]
In this case, the condition for *A* to dominate *B* is that $\int_a^x F_A(y)dy \leq \int_a^x F_B(y)dy$
for all values of *x*… [½]
… with the strict inequality holding for some value of *x*… [½]
… where *a* is the lowest return that the portfolios can possibly provide. [½]

[Max 2]

(iii)

(a) Portfolio 2 stochastically dominates portfolio 1 [½]
… at first order [½]
…because the CDF of portfolio 2 is greater than the CDF of portfolio 1 at all values of x. [1]
…or because we are using a normal distribution [½]
…and $\mu_1<\mu_2$ and $\sigma_1=\sigma_2$ [½]

(b) Portfolio 1 stochastically dominates portfolio 2 [½]
… at second order [½]
…because the integral of the CDF for portfolio will always be less than or equal to the CDF of portfolio 2. [1]
…but the CDF lines cross so neither portfolio is first order dominant. [½]
…or because we are using a normal distribution [½]
…and $\mu_1=\mu_2$ and $\sigma_1<\sigma_2$ [½]

(c) Neither portfolio 1 nor portfolio 2 dominates [½]
…because the CDFs will cross at some value of x [½]
…and so will the integral of the CDFs [½]

Or portfolio 1 might dominate portfolio 2 *[½]*
…if μ_1 is much greater than μ_2 *[½]*
…and σ_1 is only a little larger than σ_2 *[½]*

Many students scored well on parts (i) and (ii) though some lost marks for not stating the non-satiated / risk averse requirements.

In part (iii) many students answered correctly but were not always able to explain clearly why dominance did or didn't hold.

Q3

(i) $E[S_t] = S_0 e^{\mu t + \frac{1}{2} t\sigma^2}$ [1]
$Var[S_t] = {S_0}^2 e^{2\mu t + t\sigma^2}\left(e^{t\sigma^2} - 1\right)$ [1]

(ii) $Var[S_3] / E[S_3]^2 = \exp(\sigma^2(u-t)-1) = \exp(3\sigma^2 - 1)$ [1]
So $\sigma^2 = (\ln(Var[S_3] / E[S_3]^2 + 1) / 3 = (\ln(1{,}290^2 / 2{,}042^2 + 1)) / 3 = 0.112$ [1]
And $\mu = (\ln(E[S_3] / S_0) - ½ * \sigma^2 * 3) / 3 = (\ln(2{,}042 / 1{,}000) - ½ * 0.112 * 3) / 3 = 0.182$ [1]

(iii) We want $P(2{,}000 < S_5 < 2{,}500)$ [½]
$= P(2{,}000/S_0 < S_5/S_0 < 2{,}500/S_0)$ [½]
$= P(\ln(2{,}000/S_0) < \ln(S_5/S_0) < \ln(2{,}500/S_0))$ [½]
$= P((\ln(2{,}000/S_0) - 5\mu) / \sqrt{(5\sigma^2)} < Z < (\ln(2{,}500/S_0) - 5\mu) / \sqrt{(5\sigma^2)})$ [½]
$= 0.5033 – 0.3860$ [½]

= 0.1174 [½]

(iv)

(a) $\text{VaR}(X) = -t$ where $P(X < t) = p$ [1]

(b) $Expected\ shortfall = E[Max(L - X, 0)] = \int_{-\infty}^{L} (L - x)f(x)dx$ [1]

(v) Many distributions encountered in the real world are non-Normal [½]
…and it can be difficult to find a distribution function that will model the tails appropriately. [½]
Models built for core scenarios may not work well in the tails [½]
…or may be heavily dependent on the assumptions used. [½]
Often there is little data in the tails on which to base our modelling [½]
Events in the tails may be unlikely but extreme [½]
…and difficult to predict [½]
…for example a financial crisis. [½]
If we use Monte-Carlo simulation then we need a very large number of simulations to model tails sufficiently well. [½]
[Max 3]

Most students answered part (i) well here, though a common mistake was to miss out S_0 and/or t from the formulae.

Parts (ii) and (iii) led to lots of algebraic mistakes, but students using incorrect formulae from (i) were allowed full marks in (ii) and (iii) for following a correct method.

Part (v) required students to think about practical issues and only the stronger students picked up marks here.

Q4

(i) The assumptions underlying the Black-Scholes model are as follows:

- The price of the underlying share follows a geometric Brownian motion. [½]
- There are no risk-free arbitrage opportunities. [½]
- The risk-free rate of interest is constant, the same for all maturities and the same for borrowing or lending. [½]
- Unlimited short selling (that is, negative holdings) is allowed. [½]
- There are no taxes or transaction costs. [½]
- The underlying asset can be traded continuously and in infinitesimally small numbers of units. [½]

(ii) d1 = -0.0219 [1]
d2 = -0.1719 [1]
N(d1) = 0.4913 [½]
N(d2) = 0.4317 [½]

Option price = €2.03 [1]

(iii) By put-call parity (or using the Black-Scholes formula with the same d1 and d2 as above) [½]
Option price = €1.60 [½]

(iv) If the share was dividend-paying…
…the call option would be worth less… [½]
…because by holding the call option instead of the share you miss out on receiving dividends [1]
The put option would be worth more… [½]
…because by holding the share and a put option instead of cash you receive dividends [1]

[Max 2]

This question was answered well, with most students listing most (if not all) of the Black-Scholes assumptions and most students calculating the option prices correctly (barring a few algebraic slips).

In part (iv) most students stated the correct impact on the option prices, but some missed out on the marks for giving reasons.

Q5

(i) A portfolio is efficient if the investor cannot find a better one [½]
…in the sense that it has a higher expected return for the same variance or a lower variance for the same expected return. [½]

(ii) Investors are never satiated. At a given level of risk, they will always prefer a portfolio with a higher return to one with a lower return. [1]

Investors dislike risk. For a given level of return, they will always prefer a portfolio with lower variance to one with higher variance. [1]

(iii) The portfolio is inefficient [0.5]
So we can find another portfolio with the same expected return but lower risk [0.5]
Or the same risk but a higher expected return [0.5]

[Max 1]

(iv) Variance = $V = x_A^2V_A + x_B^2V_B + 2x_Ax_BC_{AB} = x_A^2V_A + (1-x_A)^2V_B + 2x_A(1-x_A)C_{AB}$
$= (V_A+V_B-2C_{AB})x_A^2 + (2C_{AB}-2V_B)x_A + V_B$ [1]
$dV/dx_A = 2(V_A+V_B-2C_{AB})x_A + 2C_{AB}-2V_B$ [1]
$dV/dx_A = 0$ iff $x_A = (V_B-C_{AB}) / (V_A+V_B-2C_{AB})$ [1]
$d^2V/dx_A^2 = 2(V_A+V_B-2C_{AB}) > 0$ because $C_{AB} <= (V_A+V_B)/2$ hence this is a minimum [1]

This question was answered well by most students. In part (iii) some students said that an inefficient portfolio is one which is not efficient, which scored no marks without further explanation.

In part (iv) many students missed the final mark for checking that the point of inflection is a minimum.

Q6

(i) Let K be the forward price. Now compare the setting up of the following portfolios at time 0:

A: one long forward contract.
B: borrow Ke^{-rT} cash and buy one share at S_0. [1]

If we hold both of these portfolios up to time T then both have a value of S_T - K at T. [1]

By the principle of no arbitrage these portfolios must have the same value at all times before T. [1]

In particular, at time 0, portfolio B has value $S_0 – Ke^{-rT}$ which must equal the value of the forward contract. This can only be zero (the value of the forward contract at t = 0) if $K = S_0e^{rT}$. [1]

(ii) $K = £12*e^{5*0.05} = £15.41$ [1]

(iii) Consider at time $t = 1$:
Portfolio A = the forward and $15.41e^{-4*0.05}$ cash
Portfolio B = one share [1]

These have equal value at $t = 5$, so must be equal at $t = 1$ by the principle of no arbitrage. [1]

So value of existing contract = $10 – 15.41e^{-4*0.05} = -£2.61$ [1]

(iv) The dividend will not affect the forward price directly… [½]
Because the forward price only depends on future expected dividends. [½]

The dividend will not directly affect the share price… [½]
Because it was not expected hence not priced in already [½]
The dividend might still affect the share price in the real world… [½]
Either negatively because cash has been transferred out of the business… [½]
Or positively because it suggests confidence in the business. [½]
[Max 2]

Most students were able to prove the forward price using replicating portfolios, and there were many alternatives to the model solution that were valid and scored marks.

Fewer students were able to calculate the value of the portfolio in part (iii).

Very few students scored marks in (iv), with most saying that the unexpected dividend would change the forward price.

Q7

(i) The factors and the effect they would have are:

- The price would decrease as the underlying share price increased.
- The price would increase as the strike price increased.
- The price would increase as the time to expiry increases or decrease as time passes. *[Either could be true depending on whether we think about time passing or a new option with a different term.]*
- The price would increase as the volatility of the underlying share increased.
- The price would decrease as interest rates increased.
- The price would increase as the dividend rate increased.

[½ per point]

(ii) dV = delta*dS + 0.5*gamma*(dS)² + theta*dt [1]
= 0.5*0.7 + 0.5*0.1*0.7^2 - 0.05*2 = 0.2745 [1]
So the new option price is approximately $15.87. [1]

(iii) Gamma describes how delta changes when the share price changes. [1]
A low value of gamma therefore implies that delta is not very sensitive to changes in the share price. [1]
So the portfolio is likely to require less rebalancing as the share price changes. [1]
This is desirable because rebalancing costs time and money and introduces risk. [1]
[Max 3]

Part (i) was answered well, with most students knowing how each factor would affect the option price.

Only the better students scored well in part (ii), with most applying delta correctly but fewer applying gamma and theta.

Part (iii) was generally answered well but not always with enough detail for three marks.

Q8

(i) (a) 1,000,000 x 1.01 = 1,010,000 [1]

(b) $200{,}000^2 = 4 \times 10^{10}$ [1]

(c) $0.5 \times 200{,}000^2 = 2 \times 10^{10}$ [1]

(d) Let the one-day portfolio return be denoted x :
$P(1{,}000{,}000\,(1 + x) < 1{,}000{,}000) = P(x < 0)$
$= P((x - 1\%)/20\% < -0.05) = P(Z < -0.05) = 0.48006$ [2]

(ii) $N{-}1(0.01) = -2.3263$ [1]
£1,000,000 × 1.01 + £1,000,000 × 20% × –2.3263 = £544,740. [1]
So the 99% Value at Risk is £1,010,000 – £544,740 = £465,260 [1]

(iii) P(not seeing a 99% VaR event in n days) = 0.99^n. [1]
So we want $0.99^n > 0.5$ [1]
So $n \log 0.99 > \log 0.5$, so $n \geq 69$ days [1]

> *Parts (i) and (ii) of this question were answered well.*
>
> *Part (iii) caused more difficulty, with very few students scoring marks and many trying to apply a normal distribution.*

Q9

(i) Assets = 750 + 3 x 50 = 900 [1]

(ii) Ruin will occur if two or more students qualify:
$P(X_1=2) = 3 \text{ x } 0.25^2 \text{ x } 0.75 = 0.1406$ [1]
$P(X_1=3) = 0.25^3 = 0.0156$ [1]
$P(X_1\geq2) = 0.1406 + 0.0156 = 0.1563$ [1]

(iii) (a) Assets = 900 + 3 x 50 = 1050 [1]
(b) Ruin will occur if all three students qualify:
$P(X_2=3) = 0.25^3 = 0.0156$ [1]

(iv) (a) Assets = 900 - 500 + 2 x 50 = 500 [1]
(b) Ruin will occur if both students qualify:
$P(X_2=2) = 0.25^2 = 0.0625$ [1]

(v) Combining the three scenarios:
$P(\text{Ruin}) = P(X_1\geq2) + P(X_1=0) \text{ x } P(X_2=3) + P(X_1=1) \text{ x } P(X_2=2)$ [1]
$= 0.1563 + 0.75^3 \text{ x } 0.0156 + 3 \text{ x } 0.25 \text{ x } 0.75^2 \text{ x } 0.0625$ [1]
$= 0.1892$ [1]

(vi) The expected payment per policy in the first exam session is 0.25 x £500 = £125 [½]
This is significantly more than the premium of £50 [½]
So the insurer should expect ruin at the first exam session if it issues more policies than its initial reserves can cover. [1]
The volatility of the insurer's portfolio will also reduce [½]
[Max 2]

This question was answered fairly well but with some slips in the algebra.

Some students tried to use a normal distribution which was not needed and scored no marks. Some also assumed that ruin occurs when the assets reach zero (rather than falling below zero) which is not the definition in the core reading but was allowable as an alternative approach.

A lot of students said in part (v) that writing more policies would help the insurer, which might often be true but is not the case here.

Q10

(i) $Sn = (1 + i_1)(1 + i_2) \ldots (1 + i_n)$ [½]

From this we obtain

$$S_n = \prod_{t=1}^{n} (1 + i_t)$$

[½]

and hence

$$E[S_n] = E\left[\prod_{t=1}^{n} (1 + i_t)\right]$$

$$= \prod_{t=1}^{n} E[(1 + i_t)]$$

[½]

since (by hypothesis) $i_1, i_2, \ldots, i_n$ are independent. [½]

For example, suppose that the yield each year has mean j and variance s^2. Then we have

$$E[S_n] = \prod_{t=1}^{n} E[1 + i_t]$$

$$= \prod_{t=1}^{n} (1 + E[i_t])$$

[½]

$$= (1 + j)^n$$

[½]

since, for each value of t, $E[i_t] = j$.

[Note – the exam paper says the investment occurs at time t=1 so an answer rolling up for n-1 years is also valid.]

(ii) And

$$(S_n)^2 = \prod_{t=1}^{n} (1 + i_t)^2$$

Hence

$$E[S_n^2] = E\left[\prod_{t=1}^{n} (1 + i_t)^2\right]$$

$$= \prod_{t=1}^{n} E[1 + 2i_t + i_t^2]$$ [½]

since (by hypothesis) $i_1, i_2, ..., i_n$ are independent. [½]

$$= \prod_{t=1}^{n} (1 + 2E[i_t] + E[i_t^2])$$

$$= (1 + 2j + j^2 + s^2)^n$$ [½]

since, for each value of t,

$$E[i_t^2] = (E[i_t])^2 + \text{var}[i_t] = j^2 + s^2$$ [½]

The variance of Sn is

$$\text{var}[S_n] = E[S_n^2] - (E[S_n])^2$$ [½]

$$= (1 + 2j + j^2 + s^2)^n - (1 + j)^{2n}$$ [½]

This question was answered fairly well, and a number of students actually answered it first in their answer book.

A common way to miss out on marks was not explaining some of the steps where we use independence of variables.

The exam question was based on an investment at time t=1 but the standard bookwork with an investment at t=0 was the most common answer and was awarded full marks.

END OF EXAMINERS' REPORT