

INSTITUTE AND FACULTY OF ACTUARIES

EXAMINATION

5 October 2020 (am)

Subject CM2A – Financial Mathematics and Loss Reserving Core Principles

Time allowed: Three hours and fifteen minutes

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator from the approved list.

If you encounter any issues during the examination please contact the Examination Team on T. 0044 (0) 1865 268 873.

- 1** (i) State how the following economic characteristics of investors can be expressed mathematically in a utility function $U(w)$ for wealth w , defining any further notation you use:
- (a) non-satiation.
 - (b) risk aversion.
 - (c) increasing absolute risk aversion.
 - (d) decreasing relative risk aversion.
- [3]

A new insurer uses a quadratic utility function, where $U(w) = w + dw^2$ for some $d < 0$.

- (ii) Derive the absolute risk aversion of the insurer's utility function. [2]
- (iii) Demonstrate whether the insurer's absolute risk aversion is decreasing, constant or increasing, relative to wealth. [1]

The insurer is about to write its first two contracts on a new policy. The contracts will be identical but independent, each with a 10% probability of a claim. Each claim costs £100 and there can be, at most, one claim per policy. The insurer's initial wealth is £250. The insurer uses the utility function with $d = -0.001$.

- (iv) Calculate the premium, p , that the insurer should charge per policy such that the insurer's expected utility of wealth after writing the policies is equal to the insurer's current utility of wealth. [5]
- (v) Comment on whether the policy might be attractive to customers at this premium. [3]

[Total 14]

- 2** The chief executive officer (CEO) of a company benefits from an executive reward plan that includes company shares currently worth €100,000. The shares currently trade at €1 each. The CEO wishes to retire in 4 years' time and hopes the share fund value at that time will be at least a target value of €150,000.

The share price S_t at time t (measured in years) follows the stochastic differential equation:

$$S_t = e^{(0.06875t + 0.25W_t)}$$

where W_t is a Standard Brownian Motion. The 'surplus amount' is defined as the difference between the share fund value in 4 years' time and the CEO's target value.

- (i) Calculate the standard deviation of the surplus amount. [4]

The CEO is considering buying put options on the shares to protect against the risk that the share price is lower than required at retirement.

- (ii) Suggest four issues that would need to be considered before deciding whether to proceed with this hedge. [2]
[Total 6]

- 3** Consider a European call option based on an underlying non-dividend paying share priced at \$34.55 with volatility of 10% per annum. The option has a strike price of \$40 and is 3 years from expiry. The risk-free force of interest is 2.5% per annum.

- (i) Calculate the value of the option using the Black–Scholes model. [4]
(ii) Explain why Theta is negative for this option. [2]
(iii) Explain why Delta for this option is positively correlated with the underlying share price. [2]
(iv) Explain, by comparing the cashflows of an option-holder and a shareholder, why the value of this option would be lower if the share was dividend-paying. [2]

Consider now a European put option on the same underlying share, with the same strike price and time to expiry.

- (v) Calculate the value of the put option. [2]
[Total 12]

- 4** A bank has received a loan application from a customer to borrow £10,000 for 3 years. The loan will be repaid by a single lump sum at the end of the 3 years with compounded interest.

The bank estimates that there is a 5% chance the customer will be declared insolvent in each future year. If the customer is declared insolvent before the loan is repaid, the bank will receive nothing.

- (i) Calculate the probability that the customer will still be solvent when the loan matures. [1]

The bank aims to achieve an annual expected return, after allowing for defaults, of 10%.

- (ii) Calculate the annual rate of interest the bank should charge on the loan, to achieve its desired expected return. [2]

The customer feels this rate of interest is too high and has proposed an alternative arrangement. They intend to use the loan to buy a car and they have offered the car as security such that the bank will receive the car and sell it if the customer defaults on the loan.

The bank estimates that it will be able to sell the car for:

- £8,000 at the end of year 1 if the customer defaults in year 1.
- £6,000 at the end of year 2 if the customer defaults in year 2.
- £4,000 at the end of year 3 if the customer defaults in year 3.

- (iii) Calculate the annual rate of interest the bank should charge on the loan to achieve its desired expected return, assuming it agrees to the customer's proposal. [3]

- (iv) State, with reasons, the type of credit risk model that the bank is using in this scenario. [1]

[Total 7]

- 5** The run-off triangle below shows cumulative claims incurred on a portfolio of general insurance policies.

<i>Accident year</i>	<i>Development year</i>		
	<i>0</i>	<i>1</i>	<i>2</i>
2017	2,440	3,294	3,788
2018	2,065	2,849	
2019	2,158		

Past and projected future inflation is given by the following index (measured at the midpoint of the relevant year).

<i>Year</i>	<i>Index</i>
2017	100
2018	105
2019	109
2020	116
2021	123

Calculate the outstanding claims using the inflation adjusted chain ladder method.

[9]

- 6** A bank sells 1,000 European put options p_t on a non-dividend paying share S_t . The initial share price is $S_0 = \$17$ and the strike price of the option is \$19 at expiry in 2 years' time. The share has volatility 15% p.a. and the risk-free force of interest is 3% p.a.

The option price is \$1.963 and the Delta of the option is -0.446 .

- (i) Calculate the initial income for the bank from selling the options. [1]
- (ii) Determine the portfolio of shares and cash the bank should hold to Delta hedge its initial position. [3]

One day later the share price has increased to \$19.

- (iii) Estimate the new option price, using the value of Delta provided. [1]
- (iv) Calculate the new option price, using the Black–Scholes formula. [4]
- (v) Comment on the reasons for differences between your answers to parts (iii) and (iv). [2]
- (vi) Explain why a low value of Vega for this derivative would be desirable to the bank. [2]

[Total 13]

7 Consider the price, D_t , at time t of a call option on an underlying stock S_t . The call option has a strike price of K and matures at time T . Let C_t be a cash account at time t ($t < T$) and r be the continuously compounded risk-free rate. T is measured in years.

(i) Define a self-financing strategy (a_t, b_t) . [1]

Consider the two portfolios below:

A: a_t units of S_t and b_t units of C_t

B: 1 unit of D_t

(ii) Construct two equations that must be satisfied so that Portfolio A is self-financing and replicates the price of Portfolio B. [2]

Consider now the scenario where:

$$S_0 = \text{£}20$$

$$T = 2$$

$$K = \text{£}20$$

$$r = 5\%$$

In the period from $t = 0$ to $t = 1$, the price can either increase by 50%, or decrease by 20%. In the period from $t = 1$ to $t = 2$, the price can either increase by 40%, or decrease by 30%.

(iii) Calculate the price of the call option at $t = 0$ using a 2-period binomial tree. [5]

(iv) Derive the portfolio of stocks and cash at $t = 0$ that replicates the option value at $t = 1$. [2]

(v) Show that the replicating portfolio from part (iv) matches the option price at $t = 1$, for the two possible share prices at $t = 1$. [2]

(vi) Comment on the limitations of using a binomial tree to set up and maintain a replicating portfolio for this option in the real world. [3]

[Total 15]

8 Consider an insurer with initial surplus U and surplus process $U(t)$ at time t .

- (i) (a) Define the surplus process for this insurer in terms of the initial surplus, premium income and the aggregate claims process, defining all notation used. [3]
- (b) State two limitations of this model.
- (ii) (a) Define the probabilities of ruin in both finite and infinite time. [3]
- (b) State the relationship between these probabilities.

An insurance company has written 500 1-year car insurance policies. Each policy charges a monthly premium of £10 over the life of the policy. You can assume for your calculations that the premium is paid continuously.

The probability of a claim on each policy in any given month is 1%, and claims are accounted for on the last day of each month. Claim amounts follow a uniform distribution between £0 and £1,000. Claims are independent and each policy can have at most one claim per month.

- (iii) Calculate the mean and standard deviation of the monthly aggregate claims. [4]

Denote by $\psi(U, t)$ the probability of ruin before time t (measured in months), given initial surplus U .

- (iv) Estimate $\psi(2,000, 1)$ by assuming that monthly claims are approximately Normally distributed. [3]
- (v) Comment on how the probability of ruin will develop for times $t > 1$. [2]
- [Total 15]

9 Consider an amount of \$1 invested at time $t = 0$ in an asset with an annual rate of return that is independently distributed each year with mean $j\%$ and standard deviation $s\%$. Let S_n be the accumulated value of this investment at time $t = n$.

(i) State the mean and variance of S_n in terms of j , s and n . [2]

An investor has \$5,000 at time $t = 0$. He is considering investing in an asset where $j = 4\%$ and $s = 20\%$.

(ii) Derive the mean and standard deviation at time $t = 10$ of the value of this investment. [2]

The investor wants to replace his kitchen at time $t = 10$ at a cost of \$8,200.

(iii) Determine the probability that the investor could invest in the asset at time $t = 0$ and have enough funds at time $t = 10$ for his new kitchen. You should assume the value of the assets at time $t = 10$ follows a Normal distribution. [2]

(iv) State the downside semi-variance of the value of the investor's assets at $t = 10$. [1]

The investor expects that from $t = 10$ onwards the price of the new kitchen will increase by 4% per year.

(v) Explain whether the investor is more likely or less likely to have sufficient assets to afford the new kitchen if he waits longer than 10 years. [2]
[Total 9]

END OF PAPER