INSTITUTE AND FACULTY OF ACTUARIES

EXAMINATION

26 September 2019 (pm)

Subject CM2A – Financial Mathematics and Loss Reserving Core Principles

Time allowed: Three hours and fifteen minutes

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*
2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*
3. *Mark allocations are shown in brackets.*
4. *Attempt all questions, begin your answer to each question on a new page.*
5. *Candidates should show calculations where this is appropriate.*

Graph paper is NOT required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator from the approved list.

1 Consider a random variable X with probability density function $f(x)$ and mean μ.

(i) Define algebraically the following risk measures for X:

(a) shortfall probability below a level L
(b) Value at Risk at the level p
(c) downside semi-variance.

[3]

An actuarial student is saving up to buy a car. He currently has £9,000 and he needs £10,000 to buy the car. There are three options for saving:

- Option 1: A bank account paying a guaranteed 10% per annum interest rate.

- Option 2: An equity fund whose returns are Gaussian with mean 15% p.a. and standard deviation 15% p.a.

- Option 3: A gamble that will return double the original investment with probability 65% or lose the entire investment otherwise.

Assume that the student can invest his entire savings in only a single investment option.

(ii) Calculate for each of these three investment options, for a one-year time horizon:

(a) The shortfall probability below £10,000
(b) The Value at Risk at the 75% level.

[5]

(iii) Recommend, with reasons, an appropriate investment strategy for the student, given the results in part (ii). [3]

[Total 11]

2 (i) State the four key features of a standard Brownian motion B_t. [4]

Consider a stochastic differential equation

$$dX_t = Y_t dB_t + A_t dt$$

where A_t is a deterministic process and Y_t is a stochastic process adapted to the natural filtration of B_t.

(ii) Write down Ito's lemma for $f(t,X_t)$, where f is a suitable function. [1]

(iii) Determine $df(t,X_t)$ where $f(t,X_t) = \exp(4\, t^2\, X_t)$. [2]
[Total 7]

3 Let the random variable $\log(1 + i_t)$ be normally distributed with mean μ and variance σ^2, where i_t is the rate of interest for year t.

(i) Explain the distribution and parameters for the variable $S_n = \prod_{t=1}^{t=n}\left(1 + i_t\right)$ for $n > 1$ including your reasoning. [3]

Consider the present value of a payment of 1 due at the end of n years denoted by: $V_n = \prod_{t=1}^{t=n}\left(1 + i_t\right)^{-1}$

(ii) Derive the distribution and parameters of V_n. [5]
[Total 8]

4 Consider a put option p_t on a dividend-paying share with strike price K and maturity at time T. The current time is t and the risk-free rate of interest is r per annum. The share has current price S_t, volatility σ and dividend yield δ.

(i) Write down the impact on the option price of each of the following changes:

(a) An increase in the share price S_t.
(b) An increase in the strike price K.
(c) A decrease in time to expiry $(T–t)$.
(d) A decrease in the dividend yield q.
(e) A reduction in the risk free rate r.
(f) An increase in the volatility σ.

[3]

(ii) Explain which of the changes in (i) move the price of a call option in the **same** direction as the price of a put option. [2]
[Total 5]

5 A non-dividend paying share is currently priced at \$80. Each year the share price will either increase by 10% or fall by 10% with equal probability. A call option is written on the share with a strike price of \$75 and expiry in two years. The risk-free force of interest is 5% per annum.

(i) Calculate the risk-neutral price of the option using a binomial tree. [4]

An individual has \$1,000 to invest. In making his investment choices he uses the utility function $U(w) = w^{0.5}$.

(ii) Calculate the investor's expected utility of wealth at time $t = 2$ if he invests the entire fund at $t = 0$ in:

(a) a bank account returning the risk-free rate. [1]

(b) the shares. [3]

(c) the call options. [3]

(iii) Explain why the relative appeal of the choices (a), (b) and (c) above is consistent with the form of the investor's utility function. [2]

[Total 13]

6 Let p_t denote the value at time t of a European put option on a non-dividend paying share S_t with maturity at time T and a strike price K. The risk-free rate of interest is r.

(i) Derive the lower bound for p_t in terms of S_t and K. [3]

(ii) Explain how the lower bound would change if p_t were an American put option. [2]

The put option p_t has the following characteristics:

- Strike price = £100
- Time to expiry 6 months.

The risk-free rate of interest is 4% per annum.

(iii) Calculate an upper bound for the value of the option p_t. [2]

(iv) Explain the conditions necessary for the option price to approach the upper bound in part (iii). [2]

[Total 9]

7 A market that satisfies the assumptions of the Capital Asset Pricing Model (CAPM) comprises n assets.

Let the random return on asset i be denoted by R_i, the expected return by r_i , and the corresponding returns for the market portfolio by R_M and r_M. Let π_i be the proportion of asset i in the market portfolio.

(i) (a) Define β_i algebraically in this market.

(b) Write down the relationship between these expected returns in this market, including a definition of any additional notation that you use.

(c) Show that $\Sigma\pi_i \beta_i = 1$.

[6]

Consider a market where $n = 4$, that is, there are four assets in the market with the following attributes:

Asset	*1*	*2*	*3*	*4*	*Risk-free Asset*
Expected return (per annum)	14%	r_2	r_3	r_4	3%
Market capitalisation	£4m	£2m	£2m	£2m	

The variance-covariance matrix (in %%) of annual returns on the four assets is as follows:

Asset	*1*	*2*	*3*	*4*
1	4	1	1	1
2	1	3	1	1
3	1	1	2	1
4	1	1	1	x^2

for some $x^2 > 0$. The variance of the return on the market portfolio is $\frac{8}{5}$ %%.

(ii) (a) Calculate the proportions of each asset in the market portfolio.

(b) Calculate β_1, β_2, β_3 and β_4.

(c) Calculate r_M, r_2, r_3 and r_4.

[7]

[Total 13]

8 (i) Describe how the Merton model can be used to value the debt issued by a company. [5]

A company has a value of €100m. Its total equity is currently worth €50m, and it has €80m nominal of debt maturing in ten years. The risk-free force of interest is 4% per annum.

(ii) Calculate the implied volatility of the value of the company to the nearest 1% using the Merton model. [4]

(iii) Calculate the value of vega for the value of the company's debt under the Merton model. [1]

(iv) Explain why a low value of vega might be desirable when using the Merton model. [2]

[Total 12]

9 (i) Write down the general form of a statistical model for a claims run-off triangle, defining all terms used. [5]

The table below shows the cumulative incurred claims on a portfolio of insurance policies.

	Development Year		
Accident Year	*1*	*2*	*3*
2016	130	180	190
2017	140	185	
2018	150		

The company decides to apply the Bornhuetter-Ferguson method to calculate the reserves, with the assumption that the Ultimate Loss Ratio is 80%. Claims are assumed to be fully run off by development year 3.

The earned premium for 2018 is 300 and the paid claims for 2018 are 100.

(ii) Calculate the reserve in respect of the accident year 2018. [6]

[Total 11]

10 An insurer writes policies that insure policyholders against losing their mobile phone. Each policy pays £500 at the end of the year if the policyholder loses their phone during the year.

The insurer has carried out research on mobile phone losses in the general population, which showed that there is a 10% chance of an individual losing their phone in a one-year period. To allow for expected claims and a profit margin the insurer sets the annual premium at £125. Premiums are paid at the start of the policy year.

The insurer writes ten policies and has no assets at the time the policies are written. The insurer assumes that the policies are independent of each other. Discounting and expenses can be ignored.

(i) Calculate the probability that the insurer is still solvent at the end of the year. [3]

After one year the insurer has received two claims.

(ii) State two factors relating to policyholder behaviour which may give rise to the insurer experiencing more claims than expected, explaining for each factor how it may arise in this case. [2]

(iii) Explain how insurers typically reduce the risk posed by the factors identified in part (ii). [2]

At the end of year one, after paying any claims but before receiving any more premiums, the insurer has assets of £250. All ten policyholders remain with the insurer for another year and all still have a 10% chance per year of losing their phone.

The insurer decides that premiums for year two will be £x per policy, but with a 25% discount for any policyholders who did not claim in year one.

(iv) Calculate the base premium, £x, that the insurer should charge to achieve a probability of ruin in year two of no more than 5%. [4]

[Total 11]

END OF PAPER