INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORTS

September 2019

Subject CM2A – Financial Mathematics and Loss Reserving

Introduction

The Examiners' Report is written by the Chief Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and using past papers as a revision aid and also those who have previously failed the subject.

The Examiners are charged by Council with examining the published syllabus. The Examiners have access to the Core Reading, which is designed to interpret the syllabus, and will generally base questions around it but are not required to examine the content of Core Reading specifically or exclusively.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report; other valid approaches are given appropriate credit. For essay-style questions, particularly the open-ended questions in the later subjects, the report may contain more points than the Examiners will expect from a solution that scores full marks.

The report is written based on the legislative and regulatory context pertaining to the date that the examination was set. Candidates should take into account the possibility that circumstances may have changed if using these reports for revision.

Mike Hammer
Chair of the Board of Examiners
September 2019

A. General comments on the *aims of this subject and how it is marked*

1. The aim of Subject CM2 is to develop the necessary skills to construct asset liability models, value financial derivatives and calculate reserves for insurance or guarantees. These skills are also required to communicate with other financial professionals and to critically evaluate modern financial theories.

2. The marking approach for CM2 is flexible in the sense that different answers to those shown in the solution can earn marks if they are relevant and appropriate. Marks for the methodology are also awarded including marks for using the right method even if an error in an earlier part of the question prevents the final answer from being correct. The marking focusses on rewarding students' understanding of the concepts, including their ability to articulate algebra and arguments clearly.

B. Comments on *student performance in this diet of the examination.*

1. Students who scored strongly were those who were able to set out their thinking or algebra clearly and explain every step. A key weakness for some students was statistical knowledge – CM2 assumes knowledge from CS1 and CS2 so it is advisable to sit those exams before attempting CM2.

2. Students performed relatively well on knowledge-based questions, although many missed the opportunity to be awarded full marks. The questions that required more thought tended to differentiate the better students.

3. Some students missed out on marks by not identifying the level of detail required, especially in question 3 which was an 'explain' question and needed more than just algebra for full marks.

C. Pass Mark

The Pass Mark for this exam was 61.

Q1
(i)
(a)

Shortfall probability $= \int_{-\infty}^{L} f(x)dx$ [1]

(b)
VaR(X) = -t where P(X < t) = p
Or also accept P(X < t) = 1 - p
[1]

(c)
$\int_{-\infty}^{\mu} (\mu - x)^2 f(x)dx$
[1]
[Total 3]

(ii)
(a)
Bank account = £9,000 x 1.1 = £9,900 hence 100% shortfall prob. [½]
Shares = P(X<(10/9-1)) where X~N(0.15,0.15^2) = 40% [1]
Gamble = shortfall if not successful hence 35% shortfall prob. [½]

(b)
Bank account = guaranteed value so VaR = £9,900 [1]
Shares = P(X<t) = 0.25 where X~N(0.15,0.15^2) => t=0.0488 => VaR = 9,000*(1+0.0488) = £9,439 [1]
Gamble = 35% chance of ending with nothing hence VaR = £9,000 [1]
[Total 5]

(iii)
The two risk measures are not conclusive and each suggests that a different investment would be best. [1]
In reality the bank account delivers nearly enough money with no risk [1]
so the student might be best to either invest wholly in the bank account and wait a little longer to buy the car [½]
or, if allowed to split the investment, invest mostly in the bank account and a little in the shares or the gamble. [½]
The student could also seek out other investments with a different risk/return profile [½]
The shortfall probability as it assumes all the student cares about is reaching £10k over one year. [½]
The student might also consider the size of any shortfall or surplus. [½]
If the student only needs £10,000 then it makes no sense to invest more than £5,000 in the gamble. [½]
[Marks available 5, maximum 3]
[Total 11]

Most students scored full marks for the bookwork in part (i).

Many students also scored well in parts (ii) and (iii), but failed to score full marks for the VaR calculations. A few different answers were allowed here though since this risk measure has more than one possible interpretation.

Q2

(i)

Bt has independent increments, i.e. $B_t - B_s$ is independent of $\{B_r, r \leq s\}$ whenever $s < t$ [1]

Bt has Gaussian increments, i.e. the distribution of $B_t - B_s$ is $N(0, t - s)$ [1]

B_t has continuous sample paths $t \to B_t$ [1]

$B_0 = 0$ [1]

(ii)

$$df(t,x) = \frac{\partial f}{\partial x} Y_t dB_t + \left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} A_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} Y_t^{\,2}\right] dt$$ [1]

(iii)

Using Ito's lemma: $df(t,X_t) = 4t^2 \exp(4t^2 X_t) Y_t dB_t + [\, 8t X_t \exp(4t^2 X_t) + 4t^2 \exp(4t^2 X_t) A_t + \tfrac{1}{2} 16 t^4 \exp(4t^2 X_t) Y_t^2 \,] dt$ [2]

[Total 7]

Most students scored well here with a significant number earning full marks. The question required only bookwork and application to a simple problem.

Q3

(i)

Assuming that the variables are independent… [1]

The sum of a set of independent normal random variables is itself a normal random variable. [1]

Hence, when the random variables $(1 + i_t)$ $(t \geq 1)$ are independent and each has a log-normal distribution with parameters μ and σ^2, [1]

the random variable Sn has a log-normal distribution with parameters $n\mu$ and $n\sigma^2$. [1]

[Marks available 4, maximum 3]

(ii)

$\log V_n = -\log(1 + i_1) - \ldots - \log(1 + i_n)$ [1]

Since, for each value of t, $\log(1 + i_t)$ is normally distributed with mean μ and variance σ^2, each term on the right hand side of the above equation is normally distributed with mean $-\mu$ and variance σ^2. [1]

Also, the terms are independently distributed. [1]

So, $\log V_n$ is normally distributed with mean $-n\mu$ and variance $n\sigma^2$. [1]

That is, V_n has log-normal distribution with parameters $-n\mu$ and $n\sigma^2$. [1]

[Total 8]

The marks suggest that students found Question 3 the hardest on the paper. Many produced a good algebraic answer and scored partial marks, but the question also required a clear explanation of some of the key steps for full marks.

Q4
(i)
(a) Lower option value [½]
(b) Higher option value [½]
(c) Lower option value [½]
(d) Lower option value [½]
(e) Higher option value [½]
(f) Higher option value [½]

(ii)
A decrease in time to expiry will reduce the price of all options because the 'optionality' becomes less valuable as the final outcome becomes more certain. [1]
An increase in volatility will increase the price of all options because the 'optionality' becomes more valuable as the final outcome becomes less certain. [1]
[Total 5]

Most students scored close to full marks here.

Q5
(i)
$q = (e^r - d) / (u - d) = (e^{0.05} - 0.9) / (1.1 - 0.9) = 0.7564$ [1]

Stock price tree: [1]

		96.8
	88	
80		79.2
	72	
		64.8

Payoff tree: [1]

21.8

4.2

0

$\text{Value} = (21.8 \times 0.7564^2 + 2 \times 4.2 \times 0.7564 \times (1 - 0.7564)) \times e^{-2 \times 0.05} = \$12.69$ [1]

(ii)

(a)

Value at t=2 = $1{,}000e^{2x0.05}$ = $1,105.17 [½]

Utility = $1{,}105.17^{0.5}$ = 33.24 [½]

(b)

She can buy 1,000 / 80 = 12.5 shares [½]

Payoff tree: [½]

1,210

990

810

Utility tree: [1]

34.79

31.46

28.46

Using real world probabilities: Value = 34.79 x 0.5^2 + 2 x 31.46 x 0.5 x (1 – 0.5) + 28.46 x $(1 – 0.5)^2$ = 31.54 [1]

(c)

She can buy 1,000 / 12.69 = 78.80 call options [½]

Payoff tree: [½]

1,719

331

0

Utility tree: [1]

41.46

18.20

0

Using real world probabilities: Value = 41.46 x 0.5^2 + 2 x 18.20 x 0.5 x (1 – 0.5) = 19.46 [1]

(iii)

A utility function of this form implies a risk averse investor. [1]

Buying the shares is riskier than investing in the bank account, and investing all the funds in call options is riskier still. [1]
We would therefore expect the shares to give a lower expected utility than cash, and the options a lower utility again. [1]
Risk averse investors require p>q, but here we have p=0.5 and q=0.7564 [1]
[Marks available 4, maximum 2]
[Total 13]

This question caused some difficulty, with some students failing to use a risk-neutral probability measure in part (i).

Conversely, part (ii) required a real-world measure and some students used risk-neutral measures. These mistakes were penalised lightly and the average score on this question was quite high.

Q6

(i)
We compare a portfolio containing the put option and the share to a portfolio containing cash worth $Ke^{-r(T-t)}$. [1]
At time T the portfolio with the option and the share will be worth at least as much as the cash, so by the principle of no arbitrage: [1]
$p_t + S_t \geq Ke^{-r(T-t)}$ [1]

(ii)
Because early exercise is always possible, we have: [1]
$p_t + S_t \geq Ke^{-r(T-t)}$ [1]

(iii)
$P_t <= Ke^{-r(T-t)} = 100e^{-0.04*0.5}$ [1]
=£98.02 [1]

(iv)
The value of the put option p_t will be maximised if the underlying asset is worthless at expiry… [1]

…which will happen if it is worthless now and has zero volatility. [1]
The option will also approach the bound if the underlying Geometric Brownian Motion has negative drift. [1]

[Marks available 3, maximum 2]

[Total 9]

This was answered well by most students, though part (i) seemed to cause the most difficulty. This required two portfolios and a 'no arbitrage' argument, whereas a number of students tried to produce an argument using only one portfolio which cannot be done.

The simpler parts (ii), (iii) and (iv) generally saw good scores.

Q7
(i)
(a)
$\beta_i = \text{Cov}(R_i, R_M)/\text{Var}(R_M)$. [1]

(b)
$r_i - r_0 = \beta i\ (r_M - r_0)$ [1½]
where r_0 is the return on the risk-free asset. [½]

(c)
Since $R_M = \Sigma\pi_i\ R_i$, it follows that
$\text{Var}(R_M) = \Sigma\pi_i\ \text{Cov}(R_i, R_M)$ [1]
and so $\Sigma\pi_i\ \beta_i = \Sigma\pi_i\ \text{Cov}(R_i, R_M)/\text{Var}(R_M)$ [1]
$= \text{Var}(R_M)/\text{Var}(R_M) = 1$ [1]

(ii)
(a)
The proportions are given by proportions of market capitalisation so that
$\pi_1 = 2/5$, $\pi_2 = \pi_3 = \pi_4 = 1/5$. [1]

(b)
$\text{Cov}(R_1, R_M) = 2/5 \times 4 + 1/5 \times 1 + 1/5 \times 1 + 1/5 \times 1 = 11/5$, so $\beta_1 = (11/5) / (8/5) = 11/8$. [1]
Similarly, $\text{Cov}(R_2, R_M) = 2/5 \times 1 + 1/5 \times 3 + 1/5 \times 1 + 1/5 \times 1 = 7/5$, so $\beta_2 = 7/8$ [½]
and $\text{Cov}(R_3, R_M) = 2/5 \times 1 + 1/5 \times 1 + 1/5 \times 2 + 1/5 \times 1 = 6/5$, so $\beta_3 = 6/8$.
[½]
Now it follows from (i)(c) (i.e. $\Sigma\pi_i\beta_i=1$) that $\beta_4 = 5/8$. [1]

(c)
We conclude that, since $r_i - r_0 = \beta_i\ (r_M - r_0)$,
$11\% = 11/8 \times (r_M - r_0)$ [0.5] so that $r_M = 11\%$ [1]
Then:

Asset number	1	2	3	4	Market
Expected return	14%	**10%**	**9%**	**8%**	11%

[½ marks each *for the three figures in bold*]
[Total 13]

Part (i) of this question was answered well by many students, though only the better students scored full marks in (i)(c).

Part (ii) caused difficulty, with some students scoring full marks but most failing to find the correct values of β. Many students still scored some marks by calculating asset and market returns (which were marked correct if the method was right).

Q8

(i)

Merton's model assumes that a corporate entity has issued both equity and debt such that its total value at time t is of F(t). F(t) varies over time as a result of actions by the corporate entity which does not pay dividends on its equity or coupons on its bonds. [1]

Part of the corporate entity's value is zero-coupon debt with a promised repayment amount of L at a future time T. At time T the remainder of the value of the corporate entity will be distributed amongst the equity holders and the corporate entity will be wound up. [1]

The corporate entity will default if the total value of its assets, F(T) is less than the promised debt repayment at time T i.e. F(T) < L. In this situation, the bond holders will receive F(T) instead of L and the equity holders will receive nothing. [1]

This can be regarded as treating the equity holders of the corporate entity as having a European call option on the assets of the company with maturity T and a strike price equal to the value of the debt. [1]

We can therefore value the bond as B(t) = F(t) – E(t) [1]

The Black-Scholes model is used to value the call option where we assume that F(t) follows geometric Brownian motion. [1]

[Marks available 6, maximum 5]

(ii)

The Merton model views the value of the equity as a call option on the company's value. In this case the strike price is €80m and the value of the underlying asset is €100m. The value of the option is €50m. [1]

For σ = 10%: d1 = 2.1287, d2 = 1.8124, value = €46.6m
For σ = 15%: d1 = 1.5509, d2 = 1.0765, value = €47.9m
For σ = 19%: d1 = 1.3375, d2 = 0.7367, value= €49.7m
For σ = 20%: d1 = 1.3015, d2 = 0.6690, value= €50.2m [3 for valid workings]
So to the nearest 1% the volatility is 20%. [1]

[Marks available 5, maximum 4]

(iii)

Vega of the bond is defined as $\partial B \partial\sigma = \partial F \partial\sigma - \partial E \partial\sigma$ and is unknown. [1]

Alternatively, from the figures above vega for the share = €0.5m$\%^{-1}$ [1]

[Marks available 2, maximum 1]

(iv)
The volatility of a company's value is not directly observable and is not easy to estimate. [1]
It is therefore desirable to use a model where the output is relatively insensitive to the volatility figure because this will give a more accurate answer even if our estimate of the volatility is wrong. [1]

[Total 12]

This question was answered well, though only the better students gave enough detail in (i) for five marks.

Part (ii) was answered well, though it was time-consuming and not all students completed it.

Only the better students identified in part (iv) that a low value of vega is desirable because volatility must be estimated and is not directly observable.

Q9

(i)
The general form for the incremental claims C_{ij} can be written as:
$C_{ij} = r_j \, s_i \, x_{i+j} + e_{ij}$ [1]

where:

r_j is the development factor for year j, representing the proportion of claim payments in year j. Each r_j is independent of the origin year i. [1]
s_i is a parameter varying by origin year, i, representing the exposure, for example the number of claims incurred in the origin year i. [1]
x_{i+j} is a parameter varying by calendar year, for example representing inflation. [1]
e_{ij} is an error term. [1]

(ii)
Development factors are:
Year 3 = 190 / 180 = 1.05556 [1]
Year 2 = (180+185) / (130+140) = 1.35185 [1]
1 – 1/f = 1 – 1 / (1.05556 x1.35185) = 0.29921 [1]
Emerging liability for 2018 = 300 x 0.8 x 0.29921 = 71.8 [1]
Reported liability = 150 [½]
Ultimate liability = 150 +71.8 = 221.8 [½]
Reserve = 221.8 – 100 = 121.8 [1]

[Total 11]

Many students struggled in part (i) here, either not providing a valid form for the model or not giving enough detail on the parameters for five marks.

Part (ii) was answered well by most students.

Q10

(i)

Insurer's assets at end of year = 10 x £125 = £1,250 [½]

Insurer is solvent if assets >=0 iff fewer than three claims received [½]

P(no claims) = 0.9^{10} = 0.3487 [½]

P(one claim) = 10 x 0.1 x 0.9^9 = 0.3874 [½]

P(two claims) = 45 x 0.1^2 x 0.9^8 = 0.1937 [½]

P(fewer than three claims) = 0.3487 + 0.3874 + 0.1937 = 0.9298 [½]

(ii)

Adverse selection: [½]

describes the fact that people who know that they are particularly bad risks are more inclined to take out insurance than those who know that they are good risks. [½]

Moral hazard: [½]

describes the fact that a policyholder may, because they have insurance, act in a way which makes the insured event more likely. [½]

Moral hazard makes insurance more expensive. It may even push the price of insurance above the maximum premium that a person is prepared to pay. [½]

[Marks available 4½, maximum 2]

(iii)

To try and reduce the problems of adverse selection insurance companies try and find out lots of information about potential policyholders. [1]

Policyholders can then be put in small, reasonably homogenous pools and charged appropriate premiums. [1]

Insurers might also ensure that the pay-out is proportionate to the loss incurred. [½]

To mitigate moral hazard insurers might apply an excess to the policy… [½]

Or award a discount to policyholders who do not claim. [½]

[Marks available 3½, maximum 2]

(iv)

Assets at end of year two = 250 + 8*0.75x + 2x = 250 + 8x [½]

From (i): P(no claims) = 0.9^{10} = 0.3487

P(one claim) = 10 x 0.1 x 0.9^9 = 0.3874

P(two claims) = 45 x 0.1^2 x 0.9^8 = 0.1937

So P(three or more claims) = 1 – 0.3487 – 0.3874 – 0.1937 = 0.0702 [½]

And newly calculated: P(three claims) = 120 x 0.1^3 x 0.9^7 = 0.0574 [½]

So P(four or more claims) = 1 – 0.3487 – 0.3874 – 0.1937 – 0.0574 = 0.0128 [½]

So to keep the probability of ruin below 5% the insurer needs enough assets at the end of year two to cover three claims. [½]

So we need assets of at least £1,500 [½]

So 250 + 8x = 1,500 [½]

So x = £156.25 [½]

[Total 11]

This question saw a good attempt by most students though some of the probability calculations in parts (i) and (iv) caused difficulty. Some students tried to use a Normal approximation which was not required and produced the wrong answers.

The bookwork in part (ii) and (iii) was generally answered well.

END OF MARKING SCHEDULE

