

# INSTITUTE AND FACULTY OF ACTUARIES

## EXAMINATION

27 September 2021 (am)

### **Subject CM2 - Financial Engineering and Loss Reserving Core Principles**

#### **Paper A**

Time allowed: Three hours and twenty minutes

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| <p>In addition to this paper you should have available the 2002 edition of the<br/>Formulae and Tables and your own electronic calculator.</p> |
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If you encounter any issues during the examination please contact the Assessment Team on  
T. 0044 (0) 1865 268 873.

**1** An investor makes decisions using the utility function  $U(w) = \ln(w)$  where  $w > 0$ .

The investor is going to invest \$100 now for a period of 1 year, and has identified the following two assets to invest in:

- Asset A is risk-free and will not change in value over the year.
- Asset B will increase in value by 50% over the year with probability 0.6 or decrease in value by 50% over the year with probability 0.4.

The investor does not make any allowance for discounting when making investment decisions. They are going to invest a proportion,  $x$ , of their wealth in Asset A and the remaining proportion,  $(1 - x)$ , in Asset B.

- (i) Construct a formula, in terms of  $x$ , for their expected utility at the end of the year. [2]
- (ii) Determine, using your result from part (i), the amount that the investor should invest in each asset to maximise their expected utility. [5]
- [Total 7]

**2** Consider an exponential distribution with parameter  $\lambda = 2$ , and a lognormal distribution with parameters  $\mu = -1.04$ ,  $\sigma = 0.833$ .

- (i) Calculate for each distribution:
- (a) the mean.
  - (b) the variance.
  - (c) the 99th percentile. [8]
- (ii) Comment on your answers to part (i) in the context of choosing a distribution for financial modelling. [2]
- (iii) Comment on why the lognormal distribution may be preferred to the exponential distribution for modelling a security price. [2]
- [Total 12]

**3** Claims on a portfolio of insurance policies arise as a Poisson process with rate  $\lambda$ .

The insurance company calculates premiums using a loading of  $\theta$  and has an initial surplus of  $U$ . The probability of ruin before time  $t$  is defined as  $\Psi(U, t)$ .

Suppose that  $\theta = 0.1$  and the claim amounts follow an exponential distribution with mean  $\mu = 0.5$ .

Calculate the numerical value of  $R$ , the adjustment coefficient. [6]

- 4 A non-dividend paying share, with price  $S_t$  at time  $t$ , has a European call option written on it with value  $c_t$  at time  $t$ . The call matures at time  $T$  and has a strike price of  $K$ . The continuously compounded risk-free rate is  $r$ .

- (i) State an upper bound for the value of the call option,  $c_t$ . [1]

Consider a portfolio containing one call option and  $Ke^{-(T-t)r}$  cash.

- (ii) Demonstrate that, at time  $T$ , the value of the portfolio will always be greater than or equal to the value of the share,  $S_T$ . [3]
- (iii) Determine, using the result in part (ii), a lower bound for the value of the call option:

$$c_t \geq S_t - Ke^{-(T-t)r}. \quad [1]$$

An investor holds an American call option on the same share. Assume that  $r > 0$ .

- (iv) Explain, using the result in part (iii), why it would never be optimal for the investor to exercise this option before its maturity date. [4]
- (v) Discuss how your answer to part (iv) would change if the share paid dividends. [2]

[Total 11]

- 5 Consider the following assets in a world where the Capital Asset Pricing Model (CAPM) holds. There are three risky assets and one risk-free asset. No other assets exist in the market.

| <i>Asset</i>    | <i>Expected return (% p.a.)</i> | <i>Total value of assets in market (\$m)</i> | <i>Beta</i> |
|-----------------|---------------------------------|--|-------------|
| Risky asset A   | 5                               | 10   | $\beta$     |
| Risky asset B   | 10                              | 50   | 1           |
| Risky asset C   | $x$                             | 20   | 2           |
| Risk-free asset | 3                               | 40   | n/a         |

- (i) Calculate the expected return on the market portfolio. [4]
- (ii) Calculate  $x$ . [1]
- (iii) Calculate  $\beta$ . [1]
- (iv) Discuss the limitations of the CAPM. [3]

[Total 9]

- 6** The annual rates of return from a particular investment, Investment A, are independently and identically distributed. Each year the distribution of  $(1 + i_t)$ , where  $i_t$  is the return earned on Investment A in year  $t$ , is log-normal with parameters  $\mu$  and  $\sigma^2$ .

The mean and standard deviation of  $i_t$  are 0.04 and 0.03, respectively.

- (i) Calculate  $\mu$  and  $\sigma^2$ . [4]

An insurance company has liabilities of \$20m to meet in 3 years' time. It currently has assets of \$17.5m, which are invested in Investment A.

- (ii) Determine the probability that the insurance company will be unable to meet its liabilities. [5]

[Total 9]

- 7** A one-period binomial tree has been constructed. In it, a stock with initial value  $S_0$  can evolve over a single time period to be worth either  $S_0u$  or  $S_0d$ , where  $u > d$ . The continuously compounded risk-free rate of return is  $r$ .

- (i) Demonstrate that, to avoid arbitrage, the relationship  $d < e^r < u$  must hold. [3]

- (ii) State the formula for the risk-neutral probability,  $q$ , of an up movement. [1]

- (iii) Demonstrate that the relationship in part (i) is equivalent to  $0 < q < 1$ . [2]

[Total 6]

**8** Consider a random variable  $X$  with probability density function  $f_X(x)$ .

- (i) Write down the formula for the following in terms of  $f_X(x)$ :
- (a) Variance
  - (b) Downside semi-variance
  - (c) Expected shortfall relative to a level.

[3]

A trader has built a Value at Risk (VaR) model of a security that fits a distribution to underlying historical data.

The modelled 1-day 99% VaR for this security is  $\$L$ , meaning that there is a 1% chance that the trader loses more than  $\$L$  in 1 working day by holding this security.

The trader is examining the effectiveness of their model over a month with 20 working days in it. They have assumed that day-on-day movements of the security are all independent from one another.

- (ii) Demonstrate that, assuming the VaR model is correct, the probability the trader loses more than  $\$L$  on at least 3 working days in the month is 0.001. [3]

Over this month, a market crash occurs. On each of 10 separate working days during the month, the security generates losses in excess of  $\$L$  per day.

- (iii) Discuss, without any further calculations, the effectiveness of VaR as a risk measure, given this information and your answer to part (ii). [4]

[Total 10]

**9** In any year, the effective interest rate per annum has mean value  $j$  and standard deviation  $s$  and is independent of the interest rates in all previous years.

Let  $S_n$  be the accumulated amount after  $n$  years of a single investment of one at time  $t = 0$ .

You are given that:

$$E[S_n] = (1 + j)^n.$$

$$\text{Var}[S_n] = (1 + 2j + j^2 + s^2)^n - (1 + j)^{2n}.$$

The interest rate per annum in any year is equally likely to be  $i_1$  or  $i_2$  ( $i_1 > i_2$ ). No other values are possible.

- (i) Derive expressions for  $j$  and  $s^2$  in terms of  $i_1$  and  $i_2$ . [3]

Consider the accumulated value at time  $t = 20$  of  $\$2$  million invested at time  $t = 0$ . This amount has an expected value of  $\$4.5$  million and a standard deviation of  $\$0.75$  million.

- (ii) Calculate the values of  $i_1$  and  $i_2$ . [6]

[Total 9]

**10** Consider a Poisson process with parameter  $\lambda$ . Let the random variable  $T_1$  denote the time of the first claim.

(i) Show that  $T_1$  follows an exponential distribution with parameter  $\lambda$ . [2]

(ii) Prove that the inter-event times between subsequent claims also follow an exponential distribution with parameter  $\lambda$ . [3]

You are given below, the formula for the ultimate probability of ruin,  $\Psi(U)$ , where individual claim amounts are exponentially distributed with mean 1, the premium loading factor is  $\theta$ , and the initial surplus is  $U$ :

$$\Psi(U) = \frac{1}{1+\theta} \exp\left(\frac{-\theta U}{1+\theta}\right)$$

(iii) Comment on what can be inferred from this result about the relationship between the ultimate probability of ruin and:

(a) the initial surplus.

(b) the premium loading factor.

[5]

[Total 10]

**11** Academic studies have shown that lemurs (primates from the island of Madagascar), are risk averse and non-satiated. A zoologist is trying to determine an appropriate utility function,  $U(w)$ , to model their behaviour in an experiment.

(i) Determine, for each of the following functions, whether the zoologist could use it as a valid utility function:

(a)  $U(w) = w + w^2$  for  $-\infty < w \leq 3$

(b)  $U(w) = \frac{(w^\gamma - 1)}{\gamma}$  for  $0 < w < \infty$  and  $\gamma < 1$

(c)  $U(w) = w - 2w^2$  for  $-\infty < w \leq 3$

[8]

The zoologist has chosen the function  $U(w) = \ln(1 + w)$ . The zoologist now carries out an experiment, and presents the lemurs with two options:

- Scenario A:  $w = 3$  or  $w = 0$  with equal probability
- Scenario B:  $w = 1.1$  with certainty.

The zoologist finds that the lemurs prefer Scenario B. Assume that the initial wealth of the lemurs is  $w = 0$ , and that they behave rationally in the experiment.

(ii) Show that the utility function the zoologist has chosen is consistent with the behaviour of the lemurs in the experiment [3]

[Total 11]

**END OF PAPER**