

INSTITUTE AND FACULTY OF ACTUARIES

EXAMINATION

22 September 2021 (am)

Subject CS2 – Risk Modelling and Survival Analysis Core Principles

Paper A

Time allowed: Three hours and twenty minutes

<p>In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator.</p>
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If you encounter any issues during the examination please contact the Assessment Team on
T. 0044 (0) 1865 268 873.

- 1** An Analyst is assessing the risks of an equity portfolio and wishes to estimate the probability that the portfolio will incur at least one daily loss exceeding 5% next month.

Explain how a generalised extreme value distribution and the block maxima method could be used to estimate this probability. [3]

- 2** Consider the illness-death continuous-time Markov model with the following four states:

- State H: healthy
- State SR: sick due to recoverable causes
- State SNR: sick due to non-recoverable causes
- State D: dead.

(i) Derive, by first writing down its likelihood function, the maximum likelihood estimator of the transition rate from State H to State SNR, defining each of the terms you use. [6]

(ii) State the asymptotic distribution of the estimator derived in part (i). [2]

[Total 8]

- 3** Consider the following time series process:

$$Y_t = e_t + \beta e_{t-1}$$

where e_t is a white noise process with variance σ^2 .

(i) Write down the autocorrelation function $\{\rho_k\}$, of Y_t , for $k \geq 0$. [2]

(ii) Determine the possible values of β for which the value of the partial autocorrelation function at lag 2, $\phi_2 = -\frac{1}{3}$. [4]

(iii) Comment on the practical suitability of this time series process for the values of β calculated in part (ii). [2]

[Total 8]

- 4 An Actuary has graduated the mortality experience of a population aged 55 to 65 years using the following formula:

$$\mu_x = \begin{cases} ax + b\exp(cx) & \text{if } x < 65 \\ d & \text{if } x = 65 \end{cases}$$

where a , b , c and d are constants and x is age in years.

The mortality experience data and the graduated rates calculated by the Actuary are shown in the table below. All data have been collated between 1 January 2020 and 31 December 2020 inclusive on an ‘age nearest birthday’ basis:

<i>Age x</i>	<i>Exposed-to-risk (years)</i>	<i>Observed deaths</i>	<i>Graduated rates</i>
55	2,737	53	0.02094
56	2,610	57	0.02357
57	2,649	86	0.02636
58	2,611	77	0.02930
59	2,449	74	0.03238
60	2,213	96	0.03555
61	2,025	79	0.03880
62	1,969	68	0.04208
63	1,900	78	0.04537
64	1,803	83	0.04860
65	1,736	105	y

Let y be the graduated rate at age 65 and let the null hypothesis be that the graduated rates are the true rates underlying the observed data.

Determine the range of values that y needs to take so that there is insufficient evidence, at the 97.5% confidence level, to reject the null hypothesis under a chi-square goodness-of-fit test.

[9]

- 5** A study was undertaken into survival rates following major heart surgery. Patients who underwent this surgery were monitored from the date of surgery until whichever of the following events occurred first:

- they died
- they left the hospital where the surgery was carried out, or
- a period of 30 days had elapsed.

- (i) State, with reasons, two forms of censoring that are present in this study and one form of censoring that is not present. [3]

The Analyst collating the results calculated the Nelson–Aalen estimate of the survival function, $S(t)$, as follows:

t (days)	$S(t)$
$0 \leq t < 5$	1.0000
$5 \leq t < 17$	0.9001
$17 \leq t < 25$	0.8456
$25 \leq t$	0.7157

- (ii) State, using the Nelson–Aalen estimate, the probability of survival for 20 days after the surgery. [1]

The Analyst also wishes to calculate the Kaplan–Meier estimate of the survival function.

- (iii) Determine the Kaplan–Meier estimate of the survival function. [5]
[Total 9]

- 6** A home insurance company's total monthly claim amounts have a mean of 250 and a standard deviation of 300. The company has estimated that it will face insolvency if the total monthly claim amounts reach or exceed 1,000 in any given month.

- (i) Determine the probability that the company faces insolvency in any given month if the company assumes that total monthly claim amounts follow the Normal distribution. [2]
- (ii) Determine the revised value of the probability in part (i) if the company assumes that total monthly claim amounts follow the two-parameter Pareto distribution. [4]
- (iii) Explain why the Normal distribution is unlikely to be a good fit for the distribution of the total monthly claim amounts for this company. [3]

An Analyst has determined that the two-parameter Pareto distribution is the best fit for the distribution of the total monthly claim amounts for this company.

- (iv) Outline, using the results from parts (i) and (ii), the potential consequences of the company assuming that the total monthly claim amounts follow the Normal distribution rather than the two-parameter Pareto distribution. [2]
[Total 11]

An Actuary is considering using the following process to model a seasonal data set:

$$(1 - B^3)(1 - (\alpha + \beta)B + \alpha\beta B^2)X_t = e_t$$

where B is the backwards shift operator and e_t is a white noise process with variance σ^2 .

A seasonal difference series is defined as follows:

$$Y_t = X_t - X_{t-3}$$

- (i) Express the equation for the original process X_t in terms of the seasonal difference series, Y_t , and the backwards shift operator B . [1]
- (ii) Determine the range of values of α and β for which the seasonal difference series, Y_t , is stationary. [2]

Let γ_k and ρ_k denote the values at lag k of the autocovariance and autocorrelation functions, respectively, of the seasonal difference series, Y_t . The first Yule–Walker equation for Y_t may be written as follows:

$$1 - (\alpha + \beta)\rho_1 + \alpha\beta\rho_2 = \frac{\sigma^2}{\gamma_0}$$

- (iii) Write down the second and third Yule–Walker equations for Y_t in terms of ρ_1 and ρ_2 . [2]

The Actuary has observed the following sample autocorrelation values for the series Y_t : $\hat{\rho}_1 = 0.5$ and $\hat{\rho}_2 = 0.2$.

- (iv) Estimate, using the equations in part (iii), the parameters α and β based on this information. [5]

[Hint: let $M = \alpha + \beta$ and $N = \alpha\beta$ and use the formula for finding the roots of a quadratic equation.]

- (v) Determine the values of the one-step ahead and two-step ahead forecasts, \hat{x}_{550} and \hat{x}_{551} , respectively, based on the parameters estimated in part (iv) and the observed values x_1, x_2, \dots, x_{549} of X_t . [4]

[Total 14]

- 8** A telecommunications company is modelling the capacity requirements for its mobile phone network. It assumes that if a customer is not currently on a call ('offline'), the probability of initiating a call in the short time interval $[t, t + dt]$ is $0.25dt + o(dt)$. If the customer is currently on a call ('online'), then it assumes that the probability of ending the call in the time interval $[t, t + dt]$ is given by $0.75dt + o(dt)$.

The following probabilities are defined:

$P_{\text{OFF ON}}(t)$ = Probability that the customer is online at time t , given that they were offline at time 0

$P_{\text{OFF OFF}}(t)$ = Probability that the customer is offline at time t , given that they were offline at time 0

$P_{\text{ON OFF}}(t)$ = Probability that the customer is offline at time t , given that they were online at time 0

$P_{\text{ON ON}}(t)$ = Probability that the customer is online at time t , given that they were online at time 0.

- (i) Explain why the status of an individual customer can be considered as a Markov jump process. [2]
 - (ii) Write down Kolmogorov's forward equation for $\frac{d}{dt}P_{\text{OFF OFF}}(t)$. [2]
 - (iii) Solve the equation in part (ii) to obtain a formula for $P_{\text{OFF OFF}}(t)$. [7]
 - (iv) Derive an expression, in terms of t , for the expected proportion of time spent online over the period $[0, t]$, given that the customer is offline at time 0. [7]
- [Total 18]

- 9 In a small country, there are only four authorised car insurance companies A, B, C and D. All car owners take out car insurance from an authorised insurance company. All policies provide cover for a period of 1 calendar year.

The probabilities of car owners transferring between the four companies at the end of each year are believed to follow a Markov chain with the following transition matrix:

$$\begin{matrix} & \begin{matrix} A & B & C & D \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} & \begin{pmatrix} 1-2\alpha-\alpha^2 & \alpha & \alpha & \alpha^2 \\ \alpha & \frac{1}{2}-\alpha & \frac{1}{2}-\alpha & \alpha \\ \alpha & \frac{1}{2}-\alpha & \frac{1}{2}-\alpha & \alpha \\ 0 & \alpha & \alpha & 1-2\alpha \end{pmatrix} \end{matrix}$$

for some parameter α .

- (i) Determine the range of values of α for which this is a valid transition matrix. [4]
- (ii) Explain whether the Markov chain is irreducible, including whether this depends on the value of α . [3]

The value of α has been estimated as 0.2.

Mary has just bought her policy from Company D for the first time.

- (iii) Determine the probability that Mary will be covered by Company D for at least 4 years before she transfers to another insurance company. [3]

James took out a policy with Company A in January 2018. Sadly, James' car was stolen on 23 December 2020.

- (iv) Determine the probability that a different company, other than Company A, covers James' car at the time it was stolen. [3]

Company A makes an offer to buy Company D. It bases its purchase price on the assumption that car owners who would previously have purchased policies from Company A or Company D would now buy from the combined company, to be called ADDA.

- (v) Write down the transition matrix that will apply after the takeover if Company A's assumption about car owners' behaviour is correct. [2]
- (vi) Comment on the appropriateness of Company A's assumption. [5]

[Total 20]

END OF PAPER