EXAMINATION

September 2006

Subject CT1 — Financial Mathematics
Core Technical

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

M A Stocker
Chairman of the Board of Examiners

November 2006

Comments

As in many recent diets, the questions requiring verbal reasoning (e.g. Question 4(i)) tended not to be well answered with candidates producing vague statements which did not demonstrate that they understood the relevant points

Please note that differing answers may be obtained from those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this.

However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

Comments on solutions presented to individual questions for this September 2006 paper are given below.

Question 1

Generally well answered. To gain full marks candidates were required to specify the difference between futures and options rather than just defining each contract separately.

Question 2

Well answered. This was a question where some candidates were penalised if answers had been rounded excessively.

Question 3

Generally well answered. Another possible solution is to use $1+j=\dfrac{1+0.6i}{1+f}=\dfrac{1+0.6\times 0.11}{1.07143}$ *which leads to the same answer.*

Question 4

For full marks in part (i), an answer should have included a description of the 'risk-free' concept (rather than just saying arbitrage profits are impossible). Many students had difficulty with part (ii).

Question 5

Full marks were given if either 365 or 365.25 days were used in the calculation. Most students scored well on this question.

Question 6

This question was well answered. For full marks, candidates were required to show detailed steps in deriving the result required including a definition of the initial terms used and a correct explanation of the relevance of the independence assumption.

Question 7

This question was poorly answered to the surprise of the examiners. Many candidates struggled to deal with the linked internal rate of return.

Question 8

Well answered.

Question 9

This question appeared to reward candidates who had a good understanding of the topic. Whilst the best candidates usually scored close to full marks on this question, weaker or less-prepared candidates often scored very badly.

Whilst the question did state that payments were made monthly, the examiners recognised that there was some potential for misinterpretation as to the frequency of the loan repayments in part (e) and took this into account. Thus students who used the formula $Xa_{\overline{30}|} = 100,000$ *with* $i^{(12)}$ *= 6% &* i *=6.168% to get an answer of £7,396 in this part were awarded full marks.*

Question 10

Generally well answered.

Question 11

This was the worst answered question on the paper by some margin with very few candidates scoring close to full marks. This may be because this type of question has not appeared in recent diets. Candidates needed to show that they could derive logically the amounts that will be paid, the real values of those amounts and their present values in real terms. Appropriate formulae then needed to be developed.

Question 12

Many candidates answered this question well although a minority scored very badly (possibly due to time pressure).

1 (i) A future is a contract binding buyer and seller to deliver or take delivery of an asset at a given price at a given time in the future. An option is a contract that gives the buyer the option to deliver or take delivery of the asset at the given price. The seller of the option must deliver/take delivery if the buyer of the option wishes to exercise the option.

(ii) Convertibles have option-like characteristics because they give the holder the option to purchase equity in a company on pre-arranged terms.

2 The accumulated value is

$$4\bar{s}_{\overline{3}|} + 2\bar{s}_{\overline{2}|} + 2\bar{s}_{\overline{1}|}$$

$$= \frac{i}{\delta}\left(4s_{\overline{3}|} + 2s_{\overline{2}|} + 2s_{\overline{1}|}\right)$$

$$= \frac{0.04}{0.039221}\left(4\times 3.1216 + 2\times 2.0400 + 2\right)$$

$$= 18.9352$$

3 (a) The money rate of return is i where $(1+i) = 11.1/10$
$i = 0.11$ or 11%

(b) The rate of inflation is f where $(1+f) = 120/112$
$f = 0.07143$ or 7.143%

(c) The net real rate of return per annum is j
where $j = \frac{0.6i - f}{1+f} = \frac{0.6\times 0.11 - 0.07143}{1.07143} = -0.005068$ or -0.5068%

4 (i) The no arbitrage assumption means that it is assumed that an investor is unable to make a risk-free trading profit.

(ii) In all states of the world, security B pays 80% of A. Therefore its price must be 80% of A's price, or the investor could obtain a better payoff by only purchasing one security and make risk-free profits by selling one security short and buying the other. The price of B must therefore be 16p.

5 (i) (a) Let the answer be t days

$3{,}600(1 + 0.06 \times t/365) = 4{,}000$

$t = 675.9$ days

(b) Let the answer be t days

$$3{,}600\left(1+\frac{0.06}{4}\right)^{4t/365} = 4{,}000$$

$(4t/365)\ln(1.015) = \ln(4{,}000/3{,}600)$

$t = 645.7$ days

(c) Let the answer be t days

$$3{,}600\left(1+\frac{0.06}{12}\right)^{12t/365} = 4{,}000$$

$(12t/365)\ln(1.005) = \ln(4{,}000/3{,}600)$

$t = 642.5$ days

(ii) (i)(a) takes longest because, under conditions of simple interest, interest does not earn interest.

6 (i) Let i_t be the (random) rate of interest in year t. Let S_{10} be the accumulation of the unit investment after 10 years:

$$E(S_{10}) = E\left[(1+i_1)(1+i_2)\ldots(1+i_{10})\right]$$

$$E(S_{10}) = E[1+i_1]E[1+i_2]\ldots E[1+i_{10}] \text{ as } \{i_t\} \text{ are independent}$$

$$E[i_t] = j$$

$$\therefore \quad E(S_{10}) = (1+j)^{10} = 1.07^{10} = 1.96715$$

(ii)
$$E\left(S_{10}^2\right) = E\left[\left[(1+i_1)(1+i_2)\ldots(1+i_{10})\right]^2\right]$$

$$= E(1+i_1)^2 E(1+i_2)^2 \ldots E(1+i_{10})^2 \text{ (using independence)}$$

$$= E\left(1+2i_1+i_1^2\right)E\left(1+2i_2+i_2^2\right)\ldots E\left(1+2i_{10}+i_{10}^2\right)$$

$$=\left[E\left(1+2i_t+i_t^2\right)\right]^{10}=\left(1+2j+s^2+j^2\right)^{10}$$

as $E\left[i_i^2\right]=V\left[i_t\right]+E\left[i_t\right]^2=s^2+j^2$

$$\therefore\ \mathrm{Var}\left[S_n\right]=\left(1+2j+s^2+j^2\right)^{10}-\left(1+j\right)^{20}$$

$$=\left(1+2\times 0.07+0.016+0.07^2\right)^{10}-\left(1.07\right)^{20}=0.5761$$

(iii) If 1,000 units had been invested, the expected accumulation would have been 1,000 times bigger. The variance would have been 1,000,000 times bigger.

7 (i) (a) $\left(1+i\right)^2=\dfrac{450}{600}\dfrac{500}{450+40}\dfrac{800}{500+100}\Rightarrow i=1.015\%$

(b) First sub-interval is first year. Money weighted rate of return is i_1 where $\left(1+i_1\right)=\dfrac{450}{600}\Rightarrow i_1=-25\%$

Second sub-interval is second year. Money weighted rate of return is i_2 where $490\left(1+i_2\right)+100\left(1+i_2\right)^{1/2}=800$

Then $\left(1+i_2\right)^{1/2}=\dfrac{-100\pm\sqrt{100^2-4\times 490\times(-800)}}{2\times 490}=\dfrac{-100\pm 1256.1847}{980}$

= 1.17978 (taking positive root)

$\left(1+i_2\right)=1.39188\Rightarrow i_2=39.188\%$

Linked internal rate of return is i
where $\left(1+i\right)^2=0.75\times 1.39188\Rightarrow i=2.1719\%$

(ii) The linked IRR is higher because it relies on two money weighted rates of return. With the calculation of the second money weighted rate of return, there is more money in the fund when the fund is performing well (in the second half of the year).

8 (i) $$150 = 100\exp\left\{\int_0^5 \left(at + bt^2\right)dt\right\} = 100\exp\left[\tfrac{1}{2}at^2 + \tfrac{1}{3}bt^3\right]_0^5 = 100\exp\left[12.5a + 41.667b\right]$$

$$230 = 100\exp\left\{\int_0^{10} \left(at + bt^2\right)dt\right\} = 100\exp\left[\tfrac{1}{2}at^2 + \tfrac{1}{3}bt^3\right]_0^{10} = 100\exp\left[50a + 333.333b\right]$$

$$\ln(1.5) = 12.5a + 41.667b$$
$$\ln(2.3) = 50a + 333.333b$$

The second expression less four times the first expression gives:

$$\ln(2.3) - 4\ln(1.5) = 166.667b \Rightarrow b = -0.0047337$$

$$a = \frac{\ln(2.3) - 333.333 \times -0.0047337}{50} = 0.0482162$$

(ii) $$100e^{10\delta} = 230 \Rightarrow 10\delta = \ln 2.3 \Rightarrow \delta = 0.08329$$

(iii) $$\text{Present Value } = \int_0^{10} 20e^{0.05t}e^{-0.08329t}dt$$

$$= \int_0^{10} 20e^{-0.03329t}dt$$

$$= 20\left[\frac{e^{-0.03329t}}{-0.03329}\right]_0^{10}$$

$$= 20 \times 8.5058 = 170.116$$

9 (a) Premiums were expected to accumulate to

$$1{,}060\ddot{s}_{\overline{30|}}^{(12)} \text{ at } 7\% = 1{,}060\frac{i}{d^{(12)}}s_{\overline{30|}} = 1{,}060 \times 1.037525 \times 94.4608 = £103{,}885.77$$

(b) Premiums would have accumulated to

$$1{,}060\ddot{s}_{\overline{30|}}^{(12)} \text{ at } 4\% = 1{,}060\frac{i}{d^{(12)}}s_{\overline{30|}} = 1{,}060 \times 1.021537 \times 56.0849 = £60{,}730.37$$

The shortfall is 100,000 – 60,730.37 = £39,269.63

(c) Accumulation will be

$$1,060\ddot{s}^{(12)}_{\overline{20}|4\%}(1.04)^{10} + 5,000\ddot{s}^{(12)}_{\overline{10}|4\%}$$

$$= 1,060\frac{i}{d^{(12)}}s_{\overline{20}|}(1.04)^{10} + 5,000\frac{i}{d^{(12)}}s_{\overline{10}|}$$

$$= 1,060 \times 1.021537 \times 29.7781 \times 1.48024 + 5,000 \times 1.021537 \times 12.0061$$

$$= £109,053.12$$

Therefore the excess is £9,053.12

(d) The investor has earned a return of 4 % by investing extra premiums in the investment policy. The investor could have obtained a lower present value of total payments on the loan by paying off part of the loan instead. This is because the interest being paid on the loan was greater than the interest he was earning on his premiums.

(e) If he had repaid the loan by a level annuity, the annual instalment would have been X where

$\frac{X}{12}a_{\overline{360}|} = 100,000$ at 0.5% (or $Xa^{(12)}_{\overline{30}|} = 100,000$ with $i^{(12)} = 6\%$ & $i = 6.168\%$)

$$X = \frac{12 \times 100,000}{a_{\overline{360}|}} = \frac{1,200,000}{166.7916} = £7,194.61$$

10 Present value of companies' and consumers' costs is (in £ million)

$$\frac{i}{\delta}(50+10)\left(v + 1.03v^2 + 1.03^2v^3 + \ldots + 1.03^{19}v^{20}\right)$$

$$= \frac{i}{\delta}60v\left(1 + 1.03v + (1.03v)^2 + \ldots + (1.03v)^{19}\right)$$

$$= \frac{i}{\delta}60v\frac{\left(1-(1.03v)^{20}\right)}{1-1.03v} = 1.019869 \times 60 \times 0.96154 \times \left(\frac{1-1.80611 \times 0.45639}{1-1.03 \times 0.96154}\right)$$

$$= 1.019869 \times 60 \times 0.96154 \times 18.27680 = 1075.383$$

Present value of costs to financial advisors (in £ million)

$$\frac{i}{\delta}\left(60v+19v^2+18v^3+\ldots+v^{20}\right)$$

$$=40\frac{iv}{\delta}+\frac{i}{\delta}\left(20v+19v^2+18v^3+\ldots+v^{20}\right)$$

$$=40\frac{iv}{\delta}+\frac{i}{\delta}\left(21a_{\overline{20|}}-Ia_{\overline{20|}}\right)=\frac{i}{\delta}\left(40v+21a_{\overline{20|}}-Ia_{\overline{20|}}\right)$$

$$=1.019869\times\left(40\times0.96154+21\times13.5903-125.1550\right)$$

$$=1.019869\times198.7029=202.651$$

Total PV of all costs = £1278.034 million

Present value of benefits (in £ million)

$$\frac{i}{\delta}\left(30v+33v^2+36v^3+\ldots+87v^{20}\right)+\frac{i}{\delta}12a_{\overline{20|}}$$

$$\frac{i}{\delta}\left(27a_{\overline{20|}}+3v+6v^2+9v^3+\ldots+60v^{20}+12a_{\overline{20|}}\right)$$

$$=\frac{i}{\delta}\left(3\left(Ia\right)_{\overline{20|}}+39a_{\overline{20|}}\right)$$

$$=1.019869\left(3\times125.1550+39\times13.5903\right)$$
$$=1.019869\times905.4867$$

= 923.478

Net present value of costs = PV(costs) – PV(benefits)
= 1278.034 – 923.478 = £354.556 million

11 (i)

- Payments guaranteed by government.
- Can be various different indexation provisions but, in general, protection is given against a fall in the purchasing power of money.
- Fairly liquid (i.e. large issue size and ability to deal in large quantities) compared with corporate issues, but not compared with conventional issues.
- Normally coupon and capital payments both indexed to increases in a given price index with a lag.
- Low volatility of return and low expected real return.
- More or less guaranteed real return if held to maturity (can vary due to indexation lag).
- Nominal return is not guaranteed.

(ii) The first coupon the investor will receive will be on 31st December 2003. The net coupon per £100 nominal will be:

$$0.8\times 1\times \text{ (Index May 2003/Index November 2001)} = 0.8\times 1\times \frac{113.8}{110}$$

In real present value terms, this is $0.8\frac{113.8}{110}\frac{v}{(1+r)^{0.5}}$

where r = 2.5% per annum and v is calculated at 1.5% (per half year)

The second coupon on 30th June 2004 per £100 nominal will be

$$0.8\times 1\times \frac{113.8}{110}(1+r)^{0.5}$$

In real present value terms, this is $0.8(1+r)^{0.5}\frac{113.8}{110}\frac{v^2}{(1+r)}$

The third coupon on 31st December 2004 per £100 nominal will be

$$0.8\times 1\times \frac{113.8}{110}(1+r)$$

In real present value terms, this is $0.8(1+r)\frac{113.8}{110}\frac{v^3}{(1+r)^{1.5}}$

Continuing in this way, the last coupon payment on 30 June 2009 per £100 nominal will be $0.8\times 1\times \frac{113.8}{110}(1+r)^{5.5}$

In real present value terms, this is $0.8(1+r)^{5.5}\frac{113.8}{110}\frac{v^{12}}{(1+r)^6}$

By similar reasoning, the real present value of the redemption payment is

$$100(1+r)^{5.5}\frac{113.8}{110}\frac{v^{12}}{(1+r)^6}$$

The present value of the succession of coupon payments and the capital payment can be written as:

$$P=\frac{1}{(1+r)^{0.5}}\frac{113.8}{110}\left(0.8\left(v+v^2+\ldots+v^{12}\right)+100v^{12}\right)$$

$$=\frac{1}{1.0124224}\frac{113.8}{110}\left(0.8a_{\overline{12}|1.5\%}+100v^{12}_{1.5\%}\right)$$

$$=1.02185\times(0.8\times 10.9075+100\times 0.83639)$$

$$=94.3833$$

12 (i) Present value of liabilities is $160,000a_{\overline{15}|}+200,000v^{10}$ at 7%

$=160,000\times 9.1079+200,000\times 0.50835$

$=£1,558,934$

(ii) Discounted mean term (DMT) of liabilities is

$$=\frac{\left(1\times 160,000\times v+2\times 160,000\times v^2+\ldots+15\times 160,000\times v^{15}\right)+200,000\times 10\times v^{10}}{160,000a_{\overline{15}|}+200,000v^{10}}$$

$$=\frac{160,000\times\left(Ia_{\overline{15}|}\right)+200,000\times 10\times v^{10}}{160,000a_{\overline{15}|}+200,000v^{10}}$$

$$=\frac{160,000\times 61.5540+200,000\times 10\times 0.50835}{1,558,934}$$

$=\dfrac{10,865,340}{1,558,934}$ = 6.9697 years (½ mark deducted for no units)

(iii) Let the nominal amounts in each security equal A and B respectively.

If the present values of assets and liabilities are to be equal then:

$$A\left(0.08a_{\overline{8|}}+v^{8}\right)+B\left(0.03a_{\overline{25|}}+v^{25}\right)=1,558,934 \quad (1)$$

If the DMTs of the assets and liabilities are equal, then:

$$\frac{A\left(0.08(Ia)_{\overline{8|}}+8v^{8}\right)+B\left(0.03(Ia)_{\overline{25|}}+25v^{25}\right)}{1,558,934}=6.9697$$

or $A\left(0.08(Ia)_{\overline{8|}}+8v^{8}\right)+B\left(0.03(Ia)_{\overline{25|}}+25v^{25}\right)=10,865,340 \quad (2)$

From (1)

$$A\left(0.08\times 5.9713+0.58201\right)+B\left(0.03\times 11.6536+0.18425\right)=1,558,934$$
$$\Rightarrow 1.059714A+0.533858B=1,558,934$$

From (2)

$$A\left(0.08\times 24.7602+8\times 0.58201\right)+B\left(0.03\times 112.3301+25\times 0.18425\right)=10,865,340$$
$$\Rightarrow 6.636896A+7.976153B=10,865,340$$

Therefore

$$6.636896\left(\frac{1,558,934-0.533858B}{1.059714}\right)+7.976153B=10,865,340$$

$$\Rightarrow B\left(7.976153-\frac{6.636896\times 0.533858}{1.059714}\right)=10,865,340-\frac{6.636896\times 1,558,934}{1.059714}$$

$$\Rightarrow B=\frac{1,101,872.85}{4.632647}=\pounds 237,850$$

$$A=\left(\frac{1,558,934-0.533858B}{1.059714}\right)=\pounds 1,351,266$$

(iv) It appears that the asset payments are more spread out than the liability payments. The third condition for immunisation is that that convexity of the assets is greater than that of the liabilities, or that the asset times are more spread around the discounted mean term than the liability times. From observation is appears likely that this condition is met.

END OF EXAMINERS' REPORT