EXAMINATION

September 2007

Subject CT1 — Financial Mathematics
Core Technical

MARKING SCHEDULE

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

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Chairman of the Board of Examiners

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Comments

Please note that different answers may be obtained from those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this.

However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

It should be noted that the rubric of the examination paper does ask for candidates to show their calculations where this is appropriate. Candidates often failed to show sufficient clarity and detail in their working and lost marks as a result.

Q1.

Well answered.

Q2.

Well answered.

Q3.

Whilst this question was generally answered well, some candidates lost marks by not stating the conclusions that arose from their calculations i.e. that neither deal was acceptable.

Q4.

This question was very poorly answered which was disappointing given that this was a bookwork question.

Q5.

Reasonably well answered but some candidates failed to obtain full marks by not stating the required assumption.

Q6.

Parts (i) and (ii) were well answered but part (iii) was a good differentiator with weaker candidates failing to recognise the correct method for calculating the gross redemption yield. As with many previous diets, many candidates in part (iv) had great difficulty in giving a clear explanation of their calculations.

Q7.

Generally well answered. Some candidates lost marks by not giving an explicit formula for $v(t)$ when $t \leq 10$.

Q8.

This question was very poorly answered to the surprise of the examiners who felt that the question should have been relatively straightforward.

Q9.

Part (i) can be done much more simply than by using the method given in this report but the calculations given would still need to be done for part (ii).

Q10.

This question was the worst answered on the paper. Part (ii) did successfully differentiate between candidates with weaker candidates appearing to struggle to apply the theory to a real-life situation.

Q11.

The first three parts were generally answered well by the candidates who attempted the question. Many struggled to complete part (iv) although it is possible that this was due to time pressure. When calculating DMTs, candidates were expected to give the answer in terms of the correct units.

1 The first investor receives the higher rate of return if:

$$\frac{97.9}{96} > \frac{100}{97.9}$$

This inequality does not hold, therefore the second investor receives the higher rate of return.

2 Start by working in half years. The half yearly effective return is i such that:

$769 = 4v + 800v - 0.3(800 - 769)v$

$769 = (804 - 240 + 230.7)v$

$v = \dfrac{769}{794.7} = 0.967661$ therefore $i = 3.3420\%$

Annual effective rate is $(1.03342^2 - 1) = 6.7957\%$

3 The annual rate of payment for the first deal is 240.

This deal is acceptable if:

$240\,\ddot{a}^{(12)}_{\overline{2}|} < 456$ at a rate of interest of 5%

$240\,\ddot{a}^{(12)}_{\overline{2}|} = 240 \times 1.8594 \times 1.026881 = 458.252$

Therefore first deal is not acceptable

The annual rate of payment on the second deal is 246.

This deal is acceptable if:

$246\,a^{(12)}_{\overline{2}|} = 246 \times 1.8594 \times 1.022715 = 467.803$

Therefore second deal is also not acceptable

4 Main characteristics of equity investments:

- Issued by commercial undertakings and other bodies.
- Entitle holders to receive all net profits of the company in the form of dividends after interest on loans and other fixed interest stocks has been paid.
- Higher expected returns than for most other asset classes …

- …but risk of capital losses
- … and returns can be variable.
- Lowest ranking form of finance.
- Low initial running yield…
- … but dividends should increase with inflation.
- Marketability varies according to size of company.
- Voting rights in proportion to number of shares held.

5 Assuming no arbitrage:
Present value of dividends is (in£):
$0.5v^{1/2}$ (at 5%) + $0.5v$ (at 6%) = 0.5(0.97590+0.94340) = 0.95965

Hence forward price is: $F = (9-0.95965)\times 1.06 = £8.5228$

6 (i) $f_{3,1}$ is such that $1.045\times f_{3,1} = 1.05^2$. Therefore $f_{3,1} = 5.5024\%$

(ii) One-year spot rate is same as one-year forward rate = 4%

Two-year spot rate is i_2 such that $(1+i_2)^2 = 1.04\times 1.0425$.
Therefore $i_2 = 4.1249\%$

Three-year spot rate is i_3 such that $(1+i_3)^3 = 1.04\times 1.0425\times 1.045$.
Therefore $i_3 = 4.2498\%$

Four year spot rate is such that $(1+i_4)^4 = 1.04\times 1.0425\times 1.045\times 1.055024$
Therefore $i_4 = 4.5615\%$

(iii) Present value of the payments from the bond is:

$P = 3(1.04^{-1} + 1.041249^{-2} + 1.042498^{-3} + 1.045615^{-4})$
$+ 100\times 1.045615^{-4}$

Therefore $P = 3(0.96154 + 0.92234 + 0.88262 + 0.83659)$
$+ 100\times 0.83659 = 94.468$

Equation of value to find the gross redemption yield from the bond is such that:

$94.468 = 3a_{\overline{4}|} + 100v^4$

Try $i = 4.5\%$

$v^4 = 0.83856$, $a_{\overline{4}|} = 3.58753$, RHS = 94.619

Try i = 5%

$v^4 = 0.82270$, $a_{\overline{4}|} = 3.5460$, RHS = 92.908

Interpolation:

Yield = 0.045 + 0.005 × (94.619 – 94.468) /(94.619 – 92.908)

= 4.544%

(iv) The yield from the bond is lower than the one-year forward rate up to time 4 because the bond can be seen to be a series of zero coupon bonds (1 year, 2 years etc.) each with lower yields than the forward rate. The gross redemption yield from the bond is, in effect, an average of spot rates that are themselves a weighted average of earlier forward rates.

7 (i) For $t \leq 10$

$$v(t) = e^{-\int_0^t 0.04+0.01s\,ds} = e^{-\left[0.04s+0.005s^2\right]_0^t} = e^{-0.04t-0.005t^2}$$

For $t > 10$

$$v(t) = v(10)e^{-\int_{10}^t 0.05\,ds} = e^{-0.9}e^{-\left[0.05s\right]_{10}^t} = e^{-0.9}e^{-0.05(t-10)} = e^{-(0.4+0.05t)}$$

(ii) (a) Present value $= 1000e^{-(0.4+0.05\times 15)} = 1000e^{-1.15} = 316.637$

(b) $1000(1-d)^{15} = 316.637 \Rightarrow d = 7.380\%$

(iii) Present value $= \int_{10}^{15} e^{-(0.4+0.05t)}\,20e^{-0.01t}\,dt$

$$= 20\int_{10}^{15} e^{-0.4}e^{-0.06t}\,dt$$

$$= 20e^{-0.4}\left[\frac{e^{-0.06t}}{-0.06}\right]_{10}^{15} = 20e^{-0.4}\left(-6.77616 + 9.14686\right) = 31.783$$

8 (i) Linked internal rate of return is found by linking the money weighted rate of return from the sub-periods.

$(\text{LIRR})^3 = 1.05 \times 1.06 \times 1.065 \times 1.03$

Therefore LIRR = 0.06879 or 6.879%

(ii) The TWRR requires the value of the fund every time a payment is made.

Size of the fund after six months is: $12.5 \times (1.05) = 13.125$
Size of the fund after one year is: $(13.125 + 6.6) \times 1.06 = 20.909$
Size of the fund after two years is: $(20.909 + 7) \times 1.065 = 29.723$
Size of the fund after three years is: $(29.723 + 8) \times 1.03 = 38.855$

The TWRR is i where i is the solution to:

$$(1+i)^3 = (13.125/12.5) \times [20.909/(13.125+6.6)] \times [29.723/(20.909+7)] \times [38.855/(29.723+8)]$$

or just use the rates of return given to give:

$(1+i)^3 = 1.05 \times 1.06 \times 1.065 \times 1.03$

giving $i = 6.879\%$

(iii) For MWRR, we need to know the size of the fund at the end of the period. We can use the values above to give:

MWRR is solution to: $12.5(1+i)^3 + 6.6(1+i)^{2.5} + 7(1+i)^2 + 8(1+i) = 38.855$

Solve by iteration and interpolation, starting with $i = 7\%$.

$i = 7\%$ gives LHS = 39.704
$i = 6\%$ gives LHS = 38.868
$i = 5.5\%$ gives LHS = 38.454

Interpolate between 5.5% and 6%.

$i = 0.055 + 0.005 \times (38.855 - 38.454)/(38.868 - 38.454) = 5.98\%$

(iv) (i) and (ii) are the same because there are no cash flows within sub-periods to "distort" the LIRR away from the TWRR. The MWRR is lower because the fund has a smaller amount of money in it at the beginning when rates of return are higher.

9 (i) $(1+i_t) \sim Lognormal(\mu, \sigma^2)$

$\ln(1+i_t) \sim N(\mu, \sigma^2)$

$\ln(1+i_t)^{10} = \ln(1+i_t) + \ln(1+i_t) + \ldots + \ln(1+i_t) \sim N(10\mu, 10\sigma^2)$

since i_t's are independent

$(1+i_t)^{10} \sim Lognormal(10\mu, 10\sigma^2)$

[½] for correct use of independence assumption

$$E(1+i_t) = \exp\left(\mu + \frac{\sigma^2}{2}\right) = 1.06$$

$$Var(1+i_t) = \exp(2\mu + \sigma^2)\left[\exp(\sigma^2) - 1\right] = 0.08^2$$

$$\frac{0.08^2}{1.06^2} = \left[\exp(\sigma^2) - 1\right] \therefore \sigma^2 = 0.0056798$$

$$\exp\left(\mu + \frac{0.0056798}{2}\right) = 1.06 \Rightarrow \mu = \ln 1.06 - \frac{0.0056798}{2} = 0.055429$$

$10\mu = 0.55429, 10\sigma^2 = 0.056798$

Let S_{10} be the accumulation of one unit after 10 years:

$$E(S_{10}) = \exp\left(0.55429 + \frac{0.056798}{2}\right) = 1.790848$$

Expected value of investment $= 2{,}000{,}000 E(S_{10}) = £3.5817m$

(ii) We require $P[S_{10} < 0.8 \times 1.790848 = 1.4327]$

$P[\ln S_{10} < \ln 1.4327]$ where $\ln S_{10} \sim N(0.55429, 0.056798)$

$$\Rightarrow P\left[N(0,1) < \frac{\ln 1.4327 - 0.55429}{\sqrt{0.056798}}\right]$$

$$\Rightarrow P[N(0,1) < -0.8171] = 0.207 \approx 21\%$$

10 (i) (a) The flat rate of interest is: $(2\times2{,}400 - 2{,}000)/(2\times2{,}000) = 70\%$

(b) The flat rate of interest is not a good measure of the cost of borrowing because it takes no account of the timing of payments and the timing of repayment of capital.

(ii) If the consumers' association is correct, then the present value of the repayments is greater than the loan at 200%

i.e. $2{,}000 < 2{,}400\dfrac{i}{d^{(12)}}a_{\overline{2}|}$

$i = 2$; $a_{\overline{2}|} = 0.44444$; $d^{(12)} = 1.04982$ gives RHS = 2,032

The consumers' association is correct.

If the banks are correct, then the present value of the payments received by the bank, after expenses, is less than the amount of the loan at a nominal (before inflation) rate of interest of $(1.01463\times1.025 - 1)$ per annum effective = 0.04.

i.e. $2{,}000 > 720\dfrac{i}{d^{(12)}}a_{\overline{2}|} + 720\dfrac{i}{d^{(12)}}a_{\overline{1.5}|} + 960\dfrac{i}{d^{(12)}}a_{\overline{1}|} - 0.3\times2{,}400\dfrac{i}{d^{(12)}}a_{\overline{2}|}$

$\dfrac{i}{d^{(12)}} = 1.021529$; $a_{\overline{2}|} = 1.8861$; $a_{\overline{1}|} = 0.9615$; $a_{\overline{1.5}|} = \dfrac{1-1.04^{-1.5}}{0.04} = 1.4283$

So RHS = $720\times1.021529\times1.8861 + 720\times1.021529\times1.4283 + 960\times1.021529\times0.9615 - 0.3\times2{,}400\times1.021529\times1.8861$

= 1,387.23+ 1,050.52 + 942.91 – 1,387.23 = 1,993.43

Therefore, the banks are also correct.

11 (i) Present value of the fund's liabilities (in £m) is:

$$100\left(v + 1.05v^2 + 1.05^2v^3 + \ldots + 1.05^{59}v^{60}\right)$$

$$= 100v\left(1 + 1.05v + (1.05v)^2 + \ldots + (1.05v)^{59}\right)$$

$$= 100v\left(\frac{1-(1.05v)^{60}}{1-1.05v}\right) = 100\times0.97087\left(\frac{1\text{-}\left(\frac{1.05}{1.03}\right)^{60}}{1\text{-}\left(\frac{1.05}{1.03}\right)}\right)$$

$$= 97.087\times111.7795 = £10{,}852m$$

(ii) Let the nominal holding of bonds = N in £m

The present value of the bonds must equal £10,852m

Therefore $0.04Na_{\overline{20|}} + Nv^{20} = 10{,}852$ at 3%

$a_{\overline{20|}} = 14.8775$, $v^{20} = 0.55368$

So $10{,}852 = 0.04N \times 14.8775 + N \times 0.55368$

$N = 10{,}852 / (0.04 \times 14.8775 + 0.55368) = £9{,}446.54m$

(iii) The numerator for the duration of the liabilities can be expressed as follows:

$$100v\,(1\times 1 + 1.05v\times 2 + 1.05^2v^2 \times 3 + \ldots + 1.05^{59}v^{59}\times 60)$$

$$= \frac{1.03}{1.05}100\,v\,(1.05v\times 1 + 1.05^2v^2\times 2 + 1.05^3v^3 \times 3 + \ldots + 1.05^{60}v^{60}\times 60)$$

The part inside the brackets can be regarded as $(Ia)_{\overline{60|}}$ evaluated at a rate of interest i such that $v = 1.05/1.03$; the discount factor outside the brackets should be evaluated at 3%

$$\frac{1.03}{1.05}100\,v = \frac{100}{1.05} = 95.2381$$

For the $(Ia)_{\overline{60|}}$ function, $v = 1.019417$; $i = -0.019048$; $(1+i)a_{\overline{60|}} = 111.7727$

$$(Ia)_{\overline{60|}} = \frac{111.7727 - 60\times 1.019417^{60}}{-0.019048} = 4118.567$$

Therefore numerator for duration is: $95.2381 \times 4118.567 = 392{,}244$

Therefore the duration is: 392,244/10,852 = 36.1 years.

(iv) The duration of the assets can be expressed as the sum of payments times time of receipt times present value factors divided by total present value.

The equation for the numerator is

$$0.04\times 9{,}446.54\,(Ia)_{\overline{20|}} + 9{,}446.54\times 20\times v^{20} \text{ at } 3\%$$

$(Ia)_{\overline{20|}} = 141.6761$, $v^{20} = 0.55368$

Numerator is: 158,141

Therefore the duration is: 158,141/10,852 = 14.6 years.

(v) Duration of the liabilities is 36.1 years. Therefore volatility of the liabilities is: 36.1/1.03 = 35. If there were a reduction in interest rates to 1.5%, the liabilities would increase in value by approximately $35\times 1.5 = 52.5\%$

Duration of the assets is 14.6 years. Therefore volatility of the assets is: 14.6/1.03 = 14.2. If there were a reduction in interest rates to 1.5%, the assets would increase in value by approximately $14.2 \times 1.5 = 21.3\%$.

The liabilities would increase in value by an additional 31.2% of their original value i.e. by £3,386 more than the value of the assets.

END OF EXAMINERS' REPORT