Subject CT1 — Financial Mathematics. Core Technical.

September 2009 examinations

EXAMINERS REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

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Chairman of the Board of Examiners

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Comments for individual questions are given with the solutions that follow.

Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

Well-prepared candidates scored well across the whole paper. However, the comments below on each question concentrate on areas where candidates could have improved their performance.

1

a. $96\ 1.05^{\ t} = 97.89 \Rightarrow\ 1.05^{\ t} = \dfrac{97.89}{96}$

$$t = \frac{\ln\ {}^{97.89}\!/_{96}}{\ln\ 1.05} = 0.400 \text{ years or } 146 \text{ days}$$

b. Second investor held the bill for 36 days. Therefore

$$97.89\left(1+\frac{36}{365}i\right) = 100 \Rightarrow i = \frac{365}{36}\left(\frac{100}{97.89}-1\right) = 21.854\%$$

This was answered well except by the very weakest candidates.

2

- Issued by corporations.
- Holders entitled to a distribution (dividend) declared from profits.
- Potential for high returns relative to other asset classes.
- Commensurate risk of capital losses.
- Lowest ranking finance issued by companies.
- Initial running yield low but has potential to increase with dividend growth.
- Dividends and capital values have the potential to grow in nominal terms during times of inflation.
- Return made up of income return and capital gains.
- Marketability depends on the size of the issue.
- Ordinary shareholders receive voting rights in proportion to their holding.

This question was not answered as well as the examiners would have expected given that the topic is standard bookwork.

3

We convert all cash flow to amounts in time 0 values:

Dividend paid at $t = 1: 10000 \times 0.041 \times \dfrac{147.7}{153.4} = 394.77$

Dividend paid at $t = 2: 10000 \times 0.046 \times \dfrac{147.7}{158.6} = 428.39$

Dividend paid at $t = 3: 10000 \times 0.051 \times \dfrac{147.7}{165.1} = 456.25$

Sale proceeds at $t = 3: 10000 \times 0.93 \times \dfrac{147.7}{165.1} = 8319.87$

$\Rightarrow$ Equation of value involving v where $v = \dfrac{1}{1+r}$
and r = real rate of return:

$$7800 = 394.77v + 428.39v^2 + 8776.17v^3 \ldots\ldots (1)$$

[To estimate r:

Approx nominal rate of return is

$$\left(4.6 + \frac{93-78}{3}\right) / 78 = 12.3\% \text{ p.a.}$$

Average inflation over 3 year period comes from

$$\left(\frac{165.1}{147.7}\right)^{1/3} - 1 = 3.8\,\% \text{ p.a.}$$

$\Rightarrow$ Approx real return: $\dfrac{1.123}{1.038} - 1 = 8.2\,\%$ p.a.]

Try $r = 8\%$, RHS of $(1) = 7699.61$

$r = 7\%$, RHS of $(1) = 7907.09$

$$r = 7\% + \frac{7907.09 - 7800}{7907.69 - 7699.61} \times 1\%$$

= 7.52 % p.a.

Some candidates seemed to struggle to derive the equation of value based on a real rate of return and multiplied (rather than divided) the payments by the increase in the inflation index.

4

(i) Let required price = *P*:

$$P = \ 1-0.2 \ \ 5a^{2}_{\overline{20}|} + 100v^{20} \text{ at } 6\%$$

$$a^{2}_{\overline{20}|} \ = \ \frac{i}{i^{\ 2}} a_{\overline{20}|} = \frac{0.06}{0.059126} 11.4699 = 11.6394; \ v^{20} = 0.311805$$

Therefore

$$P = \ 1-0.2 \ \ 5 \times 11.6394 + 100 \times 0.311805$$
$$= 46.5576 + 31.1805 = 77.7381$$

(ii) The equation of value for the gross rate of return is:

$$77.7381 = 5a^{2}_{\overline{20}|} + 100v^{20}$$

If i = 8%

$$a^{2}_{\overline{20}|} \ = \ \frac{i}{i^{\ 2}} a_{\overline{20}|} = 1.019615 \times 9.8181 = 10.0107; \ v^{20} = 0.21455$$

RHS = 50.0534 + 21.4550 = 71.5084

If i = 7%

$$a^{2}_{\overline{20}|} \ = \ \frac{i}{i^{\ 2}} a_{\overline{20}|} = 1.017204 \times 10.5940 = 10.7763; \ v^{20} = 0.25842$$

RHS = 53.8813 + 25.8420 = 79.7233

Interpolating gives $i \approx 0.07 + \frac{79.7233 - 77.7381}{79.7233 - 71.5084} \times 0.01 = 7.24\% = 7.2\%$ say

(iii) If the nominal rate of return is 7.2% per annum effective and inflation is 3% per annum effective, then the real rate of return is calculated from:

$$\left(\frac{1.072}{1.03}\right) - 1 = 4.1\%$$

This question was answered very well.

5

(i) $130 = 100\exp\left\{\int_0^5 \left(a+bt^2\right) dt\right\} = 100\exp\left[at + \tfrac{1}{3}bt^3\right]_0^5 = 100\exp\left(5a + 41.667b\right)$

$$200 = 100\exp\left\{\int_0^{10} \left(a+bt^2\right) dt\right\} = 100\exp\left[at + \tfrac{1}{3}bt^3\right]_0^{10} = 100\exp\left(10a + 333.333b\right)$$

$\ln\left(1.3\right) = 5a + 41.667b$

$\ln\left(2\right) = 10a + 333.333b$

The second expression less twice times the first expression gives:

$\ln(2) - 2\ln(1.3) = 250b \Rightarrow b = 0.0006737$

$$a = \frac{\ln(2) - 333.333 \times 0.0006737}{10} = 0.04686$$

(ii) $100\left(1+\frac{i^{(12)}}{12}\right)^{60} = 130 \Rightarrow i^{(12)} = 12\left[\left(\frac{130}{100}\right)^{1/60} - 1\right] \Rightarrow i^{(12)} = 5.259\%$ p.a.

(iii) $130e^{5\delta} = 200 \Rightarrow 5\delta = \ln\frac{200}{130} \Rightarrow \delta = 8.616\%$ p.a.

This question was answered very well.

6

(i) A future is a contract which obliges the parties to deliver/take delivery of a particular quantity of a particular asset at a particular time at a fixed price.

An option is the right to buy or sell a particular quantity of a particular asset at (or before) a particular time at a given price.

(ii) Assume no arbitrage

a. Buying the forward is exactly the same as buying the bond except that the forward will not pay coupons and the forward does not require immediate settlement.

Let the forward price = F. The equation of value is:

$$F \;=\; 97\;\; 1.06 \;\; - 3.5 \times \frac{1.06}{1.05^{\;1/2}} - 3.5$$

$$= \; 102.82 - 3.62059 - 3.5 = 95.6994$$

b. Let six month forward interest rate $= f_{0.5,0.5} = \dfrac{1.06}{1.05^{\;1/2}} - 1 = 3.4454\%$

This does not have to be expressed as a rate of interest per annum effective, though it could be.

c. $P = 2\;\; 1.05^{\;-0.5} + 102\;\; 1.06^{\;-1} = 1.9518 + 96.2264 = 98.1782$

d. Gross redemption yield is i such that

$$98.1782 = 2\;\; 1+i^{\;-0.5} + 102\;\; 1+i^{\;-1}$$

Using the formula for solving a quadratic (interpolation will do):

$1+i^{\;-0.5} = 0.97133$. Therefore, $i \approx 6\%$ (in fact 5.99%).

e. Answer is very close to 6% (the one-year spot rate) because the payments from the bond are so heavily weighted towards the redemption time in one year.

This was generally well-answered apart from part (e). A common error in parts (c) and (d) was to assume that the coupon payments were 4% per half-year.

7 .

(i) The accumulation is $1200\ddot{s}^{(12)}_{\overline{20|}}\;\; 1.06^{\;20} + 2300\ddot{s}^{(12)}_{\overline{20|}} + 100\;\; I\ddot{a}^{(12)}_{\;\overline{20|}}\;\; 1.06^{\;20}$

$$= \frac{i}{d^{(12)}}\;\; 1200 s_{\overline{20|}}\;\; 1.06^{\;20} + 2300 s_{\overline{20|}} + 100\;\; Ia_{\;\overline{20|}}\;\; 1.06^{\;20}$$

$$= 1.032211\begin{pmatrix} 1,200 \times 36.7856 \times 3.20714 + 2,300 \times 36.7856 \\ +100 \times 98.7004 \times 3.20714 \end{pmatrix}$$

$$= 1.032211\;\; 141,571.88 + 84,606.88 + 31,654.60$$

$$= 266,138$$

(ii) Let half-yearly payment = X

$$Xa_{\overline{40|}} = 266{,}138 \text{ at } 2.5\%$$

$$\Rightarrow X = \frac{266{,}138}{25.1028} = 10{,}601.94$$

Therefore, annual rate of payment = £21,203.88

(iii) Work in half-years. Discounted mean term is:

$$10{,}601.94\left(v + 2v^2 + \ldots + 40v^{40}\right)/266{,}138$$

Numerator = $10{,}601.94\ (Ia)_{\overline{40|}}$ at 2.5% per half year effective.

$$= 10{,}601.94 \times 433.3248 = 4{,}584{,}075$$

Therefore DMT = 17.26 half years or 8.63 years.

In part (i), many candidates developed the correct formula although calculation errors were common. In such cases, candidates also lost marks for not showing and explaining their working fully. Part (ii) was answered well but many candidates surprisingly had trouble calculating the DMT in part (iii). In this part, candidates often lost marks for not showing the units properly at the end of the answer; indeed, in many cases, showing the units may well have alerted candidates to possible mistakes.

2

(i) The equation of value for the borrower is $4{,}012.13a_{\overline{20|}} = 50{,}000$.

Therefore $a_{\overline{20|}} = \dfrac{50{,}000}{4{,}012.13} = 12.4622$

From inspection of tables, i = 5%

(ii) The second customer pays interest of $0.055 \times 50{,}000$ = £2,750 per annum, annually in arrear.

The annual rate of monthly payments in advance from the savings policy is X such that:

$$X\ddot{s}^{(12)}_{\overline{20|}} = 50,000 \text{ at } 4\%$$

$$\Rightarrow X s_{\overline{20|}} \frac{i}{d^{(12)}} = 50,000$$

$$\Rightarrow X = \frac{50,000}{29.7781 \times 1.021537} = £1,643.69$$

The equation of value for this borrower is:

$$50,000 = 2,750 a_{\overline{20|}} + 1,643.686 \ddot{a}^{(12)}_{\overline{20|}}$$

$$= 2,750 a_{\overline{20|}} + 1,643.686 \frac{i}{d^{(12)}} a_{\overline{20|}}$$

Try i = 6%: RHS = 51,002.41

Try i = 7%: RHS = 47,200.14

By interpolation i = 6.3%

Part (i) was well answered but weaker candidates failed to recognise the need to calculate separately the payments into the savings policy in part (ii).

3

(i) The expected annual interest rate in the first ten years is $0.3 \times 0.04 + 0.7 \times 0.06 = 0.054$. The expected interest rate in the second ten years is clearly 5.5%.

If the premium is calculated on the basis of these interest rates, then the premium will be P such that:

$$20,000 = P(1.054)^{10}(1.055)^{10}$$

$$\Rightarrow 20,000 = 2.89022P \Rightarrow P = 6,919.89$$

(ii) The expected accumulation factor in the first ten years is:

$$0.3 \times 1.04^{10} + 0.7 \times 1.06^{10} = 1.69767$$

The expected accumulation factor in the second ten years is:

$$0.5\left[(1.05)^{10} + (1.06)^{10}\right] = 1.70987$$

As they are independent, we can multiply the accumulation factors together and multiply by the premium to give an expected accumulation of: $6,919.89 \times 1.69767 \times 1.70987 = 20,087.04$.

The expected profit is 87.04.

(iii) There is an expected profit because (in general) the accumulation of a sum of money at the expected interest rate is not equal to the expected accumulation when the interest rate is a random variable.

(iv) The highest possible outcome for the accumulation factor is:

$1.06^{10} \times 1.06^{10} = 3.20714$ with probability $0.7 \times 0.5 = 0.35$

The lowest possible outcome is:

$1.04^{10} \times 1.05^{10} = 2.41116$ with probability $0.3 \times 0.5 = 0.15$.

The range is therefore: 6,919.89 (3.20714 – 2.41116) = 5,508.05.

The other two possible outcomes are:

$1.06^{10} \times 1.05^{10} = 2.91710$ with probability $0.7 \times 0.5 = 0.35$

and $1.04^{10} \times 1.06^{10} = 2.65089$ with probability $0.3 \times 0.5 = 0.15$

The mean accumulation factor is: $1.69767 \times 1.70987 = 2.90280$

The variance of the accumulation from one unit of investment is:

$0.35(3.20714-2.90280)^2 + 0.15(2.41116-2.90280)^2$
$+ 0.35\,(2.91710-2.90280)^2 + 0.15\,(2.65089-2.90280)^2$

$= 0.03241 + 0.03626 + 0.00007 + 0.00952 = 0.07826.$

Standard deviation is $\sqrt{0.07826} = 0.27976$.

Standard deviation of the accumulation of the whole premium is: $6{,}919.89 \times 0.27976 = £1{,}935.88$ which is also the standard deviation of the profit.

This was the worst answered question on the paper with many candidates not recognising that the accumulation of a sum of money at the expected interest rate is not equal to the expected accumulation when the interest rate is a random variable. The calculation of the standard deviation of the accumulation was generally only calculated correctly by the strongest candidates.

4

(i) The discounted payback period is the first time at which the accumulated profit from/net present value of the cash flows from a project is positive at a given interest rate.

It is an inappropriate decision criterion because it does not tell us anything about the overall profitability of the project.

(ii) If the internal rate of return were greater than 1.5% then the net present value of the project at 1.5% must be greater than zero. As such, there must be a discounted payback period as the discounted payback period is the first time at which the net present value is greater than zero: such a time must exist.

(iii) Returns are real rates of return and figures are in 2009 dollar terms so we are automatically working with real rather than nominal values. All figures below are in \$bn.

The net benefits from using the technology are the \$30 every three years; \$20 incurred continuously increasing at 1% per annum and \$30 per annum incurred annually in arrears.

The costs of the technology are \$440 incurred immediately and \$50 incurred annually in arrears.

The net present value of the project at 1% per annum effective is:

$$30\ \ v^3+v^6+\ldots+v^{48}\ \ +50\times 20+30a_{\overline{50|}}-\ \ 440\ +\ 50a_{\overline{50|}}$$

The 20 does not need to be discounted because the cash flows are growing at the same rate as they are being discounted.

$$=30v^3\,\frac{1-v^{48}}{1-v^3}+560-20a_{\overline{50|}}\ \text{ calculated at 1\%}$$

$$=30\times 0.97059\,\frac{1-0.62026}{1-0.97059}+560-20\times 39.1961$$

$$=375.967\ +\ 560\ -\ 783.922$$

$$=152.045$$

(iv) The net present value of the project at 4% per annum effective is:

$$30\ \ v^3+v^6+\ldots+v^{48}\ \ +20\bar{a}'_{\overline{50|}}+30a_{\overline{50|}}-\ \ 440\ +\ 50a_{\overline{50|}}$$

All are calculated at 4% except $\bar{a}'_{\overline{50|}}$ which is calculated at

$$i\ =\ \frac{1.04}{1.01}-1\ =\ 2.97\%$$

$$=30v^3\,\frac{1-v^{48}}{1-v^3}+20\,\frac{i}{\delta}\,a'_{\overline{50|}}-440-20a_{\overline{50|}}$$

$$=30\times 0.88900\,\frac{1-0.15219}{1-0.88900}+20\times 1.014779\times 25.8755-440-20\times 21.4822$$

$$= 203.704 + 525.158 - 440 - 429.644$$
$$= -140.790$$

(v) Whether the investment should go ahead would depend on the choice of the interest rate – it is clearly a crucial assumption (students could make a choice themselves and indicate whether it should go ahead on the basis of that rate but there must be some justification for the choice).

This question was also poorly answered possibly because project appraisal using real interest rates has rarely been examined in the past (and also possibly because of time pressure). Whilst some parts of the question were challenging (e.g. the treatment of the increasing costs of climate change), it was disappointing that many candidates failed to recognise that the costs of climate change no longer incurred would be a benefit of the carbon storing technology project and so failed to score many marks.

END OF EXAMINERS' REPORT