EXAMINATION

April 2006

Subject CT1 — Financial Mathematics
Core Technical

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

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Chairman of the Board of Examiners

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Comments

Individual comments are shown after each question.

General comments

As is in some recent diets, the questions requiring verbal reasoning (such as Q3(c), Q7(b), Q10(iv) and Q11(iv)) tended not to be well answered with candidates producing vague statements which did not demonstrate that they understood the relevant points.

Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this.

However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

1 Annual rate of interest is i where $\left(1-\frac{28d}{365}\right)^{-1}=(1+i)^{28/365}$

This gives $i=\left(1-\frac{28\times 0.045}{365}\right)^{-365/28}-1=4.611\%$

Comments on question 1: *This was generally well answered.*

2 We require X where:

$$600a_{\overline{n}|}^{(4)}=12X\ddot{a}_{\overline{n}|}^{(12)}\Rightarrow X=50\frac{a_{\overline{n}|}^{(4)}}{\ddot{a}_{\overline{n}|}^{(12)}}=50\frac{d^{(12)}}{i^{(4)}}$$

$$d^{(12)}=12\left(1-(1-d)^{1/12}\right)=12\left(1-e^{-\delta/12}\right)=0.099584$$

$$i^{(4)}=4\left((1+i)^{1/4}-1\right)=4\left(e^{\delta/4}-1\right)=0.101260$$

Hence $X=49.1724$ or £49.17

Comments on question 2: *Candidates were not penalised for assuming that the annuities were for a specific term even though this was not needed for the calculations.*

3 (a) $\left(1+f_{3,2}\right)^2=\frac{(1+y_5)^5}{(1+y_3)^3}=\frac{(1.035)^5}{(1.03)^3}\Rightarrow f_{3,2}=4.255\%$

(b) Par yield is yc_4 where $yc_4\left(v_{y_1}+v_{y_2}^2+v_{y_3}^3+v_{y_4}^4\right)+v_{y_4}^4=1$

Thus $yc_4\left(1.025^{-1}+1.0275^{-2}+1.03^{-3}+1.0325^{-4}\right)+1.0325^{-4}=1$

$$yc_4=\frac{0.12009}{3.71785}=3.230\%$$

(c) The par yield is equal to the gross redemption yield for a par yield bond. Coupons for the 3.5% bond are higher than for the par yield bond. Thus a lower proportion of the total proceeds are included within the redemption payment which is when spot yields/discount rates are highest. The present value of the proceeds of the 3.5% bond will be higher and so the gross redemption yield will be lower than that of the par yield bond and thus less than the par yield.

Comments on question 3: *Part (a) was answered well but some candidates struggled with the calculation of the par yield in part (b). In part (c) the marks were awarded for a clear explanation. Many candidates, who just stated their conclusion, were unable to explain their reasoning clearly and so failed to score full marks on this part.*

4 $i^{(2)} = 0.049390$

$$g\left(1-t_1\right) = 0.0625 \times 0.80 = 0.05$$

$$\Rightarrow i^{(2)} < \left(1-t_1\right) g$$

$\Rightarrow$ Capital loss on contract

$\Rightarrow$ Assume redeemed as early as possible (i.e.: after 10 years) to obtain minimum yield.

Price of stock per £100 nominal, P:

$$P = 100 \times 0.0625 \times 0.80 \times a_{\overline{10}|}^{(2)} + 100 v^{10} \text{at } 5\%$$

$$\Rightarrow P = 5\, a_{\overline{10}|}^{(2)} + 100 v^{10}$$

$$= \left(5 \times 1.012348 \times 7.7217\right) + \left(100 \times 0.61391\right)$$

$$= 39.0852 + 61.3910 = £100.4762$$

Comments on question 4: *Well answered although some candidates who recognised that the investor faced a capital loss did not recognise that this meant that the minimum yield would be obtained if the bond was redeemed at the earliest possible date.*

5 An investor can borrow £10 at the risk-free rate, buy one share for £10, enter into the forward contract to sell the share in six months time.

The initial cashflow is zero.

After one month the 50*p* dividend from the share is invested at the risk-free rate. After six months the share can be sold for £9.70, the dividend proceeds are worth $0.5e^{0.03\times 5/12}$ and the borrowing is repaid at $10e^{0.015}$. This gives a net cashflow of 9.7 $+0.5e^{0.03\times 5/12}-10e^{0.015}$ = 0.0552

The investor has made a deal with zero initial cost, no risk of future loss and a risk-free future profit.

Comments on question 5: *The majority of candidates were able to calculate the non-arbitrage forward price by use of the appropriate formula. However, marks were lost for not clearly explaining how a risk-free profit could thus be made.*

6 (a) Expected accumulated value

$$=800\left(0.25\ddot{s}_{\overline{10}|0.02}+0.55\ddot{s}_{\overline{10}|0.04}+0.2\ddot{s}_{\overline{10}|0.07}\right)$$

$$=800\left(0.25\left(s_{\overline{11}|0.02}-1\right)+0.55\left(s_{\overline{11}|0.04}-1\right)+0.2\left(s_{\overline{11}|0.07}-1\right)\right)$$

$$=800\left(\left(0.25\times 11.1687\right)+\left(0.55\times 12.4864\right)+\left(0.2\times 14.7836\right)\right)$$

$$=\left(0.25\times 8934.96\right)+\left(0.55\times 9989.12\right)+\left(0.2\times 11826.88\right)$$

$$=\pounds 10{,}093.13$$

(b) Accumulation is only over £10,000 if the interest rate is 7% p.a. which has probability 0.2

Comments on question 6: *The most poorly answered question on the paper. This model of interest rates had not been examined recently and the majority of candidates assumed instead that the interest rate changed each year (in line with previous examination questions on this topic).*

7 (a) The counterparty faces *market risk* which is the risk that market conditions will change so that the present value of the net outgo under the agreement increases.

The counterparty also faces *credit risk* which is the risk that the other counterparty will default on its payments.

(b) The company still faces the market risk since the interest rates could fall further which will make the value of the swap even more negative to the company.

The company does not currently face a credit risk since the value of the swap is positive to the other counterparty.

Comments on question 7: *Part (a) was answered well but many candidates failed to recognise in (b) that the company would not currently face credit risk in this example.*

8 (i) Main characteristics of ordinary shares:

- Issued by commercial undertakings and other bodies.
- Entitle holders to receive all net profits of the company in the form of dividends after interest on loans and other fixed interest stocks has been paid.
- Higher expected returns than for most other asset classes …
- …but risk of capital losses
- … and returns can be variable.
- Lowest ranking form of finance.
- Low initial running yield but dividends should increase with inflation.
- Marketability varies according to size of company.
- Voting rights in proportion to number of shares held.

(ii) Present value of future dividends

$$=100\times 0.25\left(1.02v+1.02\times 1.04v^2+1.02\times 1.04\times 1.06v^3+1.02\times 1.04\times 1.06^2v^4+\ldots\right)$$

$$=25\times 1.02v+25\times 1.02\times 1.04v^2\left(1+1.06v+1.06^2v^2+\ldots\right)$$

$$=25\times 1.02v+25\times 1.02\times 1.04v^2\left(\frac{1.09}{0.03}\right)$$

$$=23.3945+811.0092=834.4037=£834.40$$

(iii) Real rate of return is i such that:

$$820 = 100 \times 0.25 \times 1.02 \times \frac{100}{103} v + 100 \times 0.25 \times 1.02 \times 1.04 \times \frac{100 \times 100}{103 \times 103.5} v^2$$

$$+900 \times \frac{100 \times 100}{103 \times 103.5} v^2$$

$$= 24.7573v + 869.1150v^2$$

$$v = \frac{-24.7573 \pm \sqrt{24.7573^2 + 4 \times 869.1150 \times 820}}{2 \times 869.1150} = 0.95719$$

(taking positive root)

Hence i = 4.47%

Comments on question 8: *Despite being a bookwork question, part (i) was answered patchily with few students getting all of the required points. Part (ii) was answered well. In part (iii), it was expected that students would solve the quadratic equation. However, full credit was given to students who used interpolation methods.*

9 (i) $A(0,5) = e^{\int_0^5 0.04dt} = e^{\left[0.04t\right]_0^5} = e^{0.2} = 1.22140$

$$A(5,10) = e^{\int_5^{10} 0.008tdt} = e^{\left[0.004t^2\right]_5^{10}} = e^{0.3} = 1.34986$$

$$A(10,12) = e^{\int_{10}^{12}\left(0.005t+0.0003t^2\right)dt} = e^{\left[0.0025t^2+0.0001t^3\right]_{10}^{12}} = e^{0.1828} = 1.20057$$

Required present value

$$= \frac{1}{A(0,5)\,A(5,10)\,A(10,12)} = \frac{1}{1.22140 \times 1.34986 \times 1.20057} = \frac{1}{1.97941}$$

= 0.50520

(ii) Equivalent effective annual rate is i where $(1+i)^{12} = 1.97941 \Rightarrow i = 5.855\%$

(iii) Present Value at time $t = 0$

$$= \int_2^5 e^{-0.05t} \left(e^{-\int_0^t 0.04 ds} \right) dt = \int_2^5 e^{-0.05t} \left(e^{-0.04t} \right) dt$$

$$= \int_2^5 e^{-0.09t} dt = \left[\frac{e^{-0.09t}}{-0.09} \right]_2^5 = \frac{e^{-0.18} - e^{-0.45}}{0.09} = 2.1960$$

Comments on question 9: *Well answered.*

10 (i) Net present value of costs

$$= 5,000,000 + 3,500,000 \bar{a}_{\overline{2}|} = 5,000,000 + 3,500,000 \frac{i}{\delta} a_{\overline{2}|}$$

$$= 5,000,000 + 3,500,000 \times 1.073254 \times 1.6257 = 11,106,762$$

Net present value of benefits

$$= 450,000 v^2 \ddot{a}^{(4)}_{\overline{n-2}|} + 50,000 v^2 \left(I\ddot{a} \right)^{(4)}_{\overline{n-2}|} + S_n v^n$$

$$= 450,000 v^2 \frac{i}{d^{(4)}} a_{\overline{n-2}|} + 50,000 v^2 \frac{i}{d^{(4)}} \left(Ia \right)_{\overline{n-2}|} + S_n v^n$$

where n is the year of sale and S_n are the sale proceeds if the sale is made in year n.

If $n = 3$ the NPV of benefits

$$\begin{aligned} &= (450,000 \times 0.75614 \times 1.092113 \times 0.86957) \\ &+ (50,000 \times 0.75614 \times 1.092113 \times 0.86957) \\ &+ (16,500,000 \times 0.65752) \\ &= 323,137 + 35,904 + 10,849,080 = 11,208,121 \end{aligned}$$

Hence net present value of the project is 11,208,121 – 11,106,762 = 101,359

Note that if $n = 4$ the extra benefits in year 4 consist of an extra £1.5 million on the sale proceeds and an extra £650,000 rental income. This is clearly less than the amount that could have been obtained if the sale had been made at the end of year 3 and the proceeds invested at 15% per annum. Hence selling in year 4 is not an optimum strategy.

If n = 5 the NPV of benefits

$$\begin{aligned}
&= (450,000 \times 0.75614 \times 1.092113 \times 2.2832) \\
&+ (50,000 \times 0.75614 \times 1.092113 \times 4.3544) \\
&+ (20,500,000 \times 0.49718) \\
&= 848,450 + 179,791 + 10,192,190 = 11,220,431
\end{aligned}$$

Hence net present value of the project is 11,220,431 – 11,106,762 = 113,669

Hence the optimum strategy if net present value is used as the criterion is to sell the housing after 5 years.

(ii) If the discounted payback period is used as the criterion, the optimum strategy is that which minimises the first time when the net present value is positive. By inspection, this is when the housing is sold after 3 years.

(iii) We require

$$5,000,000 + 3,500,000\frac{i}{\delta}a_{\overline{2}|} = 450,000v^2\ddot{a}^{(4)}_{\overline{n-2}|} + 50,000v^2\,(I\ddot{a})^{(4)}_{\overline{n-2}|} + S_n v^n \text{ at } 17.5\%$$

$$\text{LHS} = 5,000,000 + 3,500,000\left(\frac{1-v^2_{0.175}}{\delta_{0.175}}\right) = 5,000,000 + 3,500,000\left(\frac{1-0.72431}{0.16127}\right)$$

$$= 10,983,227$$

$$\text{RHS} = 450,000v^2_{0.175}\left(\frac{1-v^4_{0.175}}{d^{(4)}}\right) + 50,000v^2_{0.175}\left(\frac{\ddot{a}_{\overline{4}|} - 4v^4_{0.175}}{d^{(4)}}\right) + S_6 v^6_{0.175}$$

$$d^{(4)}_{0.175} = 4\left(1 - v^{1/4}\right) = 0.15806$$

$$\ddot{a}_{\overline{4}|} = \frac{1-v^4}{d} = 3.1918$$

Therefore we have on the RHS

$$450,000 \times 0.72431 \times 3.0076 + 50,000 \times 0.72431 \times \left(\frac{3.1918 - 2.0985}{0.15806}\right) + 0.37999S_6$$

$$= 980,296 + 250,502 + 0.37999S_6$$

For equality $S_6 = \dfrac{10,983,227 - 1,230,798}{0.37999} = £25,665,000$

(iv) Reasons investor may not achieve the internal rate of return:

- Allowance for expenses when buying/selling which may be significant.
- There may be periods when the property is unoccupied and no rental income is received.
- Rental income may be reduced by maintenance expenses.
- Tax on rental income and/or sale proceeds

Comments on question 10: *A significant number of candidates assumed that the development costs amounted to £7 million per annum and subsequently found that no strategy would lead to a profit. Otherwise the calculations were performed well. In part (iv), credit was given for other valid answers. Despite this, few students scored full marks on this part.*

11 (i) Let X_A, X_B be the monthly repayments under Loans A and B respectively.

For loan A:

Flat rate of interest = 10.715%

$$= \frac{60X_A - L_A}{5L_A} = \frac{60X_A - 10000}{50000} \Rightarrow X_A = £255.96$$

For loan B:

$$L_B = 15000 = 12X_B\left(a^{(12)}_{\overline{2|}12\%} + v^2_{12\%}a^{(12)}_{\overline{3|}10\%}\right)$$

$$X_B = \frac{1,250}{\left(\frac{i}{i^{(12)}}a_{\overline{2|}}\right)_{12\%} + v^2_{12\%}\left(\frac{i}{i^{(12)}}a_{\overline{3|}}\right)_{10\%}}$$

$$= \frac{1,250}{(1.053875\times1.6901) + 0.79719(1.045045\times2.4869)}$$

$$\Rightarrow X_B = £324.43$$

Hence student's overall surplus = 600 – $X_A - X_B$ = £19.61

(ii) Effective rate of interest under loan A is *i* % where

$$= 12 \times 255.96 a_{\overline{5}|}^{(12)} = 10000 \Rightarrow a_{\overline{5}|}^{(12)} = 3.2557$$

Try i = 20%: $a_{\overline{5}|}^{(12)}$ = 3.2557

So capital outstanding after 24 months is $12 \times 255.96\ a_{\overline{3}|}^{(12)}$ at 20%

$$= 12 \times 255.96 \times 1.088651 \times 2.1065 = 7043.74$$

Capital outstanding under B is $12 \times 324.43\ a_{\overline{3}|}^{(12)}$ at 10%

$$= 12 \times 324.43 \times 1.045045 \times 2.4869 = 10118.02$$

So interest paid in month 25 under loans A and B

$$= 7043.74 \frac{i_{20\%}^{(12)}}{12} + 10118.02 \frac{i_{10\%}^{(12)}}{12} = 107.84 + 80.68 = £188.52$$

and capital repaid

$$= (255.96 - 107.84) + (324.43 - 80.68) = 148.12 + 243.75 = £391.87$$

(iii) Under the new loan the capital outstanding is the same as under the original arrangement = 17161.76.

The monthly repayment $= \left(\dfrac{255.96 + 324.43}{2} \right) = £290.20$

The effective rate of interest on the new loan A is *i* where

$$= 12 \times 290.20 a_{\overline{10}|}^{(12)} = 17161.76 \Rightarrow a_{\overline{10}|}^{(12)} = 4.9281$$

Try i = 20%: $a_{\overline{10}|}^{(12)} = 4.5642$

Try i = 15%: $a_{\overline{10}|}^{(12)} = 5.3551$

By interpolation $i = 15\% + \left(\dfrac{5.3551 - 4.9281}{5.3551 - 4.5642} \right) (20\% - 15\%) \approx 17.7\%$

Hence interest paid in month 25

$$=17161.76\frac{i^{(12)}_{17.7\%}}{12}=234.66$$

and capital repaid is £290.20 – £234.66 = £55.54

(iv) The new strategy reduces the monthly payments but repays the capital more slowly. The student could consider the following options:

- Keeping loan B and taking out a smaller new loan to repay loan A (which has the highest effective interest rate).

- Taking out the new loan for a shorter term to repay the capital more quickly.

Comments on question 11: *In part (i) some candidates struggled to deal with the flat rate of Loan A whilst others failed to deal with the change in interest rate of Loan B. Part (ii) was answered well. In part (iii), different answers for the effective rate of interest (and hence the interest paid) for the new loan could be obtained according to the actual interpolation used and full credit was given for a range of answers. If calculated exactly, the effective rate of interest is actually 17.5%. In part (iv), credit was again given for any valid strategy suitably explained.*

12 (i) We will consider three conditions necessary for immunisation

(1) $V_A = V_L$ (all expressions in terms of £m)

$$V_A = a_{\overline{5}|} + Rv^n \text{ at } 8\%$$
$$= 3.9927 + Rv^n$$

$$V_L = 3v^3 + 5v^5 + 9v^9 + 11v^{11} \text{ at } 8\%$$
$$= 15.0044$$

$$\Rightarrow Rv^n = 11.0117$$

(2) $V_A^{'} = V_L^{'}$ where $V_A^{'} = \dfrac{\partial V_A}{\partial \delta}$ & $V_L^{'} = \dfrac{\partial V_L}{\partial \delta}$

$$V_A^{'} = -(Ia)_{\overline{5}|} - nRv^n$$
$$= -11.3651 - nRv^n$$

$$V_L^{'} = -9v^3 - 25v^5 - 81v^9 - 121v^{11}$$
$$= -116.5741$$

$$\Rightarrow nRv^{n} = 105.2090$$
$$\Rightarrow n = \frac{105.2090}{11.0117} = 9.5543$$
$$\Rightarrow R = 11.0117 \times (1.08)^{9.5543} = £22.9720m$$

Alternatively:

$$V_A^{'} = V_L^{'} \text{ where } V_A^{'} = \frac{\partial V_A}{\partial i} \& V_L^{'} = \frac{\partial V_L}{\partial i}$$

$$V_A^{'} = -v(Ia)_{\overline{5|}} - nRv^{n+1}$$
$$= -11.3651v - nRv^{n+1}$$
$$= -10.5233 - nRv^{n+1}$$

$$V_L^{'} = -9v^4 - 25v^6 - 81v^{10} - 121v^{12}$$
$$= -107.9389$$

$$\Rightarrow nRv^{n+1} = 97.4156$$

$$\Rightarrow n = \frac{97.4156}{11.0117v} = 9.5543$$
$$\Rightarrow R = 11.0117 \times (1.08)^{9.5543} = £22.9720m$$

(3) $V_A^{''} > V_L^{''}$ (where $V_A^{''} = \dfrac{\partial^2 V_A}{\partial \delta^2} \& V_L^{''} = \dfrac{\partial^2 V_L}{\partial \delta^2}$)

$$V_A^{''} = \sum_{t=1}^{5} t^2 v^t + n^2 R v^n$$
$$= 40.275 + (9.5543)^2 \times 22.9720 \times v^{9.5543}$$
$$= 1045.483$$

$$V_L^{''} = 27v^3 + 125v^5 + 729v^9 + 1331v^{11}$$
$$= 1042.031$$

Alternatively (differentiating with respect to i):

$$V_A'' = \sum_{t=1}^{5} t\left(t+1\right)v^{t+2} + n\left(n+1\right)Rv^{n+2}$$

$$= v^2\sum_{t=1}^{5} t^2 v^t + v^2\left(Ia\right)_{\overline{5|}} + n\left(n+1\right)Rv^{n+2}$$

$$= 0.85734\times 40.275 + 0.85734\times 11.3651 + 9.5543\times 10.5543\times 22.9720\times v^{11.5543}$$

$$= 34.53 + 9.74 + 952.00 = 996.27$$

$$V_L'' = 3\times 3\times 4\times v^5 + 5\times 5\times 6\times v^7 + 9\times 9\times 10\times v^{11} + 11\times 11\times 12\times v^{13}$$

$$= 993.32$$

Thus $n = 9.5543, R = £22.9720m$ will satisfy all three conditions and so will achieve immunisation.

(ii) (a) Value of assets at 3% $= a_{\overline{5|}} + Rv^n = 4.5797 + 22.9720v^{9.5543} = £21.900m$

Value of liabilities at 3% = $3v^3 + 5v^5 + 9v^9 + 11v^{11} = £21.903m$

Hence fund has a deficit of approximately £3,000.

(b) Immunisation will only enable to be a fund to be protected against a *small* change in interest rates. It will not be necessarily protected against sudden large changes as in this case.

Comments on question 12: *Part (i) was answered surprisingly poorly, given that it required the same techniques as those required in previous examination questions on the same topic. Full credit was given to students who observed directly that the spread of the assets around the mean term was greater than the spread of the liabilities. Few students answered part (ii) fully and the examiners felt that students should have recognised that immunisation would not protect the fund against such a large change in interest rates even if they had not answered part (i) correctly.*

END OF EXAMINERS' REPORT