Subject CT1 — Financial Mathematics Core Technical

EXAMINERS' REPORT

April 2008

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

M A Stocker
Chairman of the Board of Examiners

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Comments

Comments on solutions presented to individual questions for this April 2008 paper are given below.

Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

Question 1 *Well answered.*

Question 2 *As has often been the case when words rather than numbers have been required, this bookwork question was answered poorly.*

Question 3 *Generally well answered, although some students treated the fees on Product B paid by the customer as a cost to the mortgage company.*

Question 4 *Well answered although many candidates' working was unclear when performing the CGT test.*

Question 5 *Part (i) was answered well but in part (ii) many candidates failed to recognise the need to calculate the 4-year spot rate before calculating the bond price.*

Question 6 *Part (i) of this question did appear to differentiate between stronger candidates who often scored very well and weaker candidates who often failed to score at all. As with many previous diets, many candidates in part (ii) had difficulty in giving a clear explanation of their results.*

Question 7 *This question was answered relatively poorly with, particularly in part (ii), candidates often appearing confused between real and money rates of interest.*

Question 8 *Most candidates managed to make a reasonable attempt at this question although marks were often lost in part (i) through a combination of calculation errors and insufficient working being shown. Candidates generally made a better attempt at the explanation required in part (ii) when compared to similar questions both on this paper and in previous diets.*

Question 9 *Well answered.*

Question 10 *Part (i) (for Option A) can be done much more simply than by using the method given in this report but the calculations given would still need to be done for part (ii). It was disappointing to see many candidates incorrectly calculate the mean accumulated value for Option B by using the mean rate of interest. Few candidates brought together the answers from (i) and (ii) to fully answer part (iii).*

1 The present value of the dividends, I, is:

$$I = 0.5v^{1/12} + 0.5v^{7/12} = 0.5\left(0.99594 + 0.97194\right) = 0.98394 \text{ calculated at } i = 5\%$$

Hence forward price is (again calculated at i = 5%):

$$F = \left(10 - 0.98394\right)\left(1+i\right)^{11/12} = 9.42845$$
$$= £9.43$$

2 (a) Eurobonds

- form of unsecured medium or long-term borrowing
- issued in a currency other than the issuer's home currency outside the issuer's home country
- pay regular interest payments and a final capital repayment at par.
- issued by large companies, governments and supra-national organisations.
- yields depend upon the issuer and issue size but will typically be slightly lower than for the conventional unsecured loan stocks of the same issuer.
- issuers have been free to add novel features to their issues in order to make them appeal to different investors.
- usually issued in bearer form

(b) Certificates of Deposit

- a certificate stating that some money has been deposited
- issued by banks and building societies
- terms to maturity are usually in the range 28 days to 6 months.
- interest is payable on maturity
- security and marketability will depend on the issuing bank
- active secondary market

3 For Product A, the annual rate of return satisfies the equation:

$$7,095.25 a_{\overline{25|}} = 100,000$$

$$\Rightarrow a_{\overline{25|}} = 14.0939$$

This equates to the value of $a_{\overline{25|}}$ at 5%. Hence the annual effective rate of return is 5%.

For Product B, the annual rate of payment is X such that:

$$X\ddot{a}^{(12)}_{\overline{25|}} = 100,000 \text{ at } 4\%$$

$$\ddot{a}^{(12)}_{\overline{25|}} = \frac{i}{d^{(12)}} a_{\overline{25|}} = 1.021537 \times 15.6221 = 15.95855$$

$$\Rightarrow X = \frac{100,000}{15.95855} = 6,266.23$$

The equation of value to calculate the rate of return from Product B is:

$$6{,}000 + 5{,}000v^{25} + 6{,}266.23\frac{i}{d^{(12)}}a_{\overline{25|}} = 100{,}000$$

Clearly the rate of return must be greater than 4%. Try 5%.

$LHS = 6{,}000 + 5{,}000 \times 0.29530 + 6{,}266.2335 \times 1.026881 \times 14.0939 = 98{,}166$

At 5% the present value of the payments is less than the amount of the loan at 5% so the rate of return must be less than 5%. Try 4%:

$LHS = 6{,}000 + 5{,}000 \times 0.37512 + 100{,}000 = 107{,}876$

Interpolate between 4% and 5% to get the effective rate of return, i:

$$i = 0.04 + 0.01\left(\frac{107{,}876 - 100{,}000}{107{,}876 - 98{,}166}\right) \approx 4.81\% \text{ (actual answer is 4.80\%)}$$

Therefore Product B charges a lower effective annual return than Product A.

4 $\left(1 + \frac{i^{(4)}}{4}\right)^4 = 1.05 \Rightarrow i^{(4)} = 0.049089$

$$g\left(1 - t_1\right) = \frac{0.07}{1.08} \times 0.75 = 0.04861$$

$$\Rightarrow i^{(4)} > \left(1 - t_1\right)g$$

$\Rightarrow$ Capital gain on contract and we assume loan is redeemed as late as possible (i.e. after 20 years) to obtain minimum yield.

Let Price of stock = P

$$P = 0.07 \times 100{,}000 \times 0.75 \times a_{\overline{20|}}^{(4)}$$

$$+\left(108{,}000 - 0.35\left(108{,}000 - P\right)\right)v^{20} \text{at } 5\%$$

$$\Rightarrow P = \frac{5250a_{\overline{20|}}^{(4)} + 70{,}200v^{20}}{1 - 0.35v^{20}}$$

$$= \frac{5250 \times 1.018559 \times 12.4622 + 70{,}200 \times 0.37689}{1 - 0.35 \times 0.37689}$$

= 107,245.38

5 Assuming no arbitrage.

(i) $i_1 = f_0$ and $(1+i_2)^2 = (1+i_1)(1+f_1)$.

Hence $a - b = 0.061$

$$\Rightarrow a = b + 0.061$$
$$(1+a-2b)^2 = 1.061 \times 1.065$$
$$\Rightarrow 1+a-2b = \sqrt{1.061 \times 1.065}$$

$$\Rightarrow b = -0.002$$
$$\Rightarrow a = 0.059$$

(ii) Firstly, find the 4-year spot rate. Consider £1 nominal:

$$1 = 0.07 \left(v_{i_1} + v_{i_2}^2 + v_{i_3}^3 + v_{i_4}^4\right) + v_{i_4}^4$$
$$= 0.07\left(1.061^{-1} + 1.063^{-2} + 1.065^{-3}\right) + 1.07 \times v_{i_4}^4$$
$$\Rightarrow (1+i_4)^4 = 1.31429212$$
$$\Rightarrow i_4 = 7.0713\% \ \ p.a$$

Let bond price per £1 nominal be P. Then

$$P = 0.05\left(v_{i_1} + v_{i_2}^2 + v_{i_3}^3 + v_{i_4}^4\right) + 1.03 v_{i_4}^4$$
$$= 0.05\left(1.061^{-1} + 1.063^{-2} + 1.065^{-3}\right) + 1.08 \times 1.070713^{-4}$$
$$= 0.9545$$

i.e. 95.45 pence per £1 nominal

6 (i) (a) The duration is:

$$\frac{500\left(v + 2v^2 + 3v^3 + \ldots + 20v^{20}\right)}{500(v + v^2 + v^3 + \ldots + v^{20})} \text{ at } 8\%$$
$$= \frac{(Ia)_{\overline{20|}}}{a_{\overline{20|}}} = \frac{78.9079}{9.8181} = 8.037 \text{ years}$$

(b) The duration is:

$$\frac{500\left[v+\left(1.08\times 2v^2\right)+\left(1.08^2\times 3v^3\right)+\ldots+\left(1.08^{19}\times 20v^{20}\right)\right]}{500\left[v+\left(1.08v^2\right)+\left(1.08^2v^3\right)+\ldots+\left(1.08^{19}v^{20}\right)\right]} \text{ at } 8\%$$

$$=\frac{v\left(1+2+3+\ldots+20\right)}{20v}=\frac{\frac{1}{2}\left(20\times 21\right)}{20}=10.5 \text{ years}$$

(ii) The duration in (i)(b) is higher because the payments increase over time so that the weighting of the payments is further towards the end of the series.

7 (i) $260=12\left(v\left(1+e\right)+v^2\left(1+e\right)^2+v^3\left(1+e\right)^3+......\right)$

where $v = \dfrac{1}{1.06}$ and e denotes inflations rate.

Then,

$260 = 12a_{\overline{\infty}|}$ at $j\%$ where $\dfrac{1}{1+j}=\dfrac{1+e}{1+i}$ i.e. $j=\dfrac{0.06-e}{1+e}$

$\Rightarrow 260=\dfrac{12}{j}$

$\Rightarrow j=0.046153846$

$\Rightarrow e=0.01324$ i.e 1.324% pa

(ii) $260=12\left(1.03v+1.03^2v^2+.....+1.03^{12}v^{12}\right)+500v^{12}$

$=12\underset{j\%}{a_{\overline{12}|}}+500\underset{i\%}{v^{12}}$ where $j=\dfrac{i-0.03}{1.03}$

Try $i=10\%$, RHS $=255.67$
$i=9\%$, RHS $=279.35$

Hence, $i=0.09+\dfrac{279.35-260}{279.35-255.67}\times 0.01$

$=0.098$

Let i' = real return

Then $\left(1+i'\right)\left(1+e\right)=1+i$

$\Rightarrow 1+i'=\dfrac{1.0982}{1.03}\Rightarrow i'=6.62\%$ pa

8 (i) Working in £000s

<u>Outlay</u>

$$Pv = 500 + 90a_{\overline{5}|} + 10(Ia)_{\overline{5}|} \ \ @11\%$$

$$a_{\overline{5}|} = \frac{1-v^5}{0.11} = 3.695897$$

$$(Ia)_{\overline{5}|} = \frac{\ddot{a}_{\overline{5}|} - 5v^5}{0.11} = \frac{1.11 \times 3.695897 - 5v^5}{0.11}$$

= 10.319900

$$\Rightarrow PV = 500 + 90 \times 3.695897 + 10 \times 10.3199$$

= 935.8297

<u>Income</u>

$$PV = 80\left(\bar{a}_{\overline{1}|} + 1.04v\,\bar{a}_{\overline{1}|} + (1.04)^2 v^2\,\bar{a}_{\overline{1}|} + \cdots \ \cdots + (1.04)^{24} v^{24}\bar{a}_{\overline{1}|}\right)$$

$$= 80\bar{a}_{\overline{1}|} \times \left[\frac{1-(1.04v)^{25}}{1-(1.04v)}\right]$$

where $\bar{a}_{\overline{1}|} = \frac{i}{\delta}.v = \frac{0.11}{\ln 1.11}.\frac{1}{1.11} = 0.949589$

$$\Rightarrow PV = 80 \times 0.949589 \times 12.74554 = 968.2421$$

PV of cost of further investment
$= 300v^{15} = 62.7013$

PV of sale $= 700v^{25} = 51.5257$

Hence *NPV* = 968.2421 + 51.5257 - 935.8297 - 62.7013

= 21.2368 (£21,237)

(ii) If interest > 11% then $\frac{1}{1+i}$ decreases.

$\Rightarrow PV$ of both income and outgo $\downarrow$

However, *PV* of outgo is dominated by initial outlay of £500k at time 0 which is unaffected.

$\Rightarrow PV$ of income decreases by more than decrease in *PV* of outgo

$\Rightarrow NPV = PV$ of income $-$ PV of outgo

would reduce (and possibly become negative)

9 (i) $$pv = 1{,}000 * \exp\left[-\int_7^{10}(0.01t - 0.04)\,dt\right] * \exp\left[-\int_5^7 (0.10 - 0.01t)\,dt\right]$$

$$= 1000 * \exp\left(-\left[\frac{0.01t^2}{2} - 0.04t\right]_7^{10}\right) * \exp\left(-\left[0.10t - \frac{0.01t^2}{2}\right]_5^7\right)$$

$$= 1000 * \exp\left(-\left[\frac{0.01*51}{2} - 0.04\times 3\right]\right) * \exp\left(-\left[0.10*2 - \frac{0.01*24}{2}\right]\right)$$
$$= 1000 * \exp(-0.255 + 0.12 - 0.20 + 0.12)$$

$$= 1000 * \exp(-0.215)$$
$= 806.54$

(ii) Required interest rate p.a. convertible monthly is given by
$$806.54\left(1 + \frac{i^{(12)}}{12}\right)^{12\times 5} = 1{,}000$$
$\Rightarrow i^{(12)} = 4.3077\%\, p.a.$ convertible monthly

(iii) Accumulated amount $= \int_0^4 100e^{0.02t} \times e^{\int_t^4 0.06dr} \times e^{\int_4^7 (0.10-0.01r)dr} \times e^{\int_7^{12}(0.01r-0.04)dr}\,dt$

$$= 100\int_0^4 e^{0.02t} \times e^{[0.06r]_t^4} \times e^{\left[0.10r - \frac{0.01r^2}{2}\right]_4^7} \times e^{\left[\frac{0.01r^2}{2} - 0.04r\right]_7^{12}}\,dt$$

$$= 100\int_0^4 e^{0.02t} e^{(0.24-0.06t)} e^{(0.30-0.165)} e^{(0.475-0.200)}\,dt$$

$$= 100e^{0.24}e^{0.135}e^{0.275}\int_0^4 e^{-0.04t}\,dt$$

$$= 100e^{0.65}\left[\frac{-e^{-0.04t}}{0.04}\right]_0^4$$
$$= 2{,}500e^{0.65}\left(1 - e^{-0.16}\right)$$

= 2,500 * 1.915540829 * 0.1478562

= 708.06

10 (i) Option A:

$$(1+i_t) \sim Lognormal\left(\mu,\sigma^2\right)$$

$$\ln(1+i_t) \sim N\left(\mu,\sigma^2\right)$$

$$\ln(1+i_t)^{10} = \ln(1+i_t)+\ln(1+i_t)+\ldots+\ln(1+i_t) \sim N\left(10\mu,10\sigma^2\right)$$

since i_t's are independent

$$(1+i_t)^{10} \sim Lognormal\left(10\mu,10\sigma^2\right)$$

$$E(1+i_t) = \exp\left(\mu+\frac{\sigma^2}{2}\right) = 1.055$$

$$Var(1+i_t) = \exp\left(2\mu+\sigma^2\right)\left[\exp\left(\sigma^2\right)-1\right] = 0.07^2$$

$$\frac{0.07^2}{1.055^2} = \left[\exp\left(\sigma^2\right)-1\right] \therefore \sigma^2 = 0.0043928$$

$$\exp\left(\mu+\frac{0.0043928}{2}\right) = 1.055 \Rightarrow \mu = \ln 1.055 - \frac{0.0043928}{2} = 0.051344$$

$$10\mu = 0.51344\,, 10\sigma^2 = 0.043928$$

Let S_{10} be the accumulation of one unit after 10 years:

$$E(S_{10}) = \exp\left(0.51344+\frac{0.043928}{2}\right) = 1.70814$$

Accumulated sum is $100E(S_{10})$ = £170.81

Option B:

The accumulated sum at the end of five years is:
100×1.06^5 = 100×1.33823 = £133.823

The expected value of the accumulated sum at the end of ten years is:

$$133.823\left(0.2\times1.01^5+0.3\times1.03^5+0.2\times1.06^5+0.3\times1.08^5\right)$$

$$=133.823(0.2\times1.05101+0.3\times1.15927+0.2\times1.33823+0.3\times1.46933)$$

$$=£169.48$$

Option A:

$$Var\left(S_{10}\right)=\exp\left(2\times0.51344+0.043928\right)\left[\exp\left(0.043928\right)-1\right]$$

$$=2.91776\times0.04491=0.13103$$

Therefore standard deviation of £100 is $100\sqrt{0.13103}=£36.20$

Option B:

Here we need to find the expected value of the square of the accumulation as follows:

$$133.823^2\left(0.2\times1.05101^2+0.3\times1.15927^2+0.2\times1.33823^2+0.3\times1.46933^2\right)$$

$$= 29{,}189.86$$

The variance of the accumulation is therefore:

$$29{,}189.86-169.48^2=£^2467.54$$

and the standard deviation is £21.62

(ii) For option A we require $P\left[S_{10}<1.15\right]$

$P\left[\ln S_{10}<\ln 1.15\right]$ where $\ln S_{10} \sim N(0.51344, 0.043928)$

$$\Rightarrow P\left[N\left(0,1\right)<\frac{\ln 1.15-0.51344}{\sqrt{0.043928}}\right]$$

$$\Rightarrow P\left[N\left(0,1\right)<-1.7829\right]=0.0373\approx4\%$$

For option B we first examine the lowest payout possible.

There is a probability of 0.2 that the amount will be $100\times1.06^5\times1.01^5$ or less which equals $133.823\times1.05101=£140.65$. Therefore the probability of a payment of less than £115 is zero.

(iii) Option A is riskier both from the perspective of having a higher standard deviation of return and also a higher probability of a very low value.

END OF EXAMINERS' REPORT