Subject CT1 — Financial Mathematics Core Technical

EXAMINERS' REPORT

September 2008

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

R D Muckart
Chairman of the Board of Examiners

November 2008

Comments

Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

Candidates appeared to be less well prepared than in previous recent diets. As has often been the case when words rather than numbers have been required, Q4 was answered relatively poorly despite only involving bookwork with a wide range of available points that could be made. Many candidates also struggled with the first part of Q2 where explanation rather than calculation was required. The remainder of the shorter questions were answered well with candidates scoring particularly highly on Q7.

The more application styled questions (especially Qs 8, 11 and 12) tended to act as a clear discriminator between stronger and weaker candidates with a significant minority of candidates scoring very few marks on these questions. By contrast, Q9 on spot and forward yields was answered relatively well compared to questions in previous diets on this topic.

1 If j = real rate of return then equation of value in real terms is:

$$95(1+j)^{91/365} = 100\frac{220}{222}$$

$$(1+j)^{91/365} = 1.04315$$

therefore j = 18.465%

2 (i) MWRR

- Requires less information compared to TWRR

But

- Affected by amount and timing of net cashflows, which may not be in the manager's control and less fair measure than TWRR
- More difficult equation to solve than TWRR
- Also: equation may not have unique (or any) solution

(ii) Let TWRR = i

Then

$$(1+i)^2 = \frac{45}{41}\times\frac{72}{57}$$
$$= 1.386392811$$
$$\Rightarrow i = 17.745\% \text{ p.a.}$$

3 (i) Consider two portfolios A and B at time 0.

Portfolio A: - buy forward at price of K
- deposit $Ke^{-\delta T}$ in risk-free asset

Portfolio B: - buy asset at price of B

Then, at maturity, both portfolios have the same value (i.e. hold the underlying asset).

Thus, by the no-arbitrage principle, both portfolios must have same value at time 0.

$$\Rightarrow \quad Ke^{-\delta T} = B \quad \Rightarrow K = Be^{\delta T}$$

(ii) $i = 2\%$ per quarter

$$\Rightarrow\ K = 200 \times (1.02)^2 - 10 \times 1.02 = 197.88$$
$$\left(\text{using } K = Be^{\delta T} - Ce^{\delta(T - t_1)}\right)$$

4 Main characteristics of commercial property investments:

- Many different types of properties available for investment, e.g. offices, shops and industrial properties.
- Return comes from rental income and from the proceeds on sale.
- Total expected return higher than for gilts
- Rents and capital values are expected to increase broadly with inflation in the long term
- Neither rental income nor capital values are guaranteed – capital values in particular can fluctuate in the short term…
- …but rental income more secure than dividends
- Rents and capital values expected to increase when the price level rises (though the relationship is far from perfect).
- Rental terms are specified in lease agreements. Typically, rents increase every three to five years, Some leases have clauses which specify upward-only adjustments of rents.
- Large unit sizes, leading to less flexibility than investment in shares
- Each property is unique…
- …. so can be difficult to value.
- Valuation is expensive, because of the need to employ an experienced surveyor
- Marketability and liquidity are poor because of uniqueness …
- …and because buying and selling incurs high costs.
- Rental income received gross of tax.
- Net rental income may be reduced by maintenance expenses
- There may be periods when the property is unoccupied, and no income is received.
- The running yield from property investments will normally be higher than that for ordinary shares.

5 Present value in first case is

$$1,200 \times \frac{i}{d^{(4)}} \times a_{\overline{10}|} = 1200 \times\ 1.024877 \times 8.1109 = £9,975.210$$

Present value in second case is:

$$2,520 \times (v^2\ +\ v^4\ + \ldots + v^{10})\ =\ 2,520 \times v^2 \times \frac{\left(1 - v^{10}\right)}{\left(1 - v^2\right)}$$

$$= 2,520 \times 0.92456 \times \frac{(1-0.67556)}{(1-0.92456)} = £10,020.01$$

Therefore first option is better for the borrower.

6 (i) Let $i_t =$ investment return for year t

Then, the expected value of the accumulation (S_{10}) is given by (in £ millions):

$$E(S_{10}) = E\left(\prod_{t=1}^{10}(1+i_t)\right)$$

$$= \prod_{t=1}^{10} E(1+i_t) \text{ using independence}$$

$$= \prod_{t=1}^{10}(1+E(i_t))$$

Now, $E(i_1) = 0.5 \times (0.07 + 0.03) = 0.05$

and for $t \neq 1$, $E(i_t) = (0.3 \times 0.02 + 0.4 \times 0.04 + 0.3 \times 0.06)$

$= 0.04$

So the expected value of the accumulation is

$1.05 \times 1.04^9 = 1.494477$ (i.e. £1,494,477)

(ii) The variance of the accumulation is

$$1,000,000^2 \times \left(E\left(S_{10}^2\right) - E(S_{10})^2\right)$$

where $E\left(S_{10}^2\right) = E\left(\prod_{t=1}^{10}(1+i_t)^2\right)$

$$= E\left(\prod_{t=1}^{10}\left(1+2i_t+i_t^2\right)\right)$$

$$= \prod_{t=1}^{10}\left(1+2E(i_t)+E\left(i_t^2\right)\right) \text{ from independence}$$

Now $E\left(i_1^{\,2}\right) = 0.5 \times \left(0.07^2 + 0.03^2\right) = 0.0029$

for $t \neq 1, E\left(i_t^{\,2}\right) = 0.3 \times 0.02^2 + 0.4 \times 0.04^2 + 0.3 \times 0.06^2$

= 0.00184

Hence,

$$E\left(S_{10}^2\right) = (1 + 0.1 + 0.0029) \times (1 + 0.08 + 0.00184)^9$$

= 2.238739

Standard deviation of the accumulation is

$$1{,}000{,}000 \times \left(2.238739 - 1.494477^2\right)^{1/2} = £72{,}646$$

(iii) The mean would remain unchanged as the expected rate of return in years 2-10 is unchanged. The variance of the rate in years 2-10 has increased and this will lead to an increase in the variance of the 10 year accumulation.

7 (i) Discounting from t = 12 to t = 5

$$v(12,5) = \exp\left(-\int_5^{12} 0.15ds\right)$$

$$= \exp\left[-0.15s\right]_5^{12} = e^{-1.05} = 0.34994$$

Discounting from t = 5 to t = 0

$$v(5,0) = \exp\left(-\int_0^5 0.05 + 0.02sds\right)$$

$$= \exp\left[-0.05s - 0.01s^2\right]_0^5 = e^{-0.5} = 0.60653$$

Hence present value of £1,000 at time t = 12

$$= 1{,}000v(12,5)v(5,0) = 1{,}000 \times 0.34994 \times 0.60653 = £212.25$$

(ii) The annual effective rate of discount is d such that:

$$1000(1-d)^{12} = 212.25$$
$$\Rightarrow d = 1 - 0.21225^{1/12} = 12.117\%$$

8 (i) Investment A: the gross rate of return per annum effective is clearly 10%. The net return is therefore $(1\text{-}0.4)\times 10\% = 6\%$ per annum effective.

Investment B: the investment will accumulate to $£1m\times 1.1^{10} = £2.5937m$ at the end of the ten years. The equation of value is:

$$1 = 2.59374(1+i)^{-10} - 0.4(2.59374\text{-}1)(1+i)^{-10}$$
$$= 1.95625(1+i)^{-10}$$
$$\Rightarrow (1+i)^{10} = 1.95625$$
$$\Rightarrow i = 6.94\%$$

Investment C: again the investment will accumulate to £2.5937m at the end of ten years. However, the indexed purchase price is subtracted from the value of the investment in this case. Thus the equation of value is:

$$1 = 2.59374(1+i)^{-10} - 0.4\left(2.59374\text{-}1\times 1.04^{10}\right)(1+i)^{-10}$$
$$= 2.5937(1+i)^{-10} - 0.4\times 2.59374(1+i)^{-10} + 0.4\times 1.04^{10}\times(1+i)^{-10}$$
$$= 2.14834(1+i)^{-10}$$
$$\Rightarrow (1+i)^{10} = 2.14834$$
$$\Rightarrow i = 7.95\%$$

(ii) All investments give a gross return of 10% per annum effective. Investment B gives a higher return than A because the tax is deferred until the end of the investment as capital gains tax is paid and not income tax. [However, candidates might note that tax is paid on the interest earned by deferral of tax]. Investment C gives a higher return than investment B because the tax is only paid on the real return over the ten year period which is lower than the nominal return.

9 (i) $103 = 6a_{\overline{3|}} + 105v^3$

try i = 6%: $a_{\overline{3|}} = 2.6730 \quad v^3 = 0.83962$
RHS = 104.1981

try i = 7%: $a_{\overline{3|}} = 2.6243 \quad v^3 = 0.81630$
RHS = 101.4573

Using linear interpolation:

$$i = 0.06 + \frac{(104.1981 - 103)}{(104.1981 - 101.4573)} \times 0.01 = 0.06437 = 6.44\%$$

(ii) Let i_n = spot yield for term n

Then

$103(1+i_1) = 111 \Rightarrow i_1 = 7.767\%$
$103 = 6(1.07767)^{-1} + 111(1+i_2)^{-2} \Rightarrow i_2 = 6.736\%$
$103 = 6(1.07767)^{-1} + 6(1.06736)^{-2} + 111(1+i_3)^{-3} \Rightarrow i_3 = 6.394\%$

(iii) First year forward rate is 7.767% (same as spot rate).

Forward rate from time one to time two is i such that:
$1.07767(1+i) = 1.06736^2 \Rightarrow i = 5.715\%$

Forward rate from time two to time three is i such that:
$1.06736^2(1+i) = 1.06394^3 \Rightarrow i = 5.713\%$

Forward rate from time one to time three is i such that:
$1.07767(1+i)^2 = 1.06394^3 \Rightarrow i = 5.714\%$

Forward rate from time zero to two and from time zero to three are the same as the respective spot rates (no additional marks for this point).

10 (i) NPV of first project in £*m* is:

$0.5\left(a_{\overline{27|}} - a_{\overline{7|}}\right) + 5v^{27} - 0.1(Ia)_{\overline{10|}} - 0.25$ at 7%
$= 0.5(11.9867 - 5.3893) + 5 \times 0.16093 - 0.1 \times 34.7391 - 0.25$
$= £0.379m$

The NPV of second project in £*m* is:

$$0.21v + 0.21(1.05)v^2 + 0.21(1.05)^2 v^3 + \ldots + 0.21(1.05)^9 v^{10} + 5.64v^{10} - 4.2$$

$$= \ 0.21v \left(\frac{1 - 1.05^{10} v^{10}}{1 - 1.05v} \right) + 5.64v^{10} - 4.2$$

$$v = 0.93458 \quad v^{10} = 0.50835$$

Therefore NPV $= 1.8055 + 5.64 \times 0.50835 - 4.2 \ = \ £0.473m$
The second project has the higher net present value at 7% per annum effective.

(ii) The second project clearly has a discounted mean term of less then ten years. However, the discounted mean term of the first project must be greater than ten years because the undiscounted incoming cash flows are less than the undiscounted outgoing cash flows after ten years.

11 (i) Working in '000s

Let X = Nominal amount of Zero Coupon Bond

Y = Nominal amount of 8% bond

$$V_L = 400v^{10} = 185.2774$$

$$V_A = 18.52774 + Xv^{12} + 0.08Ya_{\overline{16|}} + 1.1Yv^{16}$$

Then, since $V_A = V_L$ (1st condition)

$$\Rightarrow 166.74966 = 0.39711X + 0.08 \times 8.8514Y + 0.32108Y$$

$$\Rightarrow 166.74966 = 0.39711X + 1.02919Y \ldots\ldots(1)$$

2nd condition is $V_A^{'} = V_L^{'}$

$$V_L^{'} = 4000v^{10} = 1852.7740$$

$$V_A^{'} = 12Xv^{12} + 0.08(Ia)_{\overline{16|}}.Y \quad + 1.1 * 16Yv^{16}$$

$$= 4.76537X + 0.08 * 61.1154Y + 5.13727Y$$

$$\Rightarrow 1852.7740 = 4.76537X + 10.0265Y \ldots\ldots(2)$$
$$\Rightarrow 148.2429 = 2.32391Y$$

$$\left[\text{from } (2)-\frac{4.76537}{0.39711}*(1)\right]$$

Hence Y = 63,790

X = 254,583

(ii) Amount invested in X is 254,583 v^{12}

= 101,098

and amount invested in Y is:

185,277 - 18,528 - 101,098 = 65,651

(iii) The spread of the assets is clearly greater than the spread of the liability (which is a single point).

Hence, Redington's 3rd condition is satisfied and the fund is immunised.

12 (i) First 15 years:

Interest paid each month

$$=\frac{i^{(12)}}{12}\times 300,000 \text{ where } 1.085=\left(1+\frac{i^{(12)}}{12}\right)^{12}$$

$$\Rightarrow \frac{i^{(12)}}{12}=0.0068215$$

$\Rightarrow$ monthly interest = 0.0068215 $\times$ 300,000 = £2,046.45

After repayment of £150,000 after 15 years:

Interest paid each quarter

$$=\frac{i^{(4)}}{4}\times 150,000 \text{ where } 1.085=\left(1+\frac{i^{(4)}}{4}\right)^{4}$$

$$\Rightarrow \frac{i^{(4)}}{4}=0.020604$$

$\Rightarrow$ Quarterly interest = 0.020604 $\times$ 150,000 = £3,090.66

Total interest paid over the 25 years

= (2046.45 × 12 × 15) + (3090.66 × 4 × 10) = £491,987.40

(ii) 150,000 = $X\ \ddot{s}_{\overline{30}|}^{(6)}$ @ 4½ %

where $X =$ Amount paid in each 6 month period

$$\ddot{s}_{\overline{30}|}^{(6)} = \frac{(1.045)^{30} - 1}{d^{(6)}}$$

where $\dfrac{1}{1.045} = \left(1 - \dfrac{d^{(6)}}{6}\right)^6$

$\Rightarrow d^{(6)} = 0.043856$

Hence $X = \dfrac{150000}{\left[\frac{(1.045)^{30}-1}{0.043856}\right]} = \dfrac{150000}{62.5985}$ = 2396.23

$\Rightarrow$ Monthly contribution = $\dfrac{2396.23}{6}$ = £399.37 per month

(iii) Savings proceeds after 15 years:

$12 \times 399.37\ \ddot{s}_{\overline{15}|\,10\%}^{(12)}$

where $\ddot{s}_{\overline{15}|}^{(12)} = \dfrac{i}{d^{(12)}} \times s_{\overline{15}|}$

$= 1.0533781 \times 31.7725$

$= 33.46845$

Hence, savings proceeds

= 4792.44 × 33.46845 = 160,395.56

$\Rightarrow$ Loan o/s after 15 years

= 300,000 - 160,395.56 = 139,604.44

Let Y = new monthly payment

$$139{,}604.44 = 12\,Y\, a^{(12)}_{\overline{10|}\,7\%}$$

$$= 12Y\,\frac{0.07}{0.06785} \times 7.02358$$

$$\Rightarrow Y = £1{,}605.50 \text{ per month}$$

END OF EXAMINERS' REPORT