EXAMINATION

September 2005

Subject CT1 — Financial Mathematics
Core Technical

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

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Chairman of the Board of Examiners

15 November 2005

As is in some recent diets, the questions requiring descriptions of concepts, definitions or verbal reasoning (such as Q1, Q8(ii) and Q9(i)) tended not to be well answered with candidates producing vague statements which did not demonstrate that they understood the relevant points. It is important that candidates understand the subject well enough to express important topics and issues in their own words as well as in mathematical language. In 'show that' questions or questions where students are asked to derive formulae (such as Q8 part (i)) candidates are required to show detailed steps in deriving the results required in order to obtain full marks.

Please note that differing answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates were not penalised for this. However, candidates were penalised where excessive rounding had been used or where insufficient working had been shown.

1 One party agrees to pay to the other a regular series of fixed amounts for a certain term. In exchange the second party agrees to pay a series of variable amounts based on the level of a short term interest rate.

2 If f = the rate of inflation; j = the real rate of return and i = the money rate of return, then $j = (i - f)/(1 + f)$. In this case, $f = -2\%$, $i = 1\%$ and therefore $j = 3.061\%$.

3 (a) Let the answer be t days

$1{,}500(1 + 0.05 \times t/365) = 1{,}550$

$t = 243.333$ days

(b) Let the answer be t days

$1{,}500e^{0.05(t/365)} = 1{,}550$

$0.05\,(t/365) = \ln\,(1{,}550/1500)$

$t = 239.366$ days

4

$$210 = 200\exp\left\{\int_0^5 \left(a+bt^2\right)dt\right\} = 200\exp\left[at+\tfrac{1}{3}bt^3\right]_0^5 = 200\left[5a+41.667b\right]$$

$$230 = 200\exp\left\{\int_0^{10} \left(a+bt^2\right)dt\right\} = 200\exp\left[at+\tfrac{1}{3}bt^3\right]_0^{10} = 200\left[10a+333.333b\right]$$

$$\ln(1.05) = 5a + 41.667b$$
$$\ln(1.15) = 10a + 333.333b$$

The second expression less twice the first expression gives:

$$\ln(1.15) - 2\ln(1.05) = 250b \Rightarrow b = 0.0001687$$
$$a = \frac{\ln(1.15) - 333.333\times 0.0001687}{10} = 0.0083520$$

5 (i) (a) $100 \times (1 + 0.05/12)^{-12\times10} = £60.716$

(b) $100 \times (1 - 0.05/12)^{12\times10} = £60.590$

(c) $100 \times e^{-10\delta} = £60.6531$

(ii) $98.91 = 100(1 + i)^{-91/365}$

$\ln(1 + i) = (-365/91) \times \ln(98.91/100) = 0.04396$

therefore $i = 0.04494$

6 (i)

- Used for medium or long-term borrowing
- Unsecured
- Regular annual coupon payments
- Generally repayable at par
- Generally issued by large companies and on behalf of governments
- Yields depend on risk and marketability
- Generally innovative market designed to attract different types of investor
- Issued internationally (normally by a syndicate of banks)
- Can be issued in any currency (not necessarily the domestic currency of the borrower)

(ii) (a) $97 = g a_{\overline{20}|} + 100v^{20}$ at 5% per annum effective

$a_{\overline{20}|} = 12.4622$; $v^{20} = 0.37689$ therefore $97 = 12.4622g + 100 \times 0.37689$

$g = (97 - 37.689)/12.4622 = 4.75927$

(b) Duration $= \sum C_t tv^t / \sum C_t v^t$ where C_t is the amount of the cash flow at time t

$(Ia)_{\overline{20}|} = \sum tv^t$ Therefore duration of the eurobond is:

$(4.75927\,(Ia)_{\overline{20}|} + 100 \times 20v^{20})/(4.75927\,a_{\overline{20}|} + 100v^{20})$

$(Ia)_{\overline{20}|} = 110.9506$ all other values have been used in (a) above

therefore duration is:

$(4.75927 \times 110.9506 + 100 \times 20 \times 0.37689)/(4.75927 \times 12.4622 + 100 \times 0.37689) = 1281.8239/97 = 13.2147$

7 (i) Value of loan = $50v + 48v^2 + 46v^3 + 44v^4 + \ldots + 22v^{15}$

$= 52(v + v^2 + v^3 + \ldots + v^{14} + v^{15}) - 2(v + 2v^2 + 4v^3 + \ldots + 28v^{14} + 30\,v^{15})$

$= 52\,a_{\overline{15}|} - 2\,(Ia)_{\overline{15}|}$

$(Ia)_{\overline{15}|} = 67.2668$
$a_{\overline{15}|} = 9.7122$

Therefore amount of the loan is $52 \times 9.7122 - 2 \times 67.2668 = 370.501$

Candidates who derived an appropriate formula for a decreasing annuity directly or who calculated the value of the loan by summing the individual terms received full credit.

(ii) Interest component in first year is $0.06 \times 370.504 = 22.23024$; therefore capital component is $50 - 22.23024 = 27.76976$.

Capital remaining after first instalment is $370.504 - 27.76976 = 342.73424$.
Interest paid in second instalment is $0.06 \times 342.73424 = 20.56405$

Capital in second instalment is $48 - 20.56405 = 27.43595$.

(iii) At the end of the thirteenth year, the capital outstanding is:

$24v + 22v^2 = 24 \times 0.94340 + 22 \times 0.89000 = 42.2216$

The interest due in the fourteenth instalment $0.06 \times 42.2216 = 2.53330$

The capital payment is therefore $24 - 2.53330 = 21.46670$

8 (i) Let i_t be the (random) rate of interest in year t. Let S_5 be the accumulation of a single investment of 1 unit after 5 years:

$$E(S_5) = E\left[\prod_{t=1}^{5}(1+i_t)\right]$$

$$= \prod_{t=1}^{5} E\left[(1+i_t)\right]$$

as $\{i_t\}$ are independent

$$E(S_5) = E[1+i_t]^5$$

$$E[1+i_t] = (1+E[i_t]) = 1.035$$

$$\therefore \quad E(S_5) = (1.035)^5 = 1.187686$$

$$E\left(S_5^2\right) = E\left[\prod_{t=1}^{5}(1+i_t)^2\right] = \prod_{t=1}^{5} E\left[(1+i_t)^2\right] \text{ (using independence)}$$

$$= \left(E(1+i_t)^2\right)^5 = \left(E\left[1+2i_t+i_t^2\right]\right)^5 = \left(1+2E[i_t]+E\left[i_t^2\right]\right)^5$$

$$= \left(1+2E[i_t]+Var[i_t]+E[i_t]^2\right)^5$$

$$Var(S_5) = E\left(S_5^2\right) - E(S_5)^2$$

$$= \left(1+2E[i_t]+Var[i_t]+E[i_t]^2\right)^5 - E[1+i_t]^{10}$$

$$E(i_t) = 0.035$$

$$Var(i_t) = 0.03^2$$

$$\therefore \quad Var(S_5) = \left(1+2\times 0.035+0.03^2+0.035^2\right)^5 - (1.035)^{10}$$

$$= 1.416534 - 1.410598$$

$$= 0.0059356$$

Mean value of the accumulation of premiums is:

$$425000E(S_5) + 425000(1.03)^5 = (425000\times 1.187686) + (425000\times 1.15927)$$

$$= 997458$$

Standard deviation is $425000SD(S_5) = 425000\times\sqrt{0.0059356} = 32743.21$

Candidates who obtained slightly different answers by first deriving the parameters of the lognormal distribution received full credit.

(ii) Investing all premiums in the risky assets is likely to be more risky because, although there may be a higher probability of the assets accumulating to more than £1 million, the standard deviation would be twice as high so the probability of a large loss would be greater.

9 (i) Bond yields are determined by investors' expectations of future short-term interest rates, so that returns from longer-term bonds reflect the returns from making an equivalent series of short-term investments

(ii) (a) Let i_t be the spot yield over t years:

One year: yield is 8% therefore $i_1 = 0.08$
two years: $(1 + i_2)^2 = 1.08 \times 1.07$ therefore $i_2 = 0.074988$
three years: $(1 + i_3)^3 = 1.08 \times 1.07 \times 1.06$ therefore $i_3 = 0.06997$
four years: $(1 + i_4)^4 = 1.08 \times 1.07 \times 1.06 \times 1.05$ therefore $i_4 = 0.06494$

(b) Price of the bond is $5[(1.08)^{-1} + (1.074988)^{-2} + (1.06997)^{-3}]$
$+ 105 \times (1.06494)^{-4} = 13.03822 + 81.6373 = 94.67552$

Find gross redemption yield from

$94.67552 = 5\,a_{\overline{4|}} + 100v^4$

try 7%; $a_{\overline{4|}} = 3.3872$; $v^4 = 0.76290$
gives RHS = 93.226

GRY must be lower, try 6%; $a_{\overline{4|}} = 3.4651$; $v^4 = 0.79209$
gives RHS = 96.5345

interpolate between 6% and 7%.
$i = 0.07 - 0.01 \times (94.67552 - 93.226)/(96.5345 - 93.226)$
$i = 0.07 - 0.0043812 = 0.06562$

(c) Present value of the dividend is $4v$ calculated at 8% per annum effective = 3.70370.

Therefore forward price is
$F = (400 - 3.70370) \times 1.08 \times 1.07 = 457.9600$

10 (i) Price paid by first investor is P_1

$$P_1 = 4a^{(2)}_{\overline{15}|5\%} + 100v^{15}$$

$$\frac{i}{i^{(2)}} = 1.012348$$

$$v^{15} = 0.48102$$

$$a_{\overline{15}|} = 10.3797$$

$$\therefore P_1 = (4 \times 1.012348 \times 10.3797) + (100 \times 0.48102)$$

$$= 42.0315 + 48.1020 = 90.1335$$

(ii) (a) $\left(1 + \frac{i^{(2)}}{2}\right)^2 = 1.06 \Rightarrow i^{(2)} = 0.059126$

$$g(1 - t_1) = 0.04 \times 0.75 = 0.03$$

$$\Rightarrow i^{(2)} > (1 - t_1)g$$

$\Rightarrow$ Capital gain on contract

Price paid by second investor is P_2

$$P_2 = 0.75 \times 4a^{(2)}_{\overline{7}|6\%} + 100v^7_{6\%} - 0.4(100 - P_2)v^7_{6\%}$$

$$P_2\left(1 - 0.4v^7_{6\%}\right) = 0.75 \times 4a^{(2)}_{\overline{7}|6\%} + 0.6 \times 100v^7_{6\%}$$

$$\frac{i}{i^{(2)}} = 1.014782$$

$$v^7 = 0.66506$$

$$a_{\overline{7}|} = 5.5824$$

$$\therefore P_2 = \frac{(0.75 \times 4 \times 1.014782 \times 5.5824) + (60 \times 0.66506)}{1 - 0.4 \times 0.66506}$$

$$= 77.5207$$

(b) Rate of return earned by the first investor is the solution to:

$$90.1335 = 0.75 \times 4a_{\overline{8}|}^{(2)} + 77.5207v^8$$

$$i = 2\%$$

$$\frac{i}{i^{(2)}} = 1.004975$$

$$v^8 = 0.85349$$

$$a_{\overline{8}|} = 7.3255$$

$$RHS = 88.2490$$

$$i = 1.5\%$$

$$\frac{i}{i^{(2)}} = 1.003736$$

$$v^8 = 0.88771$$

$$a_{\overline{8}|} = 7.4859$$

$$RHS = 91.3575$$

$$i = 0.02 - \left(\frac{90.1335 - 88.2490}{91.3575 - 88.2490}\right) \times 0.005 = 1.697\% \approx 1.7\%$$

11 (i) (a) An equation of value expresses the equality of the present value of positive and negative (or incoming and outgoing) cash flows that are connected with an investment project, investment transaction etc.

(b) The discounted payback period from an investment project is the first time at which the net present value of the cash flows from the project is positive.

(ii) Consider first the NPV at 9% per annum effective. Working in £million.

Present value of cash outflows:

$$1.5\bar{a}_{\overline{3}|9\%} + 0.3\ddot{a}^{(4)}_{\overline{12}|9\%}v^3_{9\%} + v^3_{9\%} + 1.05v^4_{9\%} + 1.05^2 v^5_{9\%} + \ldots + 1.05^{11}v^{14}_{9\%}$$

$$= 1.5 \times 1.044354 \times 2.5313 + 0.3 \times 1.055644 \times 7.1607 \times 0.77218$$

$$+0.77218 \times \left(\frac{1-1.05^{12}v^{12}}{1-1.05v}\right) = 5.71647 + 7.60679 = 13.32326$$

Present value of cash inflows:

$$\left(\bar{a}_{\overline{6}|9\%} - \bar{a}_{\overline{3}|9\%}\right) + 1.9\left(\bar{a}_{\overline{9}|9\%} - \bar{a}_{\overline{6}|9\%}\right) + 2.5\left(\bar{a}_{\overline{15}|9\%} - \bar{a}_{\overline{9}|9\%}\right) + 8v^{15}_{9\%}$$

$$= 2.5\bar{a}_{\overline{15}|} - 0.6\bar{a}_{\overline{9}|} - 0.9\bar{a}_{\overline{6}|} - \bar{a}_{\overline{3}|} + 8v^{15}$$

$$= 1.044354\left(2.5 \times 8.0607 - 0.6 \times 5.9952 - 0.9 \times 4.4859 - 2.5313\right) + 8 \times 0.27454$$

$$= 12.6253$$

Hence NPV of project @ 9% = 12.6253 − 13.3233 = −£0.698 million
so the IRR is less than 9% p.a. effective

To find whether the discounted payback period is less than 12 years at 7% per annum effective, we need to find the NPV @ 7% of first twelve years cashflows

Present value of cash outflows:

$$1.5\bar{a}_{\overline{3}|7\%} + 0.3\ddot{a}^{(4)}_{\overline{9}|7\%}v^3_{7\%} + v^3_{7\%} + 1.05v^4_{7\%} + 1.05^2 v^5_{7\%} + \ldots + 1.05^{8}v^{11}_{7\%}$$

$$= 1.5 \times 1.034605 \times 2.6243 + 0.3 \times 1.043380 \times 6.5152 \times 0.81630$$

$$+0.81630 \times \left(\frac{1-1.05^{9}v^{9}}{1-1.05v}\right) = 5.73739 + 6.82096 = 12.55835$$

Present value of cash inflows:

$$\left(\bar{a}_{\overline{6}|7\%}-\bar{a}_{\overline{3}|7\%}\right)+1.9\left(\bar{a}_{\overline{9}|7\%}-\bar{a}_{\overline{6}|7\%}\right)+2.5\left(\bar{a}_{\overline{12}|7\%}-\bar{a}_{\overline{9}|7\%}\right)$$

$$=2.5\bar{a}_{\overline{12}|}-0.6\bar{a}_{\overline{9}|}-0.9\bar{a}_{\overline{6}|}-\bar{a}_{\overline{3}|}$$

$$=1.034605\left(2.5\times 7.9427-0.6\times 6.5152-0.9\times 4.7665-2.6243\right)$$

$$=9.3461$$

NPV is negative so the discounted payback period is more than 12 years.

Project fulfils neither the discounted payback period criterion nor the internal rate of return criterion.

END OF EXAMINERS' REPORT