EXAMINATION

April 2007

Subject CT1 — Financial Mathematics Core Technical

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

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Chairman of the Board of Examiners

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Comments

Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this.

However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

Q1.

Whilst most candidates made a good attempt at this question on basic compound interest accumulation, comparatively few students completed the question without error.

Q2.

Well answered.

Q3.

Most students answered this question well although candidates were expected to note that the sum of the geometric progression would only converge if the rate of return was below the dividend growth rate. Depending on the interpolation used, the final answer can justifiably vary from that given.

Q4.

This proved to be the most difficult question on the paper. Other related methods to determine the answers were available e.g. calculating the forward price of each contract and working out the present value of the difference in these prices.

Q5.

Well answered.

Q6.

The calculations in parts (i) and (ii) were generally well done. Again, depending on the interpolation used, the final answer can justifiably vary from that given although the examiners penalised the use of too wide a range of interpolation.
The explanation in part (iii) was very poorly handled. In such cases, the examiners are not simply looking for a statement lifted directly from the Core Reading. Instead, candidates are expected to apply the relevant theory to the actual situation described in the question.

Q7.

Generally well answered.

Q8.

This was the best answered question on the paper.

Q9.

Many candidates struggled with this question, firstly in determining when the various costs/payments would be made and then in manipulating the resulting equation(s). A common error was not to recognise that the DPP should be expressed as a whole number of months since payments at the relevant time were being made at monthly intervals. In part (ii) little credit was given for a correct conclusion without any accompanying explanation.

Q10.

This question seemed to provide a significant differentiation between candidates with many scoring well and a sizeable minority scoring very badly. This seemed surprising given that this topic is regularly examined. A common omission on part (ii)(b) was not to state whether a capital gain had been made.

Q11.

The workings for parts (i) and (ii) were often too brief (the questions said 'Derive…'). Note that the final answer in part (ii) can justifiably vary significantly according to the rounding used in intermediate calculations. Part (iii) was poorly done with many candidates assuming a lognormal distribution for this discrete example.

1 Fund after 25 years =

$$400\ddot{S}_{\overline{30}|}^{\,i^{*}\%} \times (1.03)^{20} + 400\ddot{S}_{\overline{20}|}^{\,3\%}$$

where $1 + i^* = (1.005)^6$

$$\Rightarrow i^* = 3.03775\% \text{ per ½-year}$$

$$\ddot{s}_{\overline{30}|} \text{ @ } 3.03775\% = 1.0303775 \times \left[\frac{(1.0303775)^{30} - 1}{0.0303775}\right]$$

$= 49.3215$

$$\ddot{S}_{\overline{20}|} \text{ @ } 3\% = 1.03 \times \left[\frac{(1.03)^{20} - 1}{0.03}\right] = 27.6765$$

Hence fund =

$$400 \times 49.3215 \times (1.03)^{20} + 400 \times 27.6765$$

= 35632.06 + 11070.60

= £46,702.66

2 (i) $$PV = \int_9^{12} 50\, e^{0.01t} .\ e^{-\int_0^t \delta(t)dt}\, dt$$

where

$$\int_0^t \delta(t)\,dt = \int_0^4 (0.04 + 0.01t)\,dt + \int_4^8 (0.12 - 0.01t)\,dt + \int_8^t 0.06dt$$

$$= \left[0.04t + 0.005t^2\right]_0^4 + \left[0.12t - 0.005t^2\right]_4^8 + \left[0.06t\right]_8^t$$

$$= [0.24] + [0.64 - 0.40] + [0.06t - 0.48]$$

$= 0.06t$

Hence

$$PV = \int_9^{12} 50e^{0.01t} .\ e^{-0.06t}\ dt$$

$$= \int_9^{12} 50\, e^{-0.05t}\ dt$$

$$= \left[\frac{-50}{0.05} e^{-0.05t} \right]_9^{12}$$

$= \ -548.812 + 637.628$

$= \ 88.816$

3 Let i = money rate of return

i' = real rate of return

$\Rightarrow 1+i = (1+i')(1.03)$ here

$$21.50 = (1+i)^{4/12} \cdot \left(1.10v + 1.05 \times 1.10v^2 + (1.05)^2 \times 1.10v^3 + \cdots\cdots\right)$$

$$= (1+i)^{4/12} \times 1.10v \left(\frac{1-\left(\frac{1.05}{1+i}\right)^{\infty}}{\left(1-\frac{1.05}{1+i}\right)} \right)$$

$$= 1.10\ \frac{1}{(1+i)^{8/12}} \times \frac{1}{\left(1-\frac{1.05}{1+i}\right)} \quad \text{assuming } i > 0.05$$

$$19.5455 = \frac{1}{(1+i)^{8/12}} \times \frac{1}{1-\frac{1.05}{1+i}}$$

Try i = 10% RHS = 20.6456
11% RHS = 17.2566

$$\Rightarrow i = 0.10 + \frac{20.6456 - 19.5455}{20.6456 - 17.2566} \times 0.01 = 0.10325$$

$$\Rightarrow i' \text{ comes from } 1+i' = \frac{1.10325}{1.03} \Rightarrow i' = 7.1\% \text{ p.a.}$$

4 (i) The current value of the forward price of the old contract is:

$$95\times(1.03)^{5}-5(1.03)^{-2}-6(1.03)^{-4}$$

whereas the current value of the forward price of a new contract is:

$$145-5(1.03)^{-2}-6(1.03)^{-4}$$

Hence, current value of old forward contract is:

$$145-95(1.03)^{5}=£34.87$$

(ii) The current value of the forward price of the old contract is:

$$95(1.02)^{-12}(1.03)^{5}=86.8376$$

whereas the current value of the forward price of a new contract is:

$$145(1.02)^{-7}=126.2312$$

$\Rightarrow$ current value of old forward contract is:

126.23 - 86.84 = £39.39

5 (i) Let $Y_k =$ spot rate for k year term

$P_k =$ Price per unit nominal for k year term

$Y_9 = 0.063737$

$$P_9=\left(\frac{1}{1+Y_9}\right)^{9}=0.57344$$

(ii) $Y_7 = 0.08 - 0.04\, e^{-0.1(7)} = 0.060137$

$Y_{11} = 0.08 - 0.04e^{-0.1(11)} = 0.066685$

$$\left(1+ f_{7,4}\right)^4 = \frac{\left(1+Y_{11}\right)^{11}}{\left(1+Y_7\right)^7} = \frac{\left(1.066685\right)^{11}}{\left(1.060137\right)^7}$$

= 1.35165

$\therefore$ 4-year forward rate is 7.824% at time 7.

(iii) $Y_1 = 0.04381, Y_2 = 0.04725, Y_3 = 0.05037$

$$1 = \left(Y_{c_3}\right)\left(v_{Y_1}^1 + v_{Y_2}^2 + v_{Y_3}^3\right) + v_{Y_3}^3$$

$Y_{c_3} = 0.05016$ i.e. 5.016% p.a.

6 (i) Work in £000's

MWRR is *i* such that:

$$21(1+i)^3 + 5(1+i)^2 + 8(1+i) = 38$$

Try i = 5%, LHS = 38.223
i = 4%, LHS = 37.350

By interpolation *i* = 4.74% p.a.

(ii) TWRR is *i* such that:

$$(1+i)^3 = \frac{24}{21} \times \frac{32}{29} \times \frac{38}{40} \Rightarrow i = 6.21\% \text{ p.a.}$$

(iii) MWRR is lower than TWRR because of the large cash flow on 1/7/05; the overall return in the final year is much lower than in the first 2 years, and the payment at 1/7/05 gives this final year more weight in the MWRR, but does not affect the TWRR.

7 Let PV_L be PV of liabilities, DMT_L be DMT of liabilities, C_L be convexity of liabilities.

(i) $PV_L = 87,500v^8 + 157,500v^{19}$ at 7%

$= 94,475.86$

$$\Rightarrow DMT_L = \frac{87,500 \times 8v^8 + 157,500 \times 19v^{19}}{94,475.86} \text{ at 7\%}$$

$$= \frac{1,234,857.56}{94,475.86}$$

$= 13.070615$ years

$$C_L = \frac{87,500 \times 8 \times 9v^{10} + 157,500 \times 19 \times 20v^{21}}{94,475.86} \text{ at 7\%}$$

$$= \frac{17,657,158.78}{94,475.86}$$

$= 186.895985$

(ii) Firstly, PVs should be equal:

$\Rightarrow 66,850v^4 + Xv^n = 94,475.86$ at 7%
$\Rightarrow Xv^n = 43,476.31507$

Secondly, DMTs should be equal

$\Rightarrow 66,850 \times 4v^4 + Xnv^n = 1,234,857.56$
$\Rightarrow Xnv^n = 1,030,859.38$
$\Rightarrow n = 23.710827$ years
$\Rightarrow X = 43,476.31507 \times 1.07^n$
$= 216,255.12$

Lastly, verify 3rd condition

$$C_A = \left(66,850 \times 4 \times 5v^6 + 216,255.12n\left(n+1\right)v^{(n+2)}\right) / 94,475.86$$

= 23,140,343.20/94,475.86
= 244.93393

> C_L

Hence, immunisation is achieved.

8 (i) $800,000 = P\, a_{\overline{10}|}^{8\%} = P \times 6.7101$

$\Rightarrow P = 119,223.26$

Total amount of interest = 10 × 119,223.26 – 800,000

= £392,232.60

(ii) (a) Capital o/s at start of 8th year

= 119,223.26 $a_{\overline{3}|}^{8\%} = 119,223.26 * 2.5771 = 307,250.26$

Let new payment be P' per annum, then

$$P' a_{\overline{5}|\,12\%}^{(4)} = P' * 1.043938 * 3.6048 = 307,250.26$$

$\Rightarrow P' = 81,646.28$

$\Rightarrow$ q'ly payment $= 20,411.57$

(b) Capital o/s after 7 years = 307,250.26

$\Rightarrow$ Interest in 1st q'ly payment $= 30,7250.26 * \left(\left(1.12\right)^{\frac{1}{4}} - 1\right) = 8,829.56$

$\Rightarrow$ capital component = 20,411.57 – 8,829.56 = 11,582.01

9 (i) The discounted payback period is the first point at which the present value of the income exceeds the present value of the outgoings. The present value of all payments and income up to time *t* is given by (working in £m)

$$PV = -40 - 36a_{\overline{\frac{1}{2}}|}^{(12)} - 2v^{\frac{1}{2}}a_{\overline{t-\frac{1}{2}}|}^{(12)} + 12v^{\frac{1}{2}}\ddot{a}_{\overline{t-\frac{1}{2}+\frac{1}{12}}|}^{(12)}$$

$$= -40 - 36a_{\overline{\frac{1}{2}}|}^{(12)} - 2v^{\frac{1}{2}}a_{\overline{t-\frac{1}{2}}|}^{(12)} + 12v^{\frac{1}{2}}\left(\frac{1}{12} + a_{\overline{t-\frac{1}{2}}|}^{(12)}\right)$$

$$= -40 - 36a_{\overline{\frac{1}{2}}|}^{(12)} + v^{\frac{1}{2}} + 10v^{\frac{1}{2}}\frac{1-v^{t-0.5}}{i^{(12)}}$$

$$a_{\overline{\frac{1}{2}}|}^{(12)} = \frac{1-v^{\frac{1}{2}}}{i^{(12)}} \text{ at } 10\% = \frac{1-0.9534626}{0.0956897} = 0.48634$$

$$\Rightarrow 0.56758 = 1 - v^{t-0.5}$$

$$\Rightarrow v^{t-0.5} = 0.43242$$

$$\Rightarrow t = \frac{\log(0.43242)}{\log(0.90909)} + 0.5$$

$$\Rightarrow t \geq 9.296$$

Hence, the discounted pay back period is 9 years and 4 months.

(ii) If the effective rate of interest were less than 10% p.a. then the present values of the income and outgo would both increase. However, the bigger impact would be on the present value of the income since the bulk of the outgo occurs in the early years when discounting has less effect. Hence, the DPP would decrease.

10 (i) $i^{(2)} = 0.059126$

$$g(1-t_1) = \frac{0.09}{1.10} \times 0.75 = 0.06136$$

$$\Rightarrow i^{(2)} < (1-t_1)g$$

$\Rightarrow$ No capital gain

Price of £100 nominal stock

$$= 0.75 \times 9\, a_{\overline{13}|}^{(2)} + 110 v^{13} \text{ at } 6\%$$

$$= 0.75 \times 9 \times 1.014782 \times 8.8527 + 110 \times 0.46884$$

$$= 60.639 + 51.572$$

$$= £112.21$$

(ii) (a) $i^{(2)} = 0.078461$

$$g(1-t_1) = \frac{0.09}{1.10} \times 0.90 = 0.073636$$

$$\Rightarrow i^{(2)} > (1-t_1)g$$

$\Rightarrow$ Capital gain

Price, $P = 0.90 \times 9 \times a_{\overline{11}|}^{(2)} + \left(110 - (110 - P) \times 0.35\right) v^{11}$ at 8%

$$\Rightarrow P = \frac{0.90 \times 9 \times 1.019615 \times 7.1390 + 0.65 \times 110 \times 0.42888}{1 - 0.35 \times 0.42888}$$

$$= \frac{89.62508}{0.849892} = 105.455$$

(b) No capital gain made

$$112.21 = 0.75 \times 9 \times a_{\overline{2}|}^{(2)} + 105.455 v^2$$

Try $i = 3\%$, RHS = 112.41

$i = 4\%$, RHS = 110.36

$\Rightarrow$ yield = 3% p.a. to nearest 1%

11 (i) Let $S_3 =$ Accumulated fund after 3 years of investment of 1 at time 0

i_t = Interest rate for year t

Then, fund after 3 years

$$= 80{,}000\ S_3 = 80000(1+i_1)(1+i_2)(1+i_3)$$

$$E(i_1) = \tfrac{1}{3}(0.04+0.06+0.08) = 0.06$$

$$E(i_2) = 0.75\times 0.07+0.25\times 0.05=0.065$$

$$E(i_3) = 0.7\times 0.06 + 0.3\times 0.04 = 0.054$$

Then:

$$E[80000S_3] = 80{,}000\ E[S_3]$$

$$= E\left[80{,}000(1+i_1)(1+i_2)(1+i_3)\right]$$

$$= 80{,}000\ E(1+i_1).\ E(1+i_2).\ E(1+i_3)$$

since i_t's are independent

= 80,000 × 1.06 × 1.065 × 1.054 = £95,188.85

(ii) $\text{Var}[80000S_3] = 80{,}000^2 \times \text{Var}[S_3]$

where $\text{Var}[S_3] = E\left[S_3^2\right] - \left(E[S_3]\right)^2$

$$E\left[S_3^2\right] = E\left[(1+i_1)^2(1+i_2)^2(1+i_3)^2\right]$$

$$= E\left[(1+i_1)^2\right].\ E\left[(1+i_2)^2\right].\ E\left[(1+i_3)^2\right]$$

using independence

$$= \left(1+2E[i_1]+E\left[i_1^2\right]\right).\left(1+2E[i_2]+E\left[i_2^2\right]\right)\cdot\left(1+2\ E[i_3]+E\left[i_3^2\right]\right)$$

Now,

$$E\left(i_1^2\right) = \tfrac{1}{3}\left(0.04^2 + 0.06^2 + 0.08^2\right) = 0.0038667$$

$$E\left(i_2^2\right) = 0.75 \times 0.07^2 + 0.25 \times 0.05^2 = 0.0043$$

$$E\left(i_3^2\right) = 0.7 \times 0.06^2 + 0.3 \times 0.04^2 = 0.0030$$

Hence, $E\left[S_3^2\right]$

$= (1 + 2 \times 0.06 + 0.0038667) \times (1 + 2 \times 0.065 + 0.0043) \times (1 + 2 \times 0.054 + 0.003)$

=1.41631

Hence $\text{Var}\left[80,000 S_3\right]$

$= 80,000^2 \ \text{Var}\left[S_3\right]$

$= 80,000^2 \left(1.41631 - (1.18986)^2\right)$

= 3,476,355

(iii) *Note:* 80,000 × 1.08 × 1.07 × 1.06 = 97,995 > 97,000

But, if in any year, the highest interest rate for the year is not achieved then the fund after 3 years falls below £97,000.

Hence, answer is probability that highest interest rate is achieved in each year

$$= \frac{1}{3} \times 0.75 \times 0.7 = 0.175$$

END OF EXAMINERS' REPORT