EXAMINATION

April 2005

Subject CT1 — Financial Mathematics
Core Technical

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

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Chairman of the Board of Examiners

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1 $f = S - I - Ke^{-r(T-t)}$

where:

t is the present time
T is the time of maturity of the forward contract
r is the continuously compounded risk-free rate of interest for the interval from t to T
S is the spot price of the security at time t
I is the present value, at the risk-free interest rate, of the income generated by the security during the interval from t to T
K is the delivery price of the forward contract
f is the value of a long position in the forward contract

Here, working with £100 nominal,

$S = 95$, $K = 98$, $T - t = 1$, $r = 0.052$

$$I = 2.5\left(e^{-0.046\times 0.5} + e^{-0.052\times 1}\right) = 4.81648$$

$$\Rightarrow f = 95 - 4.81648 - 98e^{-0.052} = -2.85071$$

The value of the investor's short position in a forward contract on £1 million is therefore

$$\left(\frac{1,000,000}{100}\right)\times -f = 10,000 \times 2.85071$$

$$= £28,507$$

2 MWRR: $2.2\,(1+i)^3 + 1.44(1+i) = 4.2$

Estimate $i = 6\%$, $LHS = 4.1466$

$i = 7\%$, $LHS = 4.2359$

$$\Rightarrow i = 0.06 + \frac{4.2 - 4.1466}{4.2359 - 4.1466} * 0.01$$

= 6.60% p.a. to two decimal places

Let F = Fund value before net cashflow on 31 December 2003

Then,

TWRR = 6.60% p.a. means that

$$1.066^3 = \frac{F}{2.2} * \frac{4.2}{(F + 1.44)}$$

$$\Rightarrow 0.63452 = \frac{F}{F + 1.44}$$

$$\Rightarrow 0.63452\, F + 0.63452 \text{ x } 1.44 = F$$

$$\Rightarrow F = £2.5\text{m}$$

3 (i) Work in millions:

PV of liabilities $= 9 + 12v\, \bar{a}_{\overline{1|}}$ at 9%

$$= 9 + 12v.\frac{i}{\delta}v$$

$$= 9 + 12 \times 0.91743^2 \times 1.044354$$

= 19.54811

The assets up to $(k+2)$ years from 1 January 2006 have:

$$PV = 5v^2 a^{(2)}_{\overline{k|}} = 5v^2 \frac{i}{i^{(2)}}\, a_{\overline{k|}}$$

$$= 5 \times 0.84168 \times 1.022015 \times a_{\overline{k|}}$$

$$= 4.301048\, a_{\overline{k|}}$$

With $k = 6, PV = 4.301048 \times 4.4859$

= 19.2941

The next payment of 2.5 million at k = 6.5 is made at time 8.5 and has present value = $2.5 \times v^{8.5} = 1.2018$

This would make PV of assets (20.5m) > PV of liabilities (19.5m)

$\Rightarrow$ Discounted payback period = 8.5 years.

(ii) The income of the development is received later than the costs are incurred. Hence an increase in the rate of interest will reduce the present value of the income more than the present value of the outgo. Hence the DPP will increase.

4 (i) Accumulation $= 500\, e^{\int_0^{10} \delta(s)ds}$

$$= 500\, e^{\left[\int_0^8 (0.07-0.005s)ds + \int_8^{10} 0.06ds\right]}$$

$$= 500\, e^{\left\{\left[0.07S - \frac{0.005}{2}S^2\right]_0^8 + \left[0.06S\right]_8^{10}\right.}$$

$$= 500e^{0.40+0.12}$$

$$= 841.01$$

(ii) $PV = \int_{10}^{18} 200e^{0.1t} . e^{-\int_0^t \delta(s)ds} \quad dt$

$$= \int_{10}^{18} 200e^{0.1t} .\, e^{-\left[\int_0^8 0.07-0.005s)ds + \int_8^t 0.06ds\right]}$$

$$= \int_{10}^{18} 200e^{0.1t} .\, e^{-0.40} .\, e^{0.48-0.06t} \; dt$$

$$= 200e^{0.08} \int_{10}^{18} e^{0.04t} \; dt$$

$$= \frac{200e^{0.08}}{0.04}\left[e^{0.04t}\right]_{10}^{18}$$

$$= 5000\, e^{0.08}\left[e^{0.72} - e^{0.40}\right] = 3047.33$$

5 Present Value = $5000\left(\bar{a}_{\overline{1|}} + v.\ddot{a}^{(12)}_{\overline{1|}} + v^2.\ddot{a}^{(2)}_{\overline{1|}}\right)$ at i%

where $1+i=(1.02)^4 \Rightarrow i = 8.24322\%$ p.a. effective

$$\bar{a}_{\overline{1|}} = \frac{i}{\delta}.v = \frac{0.0824322}{Ln\,1.0824322}\cdot\frac{1}{1.0824322}$$

$$= 0.9614201$$

and $\ddot{a}^{(12)}_{\overline{1|}} = (1.0824322)^{1/12}\cdot\dfrac{1-v}{i^{(12)}}$

where

$1.0824322 = \left(1+\dfrac{i^{(12)}}{12}\right)^{12} \Rightarrow i^{(12)} = 0.0794725$

$\Rightarrow \ddot{a}^{(12)}_{\overline{1|}} = 0.9645970$

and $\ddot{a}^{(2)}_{\overline{1|}} = (1.0824322)^{\frac{1}{2}}\cdot\dfrac{1-v}{i^{(2)}}$

where $1.0824322 = \left(1+\dfrac{i^{(2)}}{2}\right)^{2} \Rightarrow i^{(2)} = 0.0808000$

$\Rightarrow \ddot{a}^{(2)}_{\overline{1|}} = 0.9805844$

So $PV = 5000\left(0.9614201 + v*0.9645970 + v^2*0.9805844\right) = 13{,}447.39$

Examiners' Comment: There are other valid methods for obtaining the required answer which also received full credit.

6 (i) (a) Work in $t = 0$ monetary values

$$25000 = 10000\left(v \times \frac{170.7}{183.3} + v^2 \times \frac{170.7}{191.0} + v^3 \times \frac{170.7}{200.9}\right)$$

where $v = \frac{1}{1+i'}$ with i' = real rate of return

Try 4% RHS = 24770.94

3% RHS = 25241.25

$$\Rightarrow i = 0.03 + \frac{25241.25 - 25000}{25241.25 - 24770.94} \times 0.01$$

$= 0.0351$ i.e. 3.5%

(b) $25000 = 10000\ a_{\overline{3}|}$ at $i\%$ p.a.

$$\Rightarrow a_{\overline{3}|} = 2.5$$

From tables, $a_{\overline{3}|} = 2.5313$ at 9%

$= 2.4869$ at 10%

$$\Rightarrow i = 0.09 + \frac{2.5313 - 2.5}{2.5313 - 2.4869} * 0.01$$

$= 0.097$

i.e. 9.7% p.a.

(ii) We should find that $\frac{1+i}{1+i'} \simeq 1+e$

where e = average annual rate of inflation over the period.

Hence $\frac{1+i}{1+i^1} = \frac{1.097}{1.035} = 1.06$

which implies 6% p.a. inflation over the period

The actual average inflation rate was:

$$(1+e)^3 = \frac{200.9}{170.7} \Rightarrow e = 5.6\% \text{ p.a.}$$

The inflation rate would not be expected to be exactly 6% p.a. since the Retail Price Index is not increasing by a constant amount each year.

7 $$\left(1+\frac{i^{(4)}}{4}\right)^4 = 1.04 \Rightarrow i^{(4)} = 0.039414$$

$$g(1-t_1) = \frac{0.05}{1.03} \times 0.80 = 0.038835$$

$$\Rightarrow i^{(4)} > (1-t_1)g$$

$\Rightarrow$ Capital gain on contract

$\Rightarrow$ Assume redeemed as late as possible (ie: after 20 years) to obtain minimum yield.

Price of stock, P:

$$P = 100000 \times 0.05 \times 0.80 \times a_{\overline{20|}}^{(4)}$$

$$+\left(103000 - 0.25(103000 - P)\right)v^{20} \text{at } 4\%$$

$$\Rightarrow P = \frac{4000\, a_{\overline{20|}}^{(4)} + 77250v^{20}}{1-0.25v^{20}}$$

$$= \frac{4000 \times 1.014877 \times 13.5903 + 77250 \times 0.45639}{1 - 0.25 \times 0.45639}$$

= 102,072.25

8 (i) No, because the spread (convexity) of the liabilities would always be greater than the spread (convexity) of the assets $\Rightarrow$ 3rd Redington condition would never be satisfied.

(ii) Conditions required: (a) $V_A = V_L$

(b) $V_A^{'} = V_L^{'}$

(c) $V_A^{''} > V_L^{''}$

where differentiation can be in respect of delta or *i*. In this solution, it is in respect of delta.

(a) $V_A = 3.43v^{15} + 7.12v^{25}$ @ 7%

= 2.5550

$V_L = 4v^{19} + 6v^{21}$

= 2.5551

$\Rightarrow V_A = V_L$ (ignoring rounding)

(b) $-V'_A = 3.43 \times 15v^{15} + 7.12 \times 25v^{25}$

= 51.444

$-V'_L = 4 \times 19v^{19} + 6 \times 21v^{21}$

= 51.445

$\Rightarrow V'_A = V'_L$ (ignoring rounding)

(c) $V''_A = 3.43 \times 15^2 v^{15} + 7.12 \times 25^2 v^{25}$

= 1099.627

$V''_L = 4 \times 19^2 v^{19} + 6 \times 21^2 v^{21}$

1038.322

$\Rightarrow V''_A > V''_L$

$\Rightarrow$ all 3 conditions are satisfied.

Examiners' Comment: There are other valid methods for obtaining the required answer which also received full credit.

9 (i) From two–year stock information:

Price = $3a_{\overline{2|}} + 102v^2$ at 5.5%

= $3 * 1.84632 + 102 * 0.89845$

= 97.1811

Therefore, from one-year forward rate information,

$$97.1811 = \frac{3}{1+i_1} + \frac{3+102}{(1+i_1)(1+f_{1,1})}$$

where i_1=one-year spot rate

$f_{1,1}$= one-year forward rate from t = 1

$$\Rightarrow 97.1811 = \frac{3}{1+i_1} + \frac{105}{(1+i_1)1.05}$$

$$\Rightarrow 97.1811 = \frac{103}{1+i_1}$$

$$\Rightarrow i_1 = 5.9877\% \text{ p.a.}$$

(ii) From three-year stock information:

$$108.9 = \frac{10}{1+i_1} + \frac{10}{(1+i_i)1.05} + \frac{110}{(1+i_1)1.05(1+f_{2,1})}$$

where $f_{2,1}$=one-year forward rate from t = 2

Hence

$$108.9 = \frac{10}{1.059877} + \frac{10}{1.059877 \times 1.05} + \frac{110}{1.059877 \times 1.05 \times (1+f_{2,1})}$$

$$\Rightarrow 108.9 = 9.4351 + 8.9858 + \frac{110}{1.11287 * (1+f_{2,1})}$$

$\Rightarrow f_{2,1} = 9.245\%$ p.a.

(iii) Let $y_2\%$ p.a. be the two-year par yield

$$\Rightarrow 100 = y_2\left(\frac{1}{1+i_1}+\frac{1}{(1+i_1)(i+f_{1,1})}\right)+\frac{100}{(1+i_1)(1+f_{1,1})}$$

$$\Rightarrow 100 = y_2\left(\frac{1}{1.059877}+\frac{1}{1.059877*(1.05)}\right)+\frac{100}{1.059877\times 1.05}$$

$$\Rightarrow 100 = y_2(1.84208)+89.8577$$

$$\Rightarrow y_2 = 5.506\%\text{ p.a.}$$

10 (i) (a) Let i_t be the (random) rate of interest in year t. Let S_n be the accumulation of a single investment of 1 unit after n years:

$$E(S_n) = E\left[(1+i_1)(1+i_2)\ldots(1+i_n)\right]$$

$$E(S_n) = E[1+i_1]E[1+i_2]\ldots E[1+i_n] \text{ as } \{i_t\} \text{ are independent}$$

$$E[i_t] = j$$

$$\therefore \quad E(S_n) = (1+j)^n$$

(b) $$E\left(S_n^2\right) = E\left[\left[(1+i_1)(1+i_2)\ldots(1+i_n)\right]^2\right]$$

$$= E(1+i_1)^2\,E(1+i_2)^2\ldots E(1+i_n)^2 \text{ (using independence)}$$

$$= E\left(1+2i_1+i_1^2\right)E\left(1+2i_2+i_2^2\right)\ldots E\left(1+2i_n+i_n^2\right)$$

$$= \left(1+2j+s^2+j^2\right)^n$$

as $E\left[i_i^2\right]=V\left[i_t\right]+E\left[i_t\right]^2=s^2+j^2$

$\therefore\ \mathrm{Var}\left[S_n\right]=\left(1+2j+s^2+j^2\right)^n-\left(1+j\right)^{2n}$

(ii) (a) $E\left[\text{Interest}\right]=j=\dfrac{1}{2}\left(i_1+i_2\right)$

$Var\left[\text{Interest}\right]=s^2=E\left[\text{Interest}^2\right]-\left[E\left(\text{Interest}\right)\right]^2$

$$=\frac{1}{2}\left(i_1^2+i_2^2\right)-\left[\frac{1}{2}\left(i_1+i_2\right)\right]^2$$

$$=\frac{1}{4}\left(i_1^2+i_2^2\right)-\frac{1}{2}i_1.i_2$$

$$=\left[\frac{1}{2}\left(i_1-i_2\right)\right]^2$$

(b) $E\left[S_{25}\right]=\left(1+j\right)^{25}=5.5$

$\Rightarrow j=0.0705686$

$\mathrm{Var}\ \left[S_{25}\right]=\left(1+2j+j^2+s^2\right)^{25}-\left(1+j\right)^{50}=\left(0.5\right)^2$

$\Rightarrow\left(1+2*0.0705686+0.0705686^2+s^2\right)^{25}-\left(1.0705686\right)^{50}\qquad=0.25$

$\Rightarrow s^2=0.000377389$

Hence, $s^2=0.000377389=\dfrac{1}{4}\left(i_1-i_2\right)^2$

$\Rightarrow\quad i_1-i_2=0.0388530$ (taking positive root since $i_1>i_2$)

$i_1+i_2=2\times0.07056862\ =0.1411372$

$\Rightarrow\quad 2i_1=0.0388530+0.1411372$

$i_1=0.089995\quad\left(8.9995\%\text{p.a.}\right)$

and $i_2 = 0.051142 \quad (5.1142\%\text{p.a.})$

11 (i) Loan $= 1000\left(a_{\overline{10}|}^{5\%} + v_{5\%}^{10}\ a_{\overline{10}|}^{7\%}\right)$

$= 1000(7.7217 + 0.61391 \times 7.0236)$

$= 12033.56$

(ii) Note $\dfrac{439.52}{8790.48} = 0.05 \Rightarrow x \le 10$

$\Rightarrow 8790.48 = 1000\left(a_{\overline{11-x}|}^{5\%} + v_{5\%}^{11-x} a_{\overline{10}|}^{7\%}\right)$

$\Rightarrow 8.79048 = \dfrac{\left(1 - v^{11-x}\right)}{0.05} + v^{11-x} * 7.0236$

$\Rightarrow 8.79048 = 20 - (20 - 7.0236)v^{11-x}$

$\Rightarrow v^{11-x} = \dfrac{11.20952}{12.9764} = 0.86384$ at 5%

$\Rightarrow x = 8$

(iii) Let Y = reduced final payment
n = new total term of loan

Loan outstanding after 10 years = $1000\ a_{\overline{10}|}^{7\%}$ = £7,023.60

After change is made:

$7023.60 = 1000\ a_{\overline{n-11}|} + Yv^{n-10}$ at 5%

try n = 20 (i.e., keep to original term)

RHS = $1000 \times 7.1078 + Y \times 0.61391$

$\Rightarrow Y = -137.15$

$\Rightarrow$ doesn't work

try n = 19

RHS = $1000 \times 6.4632 + Y \times 0.64461$

$\Rightarrow Y = 869.36$

Hence:

(a) Term shortened by 1 year

(b) Final instalment = £869.36

(c) Under original terms, total interest paid is:

$20 \times 1000 - 12033.56 = 7966.44$

Under changed terms, total interest paid is:

$18 \times 1000 + 869.36 - 12033.56 = 6835.80$

$\Rightarrow$ difference = £1,130.64

END OF EXAMINERS' REPORT