INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

September 2017

Subject CT3 – Probability and Mathematical Statistics Core Technical

Introduction

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and using past papers as a revision aid and also those who have previously failed the subject.

The Examiners are charged by Council with examining the published syllabus. The Examiners have access to the Core Reading, which is designed to interpret the syllabus, and will generally base questions around it but are not required to examine the content of Core Reading specifically or exclusively.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report; other valid approaches are given appropriate credit. For essay-style questions, particularly the open-ended questions in the later subjects, the report may contain more points than the Examiners will expect from a solution that scores full marks.

The report is written based on the legislative and regulatory context pertaining to the date that the examination was set. Candidates should take into account the possibility that circumstances may have changed if using these reports for revision.

Luke Hatter
Chair of the Board of Examiners
December 2017

A. General comments on the *aims of this subject and how it is marked*

1. The aim of the Probability and Mathematical Statistics subject is to provide a grounding in the aspects of statistics and in particular statistical modelling that are of relevance to actuarial work.

2. Some of the questions in this paper admit alternative solutions from these presented in this report, or different ways in which the provided answer can be determined. All mathematically correct and valid alternative solutions or answers received credit as appropriate.

3. Rounding errors were not penalised, but candidates lost marks where excessive rounding led to significantly different answers.

4. In cases where the same error was carried forward to later parts of the answer, candidates were given full credit for the later parts.

5. In questions where comments were required, reasonable comments that were different from those provided in the solutions also received full credit where appropriate.

B. General comments on *student performance in this diet of the examination*

1. Performance was satisfactory, with most candidates demonstrating good understanding and application of core topics in probability and mathematical statistics.

2. A number of questions contained elements of the CT3 Core Syllabus that have not appeared frequently in recent examination papers (e.g. parts of Questions 2, 6, 8) – performance in these parts was less satisfactory. Candidates are advised to revise thoroughly all elements of the syllabus.

3. Answers requiring algebraic manipulations and elements of calculus contained a considerable number of mathematical errors (e.g. Question 7). Candidates are encouraged to revise relevant core mathematical topics and practise their skills as part of their preparation for the CT3 examination.

C. Pass Mark

The Pass Mark for this exam was 55.

Solutions

Q1

(i) Mean: $\frac{1}{125}(30+52+60+56+50)=\frac{248}{125}=1.984$ [1]

The median is the 63rd largest observation, that is 2. [1]

The mode is the observation with the highest frequency, that is, 1 [1]

(ii) The mean is almost identical to the median which is an indication for symmetrical data as 50% of data are below the mean. [1]

On the other hand, the mode is different. However, the frequency of the mode, 1, and the median, 2, are almost identical. Again, this is no strong sign for asymmetry. [1]

> *Generally well answered. Many candidates ignored the fact that the mean and median are almost identical, but credit was given if the answer was given in terms of the distribution being skewed, based on a graph or on values of the summary statistics.*

Q2

(i) The only possible outcomes for $\bar{X}$ when $n=2$ are 1, 1.5, 2, 2.5 and 3, [0.5]

and the probabilities are given by:

$$P\left[\bar{X}=1\right]=P\left[X_1=1 \text{ and } X_2=1\right]=0.6^2=0.36$$
$$P\left[\bar{X}=1.5\right]=2\times0.6\times0.3=0.36$$
$$P\left[\bar{X}=2\right]=2\times0.6\times0.1+0.3\times0.3=0.21$$
$$P\left[\bar{X}=2.5\right]=2\times0.3\times0.1=0.06$$
$$P\left[\bar{X}=3\right]=0.1^2=0.01$$

Marking: 0.5 for each correct probability [2.5]

(ii) For $n=50$ we can use the CLT to obtain an approximate distribution of $\bar{X}$.
[1]

We first find $E\left[\bar{X}\right]=E\left[X_1\right]=0.6+2\times0.3+3\times0.1=1.5$ [1]

And $V\left(X_1\right)=0.5^2\times0.6+0.5^2\times0.3+1.5^2\times0.1=\frac{0.9}{4}+0.225=0.45$ [1]

The variance of $\bar{X}$ is then given by $V\left(\bar{X}\right)=\dfrac{0.45}{50}=0.009$ [1]

And the approximate distribution of $\bar{X}$ is $N\left(1.5, 0.009\right)$. [1]

> *Part (i) was not well answered. Note that the question asks for the exact distribution and involves a discrete random variable. Most candidates answered in terms of an approximation to this distribution, which was essentially part (ii) of the question.*

Q3 (i) $\dfrac{(n-1)S^2}{\sigma^2} \sim X_{n-1}^2$. For small *n* the distribution is heavily skewed to the right. [1]

As *n* gets larger the distribution becomes symmetric. [1]

(ii) $V\left(\dfrac{(n-1)S^2}{\sigma^2}\right)=2(n-1)\Rightarrow V\left(S^2\right)=\dfrac{\sigma^4}{(n-1)^2}2(n-1)\Rightarrow V\left(S^2\right)=\dfrac{2\sigma^4}{(n-1)}$ [2]

> *In part (i) the quality of the answers was mixed. Many candidates failed to comment with respect to the sample size n. Part (ii) was better answered.*

Q4 Let D_O = departs on time, D_L = departs late.

Let A_E = arrives early, A_O = arrives on time and A_L = arrives late.

(i) $P\left(A_O\right)=1-0.1-0.2=0.7$ [1]

(ii) Told $P\left(A_E|D_L\right)=0 \; so \; P\left(A_E \cap D_O\right)=P\left(A_E\right)=0.1$ [1]

$P\left(A_E|D_O\right)=\dfrac{P\left(A_E \cap D_O\right)}{P\left(D_0\right)}=0.1/0.85=0.1176$ [2]

[Or $P\left(D_O|A_E\right)=\dfrac{P\left(A_E \cap D_O\right)}{P\left(A_E\right)}=1$

$P\left(A_E|D_O\right)=P\left(D_O|A_E\right)P\left(A_E\right)/P\left(D_O\right)=1*0.1/0.85=0.1176$]

(iii) $P\left(A_L \cap D_O\right) = P\left(A_L | D_O\right) P\left(D_O\right) = 0.1 * 0.85 = 0.085$ [1]

$$P(A_O \cap D_O) = P(D_O) - P(A_L \cap D_O) - P(A_E \cap D_O) = 0.85 - 0.085 - 0.1$$
$$= 0.665$$

[1]

(iv) $P\left(A_O \cap D_L\right) = P\left(A_O\right) - P\left(A_O \cap D_O\right) = 0.7 - 0.665 = 0.035$ [1]

$P(A_O | D_L) = P(A_O \cap D_L) / P(D_L) = 0.035 / 0.15 = 0.233$ [2]

(v) $P(D_O | A_L) = P(A_L \cap D_O) / P(A_L) = 0.085 / 0.2 = 0.425$ [2]

Part (i) was answered correctly by the vast majority of candidates. The rest of the question was not well answered. Most candidates showed understanding of the probabilities in the context required here, but many struggled in applying the conditional forms correctly throughout.

Q5 (i) Confidence interval (CI) is given by

$$\hat{p} \pm 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 0.55 \pm 1.96\sqrt{\frac{0.55 * 0.45}{100}} = 0.55 \pm 0.0975$$

i.e. (0.4525, 0.6475) [2]

(ii) (a) If p is proportion voting for candidate A, we want

H_0: $p = 0.5$ (or $p \le 0.5$) v. H_1: $p > 0.5$ [1]

(b) CI in part (i) is 2-sided, so cannot be used here. [1]

(iii) With $\hat{p} = 0.52$ we want the endpoints of the 95% CI to be at most 0.52 ± 0.02. This implies that

$$1.96\sqrt{\frac{0.52 * 0.48}{n}} = 0.02 \Rightarrow n = \frac{0.52 * 0.48}{(0.02/1.96)^2} = 2397.16$$

The sample size must be at least 2398. [2]

Most parts of the question were very well answered. In part (ii)(b), many answers were not adequately explained, with candidates failing to demonstrate understanding of the relation between one or two sided *tests and CIs.*

Q6 (i) The null hypothesis here is that the individual is guessing, i.e. H_0: $p = 0.5$ where p is the probability of success in a binomial(10, p) experiment. [1]

P(type I error) = P(reject null hypothesis when it is correct, i.e. when $p = 0.5$) [1]

= P(correct answers $\geq 7 \mid p = 0.5$)

$$=\sum_{x=7}^{10}\binom{10}{x}0.5^{x}(1-0.5)^{10-x}=1-0.8281=0.1719 \quad \text{(using tables)}$$ [1]

(ii) (a) P(type II error) = P(accept H_0 when it is false, i.e. when $p = 0.7$) [1]

= P(correct answers $< 7 \mid p = 0.7$) = 0.3504 [1]

(b) P(type II error) = P(accept H_0 when it is false, i.e. when $p = 0.8$)

= P(correct answers $< 7 \mid p = 0.8$) = 0.1209 [2]

(iii) The power of the test is given by $1-P$(type II error), and therefore increases as p increases. In other words, the power is higher (i.e. it is more likely to correctly reject the hypothesis of guessing) when the individual is more skilled in identifying the preparation method correctly. [3]

The question was generally well answered, with most candidates showing understanding of the concepts involved. There were some problems however with calculating the probabilities – note that these are given in the statistical tables. Credit was given to answers using a normal approximation to the binomial distribution in parts (i) and (ii) despite the sample size being small. In part (iii) all valid comments were given credit.

Q7 (i) $M_S(y)=\exp\left\{\lambda\left(M_X(y)-1\right)\right\}$ where $M_X(y)$ is the MGF of X [1]

(ii) $M_S'(y)=\lambda M_X'(y)\exp\left\{\lambda\left(M_X(y)-1\right)\right\}$ [1]

$$\Rightarrow E(S)=M_S'(0)=\lambda M_X'(0)\exp\left\{\lambda\left(M_X(0)-1\right)\right\}=\lambda\mu$$ [1]

$$M_S''(y)=\lambda M_X''(y)\exp\left\{\lambda\left(M_X(y)-1\right)\right\}+\left(\lambda M_X'(y)\right)^2\exp\left\{\lambda\left(M_X(y)-1\right)\right\}$$ [1]

$$\Rightarrow M_S''(0) = \lambda M_X''(0)\exp\{\lambda(M_X(0)-1)\} + (\lambda M_X'(0))^2 \exp\{\lambda(M_X(0)-1)\}$$
$$= \lambda(\sigma^2+\mu^2)+(\lambda\mu)^2 \qquad [1]$$

$$V(S) = M_S''(0) - (M_S'(0))^2 = \lambda(\sigma^2+\mu^2)+(\lambda\mu)^2-(\lambda\mu)^2 = \lambda(\sigma^2+\mu^2) \qquad [1]$$

> *The answer in part (i) was given in a variety of forms, and credit was given as appropriate. There were various errors or incomplete answers in part (ii), with many candidates showing knowledge of the required steps but failing to derive the correct answer mainly due to algebra and differentiation problems.*

Q8 (i) $F_Z(z) = P[z \le z] = P[\max(X_1, X_2) \le z] = P[X_1 \le z, X_2 \le z]$ [1]

and because of X_1 and X_2 being iid [1]

$$F_Z(z) = P[X_1 \le z]^2 = \left(\frac{z+\theta}{2\theta}\right)^2 \qquad [1]$$

So, $f_Z(z) = \dfrac{d}{dz}\left(\dfrac{z+\theta}{2\theta}\right)^2 = \dfrac{z+\theta}{2\theta^2}$ [1]

(ii) $E(Z) = \displaystyle\int_{-\theta}^{\theta} z\frac{z+\theta}{2\theta^2}dz = \left[\frac{z^3}{6\theta^2}+\frac{z^2}{4\theta}\right]_{-\theta}^{\theta} = \frac{\theta}{3}$ [1]

(iii) (a) $\text{bias}(\hat\theta_1) = E(\hat\theta_1) - \theta = 3E(Z) - \theta = 0$ [1]

(b) $E(Z^2) = \displaystyle\int_{-\theta}^{\theta} z^2 \frac{z+\theta}{2\theta^2}\, dz = \left[\frac{z^4}{8\theta^2}+\frac{z^3}{6\theta}\right]_{-\theta}^{\theta} = \frac{\theta^2}{3}$ [1]

$\text{MSE}(\hat\theta_1) = V(\hat\theta_1)$ [1]

$$= V(3Z) = 9V(Z) = 9\left[E(Z^2) - E^2(Z)\right] = 9\left[\frac{\theta^2}{3}-\frac{\theta^2}{9}\right] = 2\theta^2$$

[2]

(iv) (a) $\text{bias}(\hat\theta_2) = E(\hat\theta_2) - \theta = 2E(Z) - \theta = 2\dfrac{\theta}{3} - \theta = -\dfrac{\theta}{3}$ [2]

(b) $\text{MSE}(\hat{\theta}_2) = \text{bias}^2\left(\hat{\theta}_2\right) + V\left(\hat{\theta}_2\right) = \dfrac{\theta^2}{9} + V\left(2Z\right) = \dfrac{\theta^2}{9} + 4V\left(Z\right)$

$$= \frac{\theta^2}{9} + 4\frac{2\theta^2}{9} = \theta^2$$

[2]

(v) $\hat{\theta}_1$ is unbiased, but has much larger MSE compared to $\hat{\theta}_2$ (by factor of 2). [1]
On the other hand $\hat{\theta}_2$ has considerable bias, equal to a third of the true value of the parameter. [1]

> *Part (i) was not well answered with a number of candidates not attempting it at all. Note that similar questions have appeared in past examination papers in different contexts. Part (ii) was generally well answered, although there were issues with the simple integration (e.g. limits). Parts (iii), (iv) and (v) were well answered.*

Q9 (i) Overall mean: $\bar{X} = \dfrac{1}{20}\left(6 + 6.05 + 5.75 + 5.2\right) = \dfrac{23}{20} = 1.15$ [1]

Variance:

$$S^2 = \frac{1}{19}\left[\left(7.375 + 7.458 + 6.703 + 5.510\right) - 20 \times 1.15^2\right]$$

$$= \frac{1}{19}\left[27.046 - 26.45\right] = 0.031368$$ [1]

Standard deviation

$$S = \sqrt{0.031316} = 0.177$$ [1]

With the sample not being large we assume that the sample comes from a normal distribution. [0.5]

$$t_{0.025,19} = 2.093$$ [0.5]

95% Confidence interval using t result

$$\left[1.15 - t_{0.025,19}\frac{0.177}{\sqrt{20}}, 1.15 + t_{0.025,19}\frac{0.177}{\sqrt{20}}\right]$$

$$\left[1.15 - 0.0828, 1.15 + 0.0828\right] = \left[1.067,\ 1.233\right]$$ [2]

(ii) The confidence interval in part (i) does not contain 0.9. Therefore, there is a significant difference between the reaction time under alcohol intoxication and 0.9. [2]

(iii) $Y_{ij} = \mu + \tau_i + \varepsilon_{ij}$ [0.5]

where μ is the overall mean and τ_i is the treatment effect of treatment i [0.5]

and ε_{ij} being i.i.d. $N\left(0,\sigma^2\right)$ [0.5]
In particular, it is assumed that the variance is the same in all groups. [0.5]

(iv) $$SS_B = 5\times\left[(1.2-1.15)^2+(1.21-1.15)^2+(1.15-1.15)^2+(1.04-1.15)^2\right]$$

$$= 5\times\left[0.05^2+0.06^2+0.11^2\right] = 0.091$$ [2]

$$SS_R = 7.375-\frac{6.00^2}{5}+7.458-\frac{6.05^2}{5}+6.703-\frac{5.75^2}{5}+5.510-\frac{5.20^2}{5}$$

$$=27.046-\frac{1}{5}\left(36+6.05^2+5.75^2+5.2^2\right)=27.046-26.541=0.505$$ [2]

ANOVA table:

Source of variation	d.f.	SS	MSS	F
Between groups	3	0.091	0.030	0.961
Residual	16	0.505	0.032	

[3]

The value of the test statistic is very close to 1, which needs to be compared to an $F_{3,16}$ distribution (3.239 at 5% level). [1]

We can therefore conclude that regular exercising has no impact on the reaction time under alcohol intoxication. [1]

Alternatively, the critical values (at 10%) is 2.462 > 0.961, and we come to the same conclusion.

Answers were very satisfactory overall. In part (ii) many candidates performed a full test, and received full credit, but note that this was not required. There were some numerical errors in calculating the relevant sums in part (iv).

Q10 (i) In bivariate data, the response variable is a random variable whose value may be influenced by the value of the explanatory variable. [2]

NOTE: This is not defined precisely in the core reading and should be marked on the basis of understanding rather than precision.

(ii) $S_{yy} = \left(270.16 - 38.4^2 / 9\right) = 106.32$ [1]

$S_{tt} = 4976 - 202^2 / 9 = 442.22$ [1]

$S_{yt} = 1011.2 - 38.4 \times 202 / 9 = 149.33$ [1]

$r = 149.33 / \sqrt{106.32 \times 442.22} = 0.6887$ [1]

(iii) $H_0 : \rho = 0 \text{ vs } H_1 : \rho \neq 0$ [1]

$z_r = \frac{1}{2} \log \frac{1+r}{1-r} = 0.8455$ [1]

$z_r \sim N\left(z_0, \frac{1}{n-3}\right) = N\left(0, 1/6\right)$ [1]

$\text{Test statistic} = 0.8455 / \left(\frac{1}{6}\right)^{0.5} = 2.071$ [1]

Compare with $Z_{0.975} = 1.96$ [1]

Therefore reject H_0 at 5% level (but note that we do not reject H_0 at 1% level). [1]

(iv) $\beta = S_{yt} / S_{yy} = 149.33 / 106.32 = 1.405$ [1]

$\alpha = 202 / 9 - 1.405 \times \left(38.4 / 9\right) = 16.45$ [1]

$t = 16.45 + 1.405y$ [1]

The question was very well answered. In part (iii) some candidates performed a t test instead of the test based on Fisher's transformation. Also, a few candidates were not clear in their answers about the response and explanatory variables.

END OF EXAMINERS' REPORT