INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

September 2015

Subject CT3 – Probability and Mathematical Statistics Core Technical

Introduction

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and using past papers as a revision aid and also those who have previously failed the subject.

The Examiners are charged by Council with examining the published syllabus. The Examiners have access to the Core Reading, which is designed to interpret the syllabus, and will generally base questions around it but are not required to examine the content of Core Reading specifically or exclusively.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report; other valid approaches are given appropriate credit. For essay-style questions, particularly the open-ended questions in the later subjects, the report may contain more points than the Examiners will expect from a solution that scores full marks.

The report is written based on the legislative and regulatory context pertaining to the date that the examination was set. Candidates should take into account the possibility that circumstances may have changed if using these reports for revision.

F Layton
Chairman of the Board of Examiners
December 2015

A. General comments on the *aims of this subject and how it is marked*

1. The aim of the Probability and Mathematical Statistics subject is to provide a grounding in the aspects of statistics and in particular statistical modelling that are of relevance to actuarial work.

2. Some of the questions in this paper admit alternative solutions from these presented in this report, or different ways in which the provided answer can be determined. All mathematically correct and valid alternative solutions or answers received credit as appropriate.

3. Rounding errors were not penalised, unless excessive rounding led to significantly different answers.

4. In cases where the same error was carried forward to later parts of the answer, candidates were only penalised once.

5. In questions where comments were required, reasonable comments that were different from those provided in the solutions also received full credit where appropriate.

B. General comments on *student performance in this diet of the examination*

1. The performance was generally good, with most questions being well answered.

2. The pass rate was in line with previous sessions and there were a number of excellent scripts achieving very high scores.

3. In general, questions that required moderate mathematical calculus skills (e.g. differentiation) were poorly answered.

4. In some parts candidates failed to distinguish between the need for different types of test (e.g. normal *z*-test, as opposed to a *t*-test).

C. Comparative pass rates for the past 3 years for this diet of examination

Year	*%*
September 2015	68
April 2015	66
September 2014	57
April 2014	60
September 2013	64
April 2013	59

Reasons for any significant change in pass rates in current diet to those in the past:

The pass rate for this examination diet is slightly higher than the April 2015 rate, but not materially different. Variation in the pass rate between sessions is expected as different cohorts of students sit the examination – however the increasing rates in recent diets reflect stronger performance from the candidates.

Solutions

Q1 (i) New sum is (11860 – 770 – 510 + 1000 + 280) = 11860
New sum of squares: (8438200 – 770^2 – 510^2 + 1000^2 + 280^2) = 8663600

Therefore:

New sample mean = 11860/20 = 593
New standard deviation (sd) = $\{(8663600 - 11860^2/20)/19\}^{0.5} = 292.95$

(ii) Since the sum of the two new claims is the same as those replaced, the mean is the same.

However the sd has increased as the two new claims are further away from the mean as compared to the two claims in the first sample.

Part (i) was generally well answered. In part (ii) the explanation about the sd was not always convincing.

Q2 (i) We have mean = 1.6/0.2 = 8 and sd = $(1.6/0.2^2)^{0.5} = 6.325$

For the mode we need to maximise the probability density function (pdf):

$$f(y) = \frac{\lambda^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\lambda y}$$

$$\Rightarrow \log\big(f(y)\big) = \alpha\log(\lambda) - \log\big(\Gamma(\alpha)\big) + (\alpha-1)\log(y) - \lambda y$$

and

$$\frac{d}{dy}\log\big(f(y)\big) = \frac{\alpha-1}{y} - \lambda = 0 \Rightarrow y = \frac{\alpha-1}{\lambda}.$$

$$\left(\text{Also } \frac{d^2}{dy^2}\log\left(f\left(y\right)\right) = -\frac{\alpha-1}{y^2} < 0.\right)$$

So mode is at $y = \frac{\alpha-1}{\lambda} = 3$.

Therefore, $\zeta = \frac{8-3}{6.325} = 0.791$.

(ii) For unimodal symmetrical distributions, the mode will coincide with the mean and therefore deviations of this measure from 0 will indicate asymmetry.

Also, the measure is standardised by dividing with the standard deviation to make it scale-free.

> Many candidates failed to work out the mode correctly. Note that this is a typical calculus maximisation exercise. Part (ii) was not well answered, with many candidates failing to comment on the relationship between the mean and the mode, and very few mentioning the standardisation.

Q3 $\frac{S_1^2}{S_2^2} / \frac{\sigma_1^2}{\sigma_2^2} \sim F_{24,12}$ so the confidence interval is given by

$$\left(\frac{s_1^2}{s_2^2} * \frac{1}{F_{24,12}}, \frac{s_1^2}{s_2^2} * F_{12,24}\right) = \left(\frac{2.4}{1.5} * \frac{1}{3.019}, \frac{2.4}{1.5} * 2.541\right) = \left(0.530, 4.066\right)$$

> Generally very well answered.

Q4 (i) We have $\frac{\hat{\lambda}-\lambda}{\sqrt{\lambda / n}} \sim N\left(0,1\right)$ approximately, and the confidence interval is given by

$$\hat{\lambda} \pm 1.96\sqrt{\hat{\lambda} / 500} \text{ with } \hat{\lambda} = \frac{83}{500} = 0.166$$

i.e. $0.166 \pm 1.96\sqrt{0.166 / 500}$ which gives (0.130, 0.202).

(ii) The sample size is large here, so normal approximation is valid. [Equivalently $n\lambda$ is large.]

> Generally well answered. Some candidates failed to properly justify the use of the normal approximation.

Q5 (i) $$r = \frac{2606.96}{\sqrt{5116701 \times 61.44}} = 0.1470326$$

(ii) $$t = \frac{r\sqrt{25-2}}{\sqrt{1-r^2}} = \frac{0.147 \times \sqrt{23}}{\sqrt{1-0.0216}} = 0.71$$

t has t-distribution with 23 d.f. The 95% quantile is 1.714.

Since this is a two-sided test and 0.71 is within the interval [–1.714, 1.714] the null hypothesis cannot be rejected at 10% level of significance.

(Note that other significance level may also be used.)

[Alternatively, Fisher's transformation gives z = 0.695, and conclusion is the same as above.]

> Well answered. Note that the test in part (ii) is two-sided.

Q6 $$SS_R = 49\left[7^2 + 6^2 + 9^2\right) = 8{,}134$$

$$\bar{Y} = \frac{26+22+27}{3} = 25$$

$$SS_B = 50\left((26-25)^2 + (22-25)^2 + (27-25)^2\right) = 700$$

$$F_{2,147} = \frac{\frac{SS_B}{2}}{\frac{SS_R}{147}} = \frac{700}{2}\frac{147}{8134} = 6.325$$

This is clearly a rather large value since the 1% point from a $F_{2,120}$ distribution is 4.787, so the null hypothesis is rejected. We conclude that alcohol consumption is different in different areas.

[Alternatively, the following sums can be computed:

$\sum y_A = 1{,}300 \quad \sum y_B = 1{,}100 \quad \sum y_C = 1{,}350 \quad \sum y = 3{,}750$

$\sum y_A^2 = 36{,}201 \quad \sum y_B^2 = 25{,}964 \quad \sum y_C^2 = 40{,}419 \quad \sum y^2 = 102{,}584$

$SS_T = 8{,}834 \quad SS_B = 700$]

> Mixed answers. Candidates who were able to calculate correctly the various sums of squares did well. Note that the main answer provided here is computationally more efficient than the alternative answer (which most candidates preferred).

Q7 (i)

$$E\left[Y|X=1\right] = \frac{-1\times 0 + 0\times\frac{1}{4} + 1\times 0}{\frac{1}{4}} = 0$$

(ii)

$$E\left[X|Y=-1\right] = \frac{1\times 0 + 0\times\frac{1}{4}}{\frac{1}{4}} = 0, \quad E\left[X|Y=0\right] = \frac{1\times\frac{1}{4} + 0\times\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

$$E\left[X|Y=1\right] = \frac{1\times 0 + 0\times\frac{1}{4}}{\frac{1}{4}} = 0$$

(iii)

$$E(X) = E\left[E\left[X\mid Y\right]\right] = E\left[X\mid Y=-1\right]\times P(Y=-1)$$

$$+E\left[X\mid Y=0\right]\times P(Y=0)$$

$$+E\left[X\mid Y=1\right]\times P(Y=1)$$

and $E(X) = 0\times\frac{1}{4} + \frac{1}{2}\times\frac{1}{2} + 0\times\frac{1}{4} = \frac{1}{4}$

> Some reasonable answers, but generally a mixed performance. Note that the question asks candidates to "determine" the various expectations, so working needs to be shown to gain full marks.

Q8 (i) $P[X_A \geq 2] = 1 - F_A(1) = 1 - 0.98248 = 0.01752$ using tables

(ii) $P[X_A = 1]P[A] + P[X_B = 1]P[B] + P[X_C = 1]P[C]$

$= e^{-0.2} * 0.2 * 0.2 + e^{-0.1} * 0.1 * 0.2 + e^{-0.05} * 0.05 * 0.6 = 0.07938286$.

(iii) Let X^0 be the number of claims submitted last year

$$P\left[A \mid X^0 = 1\right] = \frac{P[X_A = 1]P[A]}{P\left[X^0 = 1\right]} = e^{-0.2} * 0.2 * \frac{0.2}{0.0794} = 0.4125$$

(iv) Let X^0 be the number of claims submitted last year, and X^1 be the number of claims that will be submitted in the current year.

$$P[X^1 = 1 | X^0 = 1] = P[\{X^1 = 1 \cap A | X^0 = 1\}] + P[\{X^1 = 1 \cap B | X^0 = 1\}]$$
$$+ P[\{X^1 = 1 \cap C | X^0 = 1\}]$$

The first probability is given as

$$P[\{X^1 = 1\} \cap A | X^0 = 1] = P[X^1 = 1 | A \cap \left\{X^0 = 1\right\}]P[A \mid X^0 = 1]$$
$$= P[X^1 = 1 | A]P[A \mid X^0 = 1]$$

where the last equality follows from conditional independence of X^1 from X^0 given group membership. Then

$$P[X^1 = 1 | A]P[A \mid X^0 = 1] = P[X_A = 1]\frac{P[X_A = 1]P[A]}{P\left[X^0 = 1\right]}$$
$$= (e^{-0.2} * 0.2)^2 * 0.2 / 0.0794 = 0.06754$$

Similarly

$$P[\{X^1 = 1\} \cap B | X^0 = 1] = P[X_B = 1]^2 \frac{P[B]}{P\left[X^0 = 1\right]} = 0.02062$$

$$P[\{X^1 = 1\} \cap C | X^0 = 1] = P[X_C = 1]^2 \frac{P[C]}{P\left[X^0 = 1\right]} = 0.01709$$

Thus:

$$P[X^1 = 1|X^0 = 1] = 0.06754 + 0.02062 + 0.01709 = 0.10521$$

Part (i)	Well answered, although some candidates over-complicated the answer.
Part (ii)	Generally well answered.
Part (iii)	Reasonably well answered.
Part (iv)	This was not well answered. It is a more challenging question, with other parts leading up to this. Many candidates did not attempt it.

Q9 (i) (a) Let σ denote the standard deviation of an estimate. Then we want

$$P\left(\mu - Z_{0.975}\sigma < X < \mu + Z_{0.975}\sigma\right) = 0.95$$

So for the interval width for one observation to be equal to 10 we need:

$$\left(\mu + Z_{0.975}\sigma\right) - \left(\mu - Z_{0.975}\sigma\right) = 2Z_{0.975}\sigma = 2*1.96\sigma = 10$$

$$\Rightarrow \sigma = \frac{5}{1.96} = 2.551$$

(b) Let n denote number of satellite passes. Then the estimated survey height is Normally distributed with variance σ^2 / n. As before we want

$$2\frac{1.96\sigma}{\sqrt{n}} < 1 \Rightarrow n > \left(3.92\sigma\right)^2 = 100$$

(ii) Let $\bar{X}_1$ and $\bar{X}_2$ denote the survey estimates for the two peaks.

Then under H_0: heights are the same,

$$D = \bar{X}_1 - \bar{X}_2 \sim N\left(0, 2\sigma^2 / 20\right) = N\left(0, \sigma^2 / 10\right)$$

P-value is given as:

$$P\left(D \le -1.6 \text{ or } D \ge 1.6\right) = 2P\left(D \ge 1.6\right) = 2(1 - P\left(Z < 1.6 / \left(\sigma / \sqrt{10}\right)\right)$$
$$= 2\left(1 - P\left(Z < 1.983\right)\right) = 2\left(1 - 0.976\right) = 0.048$$

Therefore reject H_0 at the 5% significance level

[Alternatively, the value of the statistic z = 1.983 can be compared with the normal quantile 1.96.]

(iii) t-test – we assume equal variance as the same system is being used.

$$s_P^2 = \frac{1}{20+20-2}\left(19*2.5^2 + 19*2.6^2\right) = 6.505$$

$$\text{test statistic} = \frac{1.6}{s_P\sqrt{\frac{2}{20}}} = \frac{1.6}{2.55\sqrt{\frac{1}{10}}} = 1.984$$

test statistic $\sim t_{20+20-2} = t_{38}$. Quantile is 2.024 at 2.5%.

So do not reject H_0: no difference in means at 5% significance level.

(iv) Both systems gave the same estimate of difference and almost the same standard deviation, with the second being lower. However the tests gave different results.

We did not reject that there was no difference for the test in part (iii) as there was greater uncertainty since we did not know the standard deviation beforehand.

Part (i) was generally well answered. As the result is given in the question, candidates needed to clearly show how to obtain it.

The performance in parts (ii) and (iii) was mixed. Many candidates failed to demonstrate understanding of which statistic must be used in each of the two tests, which was one of the main points of the question.

Q10 (i) We have

$$L(\lambda) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{X_i}}{X_i!} = \frac{e^{-n\lambda}\lambda^{\sum_i X_i}}{\prod_i X_i!} \quad \text{and}$$

$$l(\lambda) = \log(L) = -n\lambda + \log(\lambda)\sum_i X_i - \log\left(\prod_i X_i!\right)$$

$$\text{and } \frac{d}{d\lambda} l(\lambda) = 0 \Rightarrow -n + \sum_i X_i / \lambda = 0 \Rightarrow \hat{\lambda} = \sum_i X_i / n = \bar{X}$$

(ii) Using asymptotic properties of the maximum likelihood estimator (MLE) $\hat{\lambda} \sim N(\lambda, CRlb)$ approximately

with $CRlb = -\dfrac{1}{E\left(\dfrac{d^2}{d\lambda^2} l(\lambda)\right)} = -\dfrac{1}{E\left(-\sum_i X_i / \lambda^2\right)} = \dfrac{\lambda^2}{\sum_i E(X_i)} = \dfrac{\lambda}{n}$

Therefore an approximate 95% confidence interval is given by $\hat{\lambda} \pm 1.96\sqrt{\dfrac{\lambda}{n}}$

and replacing for the variance: $\hat{\lambda} \pm 1.96\sqrt{\dfrac{\hat{\lambda}}{n}}$, i.e. $\bar{X} \pm 1.96\sqrt{\dfrac{\bar{X}}{n}}$

[Could also use central limit theorem with normal approximation to Poisson]

(iii) (a) We have $X_i = 0$ with probability $e^{-\lambda}$ and $X_i > 0$ with probability $1 - e^{-\lambda}$

Therefore, likelihood is given as

$$L(\lambda) = e^{-K\lambda}\left(1 - e^{-\lambda}\right)^{n-K} \Rightarrow$$

$$l(\lambda) = \log(L) = -K\lambda + (n-K)\log\left(1 - e^{-\lambda}\right)$$

and

$$\frac{d}{d\lambda} l(\lambda) = 0 \Rightarrow -K + (n-K)\frac{e^{-\lambda}}{1 - e^{-\lambda}} = 0 \Rightarrow \hat{\lambda} = -\log\left(\frac{K}{n}\right)$$

(b) If $K = 0$ the estimate of λ is infinity, so we need $K \geq 1$.

(iv) The estimator in part (i) is based on more information, as the exact values of the data are known, whereas in part (iii) only partial information is available. Therefore the estimator in part (i) should be more reliable and is preferable.

Parts (i) and (ii) mostly well answered. In part (iii) many candidates did not use the correct likelihood form. Answers to questions involving the likelihood function of a model that may not be typical, have also been problematic in recent sessions and candidates are encouraged to practise more with this fundamental concept in statistics.

Q11 (i) Test statistic: $S^2_{\text{new}} \Big/ S^2_{\text{old}} \sim F_{24,60}$

$$S^2_{\text{new}} = \frac{1}{24}\left(\sum x_i^2 - 25\left[\frac{1}{25}\sum x_i\right]^2\right) = \frac{800 - 25*4^2}{24} = 16.67$$

$$S^2_{\text{old}} = \frac{2,200 - 300^2/61}{60} = 12.08$$

The 95% quantile of $F_{24,60}$ is 1.7 and observed value is

$$F = \frac{16.67}{12.08} = 1.38 < 1.7$$

Therefore, there is no evidence (at 5% level) to suggest that the variance for new buildings is larger.

(ii) Assuming that the two population variances are equal, we have:

$$s_p^2 = \frac{24 \times 16.67 + 60 \times 12.08}{84} = 13.39$$

$$t = \frac{\frac{100}{25} - \frac{300}{61}}{\sqrt{13.39\left(\frac{1}{25} + \frac{1}{61}\right)}} = -1.06$$

The 0.975 quantile (2-sided test) of the t_{84} distribution is between 1.98 and 2.00.

There is no evidence to suggest that the mean maintenance costs of new buildings are different from mean maintenance costs of old buildings.

[Alternatively, if we samples are considered large, we can use the z statistic:

$$z = \frac{4 - 4.92}{\sqrt{\frac{16.67}{25} + \frac{12.05}{61}}} = -0.987\,]$$

(iii) $S_{ax} = \sum a_i x_i - \frac{1}{61}\sum a_i \sum x_i = 30,000 - \frac{1}{61}4,500*300 = 7,869$

$$S_{xx} = 2,200 - 300^2/61 = 724.6$$

$$S_{aa} = 506,400 - 4,500^2 / 61 = 174,433$$

$$\hat{\rho}(A, X) = \frac{S_{ax}}{\sqrt{S_{aa}S_{xx}}} = \frac{7,869}{\sqrt{174,433*724.6}} = 0.7$$

(iv) $$\hat{\gamma} = \frac{61*30,000 - 4,500*300}{61*506,400 - 4,500^2} = \frac{480,000}{10,640,400} = 0.04511$$

Or, using the results in part (iii): $\hat{\gamma} = \dfrac{7,869}{174,433} = 0.04511$

$$\hat{\beta} = \frac{300 - 0.04511*4,500}{61} = 1.59$$

$$\left[\text{Or: } \hat{\beta} = 4.92 - 0.04511*\frac{4500}{61} = 1.59\right]$$

Generally well answered, with some errors in the calculations.

END OF EXAMINERS' REPORT