INSTITUTE AND FACULTY OF ACTUARIES

EXAMINATION

27 April 2018 (pm)

Subject CT4 – Models
Core Technical

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*

2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*

3. *You have 15 minutes of planning and reading time before the start of this examination. You may make separate notes or write on the exam paper but not in your answer booklet. Calculators are not to be used during the reading time. You will then have three hours to complete the paper.*

4. *Mark allocations are shown in brackets.*

5. *Attempt all 10 questions, beginning your answer to each question on a new page.*

6. *Candidates should show calculations where this is appropriate.*

Graph paper is NOT required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator from the approved list.

1 A Markov Chain has the following transition matrix:

$$\begin{array}{c} A \\ B \\ C \end{array} \begin{pmatrix} 0 & 0.6 & 0.4 \\ 0.75 & 0 & 0.25 \\ 0.5 & 0.5 & 0 \end{pmatrix}$$

(i) Draw a transition graph for this Markov Chain, including the transition rates. [2]

(ii) Explain whether the Markov Chain is:

(a) irreducible

(b) periodic

[2]
[Total 4]

2 A football match between two teams, Team A and Team B, is being decided by a penalty competition. Each team takes one penalty alternately. Team A goes first.

Let X_i be the total number of penalties scored by team A minus the total number of penalties scored by team B after the ith penalty has been taken. If $X_i = 2$, team A wins and the competition stops. If $X_i = -2$, team B wins and the competition stops.

(i) Determine the possible sample paths for the process X_i for $i = 1, 2, 3, 4$. [3]

Suppose the chance of team A scoring each of its penalties is 0.5, and the chance of team B scoring each of its penalties is 0.4.

(ii) Determine the distribution of X_i for $i = 2$ and $i = 3$. [3]
[Total 6]

3 (i) List the advantages of graduation:

- by parametric formula.
- with reference to a standard table.
- using a graphical method.

[4]

(ii) Outline the steps involved in graduating mortality rates with reference to a standard table. [3]
[Total 7]

4 (i) Describe what is meant by the following terms:

(a) discrete state space

(b) stochastic model

(c) continuous time model

(d) stochastic process of mixed type

[4]

(ii) Describe the factors which should be considered when deciding whether to consider time in a discrete or continuous way for a model. [3]

[Total 7]

5 The calculation of the daily unit price for a fund investing in commercial properties can be done on one of two bases:

- Bid price – reflecting the price at which the properties could be sold, allowing for the transaction costs for selling properties.
- Offer price – reflecting the price at which properties could be purchased, again allowing for the transaction costs which would apply.

Whether the fund is priced on the Bid or Offer basis depends on whether there is net investment into or redemption from the fund.

Movements between the states Bid (B) and Offer (O) pricing basis are to be modelled using a Markov Jump Process with constant transition rates from Bid to Offer of λ and Offer to Bid of μ.

(i) Give the generator matrix of the Markov Jump Process. [1]

(ii) State the distribution of holding times in each state. [2]

Let ${}_tP_s^{ij}$ be the probability that the process is in state j at time $s + t$ given that it was in state i at time s ($i, j = B, O$).

(iii) Write down Kolmogorov's forward equations for $\frac{d}{dt}\,{}_tP_s^{BB}$ and $\frac{d}{dt}\,{}_tP_s^{BO}$. [2]

The fund was being priced on a Bid price basis at time s.

(iv) Solve the Kolmogorov equations to obtain an expression for ${}_tP_s^{BB}$. [4]

[Total 9]

6 The National Statistics Office of a small, low income country wants to estimate recent death rates. A death registration system has allowed the National Statistics Office to estimate deaths by age nearest birthday for the ten-year period 1 January 2005 – 31 December 2014.

Censuses of this country are infrequent. A successful census was completed on 1 January 2015, but the previous reliable census was on 1 January 2002. Both censuses collected data on the population aged x last birthday by single years of age.

(i) Explain why the National Statistics Office should adjust the age definition in the census data to correspond with that of the deaths data. [2]

Let the census population at age x last birthday on 1 January in year t be $P_{x,t}$.

(ii) Derive an expression, in terms of the $P_{x,t}$, for the exposed to risk for the period covering the years 2005 to 2014 inclusive which the National Statistics Office could use to estimate the overall death rate at age x nearest birthday. [5]

(iii) Set out any assumptions you make in your derivation in part (ii), indicating where in the derivation they are needed. [2]

The death registration system in this country is being maintained, but the next census is not planned until 2025.

(iv) Discuss how you might estimate death rates at age x nearest birthday for the calendar years 2015 and 2016. [3]
[Total 12]

7 (i) Define a Markov Chain. [2]

The manager of a sales team keeps records of how much each of the three sales staff (Andy, Brenda and Carol) sells each week. The data suggests that the sales staff member who makes the most sales each week can be modelled using a Markov Chain with the following transition matrix:

$$\begin{array}{r} \textit{Andy} \\ \textit{Brenda} \\ \textit{Carol} \end{array} \begin{pmatrix} 0.4 & 0.3 & 0.3 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.3 & 0.5 \end{pmatrix}$$

Brenda made the most sales in the first week in April.

(ii) Calculate the probability that each member of the sales staff makes the most sales in the third week of April. [2]

(iii) Calculate the long term proportion of weeks in which each member of the sales staff makes the most sales. [4]

The manager is keen to encourage competition in the team, so he introduces an "Employee of the Week" incentive. He awards "Employee of the Week" to the member of the sales staff who makes the most sales unless this is the same employee who was awarded "Employee of the Week" last week. If last week's "Employee of the Week" makes the most sales the manager will decide which of the other two staff should be "Employee of the Week" and is equally likely to choose either.

(iv) Justify why whoever is awarded "Employee of the Week" can NOT be modelled as a Markov Chain with state space {Andy, Brenda, Carol}. [2]

(v) Identify a state space with the minimum number of states required to model the sequence of "Employees of the Week" as a Markov Chain. [2]

[Total 12]

8 (i) State why the Gompertz model is often used in analyses of human mortality. [2]

The following data are taken from an investigation of the mortality of males aged 60–70 years inclusive in a developed country.

Age (years) x	μ_x	*Deaths*	*Exposed-to-risk*
60	0.02029	49	2,415
61	0.02230	51	2,287
62	0.02466	55	2,230
63	0.02721	68	2,499
64	0.02937	70	2,383
65	0.03102	67	2,160
66	0.03194	69	2,160
67	0.03055	66	2,160
68	0.04297	84	1,955
69	0.04405	88	1,998
70	0.04749	83	1,748

(ii) Determine the parameters of the Gompertz model using the data for ages 60 and 70 years only. [3]

(iii) Test the overall fit of the model you estimated in part (ii) using data from ages 61–69 years only. [6]

(iv) Comment on your results in part (iii). [1]
[Total 12]

9 In a three-state model where the states are:

1. Healthy

2. Sick

3. Dead

let ${}_tp_x^{ij}$ be the probability that a person in state *i* at age *x* is in state *j* at age $x + t$, and let μ_{x+t}^{ij} be the transition intensity from state *i* to state *j* at age $x + t$.

(i) Show from first principles that:

$$\frac{d}{dt}\,{}_tp_x^{13} = {}_tp_x^{11}\,\mu_{x+t}^{13} + {}_tp_x^{12}\,\mu_{x+t}^{23} \qquad [4]$$

Doctors have been investigating the incidence of a sickness called Wadles. This sickness is debilitating and can last for many years. It can sometimes kill the patient, but not always. Doctors have established that if a patient recovers from Wadles they are immune to further infection from this sickness.

(ii) Sketch a diagram showing the states required for this sickness to be modelled as a Markov process and the possible transitions between the states. [2]

During the investigation, the doctors collected data on a sample of people. The data included the total length of time each person was under investigation, and the length of time each person was observed to be afflicted with Wadles. The doctors also recorded each occasion on which a person contracted Wadles, recovered from Wadles and died, noting whether the death was caused by Wadles or another reason.

(iii) Express the likelihood for the transition intensities in terms of the data collected, defining all the terms you use. [3]

(iv) Derive the maximum likelihood estimator of the death rate from Wadles. [3]
[Total 12]

10 ForLawn is a new treatment for lawns which, according to the manufacturer, kills moss within days. A study was done in a small town in rural England in which 14 residents with mossy lawns participated. Seven, chosen at random, were given ForLawn in a plain bottle and treated their lawns with it one morning. The other seven were given the most popular moss treatment already on the market, also in a plain bottle, and treated their lawns with it on the same morning. All 14 were asked to assess their lawns each morning following treatment until all the moss had gone. The study ended after 16 days, at which time any lawn which still had moss was considered to be censored.

Unfortunately a flock of sheep escaped from the fields adjacent to the town on days 5 and 8 and made such a mess of five of the lawns in the study that these lawns had to be withdrawn. These five lawns are also considered to be censored.

The table below shows how many days elapsed before all the moss disappeared for each of the 14 lawns, or until censoring. Censoring is denoted by an *.

ForLawn group:	5, 6, 8, 8*, 8*, 11, 16*
Alternative treatment group:	3, 4, 5, 5*, 5* 5*, 10

(i) Describe THREE types of censoring present in this study, giving examples of how they occur. [3]

(ii) Calculate the Kaplan-Meier estimate of the survival functions of still having moss for each of the two groups separately. [6]

(iii) Sketch the two estimated survival functions on the same graph. [2]

(iv) Comment on your results. [2]

A local statistician suggests that it would be a good idea to use Cox regression to compare the effectiveness of ForLawn with the alternative treatment. She thinks that using a dummy variable, X, with the value 1 for ForLawn and 0 for the alternative treatment would be suitable.

(v) Determine the equations for the hazard function for the lawns treated with ForLawn and those treated with the alternative treatment, defining all the terms you use. [2]

(vi) Derive the partial likelihood of the data in this Cox model. [4]

[Total 19]

END OF PAPER