EXAMINATION

2 October 2007 (am)

Subject CT6 — Statistical Methods
Core Technical

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*
2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*
3. *Mark allocations are shown in brackets.*
4. *Attempt all 10 questions, beginning your answer to each question on a separate sheet.*
5. *Candidates should show calculations where this is appropriate.*

Graph paper is not required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator.

1 An investment actuary notices that the volatility of the price of a particular asset is much higher following a significant change in the price of the asset.

Define an ARCH model and explain what particular properties of the model would make it appropriate for modelling this asset. [5]

2 Two poachers can escape from either the farmyard or the fields of an estate. Two gamekeepers are employed by the estate to catch them. The gamekeepers have three options: both patrol the farmyard (action a_1); one of them patrols the farmyard and the other patrols the fields (action a_2); both gamekeepers patrol the fields (action a_3). The poachers also have three options: both try to escape through the farmyard (option θ_1); one of them tries to escape from the farmyard and the other from the fields (option θ_2); both try to escape over the fields (option θ_3). The number of birds they manage to poach under each combination of θ_i and a_j can be found in the following table:

	a_1	a_2	a_3
θ_1	0	90	120
θ_2	75	0	75
θ_3	120	90	0

Assume the goal of the gamekeepers and the poachers is to minimise and maximise the number of birds poached, respectively.

(i) Show that θ_1 does not dominate θ_2 and vice versa. [1]

(ii) Determine the minimax solution to this problem for the gamekeepers, and state the maximum number of birds that will be poached if they adopt this strategy. [2]

(iii) Given the prior distribution $P(\theta_1) = 0.25$, $P(\theta_2) = 0.35$ and $P(\theta_2) = 0.4$, determine the Bayes solution to the problem for the gamekeepers. [3]

[Total 6]

3 The number of claims, N, in a year on a portfolio of insurance policies has a Poisson distribution with parameter λ. Claims are either large (with probability p) or small (with probability $1 - p$) independently of one another.

Suppose we observe r large claims. Show that the conditional distribution of $N - r \mid r$ is Poisson and find its mean. [7]

4 The table below gives the cumulative incurred claims by year and earned premiums for a particular type of motor policy (Figures in £000s).

Claims paid to date total £15,000,000. The ultimate loss ratio is expected to be in line with the 2003 accident year.

	Development year				
Accident Year	*0*	*1*	*2*	*3*	*Earned Premiums*
2003	3,340	3,750	4,270	4,400	4,800
2004	3,670	4,080	4,590		4,900
2005	3,690	4,290			5,050
2006	4,150				5,200

Ignoring inflation, use the Bornhuetter-Ferguson method to calculate the total reserve required to meet the outstanding claims, assuming that the claims are fully developed by the end of development year 3. [8]

5 Aggregate annual claims on a portfolio of insurance policies have a compound Poisson distribution with parameter λ. Individual claim amounts have an exponential distribution with mean 1.

The insurer calculates premiums using a loading of α (so that the annual premium is $1+\alpha$ times the annual expected claims) and has initial surplus U.

(i) Show that if the first claim occurs at time t, the probability that this claim causes ruin is $e^{-U}e^{-(1+\alpha)\lambda t}$. [3]

(ii) Show that the probability of ruin on the first claim is $\dfrac{e^{-U}}{2+\alpha}$. [4]

(iii) Show that if the insurer wishes to set α such that the probability of ruin at the first claim is less than 1% then it must choose $\alpha > 100e^{-U} - 2$. [2]
[Total 9]

6 Claim sizes (in suitable units) for a portfolio of insurance policies come from a distribution with probability density function

$$f(x) = \begin{cases} axe^{-x^2} & 0 \le x \le 2 \quad a > 0 \\ 0 & \text{otherwise} \end{cases}$$

where $a > 0$ is a constant.

(i) Find a. [2]

(ii) Show that $|f(x)| \le 1$. [2]

(iii) Random numbers have been drawn from a $U(0,1)$ distribution, and are arranged in pairs. The first three pairs are:

0.7413 and 0.4601
0.3210 and 0.6316
0.5069 and 0.0392

Using the rectangle $\{(x, y) | 0 \le x \le 2, \quad 0 \le y \le 1\}$ and the pairs of random numbers in the order given above, use the acceptance-rejection method to generate a single observation from the claim size distribution.
[3]

(iv) Estimate the average number of $U(0,1)$ variables needed to generate a simulation using the approach in (iii). [1]

(v) The results of many thousands of simulations generated in this way are to be used to compare the impact of two possible reinsurance treaties on the insurer's profitability. Explain, with a reason, whether the modeller should use the same simulated claims for each treaty in his model, or whether a new sample for each treaty should be generated. [2]
[Total 10]

7 A no claims discount system has 4 levels. The premiums paid by policyholders in each level are as follows:

Level	*Premium*
0	100%
25	75%
40	60%
50	50%

The rules for moving between the levels are as follows:

- following a claim-free year, a policyholder moves to the next higher level of discount, or remains at 50% discount
- following a year of one or more claims, a policyholder moves to the next lower discount rate or remains at 0% discount

It is assumed that claims occur according to a Poisson process with rate λ per year per policyholder, and that the equilibrium distribution has been reached.

(i) Show that the average premium paid, if the premium paid by a policyholder in level 0 is £500, may be written as

$$500\left(\frac{1+0.75k+0.6k^2+0.5k^3}{1+k+k^2+k^3}\right)$$

where $k=\dfrac{e^{-\lambda}}{1-e^{-\lambda}}$ [5]

(ii) Calculate the average premium paid by policyholders whose claim rate per year is (a) 0.12, (b) 0.24, (c) 0.36. [3]

(iii) Comment on the results in (ii), in relation to the effectiveness of the no claims discount system discriminating between good and bad drivers. [2]

[Total 10]

8 The total claim amount, S, on a portfolio of insurance policies has a compound Poisson distribution with Poisson parameter 50. Individual loss amounts have an exponential distribution with mean 75. However, the terms of the policies mean that the maximum sum payable by the insurer in respect of a single claim is 100.

(i) Find $E(S)$ and Var (S). [7]

(ii) Use the method of moments to fit as an approximation to S:

(a) a normal distribution
(b) a log-normal distribution

[3]

(iii) For each fitted distribution, calculate $P(S > 3{,}000)$. [3]
[Total 13]

9 $y_1, y_2, \ldots, y_n$ are independent, identically distributed observations with probability function given by $f\left(y_i \mid \mu\right) = \dfrac{\mu^{y_i} e^{-\mu}}{y_i!}$.

(i) Show that the log-likelihood may be written as

$$\theta \sum_{i=1}^{n} y_i - nb(\theta) + \text{ terms not depending on } \theta$$

and identify the natural parameter, θ, and the function $b(\theta)$. [3]

(ii) The fitted value for observation y_i is denoted by $\hat{y}_i$.

(a) Write down the Pearson residual for y_i, in terms of y_i and $\hat{y}_i$.

(b) Explain why Pearson residuals are usually not suitable for model checking for the Poisson distribution.

[3]

(iii) Show that the conjugate prior density function for θ is proportional to $\exp\left\{\alpha\theta - \beta e^{\theta}\right\}$, and derive the posterior distribution for this prior. [4]

(iv) Use the identity $E\left[\dfrac{\partial \log f}{\partial \theta}\right] = 0$ (for any density function f) to show that

$$E\left[b(\theta)\right] = \frac{\alpha}{\beta} \text{ and } E\left[b(\theta) \mid y_1, y_2, \ldots, y_n\right] = \frac{\alpha + \sum_{i=1}^{n} y_i}{\beta + n}$$, and comment on these results. [5]
[Total 15]

10 The time series X_t is assumed to be stationary and to follow an ARMA (2,1) process defined by:

$$X_t = 1 + \frac{8}{15} X_{t-1} - \frac{1}{15} X_{t-2} + Z_t - \frac{1}{7} Z_{t-1}$$

where Z_t are independent $N(0,1)$ random variables.

(i) Determine the roots of the characteristic polynomial, and explain how their values relate to the stationarity of the process. [2]

(ii) (a) Find the autocorrelation function for lags 0, 1 and 2.

(b) Derive the autocorrelation at lag k in the form

$$\rho_k = \frac{A}{c^k} + \frac{B}{d^k}$$

[12]

(iii) Determine the mean and variance of X_t. [3]

[Total 17]

END OF PAPER