EXAMINATION

8 October 2009 (am)

Subject CT4 — Models
Core Technical

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*

2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*

3. *Mark allocations are shown in brackets.*

4. *Attempt all 11 questions, beginning your answer to each question on a separate sheet.*

5. *Candidates should show calculations where this is appropriate.*

Graph paper is not required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator from the approved list.

1 Describe the difference between the following assumptions about mortality between any two ages, x and y ($y > x$):

- uniform distribution of deaths
- constant force of mortality

In your answer, explain the shape of the survival function between ages x and y under each of the two assumptions. [2]

2 (i) List the key steps in constructing a new actuarial model. [4]

You work for an actuarial consultancy which is taking over responsibility for a modelling process which has previously been conducted in house by a client.

(ii) Discuss the extent to which the steps required for this task differ from those listed in your answer to (i). [2]

[Total 6]

3 (i) List the data needed for the exact calculation of a central exposed to risk depending on age. [2]

An investigation studied the mortality of persons aged between exact ages 40 and 41 years. The investigation began on 1 January 2008 and ended on 31 December 2008. The following table gives details of 10 lives involved in the investigation.

Life	*Date of 40th birthday*	*Date of death*
1	1 March 2007	–
2	1 May 2007	1 October 2008
3	1 July 2007	–
4	1 October 2007	–
5	1 December 2007	1 February 2008
6	1 February 2008	–
7	1 April 2008	–
8	1 June 2008	1 November 2008
9	1 August 2008	–
10	1 December 2008	–

Persons with no date of death given were still alive when the investigation ended.

(ii) Calculate a central exposed to risk using the data for the 10 lives in the sample. [3]

(iii) (a) Calculate the maximum likelihood estimate of the hazard of death at age 40 last birthday.

(b) Hence, or otherwise, estimate q_{40}. [2]

[Total 7]

4 (i) In the context of mortality investigations describe the principle of correspondence and give an example of a situation in which it may be hard to adhere to this principle. [2]

On 1 January 2005 a country introduced a comprehensive system of death registration, which classified deaths by age last birthday on the date of death.

The government of the country wishes to obtain estimates of the force of mortality, μ_x, by single years of age x for the period between 1 January 2005 and 1 January 2008. Annual population censuses have been taken on 30 June each year since 2004, which classify the population by age last birthday. However the only copy of the data from the population census of 30 June 2006 was lost when the computer disc on which it was stored was being transferred between government departments.

Let the population aged x last birthday on 30 June in year t be denoted by the symbol $P_{x,t}$, and the number of deaths during the period of investigation of persons aged x be denoted by the symbol d_x.

(ii) Derive an expression in terms of $P_{x,t}$ and d_x which may be used to estimate μ_x. [6]

[Total 8]

5 (i) State the Markov property. [1]

A stochastic process $X(t)$ operates with state space S.

(ii) Prove that if the process has independent increments it satisfies the Markov property. [3]

(iii) (a) Describe the difference between a Markov chain and a Markov jump process.

(b) Explain what is meant by a Markov chain being irreducible.

[2]

An actuarial student can see the office lift (elevator) from his desk. The lift has an indicator which displays on which of the office's five floors it is at any point in time. For light relief the student decides to construct a model to predict the movements of the lift.

(iv) Explain whether it would be appropriate to select a model which is:

(a) irreducible
(b) has the Markov property

[3]

[Total 9]

6 The complaints department of a company has two employees, both of whom work five days per week.

The company models the arrival of complaints using a Poisson process with rate 1.25 per working day.

(i) List the assumptions underlying the Poisson process model. [2]

On receipt of a complaint, it is immediately assessed as being straightforward, of medium difficulty or complicated. 60% of cases are assessed as straightforward and 10% are assessed as complicated. The time taken in person-days' effort to prepare responses is assumed to follow an exponential distribution, with parameters 2 for straightforward complaints, 1 for medium difficulty complaints and 0.25 for complicated complaints.

(ii) Calculate the average number of person-days' work expected to be generated by complaints arriving during a five-day working week. [2]

(iii) Define a state space under which the number of outstanding complaints can be modelled as a Markov jump process. [2]

The company has a service standard of responding to complaints within a fixed number of days of receipt. It is considering using this Markov jump process to model the probability of failing to meet this service standard.

(iv) Discuss the appropriateness of using the model for this purpose, with reference to the assumptions being made. [3]
[Total 9]

7 A firm rents cars and operates from three locations — the Airport, the Beach and the City. Customers may return vehicles to any of the three locations.

The company estimates that the probability of a car being returned to each location is as follows:

	Car returned to		
Car hired from	*Airport*	*Beach*	*City*
Airport	0.5	0.25	0.25
Beach	0.25	0.75	0
City	0.25	0.25	0.5

(i) Calculate the 2-step transition matrix. [2]

(ii) Calculate the stationary distribution π. [3]

It is suggested that the cars should be based at each location in proportion to the stationary distribution.

(iii) Comment on this suggestion. [2]

(iv) Sketch, using your answers to parts (i) and (ii), a graph showing the probability that a car currently located at the Airport is subsequently at the Airport, Beach or City against the number of times the car has been rented. [3]
[Total 10]

8 A researcher is studying a certain incurable disease. The disease can be fatal, but often sufferers survive with the condition for a number of years. The researcher wishes to project the number of deaths caused by the disease by using a multiple state model with state space:

{H – Healthy, I – Infected, $D_{\text{(from disease)}}$ – Dead (caused by the disease), $D_{\text{(not from disease)}}$ – Dead (not caused by the disease)}.

The transition rates, dependent on age x, are as follows:

- a mortality rate from the Healthy state of $\mu(x)$
- a rate of infection with the disease $\sigma(x)$
- a mortality rate from the Infected state of $\upsilon(x)$ of which $\rho(x)$ relates to Deaths caused by the disease

(i) Draw a transition diagram for the multiple state model. [2]

(ii) Write down Kolmogorov's forward equations governing the transitions by specifying the transition matrix. [3]

(iii) Determine integral expressions, in terms of the transition rates and any expressions previously determined, for:

(a) $P_{HH}(x, x + t)$

(b) $P_{HI}(x, x + t)$

(c) $P_{HD\text{(from disease)}}(x, x + t)$

[5]
[Total 10]

9 An electronics company developed a revolutionary new battery which it believed would make it enormous profits. It commissioned a sub-contractor to estimate the survival function of battery life for the first 12 prototypes. The sub-contractor inserted each prototype battery into an identical electrical device at the same time and measured the duration elapsing between the time each device was switched on and the time its battery ran out. The sub-contractor was instructed to terminate the test immediately after the failure of the 8th battery, and to return all 12 batteries to the company.

When the test was complete, the sub-contractor reported that he had terminated the test after 150 days. He further reported that:

- two batteries had failed after 97 days
- three further batteries had failed after 120 days
- two further batteries had failed after 141 days
- one further battery had failed after 150 days

However, he reported that he was only able to return 11 batteries, as one had exploded after 110 days, and he had treated this battery as censored at that duration when working out the Kaplan-Meier estimate of the survival function.

(i) State, with reasons, the forms of censoring present in this study. [2]

(ii) Calculate the Kaplan-Meier estimate of the survival function based on the information supplied by the sub-contractor. [5]

In his report, the sub-contractor claimed that the Kaplan-Meier estimate of the survival function at the duration when the investigation was terminated was 0.2727.

(iii) Explain why the sub-contractor's Kaplan-Meier estimate would be consistent with him having stolen the battery he claimed had exploded. [4]

[Total 11]

10 An investigation into the mortality of men engaged in a hazardous occupation was carried out. The following is an extract from the results.

Age x	*Initial exposed-to-risk* E_x	*Observed deaths* θ_x	$\hat{q}_x$
30	950	12	0.0126
31	1,200	14	0.0117
32	1,200	16	0.0133
33	900	9	0.0100
34	1,000	11	0.0110
35	1,100	15	0.0136
36	800	10	0.0125
37	1,250	16	0.0128
38	1,400	17	0.0121

It was decided to graduate the results with reference to English Life Table 15 (males). The formula used for the graduation was $\overset{o}{q}_x = 10q_x^s$.

(i) Using a test of the overall fit of the graduated rates to the data, test the hypothesis that the underlying mortality of men in the hazardous occupation is in accordance with the graduation formula given above. [6]

(ii) Test the graduation using two other tests which detect different features of the graduation. For each test you apply:

(a) State the feature of the graduation it is designed to detect.
(b) Carry out the test.
(c) State your conclusion.

[7]
[Total 13]

11 A study was undertaken into the length of spells of unemployment among young people in a certain city. A sample of young people was monitored from the time they started to claim unemployment benefit until either they resumed work, or they moved away from the city. None of the members of the sample died during the study.

The study investigated the impact of age, sex and educational qualifications on the hazard of returning to work using the following covariates:

A a young person's age when he or she started claiming benefit (measured in exact years since his or her 16th birthday)

S a dummy variable taking the value 1 if the person was male and 0 if the person was female

E a dummy variable taking the value 1 if the person had passed a school leaving examination in mathematics, and 0 otherwise

with associated parameters β_A, β_S and β_E.

The investigators decided to use a Cox proportional hazards regression model for the study.

(i) Explain what is meant by a proportional hazards model. [3]

(ii) Explain why the Cox model is a popular model for the analysis of survival data. [3]

(iii) (a) Write down the equation of the model that was estimated, defining the terms you use (other than those defined above).

(b) List the characteristics of the young person to whom the baseline hazard applies. [3]

The results showed:

- The hazard of resuming work for males who started claiming benefit aged 17 years exact and who had passed the mathematics examination was 1.5 times the hazard for males who started claiming benefit aged 16 years exact but who had not passed the mathematics examination.

- Females who had passed the mathematics examination were twice as likely to take up a new job as were males of the same age who had failed the mathematics examination.

- Females who started claiming benefit aged 20 years exact and who had passed the mathematics examination were twice as likely to resume work as were males who started claiming benefit aged 16 years exact and who had also passed the mathematics examination.

(iv) Calculate the estimated values of the parameters β_A, β_S and β_E. [6]

[Total 15]

END OF PAPER