EXAMINATION

April 2007

Subject CT4 — Models
Core Technical

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

M A Stocker
Chairman of the Board of Examiners

June 2007

Comments

Comments on solutions presented to individual questions for this April 2007 paper are given below and further comments, where appropriate, are given in the solutions that follow.

Question 1 *This was poorly answered by most candidates.*

Question 2 *This was reasonably well answered.*
In part (iii), many candidates did not take into account that the question related to annuities.

Question 3 *This was reasonably well answered, although many candidates took no account of the particular circumstances referred to in the question.*

Question 4 *Again, this was reasonably well answered overall.*
Many candidates failed to state the correct assumptions.

Question 5 *Overall this was poorly answered,*
Many candidates did not provide a correct definition for the hazard function.
In part (ii), marks were lost by candidates who evaluated the survival function at t = 5, rather than providing the expression for $0 \le t \le 5$ *, and by those who provided graphs which were incorrectly or incompletely labelled.*

Question 6 *This was well answered by most candidates.*

Question 7 *Overall this was reasonably well answered, with the stronger candidates scoring highly.*

Question 8 *This was well answered overall.*
In part (ii), a relatively common error was to ignore the date of surgery, effectively assuming that all lives entered into the study on 1 January 2001.

Question 9 *This was reasonably well answered overall.*
As for similar questions in previous years, the main areas where candidates lost marks were: failing to provide sufficient and sufficiently clear working; failing to identify the correct degrees of freedom to be used in the chi-squared test; and failing to state relevant and clear conclusions to the tests.
Many candidates who carried out the test for individual standardised deviations failed to address the issue of outliers.
Many candidates carried out the Grouping of Signs test, which was not appropriate with so few age groups.

Question 10 *Parts (i) and (ii) were fairly well answered overall, but few candidates scored well in part (iii).*

Question 11 *This was very poorly answered by most candidates.*
The most common error in part (iii) was to give the state space as {0, 1, 2, …., N - 1, N}. Few candidates attempted part (vi).

1 **Mixed process**

(a) Is a stochastic process that operates in continuous time, which can also change value at predetermined discrete instants.

(b) The number of contributors to a pension scheme can be modelled as a mixed process with state space $S = \{1, 2, 3, ...\}$ and time interval $J = [0, \infty]$.

Counting process

(a) Is a process, X, in discrete or continuous time, whose state space is the natural numbers {0, 1, 2, …}.

$X(t)$ is a non-decreasing function of t.

(b) Number of claims reported to an insurer by time t.

2 (i) (a) Graduation by reference to a standard table would be appropriate.

There are likely to be existing standard tables which are suitable and this method is suitable for relatively small data sets.

Alternatively, graduation by parametric formula would be suitable if the volume of data was large enough. But that is unlikely to be the case here.

Graphical graduation would not be appropriate for rates for premium calculations.

(b) Assuming graduation by reference to a standard table:

- Select a suitable table, based on a similar group of lives.
- Plot the crude rates against q_x^s from the standard table to identify a simple relationship.
- Find the best-fit parameters, using maximum likelihood or least squares estimates.
- Test the graduation for goodness of fit. If the fit is not adequate, the process should be repeated.

(ii) Considerations include:

- As the premiums are for annuity policies, it is important not to overestimate the mortality rates, as the premiums would be too low.

- The rates will be based on current mortality; the company should also take into account expected future changes, especially any reductions in mortality rates.

- Premiums charged by other insurer: if rates are too high the company will fail to attract business; if too low, it may attract too much, unprofitable business.

3 Clarify the purpose of the exercise. Why does the government want forecasts of mortality? What is the period for which the forecast is wanted? Is it short (e.g. 5–10 years) or long (e.g. 50–70 years).

Consult the existing literature on models for forecasting mortality, and speak to experts in this field of application. Consider using or adapting existing models which are employed in other countries.

Establish what data are available (e.g. on past mortality trends in the country, preferably with deaths classified by age and cause of death).

On the basis of what data are available, define the model you propose to use. If the data are simple and not detailed, then a complex model is not justified. Will a deterministic or a stochastic model be appropriate in this case?

Identify suitable computer software to implement the model, or, if none exists, write a bespoke program.

Debug the program or, if existing software is used, check that it performs the operations you intend it to do.

Run the model and test the reasonableness of the output. Consider, for example, the forecast values of quantities such as the expectation of life at birth.

Test the sensitivity of the results to changes in the input parameters.

Analyse the output.

Write a report documenting the results and the model and communicate the results and the output to the government of the small country.

4 (i) For each pensioner in the investigation, the actuary would need:

Date of entry into the investigation
(the latest of date of retirement, date of xth birthday and 1 January 2005)

Date of exit from the investigation
(the earliest of date of death, date of (x+1)th birthday and 1 January 2007)

(ii) (a) The central exposed to risk of pensioners aged x nearest birthday is given by

$$E_x^c = \int_0^2 P_{x,t}$$

$$\approx \sum_0^1 \tfrac{1}{2}\left(P_{x,t} + P_{x,t+1}\right) = \tfrac{1}{2}P_{x,0} + P_{x,1} + \tfrac{1}{2}P_{x,2}$$

Where $P_{x,t}$ is the number of pensioners aged x nearest birthday at time t, measured from 1 January 2005.

This assumes that $P_{x,t}$ is linear over the calendar year.

(b) This is a life year rate interval, from age x - ½ to x+½. The age in the middle of the rate interval is x, so $\hat{\mu}$ estimates μ_x, assuming a constant force of mortality over the life year.

The estimate of μ_x is therefore given by:

$$\hat{\mu}_{65} = \frac{d_{65,2005} + d_{65,2006}}{E_{65}^c}$$

$$= \frac{16 + 23}{\left(\frac{1}{2}\times 1678 + 1720 + \frac{1}{2}\times 1622\right)} = \frac{39}{3370}$$

$$= 0.01157$$

5 (i) The hazard function is defined as

$$h(t) = \lim_{dt \to 0^+} \frac{1}{dt}\left(\Pr[T \le t + dt \mid T > t]\right).$$

(ii) (a) Since the survival function $S(t)$ is given by

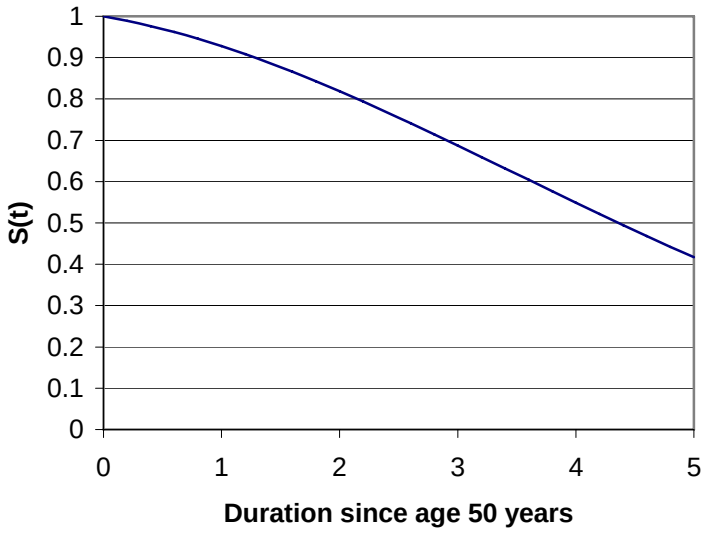
$$S(t) = \exp\left(-\int_0^t h(s)ds \right),$$

then

$$S(t) = \exp\left(-\int_0^t \left(\alpha + \beta s\right) ds \right) = \exp\left[-\alpha s - \frac{\beta s^2}{2} \right]_0^t = \exp\left[-\alpha t - \frac{\beta t^2}{2} \right]$$

where $0 \le t \le 5$.

(b) A suitable plot is shown below.



Both concave and convex plots were acceptable as this depends on parameters, α *and* β*.*

(c) If both α and β are positive, then the formula implies a force of mortality which increases with age, which is sensible for this age range.

The parameter α measures the 'level' of mortality and the parameter β measures the rate of increase with age. Varying these permits quite a wide range of forms for $S(t)$.

So the formula seems appropriate.

6 Based on the given transition diagram, the one-step transition matrix must be of the form:

$$\begin{pmatrix} a & 0 & c \\ d & e & f \\ 0 & h & i \end{pmatrix}$$

The two-step transition matrix is given by:

$$\begin{pmatrix} a & 0 & c \\ d & e & f \\ 0 & h & i \end{pmatrix} * \begin{pmatrix} a & 0 & c \\ d & e & f \\ 0 & h & i \end{pmatrix} = \begin{pmatrix} a^2 & ch & c(a+i) \\ d(a+e) & e^2+fh & cd+ef+fi \\ dh & h(e+i) & fh+i^2 \end{pmatrix}$$

$$P_{AA}^2 = 0.5625 \Rightarrow a^2 = 0.5625 \Rightarrow a = 0.75$$

Rows of transition matrix must sum to 1.

So, $a + c = 1$
and $c = 0.25$

$$P_{AB}^2 = 0.125 \Rightarrow ch = 0.125 \Rightarrow h = 0.5$$

$h + i = 1$
so i=0.5

$$P_{CC}^2 = 0.4 \Rightarrow f \times 0.5 + 0.5^2 = 0.4 \Rightarrow f = 0.3$$

$$P_{BA}^2 = 0.475 \Rightarrow d(0.75+e) = 0.475$$

Rows sum to 1 so, $d + e$ =0.7

Substitute for e:

$$d(1.45-d) = 0.475 \Rightarrow d^2 - 1.45d + 0.475 = 0$$

Solving using standard quadratic formula:

$$d = \frac{1.45 \pm \sqrt{1.45^2 - 4 \times 0.475}}{2} = \frac{1.45 \pm 0.45}{2} = 0.95 \text{ or } 0.5$$

0.95 is not possible because e would need to be negative

So $d = 0.5$ and $e = 0.2$

Transition matrix is:

$$\begin{pmatrix} 0.75 & 0 & 0.25 \\ 0.5 & 0.2 & 0.3 \\ 0 & 0.5 & 0.5 \end{pmatrix}$$

7 (i) Consider the sequence of the status of the first born child in each generation.

The state space consists of the four possible combinations of chromosomes:

Female non-carrier (FN) *or* XX
Female carrier (FC) *or* X*X
Male non-sufferer (MN) *or* XY
Male haemophiliac (MH) *or* X*Y

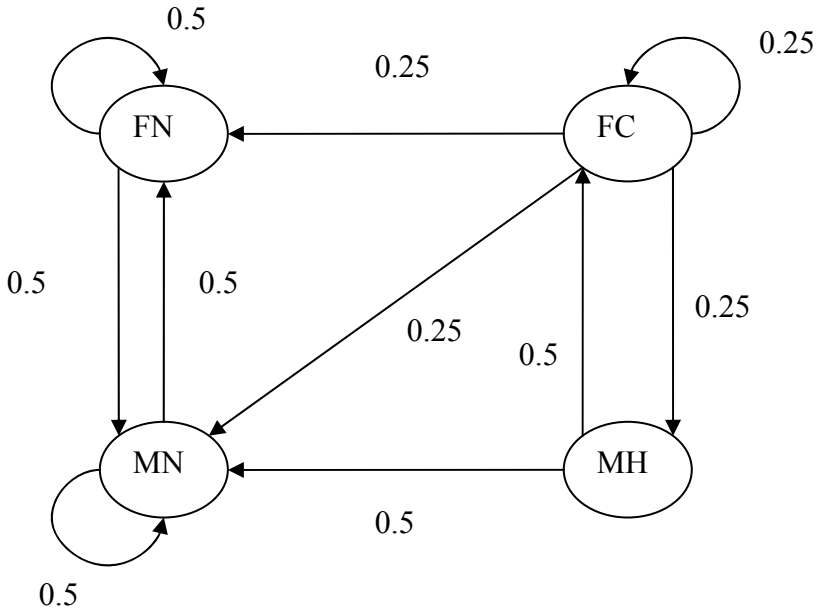
Using the assumption that there is an equal chance of either chromosome being inherited:

- A female non-carrier will lead to a female non-carrier or male non-carrier.
- A female carrier may produce:

  X*X, XX, X*Y, XY all with equal probability.
- A male non-sufferer will lead to female non-carrier or male non-carrier.
- A male haemophiliac may produce:

  X*X or XY (because his partner must provide an X) with equal probability.

The transition diagram is therefore:



Each of the transition probabilities depends only on state currently occupied, so the process possesses the Markov property.

(ii) (a) The chain is reducible because once it enters states FN or MN it cannot access FC or MH.

(b) The chain is aperiodic.
As it is reducible we need to consider each group of states. FN/MN clearly have no period, and MH/FC do not either because a loop is possible in state FC.

(iii) The transition matrix is

$$A = \begin{array}{r} \\ FN(0) \\ FC(1) \\ MN(2) \\ MH(3) \end{array} \begin{array}{cccc} FN & FC & MN & MH \\ 0.5 & 0 & 0.5 & 0 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0.5 & 0 \end{array}$$

The stationary distribution π must satisfy:

$$\pi_0 = 0.5\pi_0 + 0.25\pi_1 + 0.5\pi_2$$
$$\pi_1 = 0.25\pi_1 + 0.5\pi_3$$
$$\pi_2 = 0.5\pi_0 + 0.25\pi_1 + 0.5\pi_2 + 0.5\pi_3$$
$$\pi_3 = 0.25\pi_1$$

So,

$$\pi_1 = 0.25\pi_1 + 0.5 \times 0.25\pi_1$$
$$\Rightarrow \pi_1 = \pi_3 = 0$$
$$\Rightarrow \pi_0 = \pi_2 = 0.5$$

An alternative solution combines the states FN and MN to give a 3-state model. This was given credit.

8 (i) (a) Type I censoring is present for those lives still under observation at 31 December 2005 as the censoring times are known in advance.

(b) Interval censoring would be present if we only knew death occurred between check-ups. However, actual dates of death are known, so interval censoring is not present.

Right censoring can be seen as a special case of interval censoring (for those censored before death, we know death occurs in the interval (c_i, ∞) where c_i is the censoring time for person i).

(c) Informative censoring is not likely to be present. The censoring of lives gives us no information about future lifetimes.

(ii) The durations at which lives died or were censored are shown below. Duration is measured in years and months from the date of surgery.

Patient	*Death or censored*	*Duration*
A	death	4 years 4 months
B	death	6 months
C	death	10 months
D	death	1 year 11 months
E	death	10 months
F	censored	4 years 11 months
G	censored	4 years 10 months
H	censored	4 years 9 months
I	censored	4 years 7 months
J	censored	4 years 4 months
K	censored	4 years 4 months
L	censored	4 years 2 months
M	censored	2 years 6 months
N	censored	9 months
O	censored	4 years

The calculation of the survival function is shown in the table below. We assume that at duration 4 years 4 months, the death occurred before lives were censored.

t_j	n_j	d_j	c_j	$\hat{\lambda}_j = d_j / n_j$
0	15	0	0	0
0.5	15	1	1	1/15
0.833	13	2	0	2/13
1.917	11	1	3	1/11
4.333	7	1	6	1/7

The estimated survival function is given by, $\hat{S}(t) = \prod_{t_j \le t} \left(1 - \lambda_j\right)$. So,

t	$\hat{S}(t)$
$0.000 \le t < 0.500$	1.0000
$0.500 \le t < 0.833$	0.9333
$0.833 \le t < 1.917$	0.7897
$1.917 \le t < 4.333$	0.7179
$4.333 \le t < 5.0$	0.6154

Solutions using different assumptions (for example assuming the death at 4 years 4 months occurred after lives were censored, or assuming lives M, N and O were censored sometime within 3 months of their last check-up) were acceptable and received credit.

(iii) The probability that a patient will die within 4 years of surgery is estimated by:

$$1-\hat{S}(4) = 1 - 0.7179$$
$$= 0.2821$$

9 (i) The chi-squared test is a suitable overall test.

The test statistic is $\sum_x z_x^2$, where

$$z_x = \frac{E_x^c \hat{\mu}_{x+1/2}^f - E_x^c \tilde{\mu}_{x+1/2}^f}{\sqrt{E_x^c \tilde{\mu}_{x+1/2}^f}}.$$

$\sum_x z_x^2$ has the χ_8^2 distribution.

The calculations are shown in the table below

Age x	*Actual deaths* $E_x^c \hat{\mu}_{x+1/2}^f$	*Expected deaths* $E_x^c \tilde{\mu}_{x+1/2}^f$	z_x	z_x^2
65	30	28.4	0.3002	0.0901
66	20	30.1	-1.8409	3.3890
67	25	31.2	-1.1100	1.2321
68	40	33.5	1.1230	1.2612
69	45	34.1	1.8666	3.4842
70	50	41.8	1.2683	1.6086
71	50	46.5	0.5133	0.2634
72	45	44.5	0.0750	0.0056

$\sum_x z_x^2 = 11.3343$.

The critical value of the χ_8^2 distribution at the 5% level of statistical significance is 15.51.

Since 11.3343 < 15.51, we have no reason to reject the null hypothesis that the sex ratios of death rates among the company's pensioners are the same as those prevailing in the PMA92 and PFA92 tables.

(ii) **Standardised deviations test**

Using the individual standardised deviations test, we note that none of the z_xs exceeds 1.96 in absolute value, so there is no evidence that the sex ratios among the company's pensioners are unusual at any specific ages

Signs test

Under the null hypothesis of no difference between the company's pensioners and insured pensioners in general, the number of positive signs should have a Binomial (8, 0.5) distribution.

There are 2 negative and 6 positive signs.

The probability of obtaining 6 positive signs if the null hypothesis is true is

$$\binom{8}{6}0.5^8 = 0.1094$$

Since this is greater than 0.025 (two-tailed test), the sex ratios of death rates among the company's pensioners are not systematically higher or lower than those derived from the PMA92 and PFA92 tables.

Cumulative deviations test

The cumulative deviation

$$\sum_x (E_x^c \hat{\mu}^f_{x+1/2} - E_x^c \tilde{\mu}^f_{x+1/2}) \sim \text{Normal}\,(0, E_x^c \tilde{\mu}^f_{x+1/2}),$$

so that under the null hypothesis

$$\frac{\sum_x (E_x^c \hat{\mu}^f_{x+1/2} - E_x^c \tilde{\mu}^f_{x+1/2})}{\sqrt{\sum_x E_x^c \tilde{\mu}^f_{x+1/2}}} \sim \text{Normal}\,(0,1).$$

Using the figures in the table above we have

$$\frac{\sum_x (E_x^c \hat{\mu}^f_{x+1/2} - E_x^c \tilde{\mu}^f_{x+1/2})}{\sqrt{\sum_x E_x^c \tilde{\mu}^f_{x+1/2}}} = \frac{14.9}{\sqrt{290}} = 0.875$$

and since $|0.875| < 1.96$ using a two-tailed test, the sex ratios of death rates among the company's pensioners are not systematically higher or lower than those derived from the PMA92 and PFA92 tables.

Credit was only given for one of the Signs test and the Cumulative Deviations test as they both test for bias.

Serial correlations test (lag 1)

The calculations are shown in the tables below

$$\bar{z}^{(1)} = \frac{1}{7}\sum_{1}^{7} z_x = 0.3029, \text{ and } \bar{z}^{(2)} = \frac{1}{7}\sum_{2}^{8} z_x = 0.2707$$

Age x	$z_x - \bar{z}^{(1)}$	$z_{x+1} - \bar{z}^{(2)}$	$(z_x - \bar{z}^{(1)})(z_{x+1} - \bar{z}^{(2)})$
65	-0.0027	-2.1117	0.0057
66	-2.1439	-1.3807	2.9601
67	-1.4129	0.8523	-1.2042
68	0.8201	1.5958	1.3087
69	1.5637	0.9976	1.5598
70	0.9654	0.2425	0.2341
71	0.2103	-0.1958	-0.0412
Sum			4.8231

Age	$\left[z_x - \bar{z}^{(1)}\right]^2$	$\left[z_{x+1} - \bar{z}^{(2)}\right]^2$
65	0.0000	4.4592
66	4.5962	1.9064
67	1.9963	0.7264
68	0.6726	2.5467
69	2.4450	0.9951
70	0.9320	0.0588
71	0.0442	0.0383
Sum	10.6863	10.7310

The correlation coefficient is therefore

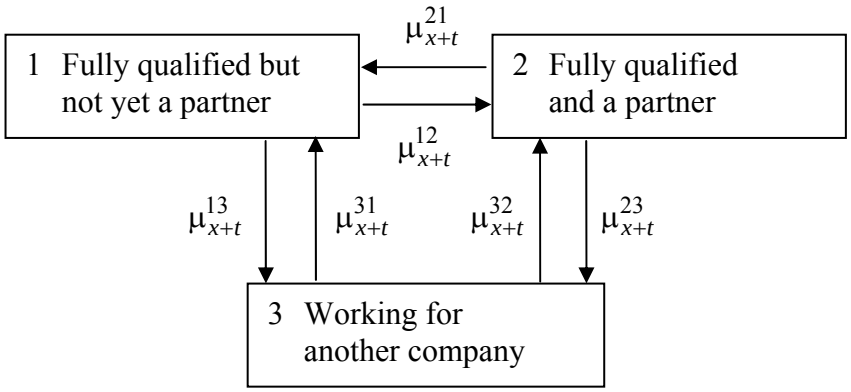
$$r_1 = \frac{4.8231}{\sqrt{(10.6863)(10.7310)}} = 0.4503$$

We test $r_1\sqrt{8} = 1.27$ against the Normal (0,1) distribution using a one-tailed test.

Since 1.27 < 1.645, we conclude that there is no evidence that the sex ratios of death rates among the company's pensioners vary with age in a way different from the ratios derived from PMA92 and PFA92.

*Note that the **Grouping of Signs test** is not appropriate with 8 ages, 6 positive and 2 negative signs.*

10 (i) (a) A suitable diagram is shown below.



(b) The chosen model ignores death among persons in the relevant age groups. Since mortality in this age group among professional people is likely to be low, this seems reasonable.

This diagram assumes that demotion is possible, i.e. some-one who has become a partner can return to non-partnership status without leaving the company.

The assumption is also made that a new employee joining from another company can do so as a partner.

Credit was given for models based on alternative assumptions, provided these were reasonable.

(ii) (a) Assume we have data on N individuals ($i = 1, ..., N$).

We should need to know for each individual:

- the total waiting time during the calendar years 1997–2006 in state (1) when aged 30 last birthday
- whether or not the individual was made a partner between exact ages 30 and 31 years during the calendar years 1997–2006 while remaining in the company.

(b) The likelihood of the data is:

$$L = \prod_{i=1}^{N} K \exp[-(\mu^{13} + \mu^{12})v_i](\mu^{12})^{d_i}$$

where

v_i is the waiting time at age 30 last birthday in state (1) for individual *i*.

d_i is an indicator variable such that $d_i = 1$ if individual *i* was made a partner while aged 30 last birthday during the period of the investigation and $d_i = 0$ otherwise.

K is a constant denoting terms that do not depend on μ^{12}.

(c) The logarithm of the likelihood is

$$\log_e L = \sum_{i=1}^{N} \log_e K - (\mu^{12} + \mu^{13})v_i + d_i \log_e \mu^{12}$$

Differentiating this with respect to μ^{12} we obtain

$$\frac{\partial \log_e L}{\partial \mu^{12}} = -\sum_{i=1}^{N} v_i + \frac{\sum_{i=1}^{N} d_i}{\mu^{12}},$$

and setting this equal to zero and solving for μ^{12} gives

$$\hat{\mu}^{12} = \frac{\sum_{i=1}^{N} d_i}{\sum_{i=1}^{N} v_i}.$$

This is the maximum likelihood estimate, as can be seen by noting that

$\frac{\partial^2 \log_e L}{(\partial \mu^{12})^2} = -\frac{\sum_{i=1}^{N} d_i}{(\mu^{12})^2}$ which must be negative.

(iii) The data on becoming a partner are classified by age last birthday, which is the same classification as used in the company's own investigation, therefore the relevant intensities will relate to the same age range.

For the correct exposed to risk we only consider those who are members of the institute but not yet partners.

Let the number of such members in the census in year *t* who were born in year *s* be $P_{t,s}$.

All persons born in year *s* would be aged *x* last birthday on 1 January in year *s*+*x*+1.

Therefore, assuming that the $P_{t,s}$ change linearly during each calendar year the correct exposed to risk for the year 1997 is

$$\frac{1}{2}(P_{1997,1956} + P_{1998,1957})$$

and the exposed to risk for the entire 10-year period of the investigation is

$$\sum_{t=1997}^{t=2006} \frac{1}{2}(P_{t,t-31} + P_{t+1,t-30}) .$$

If the number of persons becoming partners aged 30 last birthday in year *t* is θ_t, then an estimate of the relevant transition intensity is

$$\frac{\sum_{t=1997}^{t=2006} \theta_t}{\sum_{t=1997}^{t=2006} \frac{1}{2}(P_{t,t-31} + P_{t+1,t-30})} .$$

11 (i) Consider a small time interval *dt*

The probability of an arrival from the first process in time *dt* is $\lambda.dt + o(dt)$ and the probability of a arrival from the second process in time *dt* is $\mu.dt + o(dt)$.

The arrival probability for the sum of the processes in *dt* is therefore $(\lambda + \mu).dt + o(dt)$

This is by definition a Poisson process with rate $(\lambda + \mu)$.

Alternative solutions, based on the Moment Generating Function or the Probability Generating Function of a Poisson distribution were acceptable.

(ii) (a) A jump chain is formed by recording the state of a Markov jump process only at the instant when a transition has just been made.

The jump chain is in itself a Markov chain.

(b) The outcome of the jump chain can only differ from that of the standard Markov chain if the jump process enters an absorbing state.

As the jump process will make no further transitions once it enters an absorbing state, the jump chain "stops".

It is possible to model the jump chain as though transitions continue to occur but the chain continues to occupy the same state.

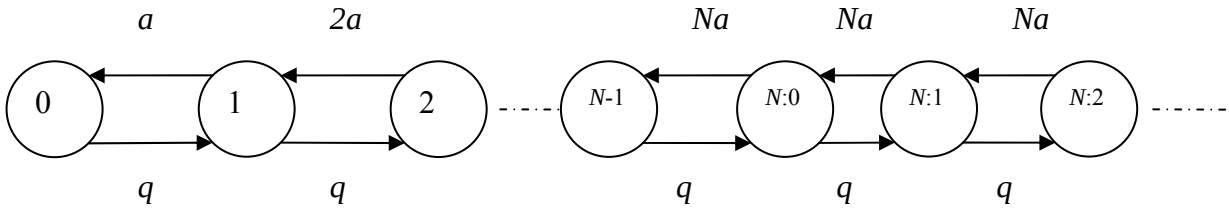
(iii) The possible states are 0 to N desks in use with no passengers queuing, and N desks in use with 0, 1, 2, ….. passengers in the queue.

When all desks are occupied and there are M passengers in the queue denote the state as N:M.

State space is:

{0, 1, 2, …., N - 1, N : 0, N : 1, N : 2, …..}

Transition diagram:



(iv) Kolmogorov forward equations in component form are:

$$\frac{d}{dt}P_0(t) = aP_1(t) - qP_0(t)$$

$$\frac{d}{dt}P_r(t) = a(r+1)P_{r+1}(t) + qP_{r-1}(t) - (ar+q)P_r(t) \qquad r+1 \le N$$

$$\frac{d}{dt}P_{N:0}(t) = aNP_{N:1}(t) + qP_{N-1}(t) - (aN+q)P_{N:0}(t)$$

$$\frac{d}{dt}P_{N:m}(t) = aNP_{N:m+1}(t) + qP_{N:m-1}(t) - (aN+q)P_{N:m}(t) \qquad m \ge 1$$

(v) Poisson process is usually suitable for arrivals at a service point.

Rate may be time inhomogeneous because passengers may aim to arrive a couple of hours before the flight — so a time-inhomogeneous Poisson process may be better.

However if the airline operates many flights this may not be an issue.

Passengers may be checked-in in family groups rather than individually.

There is likely to be a minimum time for processing a check-in due to standard security questions etc, so exponential distribution may not hold.

(vi) (a) The transition matrix is:

$$\begin{pmatrix} 0 & 1 & & & & & \\ \frac{a}{a+q} & 0 & \frac{q}{a+q} & & & & \\ & \frac{2a}{2a+q} & 0 & \frac{q}{2a+q} & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \frac{Na}{Na+q} & 0 & \frac{q}{Na+q} & \\ & & & & \frac{Na}{Na+q} & 0 & \frac{q}{Na+q} \\ & & & & & \ddots & \ddots \end{pmatrix}$$

(b) This is the probability that all the first N transitions are to the right in the transition diagram.

The probability of each transition is given by the elements in the upper half of the jump chain transition matrix in (vi)(a).

Required probability is therefore $q^{N-1}.\prod_{i=1}^{N-1}\frac{1}{ia+q}$

END OF EXAMINERS' REPORT