Faculty of Actuaries **Institute of Actuaries**

EXAMINATION

28 March 2006 (am)

Subject CT6 — Statistical Methods
Core Technical

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*

2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*

3. *Mark allocations are shown in brackets.*

4. *Attempt all 10 questions, beginning your answer to each question on a separate sheet.*

5. *Candidates should show calculations where this is appropriate.*

Graph paper is not required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator.

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1 List the main perils typically insured against under a household buildings policy. [3]

2 A No Claims Discount (NCD) system has 3 levels of discount

Level 0	no discount
Level 1	discount = p
Level 2	discount = $2p$

where $0 < p < 0.5$.

The probability of a policyholder NOT making a claim each year is 0.9.

In the event of a claim, the policyholder moves to, or remains at level 0. Otherwise, the policyholder moves to the next higher level (or remains at level 2).

The premium paid in level 0 is £1,000.

Derive an expression in terms of p for the average premium paid by a policyholder once the steady state has been reached. [6]

3 Based on the proposal form, an applicant for life insurance is classified as a standard life (1), an impaired life (2) or uninsurable (3). The proposal form is not a perfect classifier and may place the applicant into the wrong category.

The decision to place the applicant in state i is denoted by d_i, and the correct state for the applicant is θ_i.

The loss function for this decision is shown below:

	d_1	d_2	d_3
θ_1	0	5	8
θ_2	12	0	3
θ_3	20	15	0

(i) Determine the minimax solution when assigning an applicant to a category. [1]

(ii) Based on the application form, the correct category for a new applicant appears to be as an impaired life. However, of applicants which appear to be impaired lives, 15% are in fact standard lives and 25% are uninsurable. Determine the Bayes solution for this applicant. [4]

[Total 5]

4 (i) Derive the autocovariance and autocorrelation functions of the AR(1) process

$$X_t = \alpha X_{t-1} + e_t$$

where $|\alpha| < 1$ and the e_t form a white noise process. [4]

(ii) The time series Z_t is believed to follow a ARIMA(1, d, 0) process for some value of d. The time series $Z_t^{(k)}$ is obtained by differencing k times and the sample autocorrelations, $\{r_i : i = 1, \ldots, 10\}$, are shown in the table below for various values of k.

	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$
r_1	100%	100%	83%	–3%	–45%	–64%
r_2	100%	100%	66%	–12%	–5%	13%
r_3	100%	100%	54%	–11%	–4%	–3%
r_4	100%	99%	45%	–1%	6%	4%
r_5	100%	99%	37%	–3%	4%	5%
r_6	100%	99%	30%	–12%	–12%	–12%
r_7	99%	98%	27%	3%	7%	9%
r_8	99%	98%	24%	3%	0%	–4%
r_9	99%	97%	19%	3%	5%	6%
r_{10}	99%	97%	13%	–7%	–5%	–4%

Suggest, with reasons, appropriate values for d and the parameter α in the underlying AR(1) process. [4]

[Total 8]

5 (i) Let n be an integer and suppose that $X_1, X_2, \ldots, X_n$ are independent random variables each having an exponential distribution with parameter λ. Show that $Z = X_1 + \ldots + X_n$ has a Gamma distribution with parameters n and λ. [2]

(ii) By using this result, generate a random sample from a Gamma distribution with mean 30 and variance 300 using the 5 digit pseudo-random numbers.

63293 43937 08513 [5]

[Total 7]

6 An insurance company has a set of n risks ($i = 1, 2, \ldots, n$) for which it has recorded the number of claims per month, Y_{ij}, for m months ($j = 1, 2, \ldots, m$).

It is assumed that the number of claims for each risk, for each month, are independent Poisson random variables with

$$E[Y_{ij}] = \mu_{ij} .$$

These random variables are modelled using a generalised linear model, with

$$\log\mu_{ij} = \beta_i \qquad (i = 1, 2, \ldots, n)$$

(i) Derive the maximum likelihood estimator of β_i . [4]

(ii) Show that the deviance for this model is

$$2\sum_{i=1}^{n}\sum_{j=1}^{m}\left(y_{ij}\log\frac{y_{ij}}{\overline{y}_i} - (y_{ij} - \overline{y}_i)\right)$$

where $\overline{y}_i = \dfrac{1}{m}\sum_{j=1}^{m} y_{ij}$. [3]

(iii) A company has data for each month over a 2 year period. For one risk, the average number of claims per month was 17.45. In the most recent month for this risk, there were 9 claims. Calculate the contribution that this observation makes to the deviance. [3]

[Total 10]

7 (i) Let N be a random variable representing the number of claims arising from a portfolio of insurance policies. Let X_i denote the size of the ith claim and suppose that $X_1, X_2, \ldots$ are independent identically distributed random variables, all having the same distribution as X. The claim sizes are independent of the number of claims. Let $S = X_1 + X_2 + \ldots + X_N$ denote the total claim size. Show that

$$M_S(t) = M_N(\log M_X(t)).$$ [3]

(ii) Suppose that N has a Type 2 negative binomial distribution with parameters $k > 0$ and $0 < p < 1$. That is

$$P(N = x) = \frac{\Gamma(k+x)}{\Gamma(x+1)\Gamma(k)} p^k q^x \qquad x = 0, 1, 2, \ldots$$

Suppose that X has an exponential distribution with mean $1/\lambda$. Derive an expression for $M_s(t)$. [2]

(iii) Now suppose that the number of claims on another portfolio is R with the size of the ith claim given by Y_i. Let $T = Y_1 + \ldots + Y_R$. Suppose that R has a binomial distribution, with parameters k and $1 - p$, and that Y_i has an exponential distribution with mean $1/\theta$. Show that if θ is chosen appropriately then S and T have the same distribution. [6]

You may use any standard formulae for moment generating functions of specific distributions shown in the Formulae and Tables.

[Total 11]

8 An insurer has for 2 years insured a number of domestic animals against veterinary costs. In year 1 there were n_1 policies and in year 2 there were n_2 policies. The number of claims per policy per year follows a Poisson distribution with unknown parameter θ.

Individual claim amounts were a constant c in year 1 and a constant $c(1 + r)$ in year 2. The average total claim amount per policy was y_1 in year 1 and y_2 in year 2. Prior beliefs about θ follow a gamma distribution with mean α/λ and variance α/λ^2. In year 3 there are n_3 policies, and individual claim amounts are $c(1 + r)^2$. Let Y_3 be the random variable denoting average total claim amounts per policy in year 3.

(i) State the distribution of the number of claims on the whole portfolio over the 2 year period. [1]

(ii) Derive the posterior distribution of θ given y_1 and y_2. [5]

(iii) Show that the posterior expectation of Y_3 given y_1, y_2 can be written in the form of a credibility estimate

$$Z \times k + (1 - Z) \times \frac{\alpha}{\lambda} \times c(1+r)^2$$

specifying expressions for k and Z. [5]

(iv) Describe k in words and comment on the impact the values of n_1, n_2 have on Z. [3]

[Total 14]

9 (i) The general form of a run-off triangle can be expressed as:

Accident Year	*Development Year*					
	0	*1*	*2*	*3*	*4*	*5*
0	$C_{0,0}$	$C_{0,1}$	$C_{0,2}$	$C_{0,3}$	$C_{0,4}$	$C_{0,5}$
1	$C_{1,0}$	$C_{1,1}$	$C_{1,2}$	$C_{1,3}$	$C_{1,4}$	
2	$C_{2,0}$	$C_{2,1}$	$C_{2,2}$	$C_{2,3}$		
3	$C_{3,0}$	$C_{3,1}$	$C_{3,2}$			
4	$C_{4,0}$	$C_{4,1}$				
5	$C_{5,0}$					

Define a model for each entry, C_{ij}, in general terms and explain each element of the formula. [3]

(ii) The run-off triangles given below relate to a portfolio of motorcycle insurance policies.

The cost of claims paid during each year is given in the table below:

(Figures in £000s)

Accident Year	*Development Year*			
	0	*1*	*2*	*3*
2002	2,905	535	199	56
2003	3,315	578	159	
2004	3,814	693		
2005	4,723			

The corresponding number of settled claims is as follows:

Accident Year	*Development Year*			
	0	*1*	*2*	*3*
2002	430	51	24	7
2003	465	58	24	
2004	501	59		
2005	539			

Calculate the outstanding claims reserve for this portfolio using the average cost per claim method with grossing-up factors, and state the assumptions underlying your result. [9]

(iii) Compare the results from the analysis in (ii) with those obtained from the basic chain ladder method. [5]

[Total 17]

10 An insurance company has two portfolios of independent policies, on each of which claims occur according to a Poisson process. For the first portfolio, all claims are for a fixed amount of £5,000 and 10 claims are expected per annum. For the second portfolio, claim amounts are exponentially distributed with mean £4,000 and 30 claims are expected per annum.

Let S denote aggregate annual claims from the two portfolios together.

A check is made for ruin only at the end of the year.

The insurer includes a loading of 10% in the premiums, for all policies.

(i) Calculate the mean and variance of S. [4]

(ii) Use a normal approximation to the distribution of S to calculate the initial capital, u, required in order that the probability of ruin at the end of the first year is 0.01. [3]

The insurer is considering purchasing proportional reinsurance from a reinsurer that includes a loading of ξ in its premiums. The proportion of each claim to be retained by the direct insurer is $\alpha (0 \le \alpha \le 1)$.

Let S_I denote the aggregate annual claims paid by the direct insurer on the two portfolios together, net of reinsurance.

(iii) Use a normal approximation to the distribution of S_I to show that the initial capital, u', required in order that the probability of ruin at the end of the first year is 0.01 can be written as

$$u' = \alpha u + (1 - \alpha)(\xi - 0.1)E[S].$$ [6]

(iv) Show that $u > u'$, as long as $\xi < 0.476$. [3]

(v) Show that $u - u'$ decreases as ξ increases, and discuss the practical implications of this result. [3]

[Total 19]

END OF PAPER