INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

September 2017

Subject CT4 – Models
Core Technical

Introduction

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and using past papers as a revision aid and also those who have previously failed the subject.

The Examiners are charged by Council with examining the published syllabus. The Examiners have access to the Core Reading, which is designed to interpret the syllabus, and will generally base questions around it but are not required to examine the content of Core Reading specifically or exclusively.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report; other valid approaches are given appropriate credit. For essay-style questions, particularly the open-ended questions in the later subjects, the report may contain more points than the Examiners will expect from a solution that scores full marks.

The report is written based on the legislative and regulatory context pertaining to the date that the examination was set. Candidates should take into account the possibility that circumstances may have changed if using these reports for revision.

Luke Hatter
Chair of the Board of Examiners
December 2017

A. General comments on the *aims of this subject and how it is marked*

1. The aim of the Models subject is to provide a grounding in stochastic processes and survival models and their application.

2. Subject CT4 comprises five main sections:

   (1) a study of the properties of models in general, and their uses for actuaries, including advantages and disadvantages (and a comparison of alternative models of the same processes);

   (2) stochastic processes, especially Markov chains and Markov jump processes;

   (3) models of a random variable measuring future lifetime;

   (4) the calculation of exposed to risk and the application of the principle of correspondence;

   (5) the reasons why mortality (or other decremental) rates are graduated, and a range of statistical tests used both to compare a set of rates with a previous experience and to test the adherence of a graduated set of rates to the original data.

   Throughout the subject the emphasis is on estimation and the practical application of models. Theory is kept to the minimum required in order usefully to apply the models to real problems.

3. Different numerical answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may lose marks where excessive rounding has been used or where insufficient working is shown. Credit is given for valid solutions different from those shown below. Partial credit is also given to candidates submitting incomplete solutions with valid intermediate workings.

B. General comments on *student performance in this diet of the examination*

The Examiners are pleased to note that average performance was improved in some areas of the syllabus which are regularly examined, and in which performance has been disappointing in recent sessions.

There were still a large number of candidates who did not read the wording of the questions closely enough, and so lost marks on straightforward sections of the paper because they did not answer the question asked.

Question 8 on the examination paper proved to be more difficult than anticipated, a Pass Mark slightly below 60 was used.

C. Pass Mark

The Pass Mark for this examination was 58.

Solutions

Q1 (i) Rows must sum to 1.

$P_{AC} = 0$ and $P_{CB} = 0$ from transition graph.

So full transition matrix is:

$$\begin{array}{c} A \\ B \\ C \end{array}\begin{pmatrix} 0.2 & 0.8 & 0 \\ 0 & 0 & 1.0 \\ 0.6 & 0 & 0.4 \end{pmatrix}.$$ +2

[2]

(ii) Path 1 is a valid sample path. +½

All the movements between states are valid transitions from the transition graph. +½

Path 2 is not a valid sample path. +½

EITHER

There are transitions from C to B, which is not possible according to the transition graph.

OR

The process cannot stay in B. +½

[2]

[Total 4]

The vast majority of candidates correctly computed the transition matrix in part (i). In part (ii), some candidates interpreted the diagrams in the question to depict a continuous time process and therefore argued neither sample path was valid, as a Markov chain is a discrete time process. Full credit was given for this.

Q2

	State space	*Time domain*	
General random walk	Can be either	Discrete	+1
Markov jump process	Discrete	Continuous	+1
Compound Poisson process	Can be either	Continuous	+1
Markov chain.	Discrete	Discrete	+1

[4]

This question was generally well answered. The most common errors were in the description of the state space and time domains of the general random walk, and in failing to recognise that the compound Poisson process can occupy either a continuous or a discrete state space.

Q3 (i) The mean is equal to the parameter, so there are 3 calls per hour. +1

[1]

(ii) The process is memoryless so the fact that Fred has not had a call for 15 minutes is irrelevant.

Expected time until next call is 20 minutes. +1

[1]

(iii) This is the probability of zero calls in time 0.5 hours.

Using $p_j(t) = e^{-\lambda t}(\lambda t)^j / j!$

OR

Since $p_0(0.5) = \dfrac{e^{-1.5}(1.5)^0}{0!}$, +½

$p_0(0.5) = e^{-1.5} = 0.2231$. +½

[1]

(iv) The expected time that Fred is on the phone is the expected number of calls times the expected length of a call.

Per hour this is 3 calls times 7 minutes = 21 minutes. +1

So the probability that the phone is engaged is 21/60 = 0.35. +1

[2]

[Total 5]

> *Many answers to this question were poor. Basic errors were made. For example, in part (ii) many candidates argued that the mean waiting time was 20 minutes, but that as Fred had not received a call for 15 minutes he only had 5 minutes more to wait. This contradicts the memoryless property of the exponential distribution. In part (iv) a common incorrect alternative was to argue that, since Fred would (on average) be on the telephone for 3 x 7 = 21 minutes per hour, the probability that Fred would be engaged is equal to the probability that at least one call would be received in 21 minutes [exp(-21/60)] = 0.2953. This is not correct because it ignores the fact that if there is more than one call in the 21 minutes, more than one caller will find Fred engaged, and we require the probability to be calculated from the perspective of the callers, not from Fred's perspective.*

Q4 (i) For the central exposed to risk for each life we need the difference between STARTDATE and ENDDATE (in months) where:

STARTDATE = latest of 85th birthday and 1 April 2015
ENDDATE = earliest of 86th birthday, date of death, 31 March 2016

Life number	*STARTDATE*	*ENDDATE*	*Contribution to exposed to risk (months)*	
1	1 April 2015	1 August 2015	4	
2	1 April 2015	1 November 2015	7	
3	1 April 2015	1 January 2016	9	
4	1 April 2015	1 February 2016	10	
5	1 April 2015	1 March 2016	11	
6	1 April 2015	1 January 2016	9	
7	1 June 2015	1 November 2015	5	
8	1 July 2015	31 March 2016	9	
9	1 September 2015	1 March 2016	6	
10	1 January 2016	31 March 2016	3	+2

The total exposed to risk is therefore 73 months. +1

[3]

(ii) There are 3 deaths at age 85. +½

Maximum likelihood estimate is 3/73 = 0.04110 (monthly)
(36/73 = 0.49315 working annually). +½

[1]

(iii) EITHER (CONSTANT FORCE)

$q_{85} = 1 - p_{85} = 1 - \exp(-12 * 0.04110)$ +½

= 0.3893. +½

OR (ACTUARIAL ESTIMATE)

$$q_{85} \approx \frac{d_{85}}{E^c_{85} + 0.5d_{85}} = \frac{3}{(73/12) + 1.5}$$ +½

= 0.3956. +½

OR (EXACT EXPOSURE)

We add to the exposure for the deaths the duration between death and the time at which the deceased would have attained exact age 86 years.

This is 3 months for life 6, 7 months for life 7 and 6 months for life 9, a total of 16 months. +½

$$q_{85} \approx \frac{d_{85}}{E^c_{85} + (16/12)} = \frac{3}{(73+16)/12} = 0.4045.$$ +½

[1]
[Total 5]

Many candidates correctly calculated the exposed to risk in part (i). A large number of candidates did not realise that one of the deaths took place after the life's 86th birthday and so used 4 deaths rather than 3 in part (ii). It was acceptable to work in years rather than months, and credit could also be obtained for parts (i) and (ii) working in days. In part (iii), however, q_{85} is the probability of death within one year for a person at exact age 85 years, so it was not acceptable to compute the probability of death per month.

Q5 (i) Define the objectives of the modelling process. +½

Plan the modelling process and how it will be validated. +½

Collect and validate the data required. +½

Define the parameters for the model and consider appropriate parameter values +½

Define the model by capturing the essence of the real world system. +½

Involve experts on the real world system/get feedback on validity. +½

Decide on software to be used, choose random number generator, etc. +½

Write the computer program. +½

Debug the program. +½

Analyse the output. +½

Test the reasonableness of the output. +½

Consider appropriateness of response of the model to small changes in input parameters. +½

Ensure that any relevant professional guidance or standards have been complied with. +½

Document the model and ensure the results are in a format which can easily be communicated. +½
. [max. 4]

(ii) There will, by definition, be (virtually) no data about this disease yet. +½

Consideration may need to be given to using data from other diseases. +½

Expert input will be particularly important. +½

As some of the parameters may be highly uncertain and depend on the form of transmission, may need to test the robustness to a wider change in input parameters than usual. +1

Typical parameters would be the period for which a person is contagious, the probability of passing on the disease on contact, and the number of people each person would be in contact with. +1

The form of model may not need to differ from one used to study the spread of other diseases. +½

Results could be validated against the spread of other emerging diseases. +½

Careful reporting of the model findings, emphasising margins of error, may be advisable so as not to cause panic. +½

It may be relevant to consider the characteristics of the environments in which the outbreak will happen, and whether the disease is likely to affect the whole population, or some population sub-groups, defined on the basis of age, sex, etc. +½

The possibility of finding a vaccine may be an important consideration. +½

The time required to develop the model may be critical as results may be required quickly if the disease is spreading rapidly. +½

[max. 3]

[Total 7]

Part (i) of this question was very well answered, with many candidates scoring full marks. Part (ii) was more demanding, and many candidates only made a cursory attempt. The Examiners were looking for answers which revealed thought about the specific scenario described in the question. Credit was given for a wide range of points, including some not listed above. Little credit, however, was awarded to answers which were couched in general terms, without reference to the spread of a newly discovered disease. In both sections of this question, not all the points listed above were required for full credit.

Q6 (i) The statements in the question give rise to the following equations:

(1) $\exp(\beta_S + 5\beta_A + \beta_G) = 2\exp(5\beta_A)$

(2) $\exp(25\beta_A) = 0.5\exp(23\beta_A + \beta_G)$

(3) $\exp(\beta_S + 12\beta_A + \beta_G) = 1.6\exp(\beta_S + 25\beta_A)$ +2

(2) gives us

$\ln 0.5 = 2\beta_A - \beta_G$

(3) gives us

(4) $\ln 1.6 = \beta_G - 13\beta_A$

Combining these gives

$-11\beta_A = \ln 0.5 + \ln 1.6$

So $\beta_A = 0.02029$ +1

Substituting in (4) gives $\beta_G = 0.73372$ +1

(1) gives us $\beta_S = \ln 2 - \beta_G$

So $\beta_S = -0.04057$ +1

[5]

(ii) Here $h(t)$ is the hazard of symptoms disappearing so we wish to find the group with the maximum value of the hazard, or the minimum value of

$$S_t = \exp\left(-\int_0^t h_0(t) \exp(S\beta_S + A\beta_A + G\beta_G) \right),$$

the probability of still suffering from the symptoms. +1

β_S is negative, so we want $S = 0$ i.e. male. +½

β_A is positive, so we want A to be as large as possible i.e. the person to be as old as possible when the drug is administered. +1

β_G is positive so we want $G = 1$ i.e. someone who attends a gym. +½

[3]

(iii) EITHER

$$S_{\text{givenfemale}}(28) = \exp\left[-\int_0^{28} h_0(t) \exp(\beta_S + 18\beta_A + \beta_G)\, dt \right] = 0.75 \qquad +1$$

$$-\int_0^{28} h_0(t) = \frac{\ln(0.75)}{\exp(\beta_S + 18\beta_A + \beta_G)}. \qquad +1$$

The probability for the required male is

$$S_{\text{required}}(28) = \exp\left[-\int_0^{28} h_0(t) \exp(6\beta_A)\, dt \right]. \qquad +½$$

So

$$S_{\text{required}}(28) = \exp\left\{ \frac{\ln(0.75)}{\exp(\beta_S + 18\beta_A + \beta_G)} \exp(6\beta_A) \right\}. \qquad +½$$

Inserting the calculated values for β_S, β_A and β_G gives

$$S_{\text{required}}(28) = \exp(-0.11276) = 0.89337,$$

which is an 89.3% probability of still having the symptoms for the male aged 26 years when given the drug who did not attend as gym. +1

OR

$$S_{\text{givenfemale}}(28) = \exp\left[-\int_0^{28} h_0(t)\exp(\beta_S + 18\beta_A + \beta_G)\,dt\right] = 0.75$$ +1

The hazard for the given female is $h_0(t)\exp(\beta_S + 18\beta_A + \beta_G)$. +1

The hazard for the required male is $h_0(t)\exp(6\beta_A)$. +½

The ratio of the hazards is therefore $\dfrac{e^{6\beta_A}}{e^{18\beta_A+\beta_S+\beta_G}} = 0.39195$. +½

Therefore

$$S_{\text{required}}(28) = 0.75^{0.39195} = 0.89337,$$

which is an 89.3% probability of still having the symptoms for the male aged 26 years when given the drug who did not attend as gym +1

[4]

[Total 12]

Many candidates were able to formulate the equations in part (i), though a wide range of numerical errors was made when solving them. In parts (ii) and (iii), credit was given for answers that were correct given the solutions offered in part (i) for the values of β_A, β_G, and β_S. Few candidates spotted that the age related to when the drug was administered. Some candidates interpreted the question as asking for a comparison of the effectiveness of the drug among the groups of people mentioned in the bullet points in the question. Credit was given for this if the hazards for the six groups were correctly calculated and the correct group identified (43 year old males who attended the gym). Part (iii) was less well answered than the other parts. Candidates who worked only with hazards, rather than survival probabilities, received little credit.

Q7 (i) EITHER

Using the Markov assumption,

OR

The Chapman Kolmogorov equation is +½

$$p_{HH}(x,t+dt) = p_{HH}(x,t)p_{HH}(t,t+dt)$$
$$+p_{HS}(x,t)p_{SH}(t,t+dt) + p_{HD}(x,t)p_{DH}(t,t+dt)$$ +½

But $p_{DH}(t,t+dt) = 0$ or other explanation why path through *D* can be ignored +½

So:

$$p_{HH}(x,t+dt) = p_{HH}(x,t)p_{HH}(t,t+dt) + p_{HS}(x,t)p_{SH}(t,t+dt)$$ +½

Assuming that, for small *dt*

$$p_{ij}(t,t+dt) = \lambda_{ij}(t)dt + o(dt) \qquad i \neq j$$ +½

$$p_{ii}(t,t+dt) = 1 + \lambda_{ii}(t)dt + o(dt)$$

OR

$$p_{ii}(t,t+dt) = 1 - \sum_{j \neq i} \lambda_{ij}(t)dt + o(dt)$$ +½

where the λs are the instantaneous transition rates and $\lim_{dt \to 0} \frac{o(dt)}{dt} = 0$,

then substituting, we have

$$p_{HH}(x,t+dt) = p_{HH}(x,t)(1 - \sigma(t)dt - \mu(t)dt) + p_{HS}(x,t)\rho(t,C_t) + o(dt)$$ +1

so that

$$p_{HH}(x,t+dt) - p_{HH}(x,t) = p_{HH}(x,t)(-\sigma(t) - \mu(t))dt$$
$$+p_{HS}(x,t)\rho(t,C_t)dt + o(dt)$$

and hence

$$\frac{d}{dt}p_{HH}(x,t) = \lim_{dt \to 0} \frac{p_{HH}(x,t+dt) - p_{HH}(x,t)}{dt}$$ +1
$$= p_{HH}(x,t)(-\sigma(t) - \mu(t)) + p_{HS}(x,t)\rho(t,C_t)$$

[5]

(ii) The equation simplifies when considering $p_{\overline{HH}}(t)$ to

$$\frac{d}{dt}p_{\overline{HH}}(0,t) = -(\sigma(t) + \mu(t))p_{\overline{HH}}(t)$$ +½

$$\frac{1}{p_{\overline{HH}}(0,t)}\frac{d}{dt}p_{\overline{HH}}(0,t) = -(\sigma(t)+\mu(t)) = \frac{d}{dt}\ln p_{\overline{HH}}(t)\ .$$

Integrate both sides:

$$\left[\ln p_{\overline{HH}}(0,t)\right]_0^t = \int_{s=0}^{t} -(\sigma(s)+\mu(s))ds$$ +½

as $p_{\overline{HH}}(0)=1$

$$p_{\overline{HH}}(0,t) = \exp-\left(\int_{s=0}^{t}(\sigma(s)+\mu(s))ds\right)$$ +1

[2]

(iii) One method of deriving probabilities for continuous time Markov processes is by integral equations. +½

Using the law of total probability +½

we can consider the full set of possibilities for the first jump from state X or the last jump to state Y. +½

For a given time of this first/last jump, the probabilities that the jump was from each state will be in proportion to transition rates at that time. +1

By integrating across all possible times for the first/last jump we obtain the overall probability. +½

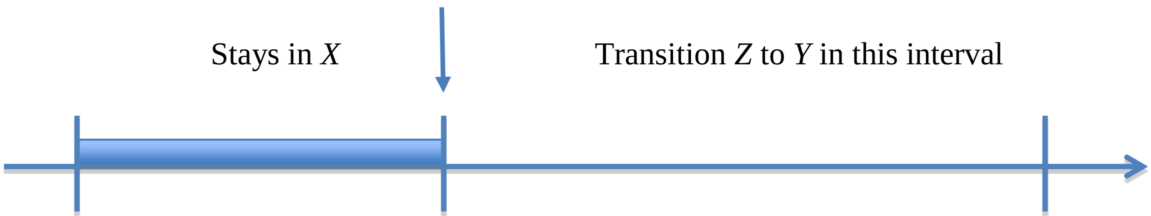
Where a probability for being in the same state at start and end is required, an additional term is needed for the probability of remaining in the same state throughout the period i.e. no jumps. +½

EITHER THE BACKWARD EQUATION

Suppose looking at $P_{XY}(s,t)$ we condition on jumps to all other states Z.

State X Jumps out of X to Z (say) State Y

Stays in X Transition Z to Y in this interval

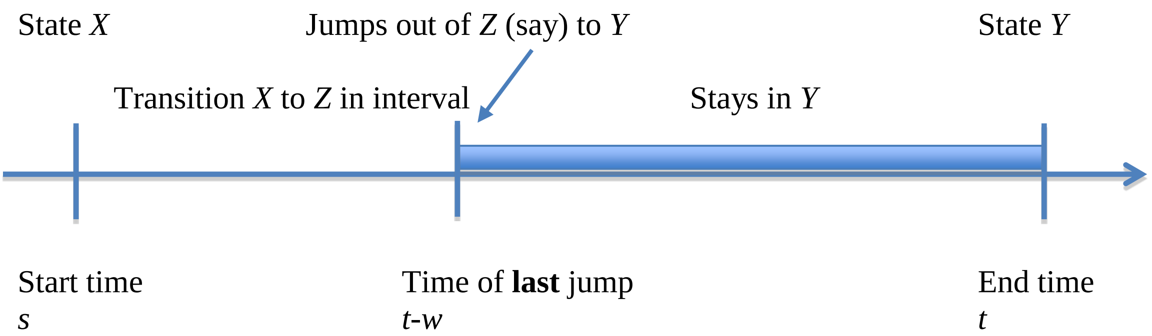


Start time Time of **first** jump End time *s* *s*+*w* *t*

+1½

OR THE FORWARD EQUATION

Suppose looking at $P_{XY}(s,t)$ we condition on jumps from all other states Z.



+1½

[max. 3]

(iv) The integrated forward equation is derived by conditioning on the last jump to state Y and the integrated backward equation from the first jump from state X. +1

(v) *For reference, the correct equation is:*

$$\Pr[X_t = H \,|\, X_s = S, C_s = w] = \int_s^t e^{-\int_s^y (\rho(u,w-s+u)+\upsilon(u,w-s+u))du} \rho(y,w-s+y) P_{HH}(y,t)dy$$

1. The term saying $\upsilon(y,w-s+y)$ is wrong: +½

   it should be $\rho(y,w-s+y)$. +½

2. The lower limit on the integral in the exponential term is wrong: +½

   it should be *s*. +½

3. The lower limit on the outer integral is wrong: +½

   it should be *s*. +½

[max. 2]
[Total 13]

This was one of the more difficult questions on the examination paper, and performance was, as expected, variable. A substantial number of candidates only attempted part (i). In part (ii) several candidates attempted to evaluate the integral by assuming the transition rates were *constant, which is incorrect as the question stated that the transition rates were dependent on time. Of those candidates who attempted part (iii), many made a good effort at the diagrams. Few candidates offered correct solutions to part (iv). It should be noted that the correct equation noted here does not match that shown in the Core Reading, Unit 4, page 17. The lower limit on the outer integral should be s rather than 0. Candidates could score credit for spotting the error in the Core Reading, but even without this, full credit could be scored, as there were two further errors to be identified.*

Q8 (i) The probability of making n claims in a year is given by

$$\frac{\lambda^n \exp(-\lambda)}{n!} \text{ where } \lambda = 0.35.$$ +½

+½

Number of claims	*Probability*	*Cost*
0	0.7047	0
1	0.2466	616.60
2	0.0432	215.81
3 or more	0.0055	41.32

+½

Giving an average cost per policy of £873.73 +½

If P is the premium paid by someone at the 0% no claims discount (NCD) level, then the premiums paying for this cost per policy are P * (1 – discount) * proportion at that level. +½

Discount Level	*Proportion of P*	*Proportion at level*	*Payment*
0	1	0.044	0.0440 P
15%	0.85	0.105	0.0893 P
30%	0.7	0.251	0.1757 P
40%	0.6	0.600	0.3600 P
		Total	0.6690 P

+1

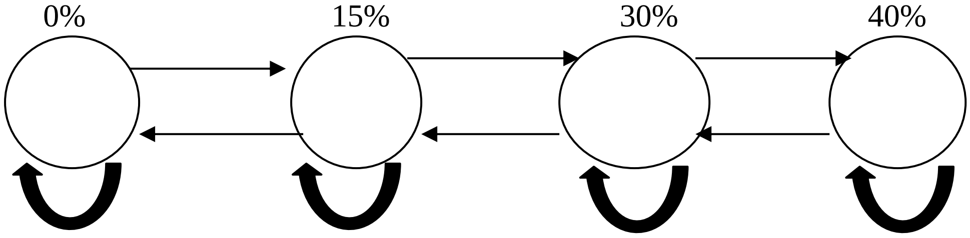
So we have 0.6690 P = £873.73, +½

giving P = £1,306.12.

So the premium at 40% NCD level is £783.67 +½

[4]

(ii)



[2]

(iii) The transition matrix for the process is

$$\begin{array}{c} 0 \\ 15 \\ 30 \\ 40 \end{array}\begin{pmatrix} 0.2953 & 0.7047 & 0 & 0 \\ 0.0487 & 0.2466 & 0.7047 & 0 \\ 0 & 0.0487 & 0.2466 & 0.7047 \\ 0 & 0 & 0.0487 & 0.9513 \end{pmatrix}.$$ +1

The stationary distribution, π, satisfies $\pi = \pi P$ +½

$\pi_1 = 0.2953\pi_1 + 0.0487\pi_2$ (1)

$\pi_2 = 0.7047\pi_1 + 0.2466\pi_2 + 0.0487\pi_3$ (2)

$\pi_3 = 0.7047\pi_2 + 0.2466\pi_3 + 0.0487\pi_4$ (3)

$\pi_4 = 0.7047\pi_3 + 0.9513\pi_4$ (4)

+1

Also $\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$. (5) +½

Working in terms of π_4:

(4) gives $0.7047\pi_3 = \pi_4(1 - 0.9513)$

$\pi_3 = 0.069068\pi_4$;

(3) gives $0.7047\pi_2 = (1 - 0.2466)\pi_3 - 0.0487\pi_4$

$0.7047\pi_2 = 0.7534(0.069068)\pi_4 - 0.0487\pi_4$

$0.7047\pi_2 = 0.003362\pi_4$

$\pi_2 = 0.00477\pi_4$;

(1) gives $\pi_1(1 - 0.2953) = 0.0487(0.00477)\pi_4$

$\pi_1 = 0.000329\pi_4$; and

(5) gives $\pi_4(0.000329 + 0.00477 + 0.069068 + 1) = 1$ +½

So $\pi_4 = 0.930954$

$\pi_3 = 0.064299$

$\pi_2 = 0.004441$

$\pi_1 = 0.000307$

+1

As before:

Discount Level	*Proportion of P*	*Proportion at level*	*Payment*
0	1	0.000307	0.00031 P
15%	0.85	0.004441	0.00377 P
30%	0.7	0.064299	0.04501 P
40%	0.6	0.930954	0.55857 P
		Total	0.60766 P

+1

0.60766 P = £873.73, so P = £1,437.85, and those at the 40% NCD level pay £862.71 +½

[6]

(iv) This may be a common feature in the market. If competitors offer it and this company does not, it may lose business. +1

The previous system may have discouraged claims if it meant that people lost their NCD. Introducing the new system may change the incidence of claims. Or the average size of claims may change (smaller ones may have gone unreported previously). +1

The one-off increase in premium when they introduce the scheme may prompt otherwise loyal customers to shop around for a better deal. +1

If the company is the first in the market to launch this option, they may win lots of new business. +1

Extra administrative costs may be incurred. +½

The protected NCD may appear unfair to policyholders as customers not making a claim can end up with the same discount as those who made a claim. +1

The new system may embody a moral hazard as it could make customers drive less carefully. +½

The new system may induce selection against the office if any new customers are more likely to make claims than those who leave the company and seek a better deal elsewhere. +½

[max. 3]

[Total 15]

This question proved to be more difficult than anticipated. Few candidates we able to apply the correct approach to calculating the *premium in part (i). Those who did calculate a premium often used the incorrect approach of multiplying £2,500 by 0.35 to obtain a cost per policy of £875. They then simply multiplied this by 0.6 to obtain the premium for someone with a no claims discount of 40%. In part (ii) a common error was to suppose that no customer could ever move down from 40% to 30%, 30% to 15% or 15% to 0%. Some candidates attempted to subdivide the 40%, 30% and 15% levels according to whether the customer made no claims or exactly one claim in the previous year. This produces a seven-state model which is inconsistent with the scenario in the question. A seven-state model can be constructed by subdividing the 40%, 30% and 15% levels according to whether the customer made no claims or one or more claims in the previous year, but it is complex. Credit was given for this if it was correct. In part (iii) it was expected that answers would be consistent with part (ii). Thus it was expected that candidates who supposed, in part (ii) that no customer could ever move down from 40% to 30%, 30% to 15% or 15% to 0% would write in part (iii) that in the long run all customers would be at the 40% discount level.*

Q9 (i) (a) Graduation by parametric formula. +½

Graduation by reference to a standard table. +½

Graphical graduation. +½

{1}

(b) **Parametric formula**

Advantages:

The resultant graduation will be sufficiently smooth provided few parameters are used. +½

It is a suitable method to produce standard tables. +½

It can be useful to fit the same formula to several experiences to give insight into the differences between experiences. +½

Disadvantages

It may be difficult to find one equation which fits at all ages. +½

Reference to a standard table

Advantages

It can be used to fit relatively small data sets where a suitable standard table exists. +½

The graduated rates should be smooth provided that a simple function is used. +½

The standard table can provide information at extreme ages where data may be scanty. +½

Disadvantages
It may be difficult to find a standard table which correctly reflects the population under investigation. +½

Graphical graduation

Advantages
It can be used for scanty data sets. +½
It enables an experienced analyst to allow for known (or likely) features of the data. +½
It can give a quick initial feel for the rates. +½

Disadvantages
The expertise may not be available. +½
The resultant figures may not be to sufficient decimal places for e.g. premium calculations +½
It is difficult to ensure smoothness of the resultant rates. +½

{2}
[max. 3]

(ii) To test for the overall goodness of fit use the χ^2 test.

The null hypothesis is that the graduated rates are the same as the true underlying rates in the block of business. +½

The test statistic $\sum_x z_x^2 \approx \chi_m^2$ where m is the degrees of freedom.

Age	*Exposed to Risk*	*Observed Deaths*	*Graduated Rates*	*Expected Deaths*	z_x	z_x^2
60	7,966	127	0.015724	125.26	0.1556	0.0242
61	7,728	139	0.017246	133.28	0.4954	0.2454
62	7,870	162	0.018921	148.91	1.0728	1.1508
63	7,622	167	0.020763	158.26	0.6949	0.4830
64	7,097	205	0.022790	161.74	3.4018	11.5720
65	7,208	179	0.025019	180.33	–0.0993	0.0099
66	6,833	185	0.027470	187.71	–0.1974	0.0390
67	6,474	212	0.030167	195.30	1.1947	1.4273
68	6,208	209	0.033134	205.70	0.2303	0.0530
69	5,914	195	0.036398	215.26	–1.3806	1.9060
				Total		16.9106

+1½

The observed test statistic is 16.91. +½

The number of age groups is 10,

but we lose three degrees of freedom, one for each parameter, +½

so $m = 7$. +½

The critical value of the χ^2 distribution with 7 degrees of freedom at the 95% significance level is 14.07. +½

Since 16.91 > 14.07, +½

we have sufficient evidence to reject the null hypothesis. +½
[5]

(iii) May fail to detect small but consistent bias. +½

For this use the signs test or cumulative deviations test over the whole age range. +½

May fail to detect a few large deviations offset by a lot of small deviations
OR
may fail to detect outliers. +½

For this use the standardised deviations test +½

The shape of the graduation may be wrong
OR
even if there is not bias over the whole range, there may be areas of the graduation where there is significant bias.
OR
there may be clumping of the signs. +½

For this use the grouping of signs test, the cumulative deviations test over sections of the age range, or the serial correlations test at lag 1. +½

The graduated rates may not be smooth. +½

For this use the third differences test. +½
[3]

(iv) **Signs test**

The null hypothesis is that the graduated rates are the same as the true underlying rates in the block of business. +½

We have 3 negative signs out of 10 ages. +½

EITHER

The probability of getting exactly 3 negative signs is equal to

$$\binom{10}{3}0.5^{10} = 0.1172$$

OR

The probability of getting 3 or fewer negative signs is 0.1719, +1

which is greater than 0.025 (two tailed test) +½

Therefore at the 95% significance level we do not reject the null hypothesis: we can say that there is no bias. +½

Cumulative Deviations test

The null hypothesis is that the graduated rates are the same as the true underlying rates in the block of business. +½

the test statistic $\dfrac{\sum_x (\text{Observed deaths} - \text{Expected deaths})}{\sqrt{\sum_x \text{Expected deaths}}}$ ~ Normal(0,1) +½

So, using the results in the table, the value of the test statistic is

$$\frac{1,780 - 1,712}{\sqrt{1,712}} = 1.6499$$ +1

Since –1.96 < test statistic < +1.96 +½

Therefore at the 95% significance level we do not reject the null hypothesis: there does not appear to be a bias. +½

Grouping of Signs test

The null hypothesis is that the graduated rates are the same as the true underlying rates in the block of business. +½

We have n_1 = 7 positive signs and n_2 = 3 negative signs. +½

There are 2 positive runs. +½

EITHER

Using the table on p. 189 of the Golden Book, we reject the null hypothesis with 1 positive run or fewer. +½

Since 2 > 1, we do not reject the null hypothesis: there do not seem to be an unduly large number of runs of consecutive ages with the same sign. +1

OR

Since

$$\Pr[1 \text{ positive run}] = \frac{\binom{6}{0}\binom{4}{1}}{\binom{10}{7}} = \frac{4}{120} = \frac{1}{30}$$

$$\Pr[2 \text{ positive runs}] = \frac{\binom{6}{1}\binom{4}{2}}{\binom{10}{7}} = \frac{36}{120} = \frac{3}{10}$$

The calculations show that Pr[1 positive run] < 0.05, but
Pr[2 positive runs] > 0.05. +½

Hence we do not reject the null hypothesis: there do not seem to be an unduly large number of runs of consecutive ages with the same sign. +1

Individual Standardised Deviations test

The null hypothesis is that the graduated rates are the same as the true underlying rates in the block of business. +½

Under the null hypothesis we would expect the individual z_xs to be distributed Normal (0,1). +½

EITHER

Only 1 in 20 $z_x s$ should have absolute magnitude greater than 1.96, and none should be outside the range –3 to +3,

OR

a table showing split of deviations, actual versus expected as below.

Range	–∞,–3	–3,–2	–2,–1	–1,0	0,1	1,2	2,3	3, +∞
Expected	0.0	0.2	1.4	3.4	3.4	1.4	0.2	0.0
Actual	0	0	1	2	4	2	0	1

+1

z_{64} = 3.40, is a definite outlier. +½

Therefore we reject the null hypothesis. +½

Test for smoothness

Age x	Graduated rates μ_x	First difference $\Delta_x = \mu_x - \mu_{x-1}$	Second difference $\Delta_x^2 = \Delta_x - \Delta_{x-1}$	Third difference $\Delta_x^3 = \Delta_x^2 - \Delta_{x-1}^2$	
60	0.015724				
61	0.017246	0.001522			
62	0.018921	0.001675	0.000153		
63	0.020763	0.001842	0.000167	0.000014	
64	0.022790	0.002027	0.000185	0.000018	
65	0.025019	0.002229	0.000202	0.000017	
66	0.027470	0.002451	0.000222	0.000020	
67	0.030167	0.002697	0.000246	0.000024	
68	0.033134	0.002967	0.000270	0.000024	
69	0.036398	0.003264	0.000297	0.000027	+2

Note that it does not matter against which ages the third differences appear in the table: they could appear against ages 60-66 or against ages 63-69.

The third differences are small and progress smoothly with age. +½

Therefore the graduation is acceptably smooth. +½

[6]

[Total 17]

This question was very well answered by many candidates, a high proportion scoring 13 or more. In part (iv), credit was given to candidates who attempted the serial correlations test with lag 1 to test whether the shape of the graduated rates were consistent with the underlying mortality. However, we do not recommend that candidates attempt this test in the examination if they have a choice. The calculations are very time-consuming and errors are likely; and other, quicker, tests for the same defect are available.

Q10 (i) The Kaplan-Meier estimator is

$$\hat{S}(t) = \prod_{t_j <= t} \left(1 - \frac{d_j}{n_j} \right)$$ +½

and the Nelson-Aalen estimator is

$$\tilde{S}(t) = \exp\left(-\sum_{t_j<=t}\frac{d_j}{n_j}\right),$$ +½

where d_j represents the number of occurrences of the event of interest <u>at duration</u> t_j, +½

and n_j represents the number exposed to the hazard at duration t_j. +½

[2]

(ii) Expanding into the individual terms:

$$\hat{S}(t) = \left(1-\frac{d_1}{n_1}\right).\left(1-\frac{d_2}{n_2}\right).......\left(1-\frac{d_j}{n_j}\right),$$ +½

and

$$\tilde{S}(t) = \exp\left(-\sum_{t_j<=t}\frac{d_j}{n_j}\right) = \exp\left(-\frac{d_1}{n_1}\right).\exp\left(-\frac{d_2}{n_2}\right)........\exp\left(-\frac{d_j}{n_j}\right).$$ +½

As each d_j/n_j must be between 0 and 1 the chart shows each term in the Nelson-Aalen estimator is no lower than the parallel in the Kaplan-Meier. +½

Hence the Nelson-Aalen estimator is always no lower than the Kaplan-Meier estimator. +½

[2]

(iii) **Interval censoring** is present because the tests only take place every three months and recurrence of eczema could occur between tests. +1

Type 1 censoring is present because it is specified in advance that the study will end after 5 years. +1

Random censoring is present as for patients who leave the study, the time of their censoring can be considered a random variable. +1

Right censoring is present for patients still free of eczema after 5 years or patients who left the study, as we do not know when the reoccurrence of eczema happened, just that it happened after a certain date. +1

Non-informative censoring could be said to be present as we have no reason to believe that those patients who left the study were any more or less likely to have the eczema recur than those who remained in the study. +1

Informative censoring could be said to be present as we could argue that those who left the study may have done so because they considered themselves cured, and were therefore less likely to suffer a recurrence than those still in the study. +1

[max. 3]

(iv) For the group continuing to receive steroid cream:

t_j	n_j	d_j	c_j	λ_j	$1-\lambda_j$
3	10	1	0	1/10	9/10
5	9	1	2	1/9	8/9
10	6	2	2	1/3	2/3
18	2	1	1	1/2	1/2

+2

The Kaplan-Meier estimate of the survival function, $S(t)_{\text{KM}}$, is

t	$S(t)_{\text{KM}}$
$0 \le t < 3$	1
$3 \le t < 5$	9/10
$5 \le t < 10$	4/5
$10 \le t < 18$	8/15
$18 \le t < 20$	4/15
+1	*+1*

+2

For the control group:

t_j	n_j	d_j	c_j	λ_j	$(1-\lambda_j)$
6	10	1	0	1/10	9/10
8	9	2	3	2/9	7/9
14	4	1	1	1/4	3/4
18	2	2	0	2/2	0

+2

The Kaplan-Meier estimate of the survival function, $S(t)_{\text{KM}}$, is:

t	$S(t)_{\text{KM}}$
$0 \le t < 6$	1
$6 \le t < 8$	9/10
$8 \le t < 14$	7/10
$14 \le t < 18$	21/40
$18 \le t < 20$	0

+2
[8]

(v) (a) In order to assess whether the risk is statistically lower a simple and quick approach would be to calculate confidence intervals around each survival function. +½

If the confidence intervals do not overlap the survival rate is statistically higher or lower at the chosen confidence level. +½

For the Kaplan Meier estimate the variance can be estimated using Greenwood's formula, +½

which is:

$$\text{Var}[\tilde{S}(t)] \approx (\tilde{S}(t))^2 \sum_{t_j \le t} \frac{d_j}{n_j(n_j - d_j)}.$$ +½

Methods such as the log-rank test or Wilcoxon's test could be used. +½

(b) In this case it is unlikely it could be shown that continuing to receive steroid cream statistically reduces the risk of recurrence, +½

as the sample size is small +½

and the survival rates do not appear markedly better for the group receiving steroid cream. +½

[max. 3]
[Total 18]

In part (i) many candidates were imprecise about the durations over which the product should be calculated for the Kaplan-Meier estimate, or over which the hazards should be summed for the Nelson-Aalen method. Vague descriptions, such as 'sum over j', lost marks. Many candidates failed to state that the d_j and n_j were the deaths and remaining risk set at duration t_j. Part (ii) was very poorly answered, with many candidates supposing that the graph in the question paper was of the Kaplan-Meier and Nelson-Aalen estimates and merely describing the graph. Such answers scored little or no credit. In part (iii) a minority of the candidates worked on the basis that each of the 20 sample members was in both groups. Good answers to part (iv) were very few. Credit was given for mentioning sensible approaches other than those listed above.

END OF EXAMINERS' REPORT