EXAMINATION

14 September 2005 (am)

Subject CT4 — Models
Core Technical

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*
2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*
3. *Mark allocations are shown in brackets.*
4. *Attempt all 13 questions, beginning your answer to each question on a separate sheet.*
5. *Candidates should show calculations where this is appropriate.*

Graph paper is not required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator.

103 Questions

A1 An insurance company has a block of in-force business under which policyholders have been given options and investment-related guarantees. A stochastic model has been developed which projects option and guarantee costs. You have used the model to estimate, for the Company Board, the probability of the insurance company having insufficient assets to honour the payouts under the policies. A Board member has asked whether there are any factors which could cause this probability to be inaccurate.

Outline the items you would mention in your response. [5]

A2 (i) In the context of a stochastic process denoted by $\{X_t : t \in J\}$, define:

(a) state space
(b) time set
(c) sample path

[2]

(ii) Stochastic process models can be placed in one of four categories according to whether the state space is continuous or discrete, and whether the time set is continuous or discrete. For each of the four categories:

(a) State a stochastic process model of that type.

(b) Give an example of a problem an actuary may wish to study using a model from that category.

[4]
[Total 6]

A3 A die is rolled repeatedly. Consider the following two sequences:

I B_n is the largest number rolled in the first n outcomes.
II C_n is the number of sixes rolled in the first n outcomes.

For each of these two sequences:

(a) Explain why it is a Markov chain.
(b) Determine the state space of the chain.
(c) Derive the transition probabilities.
(d) Explain whether the chain is irreducible and/or aperiodic.
(e) Describe the equilibrium distribution of the chain. [7]

A4 A life insurance company prices its long-term sickness policies using a three-state Markov model in continuous time. The states are healthy (*H*), ill (*I*) and dead (*D*). The forces of transition in the model are $\sigma_{HI} = \sigma$, $\sigma_{IH} = \rho$, $\sigma_{HD} = \mu$, $\sigma_{ID} = v$ and they are assumed to be constant over time.

For a group of policyholders observed over a 1-year period, there are:

23 transitions from State H to State I;
15 transitions from State I to State H;
3 deaths from State H;
5 deaths from State I.

The total time spent in State H is 652 years and the total time spent in State I is 44 years.

(i) Write down the likelihood function for these data. [3]

(ii) Derive the maximum likelihood estimate of σ. [2]

(iii) Estimate the standard deviation of $\tilde{\sigma}$, the maximum likelihood estimator of σ. [2]

[Total 7]

A5 Claims arrive at an insurance company according to a Poisson process with rate λ per week.

Assume time is expressed in weeks.

(i) Show that, given that there is exactly one claim in the time interval $[t, t + s]$, the time of the claim arrival is uniformly distributed on $[t, t + s]$. [3]

(ii) State the joint density of the holding times $T_0, T_1, \ldots, T_n$ between successive claims. [1]

(iii) Show that, given that there are n claims in the time interval $[0, t]$, the number of claims in the interval $[0, s]$ for $s < t$ is binomial with parameters n and s/t. [3]

[Total 7]

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A6 A Markov jump process X_t with state space $S = \{0, 1, 2, \ldots, N\}$ has the following transition rates:

$\sigma_{ii} = -\lambda$ for $0 \le i \le N-1$

$\sigma_{i,i+1} = \lambda$ for $0 \le i \le N-1$

$\sigma_{ij} = 0$ otherwise

(i) Write down the generator matrix and the Kolmogorov forward equations (in component form) associated with this process. [3]

(ii) Verify that for $0 \le i \le N-1$ and for all $j \ge i$, the function

$$p_{ij}(t) = e^{-\lambda t}\frac{(\lambda t)^{j-i}}{(j-i)!}$$

is a solution to the forward equations in (i). [2]

(iii) Identify the distribution of the holding times associated with the jump process. [2]

[Total 7]

A7 A time-inhomogeneous Markov jump process has state space {A, B} and the transition rate for switching between states equals $2t$, regardless of the state currently occupied, where t is time.

The process starts in state A at $t = 0$.

(i) Calculate the probability that the process remains in state A until at least time s. [2]

(ii) Show that the probability that the process is in state B at time T, and that it is in the first visit to state B, is given by $T^2 \times \exp^{-T^2}$. [3]

(iii) (a) Sketch the probability function given in (ii).

(b) Give an explanation of the shape of the probability function.

(c) Calculate the time at which it is most likely that the process is in its first visit to state B.

[6]

[Total 11]

104 Questions

B1 Describe the advantages and disadvantages of graduating a set of observed mortality rates using a parametric formula. [4]

B2 A lecturer at a university gives a course on Survival Models consisting of 8 lectures. 50 students initially register for the course and all attend the first lecture, but as the course proceeds the numbers attending lectures gradually fall.

Some students switch to another course. Others intend to sit the Survival Models examination but simply stop attending lectures because they are so boring. In this university, students who decide not to attend a lecture are not permitted to attend any subsequent lectures.

The table below gives the number of students switching courses and stopping attending lectures after each of the first 7 lectures of the course.

Lecture number	*Number of students switching courses*	*Number of students ceasing to attend lectures but remaining registered for Survival Models*
1	5	1
2	3	0
3	2	3
4	0	1
5	0	2
6	0	1
7	0	0

The university's Teaching Quality Monitoring Service has devised an Index of Lecture Boringness. This index is defined as the Kaplan-Meier estimate of the proportion of students remaining registered for the course who attend the final lecture. In calculating the Index, students who switch courses are to be treated as censored after the last lecture they attend.

(i) Calculate the Index of Lecture Boringness for the Survival Models course. [4]

(ii) Explain whether the censoring in this example is likely to be non-informative. [2]

[Total 6]

B3 A mortality investigation has been carried out over the three calendar years, 2002, 2003 and 2004.

The deaths during the period of investigation, θ_x, have been classified by age x at the date of death, where

$$x = \text{calendar year of death} - \text{calendar year of birth.}$$

Censuses of the numbers alive on 1 January in each of the years 2002, 2003, 2004 and 2005 have been tabulated and denoted by

$$P_x(2002),\ P_x(2003),\ P_x(2004) \text{ and } P_x(2005)$$

respectively, where x is the age last birthday at the date of each census.

(i) State the rate year implied by the classification of deaths, and give the ages of the lives at the beginning of the rate year. [2]

(ii) Derive an expression for the exposed to risk in terms of the $P_x(t)$ (t = 2002, 2003, 2004, 2005) which corresponds to the deaths data and which may be used to estimate the force of mortality, μ_{x+f} at age $x + f$. [4]

(iii) Determine the value of f, stating any assumptions you make. [3]

[Total 9]

B4 An investigation was carried out into the mortality of male undergraduate students at a large university. The resulting crude rates were graduated graphically. The following table shows the observed numbers of deaths at each age x, d_x, and the $\mathring{q}_x$s obtained from the graduation, together with the number of lives exposed to risk at each age.

Age x	d_x	$\mathring{q}_x$	*Exposed-to-risk*
18	6	0.0012	5,200
19	8	0.0013	5,000
20	12	0.0015	4,800
21	8	0.0017	5,000
22	9	0.0019	3,800
23	6	0.0020	3,600
24	8	0.0021	3,200

(i) Test whether the overall fit of the graduated rates to the crude data is satisfactory using a chi-squared test. [5]

(ii) Comment on your results in (i). [1]

(iii) (a) Describe three possible shortcomings in a graduation which the chi-squared test cannot detect, and

(b) State a test which can be used to detect each one. [3]

[Total 9]

B5 An investigation was carried out into the effects of lifestyle factors on the mortality of people aged between 50 and 65 years. The investigation took the form of a prospective study following a sample of several hundred individuals from their 50th birthdays until their 65th birthdays and collecting data on the following covariates for each person:

X_1 Sex (a categorical variable with 0 = female, 1 = male)

X_2 Cigarette smoking (a categorical variable with 0 = non-smoker, 1 = smoker)

X_3 Alcohol consumption (a categorical variable with 0 = consumes fewer than 21 units of alcohol per week, 1 = consumes 21 or more units of alcohol per week)

In addition, data were collected on the age at death for persons who died during the period of investigation.

In order to analyse the data, it was decided to use a Gompertz hazard, $\lambda_x = Bc^x$, where x is the duration since the start of the observation.

(i) Explain why the Gompertz hazard might be appropriate for analysing the mortality of persons aged between 50 and 65 years. [2]

(ii) Show that the substitution:

$$B = \exp(\beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3),$$

in the Gompertz model (where β_0 ... β_3 are parameters to be estimated), leads to a proportional hazards model for this particular analysis. [3]

(iii) Using the Gompertz hazard, the parameter estimates in the proportional hazards model were as follows:

Covariate	*Parameter estimate*	*Parameter*
Sex	β_1	+0.40
Cigarette smoking	β_2	+0.75
Alcohol consumption	β_3	–0.20
	β_0	–5.00
	c	+1.10

(a) Describe the characteristics of the person to whom the baseline hazard applies in this model.

(b) Calculate the estimated hazard for a female cigarette smoker aged 55 years who does not consume alcohol.

(c) Show that, according to this model, a cigarette smoker at any age has a risk of death roughly equal to that of a non-smoker aged eight years older. [6]

[Total 11]

B6 Studies of the lifetimes of a certain type of electric light bulb have shown that the probability of failure, q_0 , during the first day of use is 0.05 and after the first day of use the “force of failure”, μ_x , is constant at 0.01.

(i) Calculate the probability that a light bulb will fail within the first 20 days. [2]

(ii) Calculate the complete expectation of life (in days) of:

(a) a one-day old light bulb
(b) a new light bulb

[7]

(iii) Comment on the difference between the complete expectations of life calculated in (ii) (a) and (b). [2]

[Total 11]

END OF PAPER