# INSTITUTE AND FACULTY OF ACTUARIES

# EXAMINATION

28 April 2017 (pm)

## Subject CT6 – Statistical Methods
## Core Technical

*Time allowed: Three hours*

***INSTRUCTIONS TO THE CANDIDATE***

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*

2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*

3. *You have 15 minutes of planning and reading time before the start of this examination. You may make separate notes or write on the exam paper but not in your answer booklet. Calculators are not to be used during the reading time. You will then have three hours to complete the paper.*

4. *Mark allocations are shown in brackets.*

5. *Attempt all 10 questions, beginning your answer to each question on a new page.*

6. *Candidates should show calculations where this is appropriate.*

***Graph paper is NOT required for this paper.***

*AT THE END OF THE EXAMINATION*

*Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.*

*In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator from the approved list.*

**1** John enjoys playing two player zero-sum games.

The matrix below shows the losses to John in a particular two player zero-sum game. His strategies are denoted by I, II and III, whereas the strategies for his opponent are denoted by $a$, $b$ and $c$.

$$\begin{array}{cccc} & I & II & III \\ a & -3 & 2 & 4 \\ b & 0 & 1 & -1 \\ c & 1 & 3 & 0 \end{array}$$

(i) Explain which of John's strategies is dominated. [2]

The opponent now has the option of a fourth strategy, $d$, which results in none of John's strategies being dominated.

(ii) Suggest possible values for the strategy $d$. [1]
[Total 3]

**2** Claim amounts $X_i$ from a portfolio of insurance policies follow a gamma distribution with parameters $k$ and $\lambda_i$. Each $\lambda_i$ also follows a gamma distribution with parameters $\alpha$ and $\beta$.

(i) Show that the mixture distribution of losses is a generalised Pareto, with parameters $\alpha$, $\beta$, $k$. [4]

Claim amounts are now assumed to be exponentially distributed with parameter $\lambda_i$.

(ii) Show, using your answer to part (i), that the mixture distribution of losses is now a standard Pareto distribution with parameters $\alpha$, $\beta$. [2]
[Total 6]

**3** (i) Explain why claim amounts from general insurance policies are typically modelled using statistical distributions with heavy tails. [2]

Claim amounts on a portfolio of insurance policies are assumed to follow a Weibull distribution. A quarter of losses are below 15 and a quarter of losses are above 80.

(ii) Estimate the parameters $c$, $\gamma$ of the Weibull distribution that fit this data. [3]

(iii) Determine whether or not this Weibull distribution has a heavier tail than that of the exponential distribution with parameter $c$, by considering your estimate of $\gamma$. [2]
[Total 7]

**4** The number of claims on a portfolio of insurance policies in a given year follows a Poisson distribution with unknown mean $\lambda$. Prior beliefs about $\lambda$ are specified by a gamma distribution with mean 60 and variance 360. Over a period of three and one-third years, the total number of claims is 200.

(i) Calculate the Bayesian estimate of $\lambda$ under all-or-nothing loss. [7]

(ii) Comment on your result for part (i). [1]
[Total 8]

**5** (i) Show that the following discrete distribution belongs to the exponential family of distributions.

$$f(y;\mu) = \binom{n}{ny} \mu^{ny}(1-\mu)^{n-ny} \qquad y = 0, \frac{1}{n}, \frac{2}{n}, \ldots, 1$$

[4]

(ii) Derive expressions for the mean and variance of the distribution, $E(y)$ and $\mathrm{Var}(y)$, using your answer to part (i). [4]
[Total 8]

**6** Model A is a stationary AR(1) model, which follows the equation:

$$y_t = \mu + \alpha y_{t-1} + \varepsilon_t$$

where $\varepsilon_t$ is a standard white noise process.

(i) State two approaches for estimating the parameters in Model A. [2]

Mary, an actuarial student, wishes to revise Model A such that the error terms $\varepsilon_t$ no longer follow a Normal distribution.

(ii) Explain which of the approaches in part (i) she should now use for parameter estimation. [2]

(iii) Propose a method by which Mary will be able to calculate estimates of the parameters $\alpha$ and $\sigma^2$, with reference to any relevant equations. [3]

Mary, has now constructed Model B. She has done this by multiplying both sides of the equation above by $(1 - cB)$, where $B$ is the backshift operator, so that Model $B$ follows the equation:

$$y_t(1-cB) = (\alpha y_{t-1} + \varepsilon_t)(1-cB).$$

(iv) Explain why Model A and Model B are identical. [2]

(v) Explain for which values of $c$ Model B is stationary. [2]
[Total 11]

**7** An actuary is assessing three different insurance companies, A, B and C.

Corresponding claim amounts and number of policies are shown in the data below.

| | *Company A* | | *Company B* | | *Company C* | |
|---|---|---|---|---|---|---|
| | *\$m* | *Policies* | *\$m* | *Policies* | *\$m* | *Policies* |
| 2013 | 1.16 | 85 | 0.85 | 68 | 1.48 | 110 |
| 2014 | 1.18 | 88 | 1.02 | 82 | 1.52 | 132 |
| 2015 | 1.14 | 85 | 0.96 | 70 | 1.78 | 143 |
| 2016 | 1.32 | 92 | 0.87 | 80 | 1.92 | 165 |
| **Total** | **4.8** | **350** | **3.7** | **300** | **6.7** | **550** |

Company C has 180 policies to insure in 2017.

(i) Calculate its expected claim amount, using the assumptions underlying Empirical Bayes Credibility Theory (EBCT) Model 2. [11]

(ii) Discuss why it might be preferable to use EBCT Model 2 rather than EBCT Model 1 for this purpose. [2]
[Total 13]

**8** (i) Write down the general form of a statistical model for a claims run-off triangle, defining all terms used. [5]

The table below shows the cumulative incurred claim amounts on a portfolio of insurance policies.

| | *Development Year* | | |
|---|---|---|---|
| *Underwriting Year* | *0* | *1* | *2* |
| 2014 | 3,215 | 6,847 | 10,078 |
| 2015 | 2,986 | 7,123 | |
| 2016 | 4,167 | | |

Claims are assumed to fully run off after Development Year 2. The estimated loss ratio of both 2015 and 2016 is 91% and the respective premium income in each year is:

| | *Premium Income* |
|---|---|
| 2014 | 11,365 |
| 2015 | 12,012 |
| 2016 | 12,867 |

The total of claim amounts paid to date is 21,186 from policies written in 2014 to 2016.

(ii) Calculate the outstanding claim reserve for this portfolio using the Bornheutter-Ferguson method. [9]
[Total 14]

**9** Consider the probability density function of a Gamma distribution where:

$$f(x) = \frac{x^{\alpha-1}e^{-x}}{\Gamma(\alpha)} \qquad x > 0, \alpha > 1.$$

Consider the simpler probability density function $h(x)$ where:

$$h(x) = \beta e^{-\beta x} \;\; x > 0.$$

Let $C = \max_x \frac{f(x)}{h(x)}$.

(i) Show that $C$ is $\frac{1}{\beta\Gamma(\alpha)} x_0^{\alpha-1} e^{(\beta-1)x_0}$, where $x_0 = \frac{\alpha-1}{1-\beta}$. [4]

(ii) Construct an algorithm which outputs a random variate from $f(x)$ using $h(x)$ and the Acceptance-Rejection method. [4]

(iii) Determine the value of $\beta$ that makes the algorithm most efficient, by maximising the number of accepted values. [6]

[Total 14]

**10** Total annual claim amounts $S$ on a portfolio of insurance policies come from two independent types of policies:

Type I, which have claim amounts uniformly distributed between 3,000 and 4,000.

Type II, which have claim amounts following an Exponential distribution with mean 3,600.

Claims occur according to a Poisson process, with mean 15 per annum for Type 1 claims and mean 25 per annum for Type 2 claims.

The insurance company uses a premium loading factor of 7% and checks for ruin at the end of each year.

(i) Calculate the mean and standard deviation of $S$. [3]

(ii) Calculate the minimum initial surplus $U_m$ required such that the probability of ruin at the end of the first year is less than 0.015, using a Normal approximation for the distribution of $S$. [4]

Regulatory reforms mean the insurance company is trying to reduce this probability of ruin to less than 0.005. The insurance company is therefore purchasing proportional reinsurance from a reinsurer, who uses a premium loading factor of 17% in its premiums.

The insurance company retains a proportion $\alpha$ of each claim, and denotes by $S_I$ the aggregate annual claims it retains net of reinsurance. The insurance company continues to hold initial surplus $U_m$.

(iii) Calculate the maximum proportion $\alpha_{max}$ that the insurance company can retain in order to keep the probability of ruin less than 0.005, using a Normal approximation for the distribution of $S_I$. [6]

The insurance company is concerned that $\alpha_{max}$ is too low, reducing its profits, and intends to retain a higher proportion.

(iv) Suggest other ways in which the insurance company can reduce the probability of ruin. [3]

[Total 16]

## END OF PAPER