EXAMINATION

20 April 2007 (am)

Subject CT4 — Models
Core Technical

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*
2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*
3. *Mark allocations are shown in brackets.*
4. *Attempt all 11 questions, beginning your answer to each question on a separate sheet.*
5. *Candidates should show calculations where this is appropriate.*

Graph paper is not required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator.

1 (a) Define, in the context of stochastic processes, a:

1. mixed process
2. counting process

(b) Give an example application of each type of process.

[4]

2 An insurance company is investigating the mortality of its annuity policyholders. It is proposed that the crude mortality rates be graduated for use in future premium calculations.

(i) (a) Suggest, with reasons, a suitable method of graduation in this case.

(b) Describe how you would graduate the crude rates.

[3]

(ii) Comment on any further considerations that the company should take into account before using the graduated rates for premium calculations. [2]

[Total 5]

3 The government of a small country has asked you to construct a model for forecasting future mortality.

Outline the stages you would go through in identifying an appropriate model. [6]

4 The actuary to a large pension scheme carried out an investigation of the mortality of the scheme's pensioners over the two years from 1 January 2005 to 1 January 2007.

(i) List the data required by the actuary for an exact calculation of the central exposed to risk for lives aged x. [2]

The following is an extract from the data collected by the actuary.

Age x nearest birthday	*Number of pensioners at:*			*Deaths during:*	
	1 January 2005	*1 January 2006*	*1 January 2007*	*2005*	*2006*
63	1,248	1,312	1,290	10	6
64	1,465	1,386	1,405	13	15
65	1,678	1,720	1,622	16	23
66	1,719	1,642	1,667	22	19
67	1,686	1,695	1,601	19	25

(ii) (a) Derive an expression that could be used to estimate the central exposed to risk using the available data. State any assumptions you make.

(b) Use the data to estimate μ_{65}. State any further assumptions that you make. [4]

[Total 6]

5 (i) Define the *hazard rate*, $h(t)$, of a random variable T denoting lifetime. [1]

(ii) An investigation is undertaken into the mortality of men aged between exact ages 50 and 55 years. A sample of n men is followed from their 50th birthdays until either they die or they reach their 55th birthdays.

The hazard of death (or force of mortality) between these ages, $h(t)$, is assumed to have the following form:

$$h(t) = \alpha + \beta t$$

where α and β are parameters to be estimated and t is measured in years since the 50th birthday.

(a) Derive an expression for the survival function between ages 50 and 55 years.

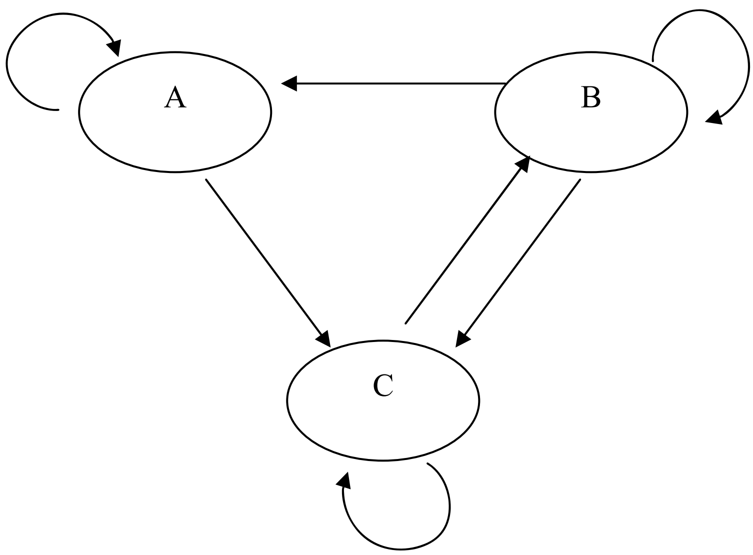
(b) Sketch this on a graph.

(c) Comment on the appropriateness of the assumed form of the hazard for modelling mortality over this age range.

[6]

[Total 7]

6 A three state process with state space {A, B, C} is believed to follow a Markov chain with the following possible transitions:



An instrument was used to monitor this process, but it was set up incorrectly and only recorded the state occupied after every two time periods. From these observations the following two-step transition probabilities have been estimated:

$P_{AA}^{2} = 0.5625$

$P_{AB}^{2} = 0.125$

$P_{BA}^{2} = 0.475$

$P_{CC}^{2} = 0.4$

Calculate the one-step transition matrix consistent with these estimates. [8]

7 Every person has two chromosomes, each being a copy of one of the chromosomes from one of their parents. There are two types of chromosomes labelled X and Y. A child born with an X and a Y chromosome is male and a child with two X chromosomes is female.

The blood-clotting disorder haemophilia is caused by a defective X chromosome (X*). A female with the defective chromosome (X*X) will not usually exhibit symptoms of the disease but may pass the defective gene to her children and so is known as a *carrier*. A male with the defective chromosome (X*Y) suffers from the disease and is known as a *haemophiliac*.

A medical researcher wishes to study the progress of the disease through the first born child in each generation, starting with a female carrier.

You may assume:

- every parent has a equal chance of passing either of their chromosomes to their children

- the partner of each person in the study does not carry a defective X chromosome; and

- no new genetic defects occur

(i) Show that the expected progress of the disease through the generations may be modelled as a Markov chain and specify carefully:

(a) the state space; and
(b) the transition diagram

[5]

(ii) State, with reasons, whether the chain is:

(a) irreducible; and
(b) aperiodic

[2]

(iii) Calculate the stationary distribution of the Markov chain. [3]
[Total 10]

8 A medical study was carried out between 1 January 2001 and 1 January 2006, to assess the survival rates of cancer patients. The patients all underwent surgery during 2001 and then attended 3-monthly check-ups throughout the study.

The following data were collected:

For those patients who died during the study exact dates of death were recorded as follows:

Patient	*Date of surgery*	*Date of death*
A	1 April 2001	1 August 2005
B	1 April 2001	1 October 2001
C	1 May 2001	1 March 2002
D	1 September 2001	1 August 2003
E	1 October 2001	1 August 2002

For those patients who survived to the end of the study:

Patient	*Date of surgery*
F	1 February 2001
G	1 March 2001
H	1 April 2001
I	1 June 2001
J	1 September 2001
K	1 September 2001
L	1 November 2001

For those patients with whom the hospital lost contact before the end of the investigation:

Patient	*Date of surgery*	*Date of last check-up*
M	1 February 2001	1 August 2003
N	1 June 2001	1 March 2002
O	1 September 2001	1 September 2005

(i) Explain whether and where each of the following types of censoring is present in this investigation:

(a) type I censoring
(b) interval censoring; and
(c) informative censoring [3]

(ii) Calculate the Kaplan-Meier estimate of the survival function for these patients. State any assumptions that you make. [7]

(iii) Hence estimate the probability that a patient will die within 4 years of surgery. [1]

[Total 11]

9 An insurance company is concerned that the ratio between the mortality of its female and male pensioners is unlike the corresponding ratio among insured pensioners in general. It conducts an investigation and estimates the mortality of male and female pensioners, $\hat{\mu}^m_{x+1/2}$ and $\hat{\mu}^f_{x+1/2}$. It then uses the $\hat{\mu}^m_{x+1/2}$ to calculate what the expected mortality of its female pensioners would be if the ratio between male and female mortality rates reflected the corresponding ratio in the PMA92 and PFA92 tables, $S_{x+1/2}$, using the formula

$$\tilde{\mu}^f_{x+1/2} = \hat{\mu}^m_{x+1/2} S_{x+1/2} .$$

The table below shows, for a range of ages, the numbers of female deaths actually observed in the investigation and the number which would be expected from the $\tilde{\mu}^f_{x+1/2}$.

Age x	*Actual deaths* $E^c_x \hat{\mu}^f_{x+1/2}$	*Expected deaths* $E^c_x \tilde{\mu}^f_{x+1/2}$
65	30	28.4
66	20	30.1
67	25	31.2
68	40	33.5
69	45	34.1
70	50	41.8
71	50	46.5
72	45	44.5

(i) Describe and carry out an overall test of the hypothesis that the ratios between male and female death rates among the company's pensioners are the same as those of insured pensioners in general. Clearly state your conclusion. [5]

(ii) Investigate further the possible existence of unusual ratios between male and female death rates among the company's pensioners, using two other appropriate statistical tests. [6]

[Total 11]

10 The members of a particular profession work exclusively in partnerships.

A certain partnership is concerned that it is losing trained technical staff to its competitors. Informal debriefing interviews with individuals leaving the partnership suggest that one reason for this is that the duration elapsing between becoming fully qualified and being made a partner is longer in this partnership than in the profession as a whole.

The partnership decides to investigate whether this claim is true using a multiple-state model with three states: (1) fully qualified but not yet a partner, (2) fully qualified and a partner, (3) working for another partnership. The period of the investigation is to be 1 January 1997 to 31 December 2006.

(i) (a) Draw and label a state-space diagram depicting the chosen model, showing possible transitions between the three states.

(b) State any assumptions implied by the diagram you have drawn and comment on their appropriateness.

[3]

(ii) (a) State what data would be required in order to estimate the transition intensity of moving from state (1) to state (2) for employees aged 30 years last birthday.

(b) Write down the likelihood of these data.

(c) Derive an expression for the maximum likelihood estimate of this transition intensity.

The investigation assumes that all transition intensities are constant within each year of age. [7]

In order to estimate the corresponding transition intensity for competitors, the partnership is compelled to rely on data kept by the relevant professional institute, of which all fully qualified individuals must be members. The institute keeps data on the numbers of members actively working on 1 January each year, classified by year of birth, according to whether or not they are partners. It also keeps data on the number of members who become partners each year, classified by age in completed years upon election to partnership.

(iii) Derive, using these data, an estimate for the profession as a whole of the corresponding transition intensity of becoming a partner among persons aged 30 years last birthday during the period of the investigation. State any assumptions you make. [5]

[Total 15]

11 (i) Consider two Poisson processes, one with rate λ and the other with rate μ. Prove that the sum of events arising from either of these processes is also a Poisson process with rate $(\lambda + \mu)$. [2]

(ii) (a) Explain what is meant by a Markov jump chain.

(b) Describe the circumstances in which the outcome of the Markov jump chain differs from the standard Markov chain with the same transition matrix. [4]

An airline has N adjacent check-in desks at a particular airport, each of which can handle any customer from that airline. Arrivals of passengers at the check-in area are assumed to follow a Poisson process with rate q. The time taken to check-in a passenger is assumed to follow an exponential distribution with mean $1/a$.

(iii) Show that the number of desks occupied, together with the number of passengers waiting for a desk to become available, can be formulated as a Markov jump process and specify:

(a) the state space; and
(b) the transition diagram

[3]

(iv) State the Kolmogorov forward equations for the process, in component form. [2]

(v) Comment on the appropriateness of the assumptions made regarding passenger arrival and the check-in process. [2]

(vi) (a) Set out the transition matrix of the jump chain associated with the airline check-in process.

(b) Determine the probability that all desks are in use before any passenger has completed the check-in process, given that no passengers have arrived at check-in at the outset. [4]

[Total 17]

END OF PAPER