Subject CT6 — Statistical Methods
Core Technical

EXAMINERS' REPORT

September 2008

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

R D Muckart
Chairman of the Board of Examiners

November 2008

Comments on individual questions

Q1 *There was a wide variety in the standard of answers to this question – it was generally well answered by the candidates who passed whilst weaker candidates struggled to relate the possible choices to the desirable characteristics of a credibility estimate.*

Q2 *This bookwork questions was generally well answered.*

Q3 *Well answered.*

Q4 *Many weaker candidates struggled to calculate α and β. Most were able to derive the posterior distribution.*

Q5 *Well answered.*

Q6 *Candidates found this (and Q7) the hardest on the paper, with very few able to make much headway. This was a little disappointing, since although the question was phrased around ARCH models it required only the standard definition on covariance and a little algebra.*

Q7 *Along with Q6, candidates found this the toughest question on the paper. Very few were able to write down the required estimator and therefore made no progress at all. Many candidates failed to recognise that the estimator had to be a function of the uniform random variables rather than a fixed number.*

Q8 *This question was a good differentiator – the candidates who passed were able to take a systematic approach to what was a fairly straightforward situation. Many weaker candidates failed to give an accurate definition of the probabilities in part (i) and a large number chose to make an approximation to the distribution in part(ii) when an exact calculation was simpler and quicker (and was what the question asked for).*

The notation used in this question differed slightly from that used in the core reading, which was unfortunate. Whilst this caused no problems for the majority of candidates, the examiners made allowance in those cases where the notation caused confusion.

Q9 *This question was generally well answered.*

Q10 *This question was well answered by the better candidates. Whilst most well prepared candidates were able to derive the relevant Yule-Walker equations, only the better candidates were able to move from these to estimates for the parameters.*

Q11 *Another good differentiator. Most candidates were able to derive the maximum likelihood estimate. Only the better candidates made much headway with part (iii).*

1 A This is an appropriate choice – the larger the value of n, the closer Z is to 1 and the more weight is placed on the data. Furthermore, high values of the variance of the prior (indicating uncertainty in the prior) lead to higher values of Z and so more weight on the sample data. Finally, high variance in the sample reduces the value of Z and places more reliance on the prior.

B This is not appropriate – the value is a constant independent of the size of the sample, whereas we would expect more weight on the sample the larger the sample.

C This is not appropriate – the greater the value of n, the lower the value of Z and the less weight is put on the sample. This is the reverse of what we would expect.

2 $C_{ij} = R_j S_i X_{i+j} + E_{ij}$

where

C_{ij} is the incremental claims from origin year i to j years ahead.
R_j is the development factor for year j independent of origin year i.
S_i is a parameter varying by origin year i, representing exposure (e.g. total claims incurred).
X_{i+j} is a parameter varying by calendar year, for example representing inflation.
E_{ij} is the error term.

3 (i) Company B can be discounted immediately because it is dominated by both of the other two options.

(ii) **Profit Table (in millions)**

	£20	£30	£40	*E[Profit]*
Current	0.6	1.6	2.6	1.5
Company A	0.4	1.65	2.9	1.525

The insurer should choose Company A.

4 Suppose that q has a $beta(\alpha,\beta)$ distribution as per the tables, and let X denote the number of failures in 2006 so that X has a $B(4500,q)$ distribution. Then

$$\begin{aligned} f(q|X) &\propto f(q)\times f(X|q) \\ &\propto q^{\alpha-1}(1-q)^{\beta-1}\times q^{x}(1-q)^{n-x} \\ &\propto q^{\alpha+x-1}(1-q)^{\beta+n-x-1} \end{aligned}$$

So the posterior distribution of q is beta with parameters $\alpha+x$ and $\beta+n-x$.

In our case, the parameters of the prior distribution are given by

$$\frac{\alpha}{\alpha+\beta}=0.015 \text{ and } \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}=0.005^2.$$

$$\alpha=0.015(\alpha+\beta)$$
$$0.985\alpha=0.015\beta$$
$$\alpha=\frac{3}{197}\beta$$

And substituting into the second equation gives

$$\frac{\frac{3}{197}\beta^2}{\left(\frac{200}{197}\beta\right)^2\left(1+\frac{200}{197}\beta\right)}=0.005^2$$
$$\frac{3}{197}=0.005^2\times\left(\frac{200}{197}\right)^2\times\left(1+\frac{200}{197}\beta\right)$$
$$591=1+\frac{200}{197}\beta$$

$$\beta=590\times\frac{197}{200}=\frac{11623}{20}=581.15$$
$$\alpha=\frac{3}{197}\times\frac{11623}{20}=8.85$$

So the revised parameters are given by:

$$\alpha_*=8.85+58=66.85$$
$$\beta_*=581.15+4500-58=5023.15$$

5 The main calculations are reported in the table:

Year	*Initial ultimate. Loss*	*r*	*f*	*1 - 1/f*	*Emerging Liability*
2006	9843.18	1.336099	1.336099	0.251552	2476.08
2007	10917.63	2.151156	2.874157=1.336099×2.151156	0.652072	7119.08

The second column (IUL) is obtained as 87% of Premium income values. The third column reports the development factors

7111/9501 = 1.336099 and (7111 + 6850) / (3541 + 2949) = 2.151156.

The emerging liability column is the product of IUL vales with those of 1 - 1/*f*.

The total emerged liability is now 2476.08 + 7119.08 = **9595.16** and the total reported liability is 9501 + 6850 + 3894 = 20245. Therefore the reserve value is 9595.16 + 20245 – 20103 = **9737.2**.

6 (i) Since e_t are independent from $X_t, X_{t-1}, \ldots$ and $\mathbf{E}(e_t) = 0$ we have that

$$\begin{aligned}
E(X_t) &= E(\mu + e_t\sqrt{\alpha_0 + \alpha_1(X_{t-1}-\mu)^2}) \\
&= \mu + E(e_t\sqrt{\alpha_0 + \alpha_1(X_{t-1}-\mu)^2}) \\
&= \mu + E(e_t)E(\sqrt{\alpha_0 + \alpha_1(X_{t-1}-\mu)^2}) \quad \text{since } e_t \text{ and } X_{t-1} \text{ are independent} \\
&= \mu + 0 \times E(\sqrt{\alpha_0 + \alpha_1(X_{t-1}-\mu)^2}) \\
&= \mu
\end{aligned}$$

The direct approach to showing that X_t *and* X_{t-s} *are uncorrelated is shown below. The crucial steps involve noting that* e_t *is independent of* X_{t-1} *as above, and that* e_t *is independent of*

$$e_{t-s}\sqrt{\alpha_0 + \alpha_1(X_{t-1}-\mu)^2}\sqrt{\alpha_0 + \alpha_1(X_{t-s-1}-\mu)^2}\,.$$

The algebra can be simplified by noting that adding a constant doesn't affect covariance, so the μ*'s can be ignored.*

$$Cov(X_t, X_{t-s}) = E(X_t X_{t-s}) - E(X_t)E(X_{t-s})$$
$$= E\left((\mu + e_t\sqrt{\alpha_0 + \alpha_1(X_{t-1}-\mu)^2})(\mu + e_{t-s}\sqrt{\alpha_0 + \alpha_1(X_{t-s-1}-\mu)^2})\right) - \mu^2$$
$$= E(\mu^2 + \mu e_t\sqrt{\alpha_0 + \alpha_1(X_{t-1}-\mu)^2} + \mu e_{t-s}\sqrt{\alpha_0 + \alpha_1(X_{t-s-1}-\mu)^2}$$
$$+ e_t e_{t-s}\sqrt{\alpha_0 + \alpha_1(X_{t-s-1}-\mu)^2}\sqrt{\alpha_0 + \alpha_1(X_{t-1}-\mu)^2}) - \mu^2$$

$$= E(\mu^2) + \mu E(e_t)E(\sqrt{\alpha_0 + \alpha_1(X_{t-1}-\mu)^2}) + \mu E(e_{t-s})E(\sqrt{\alpha_0 + \alpha_1(X_{t-s-1}-\mu)^2})$$
$$+E(e_t e_{t-s}\sqrt{\alpha_0 + \alpha_1(X_{t-s-1}-\mu)^2}\sqrt{\alpha_0 + \alpha_1(X_{t-1}-\mu)^2}) - \mu^2$$

$$= \mu^2 + \mu\times 0\times E(\sqrt{\alpha_0 + \alpha_1(X_{t-1}-\mu)^2}) + \mu\times 0\times E(\sqrt{\alpha_0 + \alpha_1(X_{t-s-1}-\mu)^2})$$
$$+E(e_t)E(e_{t-s}\sqrt{\alpha_0 + \alpha_1(X_{t-s-1}-\mu)^2}\sqrt{\alpha_0 + \alpha_1(X_{t-1}-\mu)^2}) - \mu^2$$

$$= \mu^2 + 0 + 0 + 0\times E(e_{t-s}\sqrt{\alpha_0 + \alpha_1(X_{t-s-1}-\mu)^2}\sqrt{\alpha_0 + \alpha_1(X_{t-1}-\mu)^2}) - \mu^2$$
$$= 0$$

(ii) The conditional variance of $X_t | X_{t-1}$ is

$\text{var}(X_t | X_{t-1}) = \text{var}(e_t)(\alpha_0 + \alpha_1(X_{t-1} - \mu)^2) = \alpha_0 + \alpha_1(X_{t-1} - \mu)^2$.

So the values of X_{t-1} are affecting the variance of X_t. If the same idea is applied recursively, it can be seen that the variance of X_t will be affected by the value of X_{t-s}. So X_t and X_{t-s} are not independent.

7 (i) $\hat{\theta} = \frac{1}{n}\sum_{t=1}^{n}(e^{U_t} - 1) = \left(\sum_{i=1}^{n}\frac{e^{U_t}}{n}\right) - 1$

(ii) We need to find the variance of $h(U) = e^U - 1$ where $U \sim U(0, 1)$

$$\mathbf{E}h(U) = \int_0^1 (e^x - 1)dx = e - 2$$

$$\mathbf{E}h(U)^2 = \int_0^1 (e^x - 1)^2 dx$$

$$= \int_0^1 (e^{2x} - 2e^x + 1)dx$$

$$= \frac{e^2 - 1}{2} - 2(e - 1) + 1$$

$$= \frac{e^2}{2} - 2e + 2.5$$

$\text{var}h(U) = \mathbf{E}h(U)^2 - (\mathbf{E}h(U))^2 = 0.242$ and $\text{var}\,\hat{\theta} = \dfrac{0.242}{n}$.

(iii) From the theory, the required n should satisfy

$$n \geq \frac{z_\alpha^2}{0.1^2} 0.242$$

where $P(|z| < z_\alpha) = 1 - \alpha$ with $z \sim N(0, 1)$. In our case $\alpha = 10\%$ and $z_\alpha = 1.64$ and so

$$n \geq \frac{1.64^2}{0.1^2}\ 0.242 = 65.09.$$

The answer is 66.

8 (i) Let $S(t)$ denote the insurer's surplus at time t. Then

$\psi(U) = \Pr(S(t) < 0$ for some value of $t)$ i.e. the probability of ruin at some time

$\psi(U,t) = \Pr(S(k) < 0)$ for some $k < t$ i.e. the probability that ruin occurs before time t.

(ii) (a) Immediately before the payment of any claims, the insurer has cash reserves of 1000 + 150 =1150.

The distribution of $S(1)$ is given by:

Deaths	*S(1)*	*Prob*
None	1150	$0.95 \times 0.9 = 0.855$
A only	$1150 - 1700 = -550$	$0.9 \times 0.05 = 0.045$
B only	$1150 - 400 = 750$	$0.95 \times 0.1 = 0.095$
A and B	$1150 - 2100 = -950$	$0.05 \times 0.1 = 0.005$

And the probability of ruin is given by 0.045 + 0.005 = 0.05.

(b) Assuming the surplus process ends if ruin occurs by time 1, then 2 possible values of $S(2)$ are -550 and -950.

If there are no deaths in year 1, possible values of $S(2)$ are

No deaths: 1150 + 150 = 1300

A only: 1150 + 150 – 1700 = -400
B only: 1150 + 150 – 400 = 900
A and B: 1150 + 150 – 1700 – 400 = -800

If B dies in year 1, the possible values of $S(2)$ are:

A lives: 750 + 100 = 850
A dies: 750 + 100 – 1700 = -850

The probability of ruin within 2 years is given by:

$0.05 + 0.855 \times (0.05 \times 0.9 + 0.05 \times 0.1) + 0.095 \times 0.05 = 0.0975$

Alternatively, note that ruin occurs within 2 years if and only if A dies during this time, the probability of which is $0.05 + 0.95 \times 0.05 = 0.0975$.

9 (i) The transition matrix is

$$\begin{pmatrix} p & 1-p & 0 \\ p & 0 & 1-p \\ 0 & p & 1-p \end{pmatrix}$$

The equilibrium probabilities (π_0, π_1, π_2) satisfy

$$\pi_0 = p\pi_0 + p\pi_1$$
$$\pi_1 = \pi_0\,(1-p) + p\pi_2$$
$$\pi_2 = \pi_1\,(1-p) + (1-p)\,\pi_2$$

From equations 1 and 3 above we obtain

$\pi_0 = p/(1-p)\ \pi_1$ and $\pi_2 = (1-p)/p\ \pi_1$.

Since $\pi_0 + \pi_1 + \pi_2 = 1$ we get $\pi_1 = \dfrac{p(1-p)}{p+(1-p)^2}$ together with $\pi_0 = \dfrac{p^2}{p+(1-p)^2}$

and $\pi_2 = \dfrac{(1-p)^2}{p+(1-p)^2}$

(ii) The average premium is now

$£600\left(\dfrac{p^2}{p+(1-p)^2} + 0.8\dfrac{p(1-p)}{p+(1-p)^2} + 0.5\dfrac{(1-p)^2}{p+(1-p)^2}\right)$ and for

$p = 0.1$ and $p = 0.3$ this quantity takes values £321.1 and £382 respectively.

The difference in average premiums is small given that the claim probability is three times higher, suggesting the NCD system does not discriminate well.

10 (i) ACF looks to decline exponentially suggesting an AR(p) process. While PACF becomes almost zero at lag 3 so AR(2) is a likely process here.

(ii) For p = 1 the equations are:

$$\gamma_1 = a_1\gamma_0$$
$$\gamma_0 = a_1\gamma_1 + \sigma^2$$

therefore $\hat{a}_1 = \hat{\rho}_1 = 0.854$ and $\hat{\sigma}^2 = \hat{\gamma}_0 - \hat{a}_1\hat{\gamma}_1 = \hat{\gamma}_0(1-\hat{\rho}_1^2) = 0.33917$.

(iii) For p = 2 the equations are:

$$\gamma_2 = a_1\gamma_1 + a_2\gamma_0$$
$$\gamma_1 = a_1\gamma_0 + a_2\gamma_1$$
$$\gamma_0 = a_1\gamma_1 + a_2\gamma_2 + \sigma^2$$

Solving the first two equations with respect to a_i after dividing both sizes by γ_0 we have that

$$a_2 = \frac{\rho_2 - \rho_1^2}{1-\rho_1^2}$$
$$a_1 = \rho_1(1 - a_2)$$

and with the right substitution for $\hat{\rho}_1, \hat{\rho}_2$ we get $\hat{a}_1 = 0.5679$ and $\hat{a}_2 = 0.3350$.

Then the white noise variance can now be estimated as
$\hat{\sigma}^2 = \hat{\gamma}_0 - \hat{a}_1\hat{\gamma}_1 - \hat{a}_2\hat{\gamma}_2 = 0.3011$.

(iv) Any 2 tests can be given, including:

- the turning point's test
- the "portmanteau" Ljung-Box χ^2 test
- the inspection of the values of the SACF values based on their 95% confidence intervals under the white noise null hypothesis

11 (i) Suppose X is exponentially distributed with parameter λ. Then we must show that $Y = kX$ is also exponentially distributed.

Now

$$P(Y < y) = P(kX < y)$$

$$= P(X < y/k)$$

$$= \int_0^{y/k} \lambda e^{-\lambda z} dz$$

$$= \int_0^{y} \lambda e^{-\frac{\lambda}{k}x} \frac{dx}{k} \text{ making the substitution } x = kz$$

$$= \int_0^{y} \frac{\lambda}{k} e^{-\frac{\lambda}{k}x} dx$$

Which is the distribution n function of an exponential distribution with parameter λ / k. So Y is exponentially distributed with parameter λ / k.

(ii) First note that the probability that a loss in 2006 is greater than M is given by $e^{-\lambda M}$ and likewise the probability that a loss in 2007 is greater than M is given by $e^{-\lambda M/k}$.

The likelihood of the data is given by:

$$L = C \times e^{-4\lambda M} \times \prod_i (\lambda e^{-\lambda x_i}) \times e^{-6\lambda M/k} \times \prod_j (\frac{\lambda}{k} e^{-\lambda y_i / k})$$

Where the x_i represent the claims in 2006 and y_j represent the claims in 2007.

The log-likelihood is given by

$$l = \log L = C' - 4\lambda M + 10\log\lambda - \lambda\sum x_i - 6\lambda M/k + 12\log\lambda - \lambda/k\sum y_j$$
$$= C' - 4\lambda M + 22\log\lambda - 13{,}500\lambda - 6\lambda M/k - 17{,}000\lambda/k$$

Differentiating gives

$$l' = -4M + \frac{22}{\lambda} - 13{,}500 - 6M/k - 17{,}000/k$$

And equating this to zero gives

$$-4M + 22/\hat{\lambda} - 13{,}500 - 6M/k - 17{,}000/k = 0$$

$$22/\hat{\lambda} = 13{,}500 + 17{,}000/k + 4M + 6M/k$$

$$\hat{\lambda} = \frac{22}{13{,}500 + 17{,}000/k + 4M + 6M/k}$$

We can check this is a maximum by noting that $l'' = -\frac{22}{\lambda^2} < 0$

(iii) (a) $\hat{\lambda} = \dfrac{22}{13{,}500 + 17{,}000/1.1 + 4\times 1{,}600 + 6\times 1{,}600/1.1} = 0.000499$

(b) Expected payment per claim for 2006 is given by:

$$\int_0^M x\lambda e^{-\lambda x}dx + M\int_M^\infty \lambda e^{-\lambda x}dx = \left[-xe^{-\lambda x}\right]_0^M + \int_0^M e^{-\lambda x}dx + M\left[-e^{-\lambda x}\right]_M^\infty$$

$$= -Me^{-\lambda M} + \left[-\frac{1}{\lambda}e^{-\lambda x}\right]_0^M + Me^{-\lambda M}$$

$$= \frac{1}{\lambda}(1 - e^{-\lambda M}) \quad (*)$$

$$= \frac{1}{0.000499}(1 - e^{-0.000499\times 1600})$$

$$= 1102.107$$

We know that claims for 2008 will have an exponential distribution with parameter $\mu = \lambda/1.05\times 1.1$. We need to choose the retention R so that

$$1102.107 = \int_0^R x\mu e^{-\mu x}dx + R\int_R^\infty \mu e^{-\mu x}dx$$

$$= \frac{1}{\mu}(1 - e^{-\mu R}) \quad \text{using the result from (*)}$$

$$= \frac{1.05\times 1.1}{0.0004999}\left(1 - e^{-\frac{0.000499R}{1.05\times 1.1}}\right)$$

And so

$$0.476148392 = 1 - e^{-0.00043203463R}$$

$$R = -\frac{1}{0.00043203462}\log(1 - 0.476148392)$$

=1496.52

END OF EXAMINERS' REPORT