Subject CT4 — Models
Core Technical

EXAMINERS' REPORT

September 2008

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

R D Muckart
Chairman of the Board of Examiners

November 2008

Comments

Comments on solutions presented to individual questions for the September 2008 paper are given below.

Q1 *This standard bookwork question was fairly well answered. Some candidates simply wrote down a list of steps in the development of the model, rather than answering the question that was set.*

Q2 *This straightforward question was well answered. Some candidates were vague about emphasising that continuous time models are applied to problems which require continuous monitoring.*

Q3 *Answers to this question were very poor. Many candidates did not go beyond making the point that the Binomial model is hard to extend to multiple decrements, whereas the multiple state model extends quite naturally.*

Q4 *Only a minority of candidates answered this question using the approach intended. Many tried to do a chi-squared test comparing the observed and expected numbers of births. This received some credit, especially when candidates combined the months into half-years, or thirds of a year, before performing the chi-squared test, so that the expected values in each cell were greater than 5.*

Q5 *This straightforward question was very well answered. The most common error was in reducing the number of degrees of freedom below 6. This is incorrect in this case, because the comparison is between an observed experience and a pre-existing experience, not between crude rates and graduated rates.*

Q6 *As with many exposed-to-risk questions, answers to this question were disappointing. In part (iii), few candidates realised that* $\hat{q}_{39\frac{1}{2}}$ *was required to estimate* q_{40}.

Q7 *Part (ii) of this question was standard bookwork, but was nevertheless answered in a brief or cursory fashion by many candidates. On the other hand, a good number of candidates were able to make the points required in part (i).*

Q8 *This question was very well answered, with many candidates scoring full marks. Some candidates were penalised in part (iii) for simply calculating the stationary distribution and not stating explicitly which of the numbers represented the long-run probability of being in discount level 2.*

Q9 *Answers to this question were disappointing, especially to part (iii). In part (ii), candidates who included additional transitions between states 2 and 3, and between states 2 and 1, were not penalised. However, such candidates were expected to produce answers to part (iii) which were consistent with the transition diagram they had drawn in part (ii).*

Q10 *Part (i) of this question was very disappointingly answered, as the required definition is in the Core Reading. Most candidates were able to compute the estimated survival function in part (ii). Some candidates interpreted the question as meaning that the* *numbers contacted at any duration include those reconvicted prior to that duration, so that those reconvicted must be subtracted from those contacted to obtain the relevant n_j. These candidates were given credit for part (ii). In part (iii) many candidates correctly calculated the variance of the integrated hazard but then incorrectly used this variance to compute a confidence interval around the survival function, rather than first computing the confidence interval around the integrated hazard and then using the formula $S(x) = \exp(-\Lambda_x)$ to convert this into a confidence interval around S(6).*

Q11 *Answers to this question were disappointing. Many candidates were able to answer parts (i) and (ii) reasonably well, but made little or no attempt at the remaining sections.*

Q12 *Answers to this question varied widely, but overall were disappointing. There was a large variation by centre, with average scores for some centres being several marks higher than for other centres. Perhaps this is the result of different training and education materials being used in different locations? While most candidates could write down the formula* $m_x = \dfrac{q_x}{\int_0^1 {}_t p_x dt}$ *and the formulae required to answer part (ii), it was clear from the answers to parts (iii) and (iv) that understanding of what these formulae mean was very shaky.*

1 **Instructions on how to run the model**

Tests performed to validate the output of the model.
Definition of input data.
Any limitations of the model identified (e.g. potential unreliability).
Basis on which the form of the model chosen (e.g. deterministic or stochastic)
References to any research papers or discussions with appropriate experts.
Summary of model results.
Name and professional qualification.
Purpose or objectives of the model.
Assumptions underlying the model.
How the model might be adapted or extended.

2 **Discrete time, discrete state space**

Counting process, random walk, Markov chain
No claims bonus in motor insurance.

Continuous time, discrete state space

Counting process, Poisson process, Markov jump process
Healthy-sick-dead model in sickness insurance

Discrete time, continuous state space

General random walk, ARIMA time series model, moving average model
Share price at end of each day

Continuous time, continuous state space

Compound Poisson process, Brownian motion, Ito process, white noise
Value of claims reaching an insurance company monitored
continuously

3 (a) Both models produce consistent and unbiased estimators.

The estimate of q_x made using the Binomial model will have a higher variance than that made using the multiple-state model, though the difference is tiny if the forces of mortality are small.

If data on exact ages at entry into and exit from observation are available, the multiple state model is simpler to apply. The Binomial model requires further assumptions (e.g. uniform distribution of deaths).

The Binomial model also does not use all the information available if exact ages at entry into and exit from observation are available.

However, if the forces of mortality are small, both models will give very similar results.

(b) The multiple state model can simply be extended

The estimators have the same form and the same statistical properties as in the classic life table.

The Binomial model is hard to extend to several causes of death. Although the life table as a computational tool can be extended, the calculations are more complex and awkward than those in the multiple-state model.

4 (i) Suppose that the number of births each month, B, is the outcome of a Poisson process with a rate $\lambda = 1.5$.

The probability of obtaining b births per month is given by the formula $\Pr[B=b] = \dfrac{\exp(-1.5).1.5^b}{b!}$

Therefore we have

b	$\Pr[B=b]$
0	0.223
1	0.335
2	0.251
3	0.126
4	0.047
5	0.014
6+	0.004

Therefore, if the number of births per month is the outcome of a Poisson process with a rate of 1.5 per month the probability of obtaining 5 or more births in a single month is 0.014 + 0.004 = 0.018.
EITHER This is very small OR this is < 0.05

which suggests that the historian may be correct to suspect something unusual about July 1637.

But only July has a number of births more than 5, and at the 5% level of statistical significance we expect 1 month in 20 to have such a large number, then unless we have a prior expectation that July is unusual, we should be cautious before accepting the historian's suggestion.

(ii) The assumption that births follow a Poisson process is unlikely to be entirely realistic

EITHER because of the occurrence of multiple births (twins and triplets)
OR because births tend to occur seasonally
OR because the process might be time inhomogeneous.

5 Using the chi-squared test (a suitable overall test).

If $z_x = \dfrac{\text{actual deaths - expected deaths}}{\sqrt{\text{expected deaths}}}$, then the test statistic is $\sum_x {z_x}^2 \sim \chi^2_m$,
where m is the number of ages, which in this case is 6.

The calculations are shown below.

Age x	z_x	${z_x}^2$
18	0.9487	0.9
19	0.8660	0.75
20	0	0
21	2.3094	5.3333
22	1.4142	2
23	1.3416	1.8

Therefore the value of the test statistic is 10.783.

The critical value of the chi-squared distribution at the 5% level of significance with 6 degrees of freedom is 12.59.

Since 10.783 < 12.59 there is insufficient evidence to reject the hypothesis that the mortality rate of men in the University is the same as that of the national population.

6 (i) Age label changes on the receipt of the premium on the policy anniversary so this is a policy year rate interval.

Policyholders' ages range from x to x+1 at start of the rate interval.

(ii) Central exposed to risk $E_x^c = \int_{t=0}^{4} P_{x,t}dt \approx \frac{1}{2}\sum_{t=0}^{3}(P_{x,t} + P_{x,t+1})$

Approximation assumes population changes linearly over each year during the period of investigation.

Initial exposed to risk $E_x \approx \frac{1}{2}\sum_{t=0}^{3}(P_{x,t} + P_{x,t+1}) + \frac{1}{2}d_x$,

assuming deaths are uniform over the rate interval OR deaths occur on average half way through the rate interval.

(but NOT deaths are uniform over the "year", or occur on average half way through the "year")

(iii) $\hat{q}_x = \frac{d_x}{E_x}$ estimates q_x for the average age at the start of the rate interval.

Assuming birthdays are uniformly distributed across <u>policy</u> years,

the average age at the start of the rate interval is x+½, so we require $\hat{q}_{39\frac{1}{2}}$ to estimate q_{40}.

Assuming $\hat{q}_{39\frac{1}{2}} = \frac{1}{2}[\hat{q}_{39} + \hat{q}_{40}]$ we have

$$\hat{q}_{39} = \frac{28}{\frac{1}{2}(10536 + 11005) + \frac{1}{2}*28} = 0.002596$$

$$\hat{q}_{40} = \frac{36}{\frac{1}{2}(10965 + 10745) + \frac{1}{2}*36} = 0.003311$$

and hence our estimate of q_{40} is 0.5[0.002596 + 0.003311) = 0.002954.

7 (i) Individual life offices are likely to have their systems set up to provide information on a "by policy" basis.

When data from different offices is pooled, it would not be practicable to establish whether an individual held policies with other companies.

(ii) If the mortality rate is q_x then since the lives are independent the number of deaths $\mathbf{D}_i$ will be distributed Binomial$\left(q_x, \pi_i N\right)$

So $\sum_i \mathbf{C}_i = \sum_i i\mathbf{D}_i$.

Hence $\text{Var}[\mathbf{C}] = \text{Var}\left[\sum_i \mathbf{C}_i\right] = \text{Var}\left[\sum_{\text{i}} i\mathbf{D}_i\right] = \sum_i i^2 \text{Var}\left[\mathbf{D}_i\right]$

by independence of deaths

$= \sum_i i^2 \pi_i N q_x (1 - q_x)$

If instead there were $\sum_i i\pi_i N$ independent policies/lives the variance would be additive so:

$\text{Var}\left[\mathbf{C}'\right] = \sum_i i\pi_i N q_x (1 - q_x)$

So the variance is increased by the ratio $\dfrac{\sum_i i^2 \pi_i}{\sum_i i\pi_i}$

(iii) If the proportions of lives holding *i* policies were known, the variance ratio could be allowed for in statistical tests

by using the ratio to adjust the variance upwards.

However, the variance ratio is unlikely to be known exactly.

Special investigations may be performed from time to time to estimate the variance ratios by matching up policyholders, which could then be applied to subsequent mortality investigations.

8 (i) The transition matrix of the process is

$$P = \begin{pmatrix} 0.15 & 0.85 & 0 & 0 \\ 0.15 & 0 & 0.85 & 0 \\ 0.03 & 0.12 & 0 & 0.85 \\ 0 & 0.03 & 0.12 & 0.85 \end{pmatrix}$$

(ii) (a) For the one year transition, $p_{22} = 0,$ as can be seen from above (or is obvious from the statement).

(b) The second order transition matrix is

$$\begin{pmatrix} 0.15^2+0.85\times0.15 & 0.85\times0.15 & 0.85^2 & 0 \\ 0.15^2+0.85\times0.03 & 0.85\times0.15+0.85\times0.12 & 0 & 0.85^2 \\ 0.03\times0.15+0.12\times0.15 & 0.85\times0.03\times2 & 0.85\times0.12\times2 & 0.85^2 \\ 0.03\times0.15+0.12\times0.03 & 0.12^2+0.85\times0.03 & 0.85\times0.03+0.85\times0.12 & 0.12\times0.85+0.85^2 \end{pmatrix}$$

$$= \begin{pmatrix} 0.15 & 0.1275 & 0.7225 & 0 \\ 0.048 & 0.2295 & 0 & 0.7225 \\ 0.0225 & 0.051 & 0.204 & 0.7225 \\ 0.0081 & 0.0399 & 0.1275 & 0.8245 \end{pmatrix}$$

hence the required probability is 0.2295.

(iii) In matrix form, the equation we need to solve is $\pi P = \pi$, where π is the vector of equilibrium probabilities.

This reads

$$\begin{aligned} 0.15\pi_1 + 0.15\pi_2 + 0.03\pi_3 \qquad\qquad &= \pi_1 \quad (1) \\ 0.85\pi_1 + \qquad\quad + 0.12\pi_3 + 0.03\pi_4 &= \pi_2 \quad (2) \\ +0.85\pi_2 \qquad\qquad + 0.12\pi_4 &= \pi_3 \quad (3) \\ 0.85\pi_3 + 0.85\pi_4 &= \pi_4 \quad (4) \end{aligned}$$

Discard the first of these equations and use also the fact that $\sum_{i=1}^{4} \pi_i = 1$.

Then, we obtain first from (4) that $0.85\pi_3 = 0.15\pi_4$
or, that $\pi_4 = 17\pi_3 / 3$

Substituting in (3) this gives

$$0.85\pi_2 + 0.12 \times \frac{17}{3}\pi_3 = \pi_3 \Rightarrow \pi_3 = 2.65625\pi_2$$

(2) now yields that

$$0.85\pi_1 = \pi_2 - 0.12\pi_3 - 0.03\pi_4 = \frac{1}{2.65625}\pi_3 - 0.12\pi_3 - 0.17\pi_3 = 0.0865\pi_3,$$

so that finally we get $\pi_1 = 0.10173\pi_3$.

Using now that the probabilities must add up to one, we obtain

$$\pi_1 + \pi_2 + \pi_3 + \pi_4 = (0.10173 + 0.3765 + 1 + 5.666)\pi_3 = 1,$$

or that $\pi_3 = 0.13996$.

Solving back for the other variables we get that

$$\pi_1 = 0.01424, \quad \pi_2 = 0.05269, \quad \pi_4 = 0.79311$$

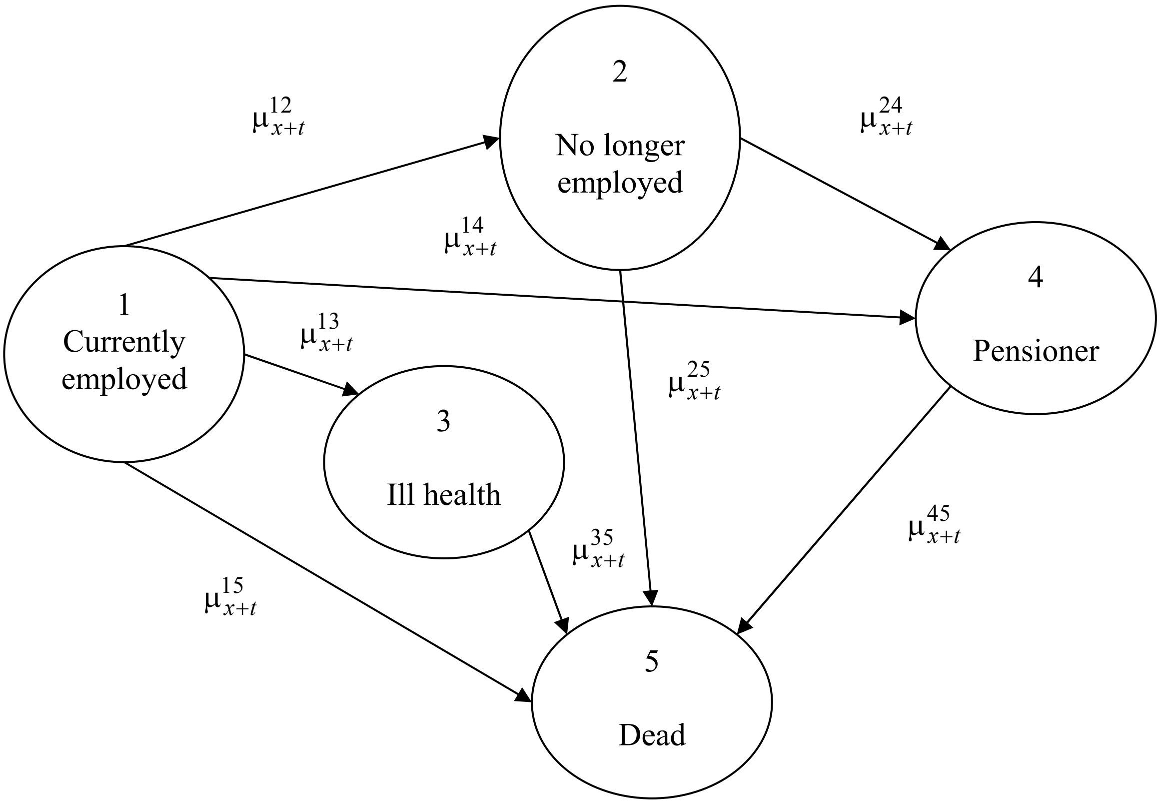
The long-run probability that the motorist is in discount level 2 is therefore 0.05269.

9 (i) The state space is discrete with states as given in the question.

The process operates in continuous time.
However, at the compulsory scheme retirement
age of 65 there is a discrete step change.

This is sometimes described as a mixed process.

(ii)



(iii) (a) For $x + t < 65$

$$\partial/\partial t\ {}_t p_x^{11} = -(\mu_{x+t}^{12} + \mu_{x+t}^{13} + \mu_{x+t}^{15})\,{}_t p_x^{11}$$

$$\partial/\partial t\ {}_t p_x^{12} = \mu_{x+t}^{12} \cdot {}_t p_x^{11} - \mu_{x+t}^{25} \cdot {}_t p_x^{12}$$

$$\partial/\partial t\ {}_t p_x^{13} = \mu_{x+t}^{13} \cdot {}_t p_x^{11} - \mu_{x+t}^{35} \cdot {}_t p_x^{13}$$

$$\partial/\partial t\ {}_t p_x^{15} = \mu_{x+t}^{15} \cdot {}_t p_x^{11} + \mu_{x+t}^{25} \cdot {}_t p_x^{12} + \mu_{x+t}^{35} \cdot {}_t p_x^{13}$$

and ${}_t p_x^{14}$ is zero.

(b) For $x + t = 65$

${}_t p_x^{11}$ and ${}_t p_x^{12}$ become 0 at $x + t = 65 + \delta$

$${}_{t+\delta} p_x^{14} = {}_{t-\delta} p_x^{11} + {}_{t-\delta} p_x^{12}$$

(c) For $x + t > 65$

$${}_t p_x^{11} = {}_t p_x^{12} = 0$$

$$\partial/\partial t \; {}_t p_x^{13} = -\mu_{x+t}^{35} \cdot {}_t p_x^{13}$$

$$\partial/\partial t \; {}_t p_x^{14} = -\mu_{x+t}^{45} \cdot {}_t p_x^{14}$$

$$\partial/\partial t \; {}_t p_x^{15} = \mu_{x+t}^{35} \cdot {}_t p_x^{13} + \mu_{x+t}^{45} \cdot {}_t p_x^{14}$$

10 (i) EITHER

The hazard rate at duration x is given by

$$\lim_{dt \to 0} \frac{\Pr[X \le x + dt \mid X > x]}{dt}.$$

OR

In discrete time, the hazard rate at duration x is given by, $\Pr[X = x \mid X \ge x]$.

OR

The hazard rate at duration x is given by $h(x) = -\dfrac{1}{S(x)} \dfrac{d}{dx}[S(x)]$,

where $S(x)$ is the survival function defined as $\Pr[X > x]$.

(ii) The integrated hazard, Λ_x, is estimated as follows:

x_j	n_j	d_j	c_j	$\dfrac{d_j}{n_j}$	$\Lambda_x = \sum_{x_j \le x} \dfrac{d_j}{n_j}$
0	100	0	0	0	0
1	100	0	3	0	0
2	97	0	2	0	0
3	95	4	1	4/95 = 0.0421	0.0421
4	90	3	2	3/90 = 0.0333	0.0754
5	85	5	0	5/85 = 0.0588	0.1343
6	80	0	80	0	0.1343

The survival function $S(x)$ is given by $\exp(-\Lambda_x)$, so that we have

x	$S(x)$
$0 \le x < 3$	1.0000
$3 \le x < 4$	0.9588
$4 \le x < 5$	0.9274
$5 \le x$	0.8744

(iii) Confidence intervals around the integrated hazard may be estimated using the formula

$$Var[\tilde{\Lambda}_x] = \sum_{x_j \le x} \frac{d_j(n_j - d_j)}{n_j^3}$$

Applying this to the data gives

x_j	n_j	d_j	$\frac{d_j(n_j-d_j)}{n_j^3}$	$\sum_{x_j \le x} \frac{d_j(n_j-d_j)}{n_j^3}$
0	100	0	0	0
1	100	0	0	0
2	97	0	0	0
3	95	4	0.000425	0.000425
4	90	3	0.000358	0.000783
5	85	5	0.000651	0.001434
6	80	0	0	0.001434

95 per cent confidence intervals around the integrated hazard at duration 6 can therefore be computed as

$$\hat{\Lambda}_6 \pm 1.96\sqrt{\mathrm{var}\,\hat{\Lambda}_6}$$

$$= 0.1343 \pm 1.96\sqrt{0.001434}$$

$$= (0.1343 - 0.0742,\ 0.1343 + 0.0742)$$

$$= (0.0601,\ 0.2085).$$

THEN EITHER

The estimated survival function, $\hat{S}(x)$ is given by $\exp(-\hat{\Lambda}_x)$,

so that the 95 per cent confidence interval for $\hat{S}(x)$ is

[exp(−0.0601), exp(−0.2085)]

which is (0.9417, 0.8118).
In the previous investigation the probability that a prisoner would not be reconvicted within 6 months of release was 1 – 0.2 = 0.8.

Since the 95 per cent confidence interval around $\hat{S}(x)$ in the current investigation does not include the value 0.8, and our estimate of $\hat{S}(x) > 0.8$ we conclude that the rate of reconviction has declined since the previous investigation.

OR

In the previous investigation the probability that a prisoner would not be reconvicted within 6 months of release was 1 – 0.2 = 0.8 – i.e. $S(6) = 0.8$

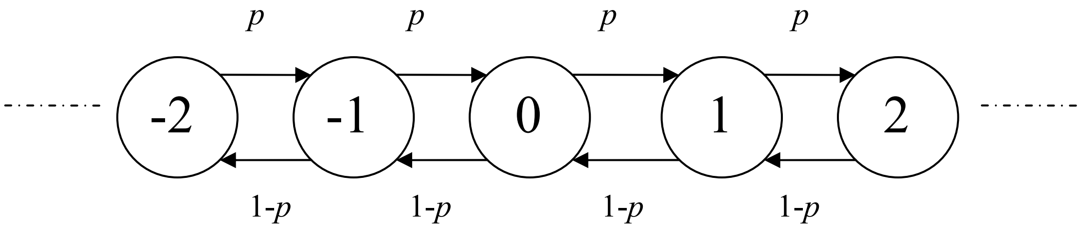
Since $S(x) = \exp(-\Lambda_x)$, the value of Λ_6 corresponding to $S(6) = 0.8$ is

$\Lambda_6 = -\log_e(0.8) = 0.2231$.

Since this is higher than the upper limit in the range (0.0601, 0.2085) we conclude that the rate of reconviction has declined since the previous investigation.

11 (i) State space is the set of integers $\mathbb{Z}$.

Transition graph:



(ii) (a) The process is not aperiodic

<u>because it has period 2</u>:
for example, starting from an even number the
process is only even after an even number of steps

(b) The process is irreducible

as the probabilities of X_n increasing and decreasing by 1 are both non-zero so any state can be reached.

(c) No stationary distribution will exist because the state space is infinite.

(iii) Suppose there are u upward movements.

Then there must be $m - u$ downward movements,

and $u - (m - u) = j - i$

So $u = \dfrac{m + j - i}{2}$.

(iv) The maximum number of upward steps is m so the transition probability is zero if $j - i > m$.

As the chain is periodic with period 2, it can only occupy state j after m steps if $m + j - i$ is even.

If $m + j - i$ is even and $j - i \le m$ then there must be u upward jumps and $(m - u)$ downward jumps.

These can be ordered in $\binom{m}{u}$ ways.

So the transition probabilities are:

$$p_{ij}^{(m)} = \begin{cases} \begin{pmatrix} m \\ u \end{pmatrix} p^u (1-p)^{m-u} & \text{if } j-i \le m \text{ and } m+j-i \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

(v) EITHER

In both cases the transition probabilities are unaltered unless $X_i = 0$.

(a) Reflecting boundary implies $P[X_{i+1} = 1 \mid X_i = 0] = 1$ (or $p_{01}^{(1)} = 1$)

(b) Absorbing boundary implies $P[X_{i+1} = 0 \mid X_i = 0] = 1$ (or $p_{00}^{(1)} = 1$)

OR

A matrix solution for the transition probabilities is acceptable

Reflecting:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots \\ 1-p & 0 & p & 0 & 0 & \dots \\ 0 & 1-p & 0 & p & 0 & \dots \\ 0 & 0 & 1-p & 0 & p & \dots \\ 0 & 0 & 0 & 1-p & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

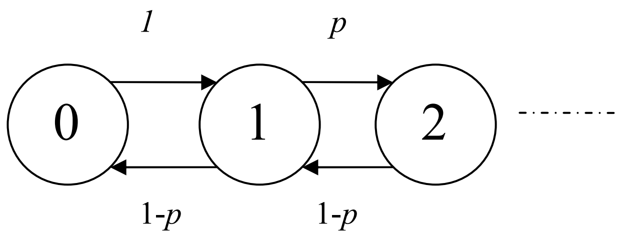
Absorbing:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 1-p & 0 & p & 0 & 0 & \dots \\ 0 & 1-p & 0 & p & 0 & \dots \\ 0 & 0 & 1-p & 0 & p & \dots \\ 0 & 0 & 0 & 1-p & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

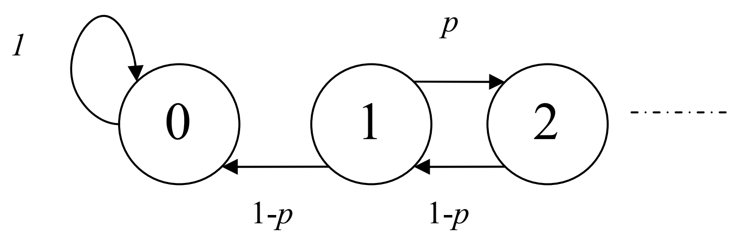
OR

A diagrammatic solution is also acceptable:

Reflecting



Absorbing:



(vi) In both cases the zero transition probabilities remain zero as the period is still 2 where relevant.

If i is sufficiently above 0 then conditions at zero will not be relevant and all the m-step transition probabilities will remain the same. (This applies if $m < i$.)

Otherwise

In (a) some sample paths which would have taken X below zero will be reflected, increasing the probability of reaching j at step m.

So the m-step transition probabilities would increase.

In (b) any sample path which reaches zero would no longer be able to access state j

so the transition probabilities would decrease.

12 (i) q_x is the probability that a life aged exactly x will die before reaching exact age x+1, and is called the initial rate of mortality.

m_x is called the central rate of mortality and represents the probability that a life alive between the ages of x and x+1 dies

They are related by:

$$m_x = \frac{q_x}{\int_0^1 {}_t p_x dt}$$

(ii) (a) Uniform distribution of deaths (UDD)

$${}_t q_x = t * q_x$$

(b) Constant force of mortality (CFM)

$${}_t q_x = 1 - e^{-\mu * t}$$

(c) Balducci assumption

$${}_{1-t} q_{x+t} = (1-t) * q_x$$

(iii) (a) **UDD**

$$\int_0^1 {}_t p_x dt = \int_0^1 (1 - 0.1t) dt = 1 - 0.1 \left[\frac{t^2}{2} \right]_0^1 = 0.95$$

(or other reasoning why exposure is 0.95 under UDD)

m_x = 0.1/0.95 = 0.105263

(b) **CFM**

μ given by:

$$1 - e^{-\mu} = 0.1$$

$$\mu = -\ln 0.9 = 0.1053605$$

EITHER

If force of mortality constant over $[x, x+1]$ then central rate must be equal to the force μ

so $m_x = 0.1053605$

OR

$$\int_0^1 {}_t p_x dt = \int_0^1 (1-(1-e^{-\mu t}))dt = -\frac{1}{\mu}\left[e^{-\mu t}\right]_0^1 = \frac{1}{\mu}(1-e^{-\mu}) = 0.949122$$

$m_x = 0.1/0.949122 = 0.1053605$

(c) **Balducci**

For consistency, observe that ${}_1 p_x = {}_t p_x \cdot {}_{1-t} p_{x+t}$

So

$${}_t p_x = \frac{{}_1 p_x}{{}_{1-t} p_{x+t}} = \frac{0.9}{1 - {}_{1-t} q_{x+t}} = \frac{0.9}{0.9+0.1t}$$

$$\int_0^1 {}_t p_x dt = \int_0^1 \frac{0.9}{0.9+0.1t} dt = \frac{0.9}{0.1}\left[\ln(0.9+0.1t)\right]_0^1 = -9\ln 0.9 = 0.9482446$$

So $m_x = 0.1/0.9482446 = 0.1054580$

(iv) The Balducci assumption implies a decreasing mortality rate over $[x, x+1]$ and UDD an increasing mortality rate.

CFM is obviously constant

For a given number of deaths over the period, the estimated exposure would be highest if we assumed an increasing mortality rate.

We would expect the central rate to be highest for that with the lowest estimate exposure, hence Balducci > CFM > UDD is the expected order.

END OF EXAMINERS' REPORT