INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

April 2016 (with mark allocations)

Subject CT6 – Statistical Methods
Core Technical

Introduction

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and using past papers as a revision aid and also those who have previously failed the subject.

The Examiners are charged by Council with examining the published syllabus. The Examiners have access to the Core Reading, which is designed to interpret the syllabus, and will generally base questions around it but are not required to examine the content of Core Reading specifically or exclusively.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report; other valid approaches are given appropriate credit. For essay-style questions, particularly the open-ended questions in the later subjects, the report may contain more points than the Examiners will expect from a solution that scores full marks.

The report is written based on the legislative and regulatory context pertaining to the date that the examination was set. Candidates should take into account the possibility that circumstances may have changed if using these reports for revision.

F Layton
Chair of the Board of Examiners
June 2016

A. General comments on the *aims of this subject and how it is marked*

1. The aim of the Statistical Methods subject is to provide a further grounding in mathematical and statistical techniques of particular relevance to financial work.

2. Errors carried over are only penalised once.

3. Generally arithmetic errors are not treated as harshly as method errors.

4. Markers exercise judgement when answers are partly correct and can award partial marks if appropriate. In particular, where a candidate has not used the method in the marking schedule, but has shown some understanding by their working, some credit is given.

5. Errors just due to rounding are not penalised unless the rounding is excessive (e.g. rounding an interim step to just 2 sig fig, say) and significantly compromises accuracy.

B. General comments on *student performance in this diet of the examination*

Well prepared candidates had plenty of opportunity to score well on this examination, particularly given that two of the longer questions (7 and 9) were relatively straightforward. A number of candidates fell just short, and it was clear that for some the bookwork underlying question 8 part (i) and question 10 part (i) had not been revised thoroughly. Question 10 was challenging and only the best prepared candidates scored well on the later parts.

C. Pass Mark

The Pass Mark for this exam was 60%.

Solutions

Q1 (i) $m = P(x \le m) = \dfrac{1}{2}$ [1]

$\text{CDF} = 1 - \left(\dfrac{\lambda}{\lambda + x}\right)^{\alpha}$ so $\left(\dfrac{\lambda}{\lambda + m}\right)^{\alpha} = \tfrac{1}{2}$ [1]

So $m = \lambda\left(2^{1/\alpha} - 1\right)$ [1]

[Total 3]

(ii) median $= 3\,(\sqrt{2} - 1) = 1.2426$ [1]
mean $= 3/(2 - 1) = 3$ [1]

So this distribution has positive skew (mean > median) [1]
This is common for a Pareto distribution (or any relevant comment) [1]

[Max 3]

[TOTAL 6]

> This straightforward question was well answered by most candidates, although a few erroneously used the formula for the coefficient of skewness, which is not applicable in this case.

Q2 (i) Mean = $\frac{1}{4}\lambda_1 + \frac{3}{4}\lambda_2 = 112.5$ [1]

$E(X^2) = \frac{1}{4}(\lambda_1{}^2 + \lambda_1{}^2) + \frac{3}{4}(\lambda_2{}^2 + \lambda_2{}^2) = 25350$ [1]

so variance = 25350 – 112.5^2 = 12,694 [1]

[Total 3]

(ii) For an exponential, mean = standard deviation and they are pretty close, so yes this is a good approximation. [2]

Alternatively

In general, the sum of two exponentials is not exponential, so this is not a good approximation. [2]

[Max 2]

[TOTAL 5]

> This question was relatively poorly answered, with only the better candidates being able to derive the variance. This is disappointing given how often this type of question occurs.

Q3 Let X be the number of correct predictions, so $X \sim \text{Bin}(10, p)$ [1]

$P(p = 1/6 \mid X = 4) = P(p = 1/6 \text{ and } X = 4) / P(X = 4)$
$= P(X = 4 \mid p = 1/6) * P(p = 1/6) / P(X = 4)$ [1]

$P(X = 4 \mid p = 1/6) = C_4^{10}\,(1/6)^4\,(5/6)^6 = 0.0542\ 659 \ldots$ [1]

$P(X = 4 \mid p = 2/6) = C_4^{10}\,(2/6)^4\,(4/6)^6 = 0.227\ 607\ 580\ldots$ [1]

$P(X = 4) = P(p = 1/6) * P(X = 4|p = 1/6) + P(p = 2/6) * P(X = 4|p = 2/6)$
$= 2/3 * 0.054\ 265\ 9 \ldots + 1/3 * 0.227\ 607\ 58$
$= 0.112\ 046 \ldots$ [1]

So posterior prob = 0.0542659 * (2/3)/0.112046 = 0.323 [1]

[TOTAL 6]

Candidates who were well prepared on Bayes' Theorem and previous exam questions on this topic did very well here, although a number of candidates struggled.

Q4 (i) Algorithm

(1) Simulate one U from $U(0,1)$ [1]

(2) If $0 < U \leq \frac{1}{3}, Y = 1; \frac{1}{3} < U \leq \frac{2}{3}, Y = 2;$ else $Y = 3$ [1]

[Total 2]

(ii) Similarly for generating samples from X

Algorithm

Simulate one U from $U(0,1)$ [1]

If $0 \leq U < \frac{1}{2}$, X=1 ; if $\frac{1}{2} \leq U < \frac{5}{6}$, $X = 2$, else $X = 3$ [1]

[Total 2]

(iii) The rejection here applied in the same way as in the continuous case we need to calculate again

$$M = \max\left\{\frac{\frac{1}{2}}{\frac{1}{3}}, \frac{\frac{1}{3}}{\frac{1}{3}}, \frac{\frac{1}{6}}{\frac{1}{3}}\right\} = \max\left\{\frac{3}{2}, 1, \frac{3}{6}\right\} = \frac{3}{2}$$ [1]

Hence $\frac{f(X)}{M\,g(x)} = \left\{1, \frac{2}{3}, \frac{1}{3}\right\}$ if $X = 1, 2, 3$ respectively. [1]

Algorithm

1 – Simulate Y as in (i). [1]
2 – Sample U from $U(0,1)$ and then take.

If $Y = 1$ $X = Y$ [½]

$U < 2/3$ and if $Y = 2$ take $X = Y$ [½]

If $U < 1/3$ and if $Y = 3$ take $X = Y$ [½]

Otherwise start again. [½]

The average number of samples from Y needed for a single sample from X is of the order $M = 3/2$, i.e. 1.5 samples on average. [1]

[Total 6]

[TOTAL 10]

> Most candidates scored well on parts (i) and (ii), although only the better prepared candidates were able to apply the acceptance-rejection method to part (iii).

Q5 (i) There is normally a delay between incidents leading to claim and the insurance pay out [1]

Insurance companies need to estimate future claims for their reserve [1]

It makes sense to use historical data to infer future patterns of claims [1]

[Max 2]

(ii) Cumulate claims

Policy Year	*0*	*1*	*2*	*3*
2011	4,657	8,097	9,028	9,600
2012	6,089	11,364	12,745	13,553
2013	5,623	10,422		12,399
2014	7,224			15,690

DF 2,3 = 9600/9028 = 1.063 358 [1]

DF 1,2 = (9028 + 12745) / (8097 + 11364) = 1.118 802 [1]

DF 0,1 = (8097 + 11364 + 10422) / (4657 + 6089 + 5623) = 1.825 585 [1]

Find expected claims 12,745 * 1.063 358 = 13,553 [1]

10,422 * 1.118 802 * 1.063358 = 12,399 [1]

7,224 * 1.825 585 * 1.118802 * 1.063358 = 15,690 [1]

So claim reserve =
(13,553 – 12,745) + (12,399 – 10,422) + (15,690 – 7,224) = 11,250 (4sf) [1]

Assuming that claims incurred are equal to claims paid

[Total 7]

[TOTAL 9]

> This straightforward question on chain ladders was very well answered.

Q6 (i) There are two possible outcomes of the observation and two possible choices hence $2 \times 2 = 4$ decision functions.

Decision Fn/Stock	*Outperforms*	*Underperforms*
d1	Long	Long
d2	Long	Short
d3	Short	Long
d4	Short	Short

[½ for each row]
[Max 2]

(ii) If good and invest make 100% of €1m = €1m
If bad and invest lose 75% of €1m = –€0.75m [1]

R(d1|Good) = 1m
R(d1|Bad) = –0.75m [1]

R(d2|Good) = 1m * 60% outperformance – 0.5m * 40% = 0.4m
R(d2|Bad) = –.75m * 40% + 0.5m * 60% = 0m [1½]

R(d3|Good) = –0.5m * 60% + 1m * 40% = 0.1m
R(d3|Bad) = 0.5m * 40% –.75m * 60% = –0.25m [1½]

R(d4|Good) = –0.5m
R(d4|Bad) = 0.5m [1]
[Total 6]

(iii) We need to determine the expectation of each Risk function
D2 dominates D3 [½]

E(R(d1)) = 1/3 * 1m – 2/3 * .75m = –1/6m
E(R(d2)) = 1/3 * 2/5m = 2/15m
E(R(d4)) = 1/3 * – 1/2m + 2/3 * 1/2m = 1/6m [1½]

So d4 is the optimal decision function under the Bayes criterion. [1]
[Total 3]
[TOTAL 11]

Apart from question 10, this was the most challenging question on the paper. Candidates who were able to identify the decision functions and set up the problem correctly generally did well, but many candidates struggled to formulate their answers.

Q7 (i) $\Psi(U) = P(U(t) < 0)$, for some $t, 0 < t < \infty$ [1]

$\Psi(U,t) = P(U(\tau) < 0)$, for some $\tau, 0 < \tau \le t$ [1]

$\Psi(U,1) = P(U(t) < 0)$, for some $t, 0 < t \le 1$ [1]

[Total 3]

(ii) λ, μ, σ^2, θ and initial surplus U [½ each – Max 2]

(iii) higher λ increases $\Psi(U, t)$ as the process is faster – claims and premiums come in quicker

higher μ increases $\Psi(U, t)$ as claims amounts are larger, relative to the surplus held

higher θ reduces $\Psi(U, t)$ as premiums increase at a quicker rate, so more of a buffer

higher U reduces $\Psi(U, t)$ as more of a buffer to withstand claims

higher σ^2 will typically increase $\Psi(U, t)$, assuming that expected premiums are higher than expected claims, since the likelihood of more extreme claims increase, [but may reduce $\Psi(U, t)$ if expected claims are higher than expected premiums.]

[1 each – Max 4]

(iv) Advantage – easier to measure, more useful for reporting [1]

Disadvantage – less information, artificial, can miss time when ruin occurs [1]

[Total 2]

[TOTAL 11]

This straightforward question on Ruin Theory was very well answered by most candidates.

Q8 (i)

$$I = \int_a^b \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(\ln x - \mu)^2} dx$$

Put y ln$x$, so $dy = \frac{1}{x}dx$ and $dx = e^y dy$ [1]

$$I = \int_{\ln a}^{\ln b} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y-\mu)^2} e^y dy$$ [½]

$$= \int_{\ln a}^{\ln b} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(y^2 - 2\mu y + \mu^2 - 2y\sigma^2)\right\} dy$$ [½]

$$= \int_{\ln a}^{\ln b} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}((y-\mu-\sigma^2)^2 - 2\mu\sigma^2 - \sigma^4)\right\} dy$$ [1]

$$= e^{\mu+½\sigma^2} \int_{\ln a}^{\ln b} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{1}{2\sigma^2}(y-\mu-\sigma^2)^2} dy$$ [½]

$$= e^{\mu+½\sigma^2} \left(\Phi\left(\frac{\ln b-\mu-\sigma^2}{\sigma}\right) - \Phi\left(\frac{\ln a-\mu-\sigma^2}{\sigma}\right)\right)$$ [½]

[Total 4]

(ii) $e^{(\mu+\sigma^2/2)} = 300$

$e^{(2\mu+\sigma^2)}(e^{\sigma^2}-1) = 400^2$

$(e^{\sigma^2}-1) = \frac{400^2}{300^2} \rightarrow \sigma^2 = \ln\left(\frac{25}{9}\right)$ [1]

$\mu = \ln 300 - \sigma^2/2 = \ln 180$ [1]

Average claim payable is

$E(X|X<500) + 500 * P(X \geq 500)$ [1]

$E(X|X<500) = 300 * \Phi\left(\frac{\ln 500 - \mu - \sigma^2}{\sigma}\right) = 300 * \Phi(0) = 150$ [1]

$P(X \geq 500) = 1 - \Phi\left(\frac{\ln 500 - \mu}{\sigma}\right) = 0.1561$ [1]

So average claim is 150 + 500 * 0.1561 = 228.0 (4sf) [1]
[Total 6]

(iii) The insurance company's expected claims would increase by less than 10%, [½]
since the chances of high claims has increased due to the standard deviation remaining the same, hence the reinsurer will pick up a greater share of the claims. [1½]
[Total 2]
[TOTAL 12]

Candidates who had learned the bookwork underlying part (i) were able to score well here. Most candidates did well on parts (ii) and (iii).

Q9 (i) The lag polynomial here is $1 - 0.6L - 0.16L^2 = (1 - 0.8L)(1 + 0.2L)$ with roots 1.25 and –5 therefore it is stationary. [2]

Hence an ARMA(2,0) process. [1]
[Total 3]

(ii) From the stationarity condition then

$$E(Y_t) = \mu = 1 + 0.6\mu + 0.16\mu + 0$$ [1]

$$\mu = \frac{1}{1-0.76} = \frac{1}{0.24}$$ [1]
[Total 2]

(iii) From the Yule-Walker equations for autocorrelation function values and for lags 1, 2, 3, … we have that $\rho_k = 0.6\rho_{k-1} + 0.16\rho_{k-2}$. [1]

In particular, for $k = 1$,

$$\rho_1 = 0.6\rho_0 + 0.16\rho_1 = 0.6 + 0.16\rho_1$$

or $$\rho_1 = \frac{0.6}{1-0.16} = \frac{0.6}{0.84} = 0.7143 = 5/7$$ [1]

For $k = 2$ we have that

$$\rho_2 = 0.6\rho_1 + 0.16\rho_0 = 0.16\rho_1 + 0.6 = \frac{0.6^2}{0.84} + 0.16$$
$$= 0.5886 = 103/175.$$ [1]

For $k = 3$

$$\rho_3 = 0.6\rho_2 + 0.16\rho_1 = 0.6 * 0.5885714 + 0.16 * 0.7142857$$
$$= 0.4674 = 409/875$$ [1]

For $k = 4$

$$\rho_4 = 0.6\rho_3 + 0.16\rho_2 = 0.6 * 0.4674286 + 0.16 * 0.5884714$$
$$= 0.3746 = 1639/4375$$ [1]

For the partial autocorrelation function we have that

$$\psi_1 = \rho_1 = 0.7143 = 5/7$$ [½]

$$\psi_2 = \frac{\rho_2 - \rho_1^2}{1-\rho_1^2} = 0.16 = 4/25 \qquad [1]$$

and $\psi_3 = \psi_4 = 0$ since Y_t is AR(2). [½]

[Total 7]

[TOTAL 12]

This straightforward question on time series was well answered by many candidates.

Q10 (i) The general form for the

$$f(x) = \exp\left[\frac{x\theta - b(\theta)}{a(\varphi)} + c(x,\varphi)\right] \qquad [1]$$

where *a, b*, *c*, are functions and θ and φ are called the natural and scale parameters respectively.

Since

$$1 = \int f(x)dx = \int \exp\left[\frac{x\theta - b(\theta)}{a(\varphi)} + c(x,\varphi)\right]dx$$

then differentiating with respect to θ we have that

$$\partial \frac{\int \exp\left[\frac{x\theta - b(\theta)}{a(\varphi)} + c(x,\varphi)\right]dx}{\partial\theta} = \frac{1}{a(\varphi)}\int (x - b'(\theta))f(x)dx = 0. \qquad [2]$$

Hence $E(X) = \int xf(x)dx = \int b'(\theta)f(x)dx = b'(\theta).$ [1]

Similarly differentiating again w.r.t. θ both sides of $\int (x - b'(\theta))f(x)dx = 0$ we have

$$\int \left[\frac{1}{a(\varphi)}(x - b'(\theta))^2 - b''(\theta)\right] f(x)dx = 0 \qquad [1]$$

hence

$$\int (x - b'(\theta))^2 f(x)dx = \text{Var}(X) = a(\varphi)\int b''(\theta)f(x)dx = a(\varphi)b''(\theta). \qquad [1]$$

[Total 6]

(ii) Differentiating further w.r.t. θ both sides of this identity

$$\int \left[\frac{1}{a(\varphi)}(x - b'(\theta))^2 - b''(\theta) \right] f(x)dx = 0$$

we have

$$\frac{1}{a(\varphi)} \int \left[\frac{1}{a(\varphi)}(x - b'(\theta))^2 - b''(\theta) \right] (x - b'(\theta)) f(x)dx$$

$$+ \int \left[\frac{2}{a(\varphi)}(x - b'(\theta))\, b''(\theta) - b'''(\theta) \right] f(x)dx = 0 \qquad [1½]$$

Since

$$\int (x - b'(\theta)) f(x)dx = 0 = \int b''(\theta)(x - b'(\theta)) f(x)dx$$

$$\frac{1}{a(\varphi)^2} \int (x - b'(\theta))^3 f(x)dx = 0 + \int b'''(\theta) f(x)dx$$

$$= 0 + b'''(\theta) \qquad [1½]$$

Therefore

$$E(X - E(X))^3 = \int (x - b'(\theta))^3 f(x)dx = a(\varphi)^2\ b'''(\theta) \qquad [1]$$

[Total 4]

(iii) $$f(x) = \exp\left[\frac{x\theta - b(\theta)}{a(\varphi)} + c(x, \varphi) \right]$$

$$= \exp\left[\left(-\frac{x}{\mu} - \log\mu \right)\alpha + (\alpha - 1)\log x + \alpha\log\alpha - \log\Gamma(\alpha) \right] \qquad [1]$$

Hence

$$\theta = -\frac{1}{\mu} \qquad [½]$$

$$\varphi = \alpha \qquad [½]$$

$$a(\varphi) = \frac{1}{\varphi} = \frac{1}{\alpha} \qquad [½]$$

$$b(\theta) = -\log(-\theta) \qquad [1]$$

$c(x, \varphi) = (\varphi - 1)\log x + \varphi\log\varphi - \log\Gamma(\varphi)$ [½]

[Total 4]

(iv) $b(\theta) = -\log(-\theta)$ hence $b'(\theta) = -\dfrac{1}{\theta} = \mu$ and $b''(\theta) = \dfrac{1}{\theta^2} = \mu^2$ [1½]

$E(X - E(X))^2 = a(\varphi)\,b''(\theta) = \dfrac{\mu^2}{\alpha}$. [1]

Similarly $b'''(\theta) = \dfrac{-2}{\theta^3} = 2\mu^3$, hence $E\left(X - E\left(X\right)\right)^3 = a\left(\phi\right)^2 b'''\left(\theta\right) = \dfrac{2\mu^3}{\alpha^2}$. [1½]

[Total 4]

[TOTAL 18]

The hardest and most challenging question on the paper. Most candidates were able to score well on part (ii), but only the best prepared candidates scored well on the whole question. Full credit was given for alternative solutions, including the use of MGFs and CGFs.

END OF EXAMINERS' REPORT