EXAMINATION

7 October 2009 (am)

Subject CT6 — Statistical Methods
Core Technical

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*

2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*

3. *Mark allocations are shown in brackets.*

4. *Attempt all 10 questions, beginning your answer to each question on a separate sheet.*

5. *Candidates should show calculations where this is appropriate.*

Graph paper is not required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator from the approved list.

1 Consider the stationary autoregressive process of order 1 given by

$$Y_t = 2\alpha Y_{t-1} + Z_t, \qquad |\alpha| < 0.5$$

where Z_t denotes white noise with mean zero and variance σ^2.

Express Y_t in the form $Y_t = \sum_{j=0}^{\infty} a_j Z_{t-j}$ and hence or otherwise find an expression for the variance of Y_t in terms of α and σ. [4]

2 An insurance company has a portfolio of two-year policies. Aggregate annual claims from the portfolio follow an exponential distribution with mean 10 (independently from year to year). Annual premiums of 15 are payable at the start of each year. The insurer checks for ruin only at the end of each year. The insurer starts with no capital. Calculate the probability that the insurer is not ruined by the end of the second year. [5]

3 The loss function under a decision problem is given by:

	θ_1	θ_2	θ_3
d_1	10	15	5
d_2	8	20	15
d_3	12	15	10
d_4	5	23	8

where d_1, d_2, d_3 and d_4 are the possible decisions and θ_1, θ_2 and θ_3 are the possible states of nature.

(i) State which decision can be discounted immediately. [1]

(ii) Determine the minimax solution to the problem. [2]

(iii) Determine the Bayes criterion solution to the problem given that $P(\theta_1) = 0.4$, $P(\theta_2) = 0.25$ and $P(\theta_3) = 0.35$. [2]

[Total 5]

4 A portfolio consists of k independent travel insurance policies. Each policy covers the policyholder's trips over one year. For policy i, the number of claims in the jth month of the covered year, Y_{ij}, is assumed to have a distribution given by

$$P(Y_{ij} = y) = \theta_{ij}(1-\theta_{ij})^y \quad \text{for } y = 0,1,2,\ldots$$

where θ_{ij} are unknown constants between 0 and 1.

(i) Write down the likelihood function and obtain the maximum likelihood estimate for the parameters θ_{ij}. [3]

(ii) Show that $P(Y_{ij} = y)$ can be written in exponential family form and suggest its natural parameter. [2]

(iii) Suppose that θ_{ij} depends on the temperature x_j recorded in the jth month. Explain why it is not appropriate to set $\theta_{ij} = \alpha + \beta x_j$. Suggest another relationship between θ_{ij} and $\alpha + \beta x_j$ that might be used. [3]

[Total 8]

5 The following claim amounts are believed to come from a lognormal distribution with unknown parameters μ and σ^2:

50, 87, 103, 119, 126, 154, 183, 203

Estimate the parameters μ and σ^2 using:

(i) the method of moments; [5]

(ii) the method of percentiles, using the upper and lower quartiles. [5]

[Total 10]

6 The following data is observed from n = 500 realisations from a time series:

$$\sum_{i=1}^{n} x_i = 13153.32 \ , \ \sum_{i=1}^{n} (x_i - \overline{x})^2 = 3153.67 \ \text{and} \ \sum_{i=1}^{n-1} (x_i - \overline{x})(x_{i+1} - \overline{x}) = 2176.03 \, .$$

(i) Estimate, using the data above, the parameters μ, a_1 and σ from the model

$$X_t - \mu = a_1(X_{t-1} - \mu) + \varepsilon_t$$

where ε_t is a white noise process with variance σ^2. [7]

(ii) After fitting the model with the parameters found in (i), it was calculated that the number of turning points of the residuals series $\hat{\varepsilon}_t$ is 280.

Perform a statistical test to check whether there is evidence that $\hat{\varepsilon}_t$ is not generated from a white noise process. [3]
[Total 10]

7 The transition rules for moving between the three levels 0%, 35% and 50% of a No Claims Discount system are as follows:

If no claim is made in a year, the policyholder moves to the next higher level of discount, or remains at the 50% level. When at the 0% or 35% level, the policyholder moves to (or remains at) the 0% level when one or more claims is made during the year. When at the 50% level of discount, the policyholder moves to the 35% level if exactly one claim is made during the year, or moves to the 0% level if two or more claims are made during the year.

It is assumed that the number of claims X made each year has a geometric distribution with parameter q such that

$$P(X = x) = q^x(1-q) \, , \ x = 0,1,2,\ldots$$

The full premium is 350.

(i) (a) Write down the transition matrix.

(b) Verify that the equilibrium distribution (in increasing order of discount) is of the form:

$$(kq^2(2-q), kq(1-q), k(1-q)^2)$$

for some constant k. Express k in terms of q. [8]

(ii) The value of the expected premium in the stationary state paid by "low risk" policyholders (with $q = 0.05$) is 178.51.

(a) Calculate the corresponding figure paid by "high risk" policyholders (with $q = 0.1$).

(b) Comment on the effectiveness of the No Claims Discount system.

[4]
[Total 12]

8 The cumulative incurred claims for an insurance company for the last four accident years are given in the following table:

	Development year			
Accident year	*0*	*1*	*2*	*3*
2005	96	136	140	168
2006	100	156	160	
2007	120	130		
2008	136			

It can be assumed that claims are fully run off after three years. The premiums received for each year from 2005 to 2008 are 175, 181, 190 and 196 respectively.

Calculate the reserve at the end of year 2008 using:

(a) The basic chain ladder method.

(b) The Bornhuetter-Ferguson method.

[12]

9 A certain proportion p of electrical gadgets produced by a factory is defective. Prior beliefs about p are represented by a Beta distribution with parameters α and β. A sample of n gadgets is inspected, and k are found to be defective.

(i) Explain what is meant by a conjugate prior distribution. [1]

(ii) Derive the posterior distribution for beliefs about p. [3]

(iii) Show that if $X \sim \text{Beta}(\alpha, \beta)$ with $\alpha > 1$ then $E\left(\frac{1}{X}\right) = \frac{\alpha+\beta-1}{\alpha-1}$. [3]

(iv) It is required to make an estimate d of p. The loss function is given by

$$L(d,p) = \frac{(d-p)^2}{p}.$$

Determine the Bayes estimate d^* of p. [4]

(v) Determine a parameter Z such that d^* can be written as

$$d^* = Z \times \frac{k}{n} + (1-Z) \times \frac{1}{\mu}$$

where μ is the prior expectation of $1/p$. [2]

(vi) Under quadratic loss, the Bayes estimate would have been $\frac{\alpha+k}{\alpha+\beta+n}$.
Comment on the difference in the two Bayes' estimates in the specific case where $\alpha = \beta = 3$, $k = 2$ and $n = 10$. [2]

[Total 15]

10 The total number of claims N on a portfolio of insurance policies has a Poisson distribution with mean λ. Individual claim amounts are independent of N and each other, and follow a distribution X with mean μ and variance σ^2. The total aggregate claims in the year is denoted by S. The random variable S therefore has a compound Poisson distribution.

(i) Derive an expression for the moment generating function of S in terms of the moment generating function of X. [4]

(ii) Derive expressions for the mean and variance of S in terms of λ, μ and σ. [6]

For a particular type of policy, individual losses are exponentially distributed with mean 100. For losses above 200 the insurer incurs an additional expense of 50 per claim.

(iii) Calculate the mean and variance of S for a portfolio of such policies with $\lambda = 500$. [9]

[Total 19]

END OF PAPER