EXAMINATION

13 April 2005 (am)

Subject CT4 — Models
Core Technical

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*
2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*
3. *Mark allocations are shown in brackets.*
4. *Attempt all 13 questions, beginning your answer to each question on a separate sheet.*
5. *Candidates should show calculations where this is appropriate.*

Graph paper is not required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator.

103 Questions

A1 (i) Define each of the following examples of a stochastic process

(a) a symmetric simple random walk
(b) a compound Poisson process

[2]

(ii) For each of the processes in (i), classify it as a stochastic process according to its state space and the time that it operates on. [2]

[Total 4]

A2 You have been commissioned to develop a model to project the assets and liabilities of an insurer after one year. This has been requested following a change in the regulatory capital requirement. Sufficient capital must now be held such that there is less than a 0.5% chance of liabilities exceeding assets after one year.

The company does not have any existing stochastic models, but estimates have been made in the planning process of "worst case" scenarios.

Set out the steps you would take in the development of the model. [6]

A3 Let Y_1, Y_3, Y_5, …, be a sequence of independent and identically distributed random variables with

$$P\left(Y_{2k+1}=1\right)=P\left(Y_{2k+1}=-1\right)=\frac{1}{2}, \qquad k=0,\ 1,\ 2,...$$

and define $Y_{2k}=Y_{2k+1}/Y_{2k-1}$ for k = 1, 2, ….

(i) Show that $\{Y_k : k=1,2,...\}$ is a sequence of independent and identically distributed random variables.

Hint: You may use the fact that, if X, Y are two variables that take only two values and $E\left(XY\right)=E\left(X\right)E(Y)$, then X, Y are independent. [4]

(ii) Explain whether or not $\{Y_k : k=1,2,...\}$ constitutes a Markov chain. [1]

(iii) (a) State the transition probabilities $p_{ij}(n)=P\left(Y_{m+n}=j\,|\,Y_m=i\right)$ of the sequence $\{Y_k : k=1,2,...\}$.

(b) Hence show that these probabilities do not depend on the current state and that they satisfy the Chapman-Kolmogorov equations.

[3]

[Total 8]

A4 Marital status is considered using the following time-homogeneous, continuous time Markov jump process:

- the transition rate from unmarried to married is 0.1 per annum
- the divorce rate is equivalent to a transition rate of 0.05 per annum
- the mortality rate for any individual is equivalent to a transition rate of 0.025 per annum, independent of marital status

The state space of the process consists of five states: Never Married (NM), Married (M), Widowed (W), Divorced (DIV) and Dead (D).

P_x is the probability that a person currently in state x, and who has never previously been widowed, will die without ever being widowed.

(i) Construct a transition diagram between the five states. [2]

(ii) Show, by general reasoning or otherwise, that P_{NM} equals P_{DIV}. [1]

(iii) Demonstrate that:

$$P_{NM} = \frac{1}{5} + \frac{4}{5} \times P_M$$

$$P_M = \frac{1}{4} + \frac{1}{2} \times P_{DIV}$$

[2]

(iv) Calculate the probability of never being widowed if currently in state NM. [2]

(v) Suggest two ways in which the model could be made more realistic. [1]

[Total 8]

A5 A No-Claims Discount system operated by a motor insurer has the following four levels:

Level 1: 0% discount
Level 2: 25% discount
Level 3: 40% discount
Level 4: 60% discount

The rules for moving between these levels are as follows:

- Following a year with no claims, move to the next higher level, or remain at level 4.
- Following a year with one claim, move to the next lower level, or remain at level 1.
- Following a year with two or more claims, move back two levels, or move to level 1 (from level 2) or remain at level 1.

For a given policyholder the probability of no claims in a given year is 0.85 and the probability of making one claim is 0.12.

$X(t)$ denotes the level of the policyholder in year t.

(i) (a) Explain why $X(t)$ is a Markov chain.
(b) Write down the transition matrix of this chain. [2]

(ii) Calculate the probability that a policyholder who is currently at level 2 will be at level 2 after:

(a) one year
(b) two years
(c) three years

[3]

(iii) Explain whether the chain is irreducible and/or aperiodic. [2]

(iv) Calculate the long-run probability that a policyholder is in discount level 2. [5]

[Total 12]

A6 An insurance policy covers the repair of a washing machine, and is subject to a maximum of 3 claims over the year of coverage.

The probability of the machine breaking down has been estimated to follow an exponential distribution with the following annualised frequencies, λ:

	1/10	If the machine has not suffered any previous breakdown.
$\lambda =$ }	1/5	If the machine has broken down once previously.
	1/4	If the machine has broken down on two or more occasions.

As soon as a breakdown occurs an engineer is despatched. It can be assumed that the repair is made immediately, and that it is always possible to repair the machine.

The washing machine has never broken down at the start of the year (time $t = 0$).

$P_i(t)$ is the probability that the machine has suffered i breakdowns by time t.

(i) Draw a transition diagram for the process defined by the number of breakdowns occurring up to time t. [1]

(ii) Write down the Kolmogorov equations obeyed by $P_0'(t), P_1'(t)$ and $P_2'(t)$. [2]

(iii) (a) Derive an expression for $P_0(t)$ and

(b) demonstrate that $P_1(t) = e^{-\frac{t}{10}} - e^{-\frac{t}{5}}$. [3]

(iv) Derive an expression for $P_2(t)$. [3]

(v) Calculate the expected number of claims under the policy. [3]
[Total 12]

104 Questions

B1 (i) Write down the equation of the Cox proportional hazards model in which the hazard function depends on duration t and a vector of covariates $\boldsymbol{z}$. You should define all the other terms that you use. [2]

(ii) Explain why the Cox model is sometimes described as "semi-parametric". [1]
[Total 3]

B2 Show that if the force of mortality μ_{x+t} $(0 \le t \le 1)$ is given by

$$\mu_{x+t} = \frac{q_x}{1 - tq_x},$$

this implies that deaths between exact ages x and $x + 1$ are uniformly distributed. [4]

B3 An investigation of mortality over the whole age range produced crude estimates of q_x for exact ages x from 2 years to 93 years inclusive. The actual deaths at each age were compared with the number of deaths which would have been expected had the mortality of the lives in the investigation been the same as English Life Table 15 (ELT15). 53 of the deviations were positive and 39 were negative.

Test whether the underlying mortality of the lives in the investigation is represented by ELT15. [5]

B4 A life insurance company has investigated the recent mortality experience of its male term assurance policy holders by estimating the mortality rate at each age, q_x. It is proposed that the crude rates might be graduated by reference to a standard mortality table for male permanent assurance policy holders with forces of mortality $\mu^s_{x+\frac{1}{2}}$, so that the forces of mortality $\mathring{\mu}_{x+\frac{1}{2}}$ implied by the graduated rates $\mathring{q}_x$ are given by the function:

$$\mathring{\mu}_{x+\frac{1}{2}} = \mu^s_{x+\frac{1}{2}} + k,$$

where k is a constant.

(i) Describe how the suitability of the above function for graduating the crude rates could be investigated. [2]

(ii) (a) Explain how the constant k can be estimated by weighted least squares.

(b) Suggest suitable weights.

[4]

(iii) Explain how the smoothness of the graduated rates is achieved. [1]

[Total 7]

B5 A study of the mortality of 12 laboratory-bred insects was undertaken. The insects were observed from birth until either they died or the period of study ended, at which point those insects still alive were treated as censored.

The following table shows the Kaplan-Meier estimate of the survival function, based on data from the 12 insects.

t (weeks)	*S(t)*
$0 \le t < 1$	1.0000
$1 \le t < 3$	0.9167
$3 \le t < 6$	0.7130
$6 \le t$	0.4278

(i) Calculate the number of insects dying at durations 3 and 6 weeks. [6]

(ii) Calculate the number of insects whose history was censored. [1]

[Total 7]

B6 An investigation into mortality collects the following data:

θ_x = total number of policies under which death claims are made when the policyholder is aged x last birthday in each calendar year

$P_x(t)$ = number of in-force policies where the policyholder was aged x nearest birthday on 1 January in year t

(i) State the principle of correspondence. [1]

(ii) Obtain an expression, in terms of the $P_x(t)$, for the central exposed to risk, E_x^c, which corresponds to the claims data and which may be used to estimate the force of mortality in year t at each age x, μ_x. State any assumptions you make. [4]

(iii) Comment on the effect on the estimation of the fact that the θ_x relate to claims, rather than deaths, and the $P_x(t)$ relate to policies, not lives. [4]

[Total 9]

B7 An investigation took place into the mortality of pensioners. The investigation began on 1 January 2003 and ended on 1 January 2004. The table below gives the data collected in this investigation for 8 lives.

Date of birth	*Date of entry into observation*	*Date of exit from observation*	*Whether or not exit was due to death (1) or other reason (0)*
1 April 1932	1 January 2003	1 January 2004	0
1 October 1932	1 January 2003	1 January 2004	0
1 November 1932	1 March 2003	1 September 2003	1
1 January 1933	1 March 2003	1 June 2003	1
1 January 1933	1 June 2003	1 September 2003	0
1 March 1933	1 September 2003	1 January 2004	0
1 June 1933	1 January 2003	1 January 2004	0
1 October 1933	1 June 2003	1 January 2004	0

The force of mortality, μ_{70}, between exact ages 70 and 71 is assumed to be constant.

(i) (a) Estimate the constant force of mortality, μ_{70}, using a two-state model and the data for the 8 lives in the table.

(b) Hence or otherwise estimate q_{70}.

[7]

(ii) Show that the maximum likelihood estimate of the constant force, μ_{70}, using a Poisson model of mortality is the same as the estimate using the two-state model. [5]

(iii) Outline the differences between the two-state model and the Poisson model when used to estimate transition rates. [3]

[Total 15]

END OF PAPER