EXAMINATION

April 2005

Subject CT4 — Models (includes both 103 and 104 parts) Core Technical

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

M Flaherty
Chairman of the Board of Examiners

15 June 2005

EXAMINERS' COMMENTS

Comments on solutions presented to individual questions for this April 2005 paper are given below:

103 Part

Question A1 *This was reasonably well answered.*
Descriptive (rather than formulaic) answers to part (i) were given equal credit. Very few candidates correctly identified the state space for the compound Poisson process in part (ii).

Question A2 *This was reasonably well answered.*
Marks were lost by candidates who did not provide sufficient detail or did not provide enough distinct points. Some candidates attempted to define the model they would adopt, rather than the stages in the modelling process.

Question A3 *This was very poorly attempted by most candidates.*
Very few candidates provided any real attempt at part (i). The examiners were looking here for a demonstration of pairwise (not mutual) independence, and the hint should have made this clear.
In part (ii), most candidates wrongly stated that the sequence was Markov.
Many candidates did not attempt part (iii); this may be because of the failure to make any progress in part (i), although it should be noted that subsequent parts of the question did not depend on correctly answering part (i).

Question A4 *This was well answered overall.*
In part (i), some candidates did not allow for re-marriage from the divorced or widowed states, which then caused them problems in part (ii).
Candidates lost marks in part (iii) if they did not provide sufficient explanation of their steps.

Question A5 *This was very well answered, with the majority of candidates scoring highly.*

Question A6 *Overall this was not well answered, but the better candidates did score well.*
Many candidates produced good answers to part (i) to (iv). In part (iii), a number of candidates did not verify that the boundary conditions were satisfied.
Some candidates struggled with part (v) and a significant number did not attempt this part of the question.

104 Part

Question B1 *This was well answered overall.*
Most candidates answered part (i) well, but many then struggled to express clearly what was required in part (ii).

Question B2 *This was very poorly answered.*
Many candidates did not seem to know how to start this, with a significant number starting with the uniform distribution assumption and working backwards.

Question B3 *This was well answered overall. Many candidates included a continuity correction. This was not necessary, as there were 92 ages, but candidates who did so received full credit if they used it correctly.*

Question B4 *This was not well answered.*
In part (i) significant numbers of candidates talked about general goodness of fit tests. This did not receive credit, as it was the appropriateness of the linear form of the function that we were looking for, before doing the graduation. Goodness-of-fit tests come later, after the graduation has been done, and were not part of this question.
In parts (i) and (ii), many candidates considered the graduated rates rather than the crude rates, for example plotting $\overset{\circ}{m}_{x+\frac{1}{2}}$ *against* $\mu^{s}_{x+\frac{1}{2}}$ *and this was penalised.*

Question B5 *This was well answered.*
Some candidates assumed that there was no censoring until the end of the investigation. This led to a non-integer number of deaths, which should have indicated an error, but few of these candidates realised this.

Question B6 *Most candidates correctly answered part (i).*
As with similar questions in previous years, part (ii) was not well answered. Many candidates lost marks by not providing sufficient explanation of their working.
In part (iii), most candidates mentioned the "variance ratio" and gave the formula from the gold book, but many did not provide a good explanation of what this meant in practice.

Question B7 *This was reasonably well answered overall.*
In part (i), candidates were asked to "estimate", so some indication of how they reached their answer was required for full credit.

103 Part

A1 (i) (a) Let $Y_1, Y_2, \ldots, Y_j, \ldots$, be a sequence of independent and identically distributed random variables with

$$P\left(Y_j = 1\right) = P\left(Y_j = -1\right) = \frac{1}{2}$$

and define

$$X_n = \sum_{j=1}^{n} Y_j$$

Then $\left\{X_n\right\}_{n=1}^{\infty}$ constitutes a symmetric simple random walk.

(b) Let N_t be a Poisson process, $t \geq 0$ and let $Y_1, Y_2, \ldots, Y_j, \ldots$, be a sequence of i.i.d. random variables. Then a compound Poisson process is defined by

$$X_t = \sum_{j=1}^{N_t} Y_j, \qquad t \geq 0.$$

(ii) (a) A simple random walk operates on discrete time and has a discrete state space (the set of all integers, Z).

(b) A compound Poisson process operates on continuous time.

It has a discrete or continuous state space depending on whether the variables Y_j are discrete or continuous respectively.

A2

- Review the regulatory guidance.
- Define the scope of the model, for example which factors need to be modelled stochastically.
- Plan the development of the model, including how the model will be tested and validated.
- Consider alternative forms of model, and decide and document the chosen approach. Where appropriate, this may involve discussion with experts on the underlying stochastic processes.

- Collect any data required, for example historic losses or policy data.

- Choose parameters. For economic factors should be able to calibrate to market data. For other factors e.g. expenses, claim distributions need to discuss with staff.

- Existing "worst case" scenarios. Discuss with staff who made the estimates, especially to gauge views on the probability of events occurring.

- Decide on the software to be used for the model.

- Write the computer programs.

- Debug the program, for example by checking the model behaves as expected for simple, defined scenarios.

- Review the reasonableness of the output. May include:

  – median outcomes (how do these compare with business plans)
  – what probability is assigned to "worst case" scenarios

- Test the sensitivity of the model to small changes in parameters.

- Calculate the capital requirement.

- Communicate findings to management. Document.

Other suitable points were given credit, including:

- Validate data.
- Run model on historic data to compare model's predictions with previous observations.
- Review parameters that have greatest effect on outputs.
- Present range of capital requirements for differing parameter inputs.

A3 (i) It is clear that Y_{2k} can only take two values, ±1, with probabilities

$$P\left(Y_{2k}=1\right)=P\left(Y_{2k+1}=Y_{2k-1}=+1\right)+P\left(Y_{2k+1}=Y_{2k-1}=-1\right)=\frac{1}{2}$$

and

$$P\left(Y_{2k}=-1\right)=$$
$$P\left(Y_{2k+1}=+1,Y_{2k-1}=-1\right)+P\left(Y_{2k+1}=-1,Y_{2k-1}=+1\right)=\frac{1}{2}$$

so that they have the same distribution as Y_{2k+1}.

To show that Y_{2k}, Y_{2k+1} are independent, we observe first that

$E(Y_{2k}) = E(Y_{2k+1}) = 0.$

Next,

$$E\left(Y_{2k}Y_{2k+1}\right) =$$
$$\frac{1}{2}E\left(Y_{2k}Y_{2k+1} \mid Y_{2k-1} = 1\right) + \frac{1}{2}E\left(Y_{2k}Y_{2k+1} \mid Y_{2k-1} = -1\right)$$

But

$$E\left(Y_{2k}Y_{2k+1} \mid Y_{2k-1} = 1\right) = 1\times 1 + 0\times(-1) = 1,$$

and similarly $E\left(Y_{2k}Y_{2k+1} \mid Y_{2k-1} = -1\right) = -1,$ which yields that

$$E\left(Y_{2k}Y_{2k+1}\right) = \frac{1}{2}\times 1 + \frac{1}{2}\times(-1) = 0.$$

Since

$$E\left(Y_{2k}\right) = E\left(Y_{2k+1}\right) = E(Y_{2k}Y_{2k+1})$$

it now follows from the hint that Y_{2k}, Y_{2k+1} are independent.

For the proof to be complete, we need to show that Y_{2k}, Y_{2m} are also independent for all *k, m*. This is obvious from the statement for all *k, m* except when *m* = *k* + 1 or *m* = *k* - 1. For this case, we could either argue as above or simply state that it is obvious by symmetry.

(ii) The sequence $\{Y_k : k = 1,2,...\}$ is not Markov; for instance

$$P\left(Y_{2k+1} = -1 \mid \ Y_{2k} = 1\right) = \frac{1}{2}$$
but
$$P\left(Y_{2k+1} = -1 \mid Y_{2k} = 1,\ Y_{2k-1} = 1\right) = 0.$$

(iii) (a) Since the Y_k are pairwise independent, we see that for all *i, j, m, n,*

$$p_{ij}(n) = P\left(Y_{m+n} = j \mid \ Y_m = i\right) = \frac{1}{2}.$$

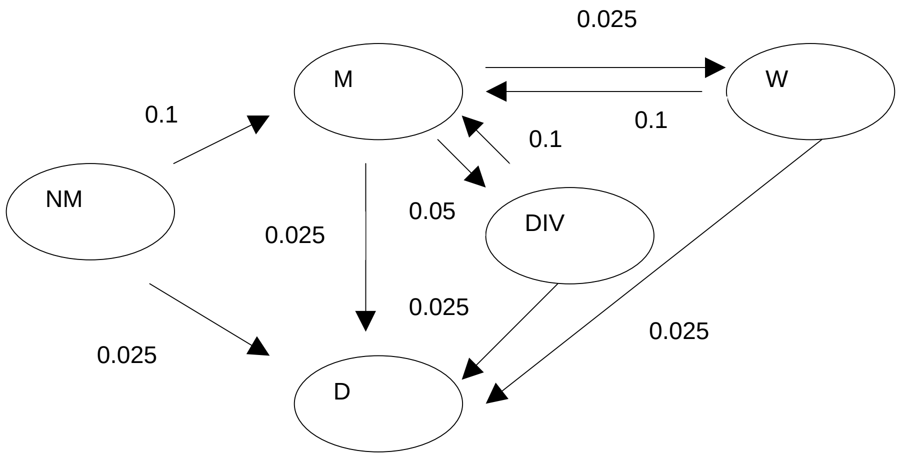
(b) The probabilities do not depend on the current state as they are all ½

Using the result in (a) we therefore see that

$$\sum_{k\in\{-1,1\}} p_{ik}(n)p_{kj}(r) = \frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}=\frac{1}{2}$$
$$= p_{ij}(n+r).$$

which shows that the Chapman – Kolmogorov equations are satisfied although $\{Y_k : k = 1,2,...\}$ is not Markov.

A4 (i)



(ii) The transitions out of the divorced state are to the same states, and with the same transition probabilities, as the transitions out of state NM. Therefore the probability of ever reaching state W is the same from both states.

Alternatively, this could be shown by producing the equation conditioning on the first move out of DIV, as in part (iii), and showing this is identical to that for P_{NM} .

(iii) Conditioning on the first move out of each state:

$$P_{NM} = \frac{0.025}{0.125}\times P_D + \frac{0.1}{0.125}\times P_M$$

$$P_M = \frac{0.025}{0.1}\times P_D + \frac{0.05}{0.1}\times P_{DIV} + \frac{0.025}{0.1}\times P_W$$

As $P_D = 1$ and $P_W = 0$, these give

$$P_{NM} = \frac{0.025}{0.125} + \frac{0.1}{0.125} \times P_M = \frac{1}{5} + \frac{4}{5} \times P_M$$

$$P_M = \frac{0.025}{0.1} + \frac{0.05}{0.1} \times P_{DIV} = \frac{1}{4} + \frac{1}{2} \times P_{DIV}$$

as required.

(iv) Using $P_{NM} = P_{DIV}$ in the above equations gives:

$$P_{NM} = \frac{1}{5} + \frac{4}{5} \times \left(\frac{1}{4} + \frac{1}{2} \times P_{NM} \right)$$

$$\Rightarrow \left(1 - \frac{2}{5}\right) \times P_{NM} = \frac{2}{5}$$

$$\Rightarrow P_{NM} = \frac{2}{3}$$

(v)

- Make mortality and marriage rates age dependent.
- Divorce rate dependent on duration of marriage.
- Divorce rate dependent on whether previously divorced.
- Make mortality rate marital status-dependent.

Other sensible suggestions received credit.

A5 (i)(a) It is clear that $X(t)$ is a Markov chain; knowing the present state, any additional information about the past is irrelevant for predicting the next transition.

(b) The transition matrix of the process is

$$P = \begin{pmatrix} 0.15 & 0.85 & 0 & 0 \\ 0.15 & 0 & 0.85 & 0 \\ 0.03 & 0.12 & 0 & 0.85 \\ 0 & 0.03 & 0.12 & 0.85 \end{pmatrix}$$

(ii)(a) For the one year transition, $p_{22} = 0$,
as can be seen from above (or is obvious from the statement).

(b) The possible transitions, and relevant probabilities are:

$$2 \to 1 \to 2: \quad 0.15 \times 0.85 = 0.1275$$
$$2 \to 3 \to 2: \quad 0.85 \times 0.12 = 0.102$$

The required probability is 0.1275 + 0.102 = 0.2295

Alternatively

The second order transition matrix is

$$P^2=\begin{pmatrix} 0.15^2+0.85\times0.15 & 0.85\times0.15 & 0.85^2 & 0 \\ 0.15^2+0.85\times0.03 & 0.85\times0.15+0.85\times0.12 & 0 & 0.85^2 \\ 0.03\times0.15+0.12\times0.15 & 0.85\times0.03\times2 & 0.85\times0.12\times2 & 0.85^2 \\ 0.03\times0.15+0.12\times0.03 & 0.12^2+0.85\times0.03 & 0.85\times0.03+0.85\times0.12 & 0.12\times0.85+0.85^2 \end{pmatrix}$$

$$=\begin{pmatrix} 0.15 & 0.1275 & 0.7225 & 0 \\ 0.048 & 0.2295 & 0 & 0.7225 \\ 0.0225 & 0.051 & 0.204 & 0.7225 \\ 0.0081 & 0.0399 & 0.1275 & 0.8245 \end{pmatrix}$$

Hence the required probability is 0.2295.

(c) The possible transitions, and relevant probabilities are:

$2\rightarrow1\rightarrow1\rightarrow2:\quad 0.15\times0.15\times0.85=0.019125$
$2\rightarrow3\rightarrow1\rightarrow2:\quad 0.85\times0.03\times0.85=0.021675$
$2\rightarrow3\rightarrow4\rightarrow2:\quad 0.85\times0.85\times0.03=0.021675$

The required probability is
0.019125 + 0.021675 + 0.021675 = 0.062475

Alternatively

The relevant entry from the third-order transition matrix equals

$0.15\times0.1275+0.85\times0.051=0.062475.$

(iii) The chain is irreducible as
any state is reachable from any other.

It is also aperiodic;
If currently at either state 1 or 4, it can remain there. This is not true for states 2 and 3, however these are also aperiodic states since the chain may return e.g. to state 2 after 2 or 3 transitions.

(iv) In matrix form, the equation we need to solve is $\pi P = \pi$, where π is the vector of equilibrium probabilities.

This reads

$$\begin{array}{lllll} 0.15\pi_1 + 0.15\pi_2 + 0.03\pi_3 & & & = \pi_1 & (1) \\ 0.85\pi_1 + & +0.12\pi_3 + 0.03\pi_4 & & = \pi_2 & (2) \\ \quad +0.85\pi_2 & & +0.12\pi_4 & = \pi_3 & (3) \\ & 0.85\pi_3 + 0.85\pi_4 & & = \pi_4 & (4) \end{array}$$

Discard the first of these equations and use also that $\sum_{i=1}^{4} \pi_i = 1$. Then, we obtain first from (4) that $0.85\pi_3 = 0.15\pi_4$ or, that $\pi_4 = 17\pi_3 / 3$

Substituting in (3) this gives

$$0.85\pi_2 + 0.12 \times \frac{17}{3}\pi_3 = \pi_3 \Rightarrow \pi_3 = 2.65625\pi_2$$

(2) now yields that

$$\begin{aligned} 0.85p_1 &= p_2 - 0.12p_3 - 0.03p_4 \\ &= \frac{1}{2.65625}p_3 - 0.12p_3 - 0.17p_3 = 0.0865p_3, \end{aligned}$$

so that finally we get $\pi_1 = 0.10173\pi_3$.

Using now that the probabilities must add up to one, we obtain

$\pi_1 + \pi_2 + \pi_3 + \pi_4 = (0.10173 + 0.3765 + 1 + 5.666)\pi_3 = 1,$

or that $\pi_3 = 0.13996$.

Solving back for the other variables we get that

$\pi_1 = 0.01424, \quad \pi_2 = 0.05269, \quad \pi_4 = 0.79311$

The long-run probability that the motorist is in discount level 2 is therefore 0.05269.

A6 (i)

$$\text{No Breakdowns} \xrightarrow{1/10} \text{One Breakdown} \xrightarrow{1/5} \text{Two Breakdowns} \xrightarrow{1/4} \text{Three Breakdowns} \cdots$$

(ii) $$P_0'(t) = -\frac{1}{10} \times P_0(t)$$

$$P_1'(t) = \frac{1}{10} \times P_0(t) - \frac{1}{5} \times P_1(t)$$

$$P_2'(t) = \frac{1}{5} \times P_1(t) - \frac{1}{4} \times P_2(t)$$

(iii)(a) Dividing the first equation by $P_0(t)$:

$$\frac{d}{dt}\left[\ln P_0(t)\right] = -\frac{1}{10}$$

Hence, using the boundary condition $P_0(0) = 1$

$$P_0(t) = e^{\frac{-t}{10}}$$

(b) Substitute into the second equation above to obtain

$$P_1'(t) = \frac{1}{10} e^{-\frac{t}{10}} - \frac{1}{5} * P_1(t)$$

Using an integrating factor $\mathrm{e}^{\frac{t}{5}}$, we get

$$\mathrm{e}^{\frac{t}{5}} \times \left[P_1'(t) + \frac{1}{5} P_1(t)\right] = \frac{1}{10} \times \mathrm{e}^{-\frac{t}{10}+\frac{t}{5}}$$

$$\Rightarrow \frac{d}{dt}\left[e^{\frac{t}{5}} \times P_1(t)\right] = \frac{1}{10} \times e^{\frac{t}{10}}$$

$$\Rightarrow e^{\frac{t}{5}} \times P_1(t) = e^{\frac{t}{10}} + const$$

$$\Rightarrow P_1(t) = e^{-\frac{t}{10}} + const \times e^{-\frac{t}{5}}$$

$$\Rightarrow P_1(t) = \exp^{-\frac{t}{10}} - \exp^{-\frac{t}{5}}$$

using boundary condition $P_1(0) = 0$

Alternatively

Differentiate the suggested solution and verify it obeys the second equation.

And that the boundary condition is satisfied.

(iv) Proceeding in a similar way with the equation for $P_2(t)$

$$P_2'(t) = \frac{1}{5}\exp^{-\frac{t}{10}} - \frac{1}{5}\exp^{-\frac{t}{5}} - \frac{1}{4} * P_2(t)$$

$$\frac{d}{dt}\left[\exp^{\frac{t}{4}} \times P_2(t)\right] = \frac{1}{5} \times (\exp^{\frac{3}{20}t} - \exp^{\frac{1}{20}t})$$

$$\exp^{\frac{t}{4}} \times P_2(t) = \frac{4}{3} \times \exp^{\frac{3}{20}t} - 4 \times \exp^{\frac{1}{20}t} + \frac{8}{3}$$

$$P_2(t) = \frac{4}{3}[\exp^{-\frac{t}{10}} - 3 \times \exp^{-\frac{t}{5}} + 2 \times \exp^{-\frac{t}{4}}]$$

(v) $$\text{Expected Claims} = 1 \times P_1(1) + 2 \times P_2(1) + 3 \times \sum_{i=3}^{\infty} P_i(1)$$

$$= P_1(1) + 2 \times P_2(1) + 3 \times \left(1 - P_0(1) - P_1(1) - P_2(1)\right)$$

$$P_0(1) = \exp^{-1/10} = 0.905$$

$$P_1(1) = \exp^{-1/10} - \exp^{-1/5} = 0.0861$$

$$P_2(1) = \frac{4}{3}[\exp^{-\frac{1}{10}} - 3 \times \exp^{-\frac{1}{5}} + 2 \times \exp^{-\frac{1}{4}}] = 0.00832896$$

Substituting these values gives:

Expected Claims = 0.1049

104 Part

B1 (i) If the hazard for life i is $\lambda(t; z_i)$, then

$$l\ (t; z_i) = l\ _0(t)\ \exp(b\, z_i^T)\ ,$$

where $\lambda_0(t)$ is the baseline hazard,

and β is a vector of regression parameters.

(ii) The model is semi-parametric because is possible to estimate β from the data without estimating the baseline hazard.

Therefore the baseline hazard can have any shape determined by the data.

B2 Since

$$_t p_x = \exp\left(-\int_0^t \mu_{x+s} ds \right),$$

$$_t q_x = 1 - {}_t p_x = 1 - \exp\left(-\int_0^t \mu_{x+s} ds \right).$$

Substituting for μ_{x+s} produces

$$_t q_x = 1 - \exp\left[-\int_0^t \frac{q_x ds}{1 - sq_x} \right]$$

Performing the integration we have

$$\begin{aligned} {}_t q_x &= 1 - \exp\left(-\left[-\log(1 - sq_x) \right]_0^t \right) \\ &= 1 - \exp\left(-\left[-\log(1 - tq_x) + \log 1 \right] \right) \\ &= 1 - \exp\left(-\left[-\log(1 - tq_x) \right] \right) \\ &= 1 - \exp\left(\log(1 - tq_x) \right) \\ &= 1 - (1 - tq_x) \\ &= tq_x. \end{aligned}$$

This is the assumption of a uniform distribution of deaths and implies that deaths between exact ages x and $x + 1$ are uniformly distributed.

B3 The null hypothesis is that the observed rates are a sample from a population in which English Life Table 15 represents the true rates.

If the null hypothesis is true, then the observed number of positive deviations, P, will be such that P ~ Binomial (92, ½).

We use the normal approximation to the Binomial distribution because we have > 20 ages

This means that, approximately, P ~ Normal (46, 23).

The z-score associated with the probability of getting 53 positive deviations if the null hypothesis is true is, therefore

$$\frac{53-46}{\sqrt{23}} = \frac{7}{4.79} = 1.46\,.$$

We use a two-tailed test, since both an excess of positive and an excess of negative deviations are of interest.

Using a 5 % significance level, we have -1.96 < 1.46 < +1.96.

(Alternatively, the p-value of the test statistic could be calculated.)

This means we have insufficient evidence to reject the null hypothesis.

B4 (i) The suitability of a linear relationship between $\mu^s_{x+\frac{1}{2}}$ and $\overset{\circ}{\mu}_{x+\frac{1}{2}}$ could be investigated by plotting $-\log(1-q_x)$ against $-\log(1-q^s_x)$ or by plotting $\mu_{x+\frac{1}{2}}$ against $\mu^s_{x+\frac{1}{2}}$ and looking for a linear relationship.

An approximately linear relationship will suffice.

If data are scarce, too close a fit is not to be expected, especially at extreme ages.

(ii) (a) We can work with either q_x^s or $\mu^s_{x+\frac{1}{2}}$.

The value of k which minimises either

$$\sum_x w_x (q_x - \overset{\circ}{q}_x)^2$$

or

$$\sum_x w_x \left(\mu_{x+\frac{1}{2}} - \overset{\circ}{\mu}_{x+\frac{1}{2}} \right)^2$$

should be found (note that the summations are over all relevant ages x)

At each age there will be a different sample size or exposed to risk, E_x. This will usually be largest at ages where many term assurances are sold (e.g. ages 25 to 50 years) and smaller at other ages.

(b) The estimation procedure should pay more attention to ages where there are lots of data. These ages should have a greater influence on the choice of k than other ages.

This implies weights $w_x \propto E_x$.
A suitable choice would be

$$w_x = \frac{1}{\text{var}\, q_x} \text{ or } w_x = \frac{1}{\text{var}\, \mu_{x+\frac{1}{2}}} \text{ or } w_x = E_x$$

(iii) The graduated forces of mortality are a linear function of the forces in the standard table.

Since the forces in the standard table should already be smooth, a linear function of them will also be smooth.

B5 (i) Consider the durations t_j at which events take place.
Let the number of deaths at duration t_j be d_j and the number of insects still at risk of death at duration t_j be n_j.

At $t_j = 1$, $S(t)$ falls from 1.0000 to 0.9167.

Since the Kaplan-Meier estimate of $S(t)$ is

$$S(t) = \prod_{t_j \le t} (1 - \lambda(t_j)),$$

we must have $0.9167 = 1 - \lambda(1)$,

so that $\lambda(1) = 0.0833$.

Since $\lambda(1) = \dfrac{d_1}{n_1}$, then we have $\dfrac{d_1}{n_1} = 0.0833$,

and, since all 12 insects are at risk of dying at t_j = 1, we must therefore have d_1 = 1 and n_1 = 12.

Similarly, at t_j = 3, we must have $0.7130 = 0.9167(1 - \lambda(3))$

so that $\lambda(3) = \dfrac{0.9167 - 0.7130}{0.9167} = 0.222 = \dfrac{d_3}{n_3}$.

Since we can have at most 11 insects in the risk set at t_j = 3, we must have d_3 = 2 and n_3 = 9.
Similarly, at t_j = 6, we must have $0.4278 = 0.7130(1 - \lambda(6))$,

so that $\lambda(6) = \dfrac{0.7130 - 0.4278}{0.7130} = 0.400 = \dfrac{d_6}{n_6}$.

Since we can have at most 7 insects in the risk set at t_j = 6, we must have d_6 = 2 and n_6 = 5.

Therefore 2 insects died at duration 3 weeks and 2 insects died at duration 6 weeks.

Alternatively

Some candidates worked back to produce a table in the usual format, as follows; this received full credit.

t	$S(t) = \prod(1 - \lambda_t)$	λ_t	n_t	d_t	c_t
0	1.0000	0	12	0	
1	0.9167	0.0833	12	1	2
3	0.7130	0.22	9	2	2
6	0.4278	0.4	5	<u>2</u>	<u>3</u>
				5	7

(ii) Summing up the number of deaths we have
total deaths = $d_1 + d_3 + d_6 = 1 + 2 + 2 = 5$.

Since we started with 12 insects, the remaining 7 insects' histories were right-censored.

B6 (i) The principle of correspondence states that a life alive at time t should be included in the exposure at age x at time t if and only if were that life to die immediately, he or she would be counted in the deaths data θ_x at age x.

(ii) $P_x(t)$ is the number of policies under observation aged x nearest birthday on 1 January in year t.

To correspond with the claims data, we wish to have policies classified by age last birthday.

Let the number of policies aged x last birthday on 1 January in year t be $P'_x(t)$. Then, assuming that birthdays are evenly distributed,

$$P'_x(t) = \frac{1}{2}\left[P_x(t) + P_{x+1}(t)\right].$$

The central exposed to risk is then given by

$$E_x^c = \int_0^1 P'_x(t)dt\,.$$

Using the trapezium approximation this is

$$E_x^c \approx \frac{1}{2}\left[P'_x(t) + P'_x(t+1)\right],$$

and, substituting for the $P'_x(t)$ in terms of $P_x(t)$ from the equation above produces

$$E_x^c \approx \frac{1}{2}\left[\frac{1}{2}\left[P_x(t) + P_{x+1}(t)\right] + \frac{1}{2}\left[P_x(t+1) + P_{x+1}(t+1)\right]\right].$$

(iii) The principle of correspondence still holds, because we are dealing with claims and policies: one policy can only lead to one claim.

However, because one life may have more than one policy it is possible that two distinct death claims are the result of the death of the same life.

Therefore claims are not independent, whereas deaths are.

The effect of this is to increase the variance of the number of claims (compared to the situation in which each life has one and only one policy) by the ratio

$$\frac{\sum_i i^2 \pi_i}{\sum_i i \pi_i},$$

where π_i is the proportion of the lives in the investigation owning i policies (i = 1, 2, 3, ...).

Typically the ratio will vary for each age x.

B7 (i)(a) The two-state estimate of μ_{70} is $\frac{d_{70}}{v_{70}}$, where v_{70} is the total time the members of the sample are under observation between exact ages 70 and 71 years.

$$v_{70} = \sum_i v_{70,i},$$

where $v_{70,i}$ is the duration that sample member i is under observation between exact ages 70 and 71 years.

For each sample member, $v_{70,i}$ = ENDDATE – STARTDATE

where ENDDATE is the earliest of the date at which the observation of that member ceases and the date of the member's 71st birthday,
and STARTDATE is the latest of the date at which observation of that member begins and the date of the member's 70th birthday.

The table below shows the computation of v_{70}.

i	*Date obs. begins*	*Date of 70th birthday*	*Date obs. ends*	*Date of 71st birthday*	$v_{70,i}$ *(years)*
1	1/1/2003	1/4/2002	1/1/2004	1/4/2003	0.25
2	1/1/2003	1/10/2002	1/1/2004	1/10/2003	0.75
3	1/3/2003	1/11/2002	1/9/2003	1/11/2003	0.5
4	1/3/2003	1/1/2003	1/6/2003	1/1/2004	0.25
5	1/6/2003	1/1/2003	1/9/2003	1/1/2004	0.25
6	1/9/2003	1/3/2003	1/1/2004	1/3/2004	0.3333
7	1/1/2003	1/6/2003	1/1/2004	1/6/2004	0.5833
8	1/6/2003	1/10/2003	1/1/2004	1/10/2004	0.25

Therefore $v_{70} = \sum_i v_{70,i}$ = 3.167.

We observed two deaths (members 3 and 4), so

$$\hat{\mu}_{70} = \frac{2}{3.167} = 0.6316 .$$

(b) $\hat{q}_{70} = 1 - \exp(-\hat{\mu}_{70})$

$$= 1 - \exp(-0.6316) = 1 - 0.5318 = 0.4682.$$

(ii) The contributions to the Poisson likelihood made by each member are proportional to the following

Member	
1	exp(-0.25 μ_{70})
2	exp(-0.75 μ_{70})
3	μ_{70} exp(-0.5 μ_{70})
4	μ_{70} exp(-0.25 μ_{70})
5	exp(-0.25 μ_{70})
6	exp(-0.3333 μ_{70})
7	exp(-0.5833 μ_{70})
8	exp(-0.25 μ_{70})

The total likelihood, *L*, is proportional to the product

$$L \propto [\exp(-3.167\mu_{70})](\mu_{70})^2 .$$

Then

$$\log L = -3.167\mu_{70} + 2\log\mu_{70}$$

so that

$$\frac{d\log L}{d\mu_{70}} = -3.167 + \frac{2}{\mu_{70}} .$$

Setting this equal to zero and solving for μ_{70} produces the maximum likelihood estimate,
which is 2/3.167 = 0.6316

Since $\frac{d^2\log L}{d\mu_{70}^{\ 2}} = -\frac{2}{\mu_{70}^{\ 2}}$, which is always negative, we definitely have a maximum.

This is the same as the estimate from the two-state model.

(iii) The Poisson model is not an exact model, since it allows for a non-zero probability of more than n deaths in a sample of size n.

The variance of the maximum likelihood estimator for the two-state model is only available asymptotically, whereas that for the Poisson model is available exactly in terms of the true μ.

The two-state model extends to processes with increments, whereas the Poisson model does not.

The Poisson model is a less satisfactory approximation to the multiple state model when transition rates are high.