INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

September 2016

Subject CT4 – Models
Core Technical

Introduction

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and using past papers as a revision aid and also those who have previously failed the subject.

The Examiners are charged by Council with examining the published syllabus. The Examiners have access to the Core Reading, which is designed to interpret the syllabus, and will generally base questions around it but are not required to examine the content of Core Reading specifically or exclusively.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report; other valid approaches are given appropriate credit. For essay-style questions, particularly the open-ended questions in the later subjects, the report may contain more points than the Examiners will expect from a solution that scores full marks.

The report is written based on the legislative and regulatory context pertaining to the date that the examination was set. Candidates should take into account the possibility that circumstances may have changed if using these reports for revision.

Luke Hatter
Chair of the Board of Examiners
December 2016

A. General comments on the *aims of this subject and how it is marked*

1. The aim of the Models subject is to provide a grounding in stochastic processes and survival models and their application.

2. Subject CT4 comprises five main sections:

   (1) a study of the properties of models in general, and their uses for actuaries, including advantages and disadvantages (and a comparison of alternative models of the same processes);

   (2) stochastic processes, especially Markov chains and Markov jump processes;

   (3) models of a random variable measuring future lifetime;

   (4) the calculation of exposed to risk and the application of the principle of correspondence;

   (5) the reasons why mortality (or other decremental) rates are graduated, and a range of statistical tests used both to compare a set of rates with a previous experience and to test the adherence of a graduated set of rates to the original data.

   Throughout the subject the emphasis is on estimation and the practical application of models. Theory is kept to the minimum required in order usefully to apply the models to real problems.

3. Different numerical answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may lose marks where excessive rounding has been used or where insufficient working is shown. Credit is given for valid solutions different from those shown below. Partial credit is also given to candidates submitting incomplete solutions with valid intermediate workings.

B. General comments on *student performance in this diet of the examination*

1. The performance of candidates in this diet was weaker than in previous sittings. The examination paper was considered to be of slightly less difficulty than the April 2016 paper, and a slightly higher Pass Mark was therefore used.

2. One or two questions (or parts of questions) on this examination paper presented simple applications in an unfamiliar way. A substantial number of candidates made little or no attempt at these questions. This suggests that they had learned standard applications by doing examples without understanding the concepts underlying them, and hence when faced with a test of these concepts which did not use one of the examples they had learned were unable to think through what was required.

> 3. Candidates' knowledge of bookwork in certain areas was less convincing than at previous sessions.
>
> 4. A disappointingly large number of candidates did not read the wording of the questions closely enough, and so lost marks on straightforward sections of the paper because they did not answer the question asked..

C. Pass Mark

The Pass Mark for this exam was 58.

Solutions

Q1

We can calculate the Maximum Likelihood Estimate (MLE) of the transition intensities directly using the two-state model, [½]

whereas the Binomial model requires additional assumptions. [½]

The variance of the Binomial estimator is greater than that of the estimate from the two-state model, though the difference is tiny unless the transition intensities are large. [1]

The MLE in the two-state model is consistent and unbiased, [½]

whereas the Binomial estimate is only consistent and unbiased if lives are observed for exactly one year, which is rarely the case. [½]

The two-state model is easily extended to encompass increments and additional decrements, whereas the Binomial model is not. [1]

The two-state model uses the exact times of the transitions, whereas the Binomial model only uses the number of transitions. [1]
[Max 3]

> Not all the points listed above were required for full credit. This was one of the bookwork questions on which performance was relatively weak. Many candidates were only able to make one or two of the points listed above.

Q2

A Markov Chain 1

Irreducible [½]
Aperiodic [½]

B Markov Chain 2

Irreducible [½]
Periodic with period 2 [½]

C Markov Chain 3

Reducible [½]
Aperiodic [½]
[Total 3]

> Many candidates did well on this question. The most common errors were to regard Markov Chain 1 as periodic (this is incorrect because return to any state is possible in 2 or 3 steps, and 2 and 3 have no common factor higher than 1), and to regard Markov Chain 2 as being periodic with period 4 rather than 2.

Q3

All three processes have a discrete state space. [1]

A Markov Chain and Markov Jump Chain both operate in discrete time but a Markov jump Process operates in continuous time. [1]

All have the Markov property which is [½]

EITHER that the future development of the process can be predicted from its present state alone, without reference to its past history.

OR that

$$P[X_t \in A \mid X_{s_1} = x_1,\ X_{s_2} = x_2, ...,\ X_{s_n} = x_n, X_s = x] = P[X_t \in A \mid X_s = x]$$

for all times $s_1 < s_2 < ... < s_n < s < t$, all states $x_1, x_2, ..., x_n, x$ in S and all subsets A of S. [½]

EITHER If a Markov Jump Process X is examined only at the times of its transitions, the resulting process is called the Jump Chain associated with X.

OR for a Jump Process X the Jump Chain X shows the states visited by X, taking an identical path through the state space. [½]

The Jump Chain obeys the Markov Property and behaves as a Markov Chain except when the Jump Chain encounters an absorbing state. From that time it makes no further transitions, implying that time stops for the Jump Chain. [1]

The Jump Chain associated with X takes the same path through the state space as X does. However questions about the times taken to visit a state are likely to have different answers for X and for the Jump Chain associated with X. [1]

The Markov Jump Chain and the Markov Chain are expressed in terms of probabilities whereas the Markov Jump Process is expressed in terms of rates. [½]

The Markov Chain can have loops in each state, the Markov Jump process cannot and the Markov Jump Chain only has loops on absorbing states. [½]
[Max 4]

Not all the above was required for full credit. Many candidates correctly identified the fact that all three processes operated in discrete state space, but a large proportion thought that the Markov Jump Chain was a continuous time process. A Markov Jump Chain is a Markov Chain in its own right, and hence operates in discrete time.

Q4

(i) The expected annual outgo is λZ. [½]

The premium is 50% more than this, hence $1.5\lambda Z$. [½]

(ii) Suppose the next claim happens at time T.

Then the company is unable to pay the claim if:

$S + 1.5\lambda ZT < Z$. [1]

This implies that

$$T < \frac{Z-S}{1.5\lambda Z} = \frac{1}{1.5\lambda}\left(1 - \frac{S}{Z}\right).$$ [½]

We are told claims arrive in accordance with a Poisson process so

$$\mathrm{P}(t \le T) = 1 - \exp(-\lambda T) \;\; = 1 - \exp\left(-\frac{1}{1.5}\left(1 - \frac{S}{Z}\right)\right), \text{ as required.} \qquad [1½]$$

[Total 4]

> This was an unfamiliar application of a Poisson process. Many candidates did not attempt this question. Of those that did, most only scored credit for part (i). Few candidates seemed to know how to attempt part (ii), especially the idea of examining the time of the next claim.

Q5

(i) We believe that mortality varies smoothly with age (and evidence from large experiences supports this belief). [½]

Therefore the crude estimates of mortality at any age contains information about mortality at adjacent ages and [½]
by smoothing the experience we can make use of data at adjacent ages to improve the estimate at each age. [1]

This reduces sampling (or random) errors. [½]

The mortality experience may be used in financial calculations. [½]

Irregularities, jumps and anomalies in financial quantities (such as premiums for life assurance contracts) are hard to justify to customers. [½]
[Max 3]

(ii) (a) **Female population of a large European country**

By parametric formula, [½]

because the experience is large
OR because the graduated rates may be used to form a new standard table for the country. [½]

(b) **Mortality of rhinoceroses in the safari parks of South Africa**

Graphical, [½]

because no suitable table is likely to exist and the experience is small. [½]

(c) **Members of a pension scheme of a large company**

With reference to a standard table, [½]

because there are many suitable tables in existence
OR it will provide help at high ages where data are scarce. [½]
[Total 6]

Answers to this question were often weak, especially part (i) which was standard bookwork. In part (ii) many candidates recommended graduation with reference to a standard table for experience (a). While this is possible, the use of a parametric formula is better in this case. It was common also for candidates to recommend using a parametric formula for experience (c). Again, while this is possible, making reference to one of the many standard tables available for pensioners would be better in this case. In part (ii), even where the most suitable method of graduation was not chosen, credit was given for sensible reasoning behind the method that was chosen.

Q6

(i) The calculations are shown in the table below.

t_j	N_j	d_j	c_j	d_j / N_j	$\sum (d_j / N_j)$
2	*20*	*0*	*1*	*0*	*0*
3	19	1	0	1/19	0.0526
5	18	1	0	1/18	0.1082
8	17	1	0	1/17	0.1670
9	16	1	0	1/16	0.2295
13	*15*	*0*	*2*	*0*	*0.2295*
14	13	1	0	1/13	0.3064
15	12	2	0	2/12	0.4731

[3½]

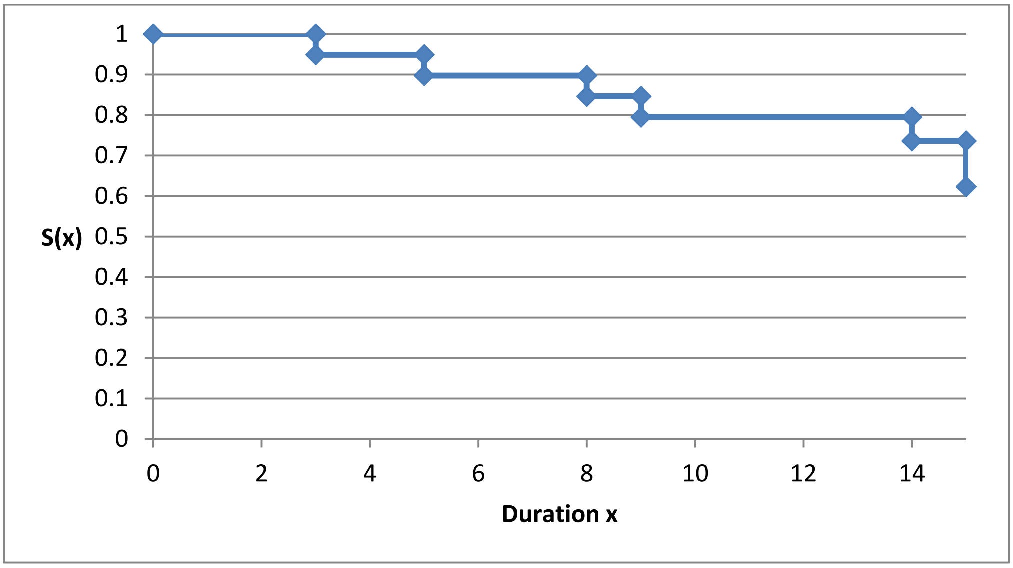
The Nelson-Aalen estimator of $S(x)$ is $\hat{S}(x) = \exp\left(-\sum_{t_j \le x} \frac{d_j}{N_j}\right)$. [½]

So we have

Range	$\hat{S}(x)$
$0 \le x < 3$	1.0000
$3 \le x < 5$	0.9487
$5 \le x < 8$	0.8974
$8 \le x < 9$	0.8462
$9 \le x < 14$	0.7949
$14 \le x < 15$	0.7361
$x = 15$	0.6231

[2]

(ii) A suitable sketch is shown below.



[2]
[Total 8]

Many candidates scored highly on this straightforward application of the Nelson-Aalen estimator of the survival function. Some candidates assumed that events happened at the beginning of each day rather than the end. These candidates could score full credit (the only change to the solution above is that the t_js for the events should be 2, 4, 7, 8, 13 and 14). A common error was to quote the estimated $S(t)$ for a duration more than 15 days. This is incorrect because we have no information about what happened after 15 days.

Q7

(i) The objectives of the modelling exercise [½]

The validity of the model for the purpose to which it is to be put [½]

The validity of the data to be used [½]

The validity of the assumptions used [½]

The possible errors associated with the model OR the fact that the parameters used are not a perfect representation of the real world situation being modelled [½]

The impact of correlations between the random variables (or input variables) that "drive" the model [½]

The extent of correlations between the various results produced from the model [½]

The current relevance of models written and used in the past [½]

The credibility of the data input. [½]

The credibility of the results output [½]

The dangers of spurious accuracy [½]

The costs of buying or constructing, and of running the model [½]

Ease of use and availability of suitable staff to use it [½]

The risk of the model being used incorrectly or with wrong inputs [½]

The ease with which the model and its results can be communicated [½]

Compliance with the relevant regulations [½]

The existence of clear documentation [½]

[Max 4]

(ii) **The objectives of the modelling exercise**

The validity of the model for the purpose to which it is to be put

The model is not hugely valid as it does not address the number of schools directly, for example by dividing the number of pupils by average school size or considering when existing schools may become obsolete, the presence of competition etc. [1]

The validity of the data to be used

The Central Statistical Office data will be fine, but that gained from the newspaper will be of limited validity as estimates of future migrants arriving may be heavily skewed by the political bias of the newspaper. Estimates of birth rates and migration rates are generally valid data for this exercise. [1]

The validity of the assumptions used
Straight line projection is dubious over 40 years, especially on immigration numbers. [1]

The possible errors associated with the model OR the fact that the parameters used are not a perfect representation of the real world situation being modelled
The total number of school children in 40 years' time is very susceptible to errors in the parameters, for example the difference between straight line projection following a baby boom will give a rapidly increasing number, whereas if the baby boom is over, the numbers may decline. [1]

The impact of correlations between the random variables that "drive" the model.
It is quite likely that the estimate of new arrivals and the children per household of new arrivals will be biased in the same direction, i.e. both overstated or understated. [1]

The extent of correlations between the various results produced from the model.
If you overestimate the number of children in the education system in, say 5 years' time, you will most likely overestimate the number of children in, say, 30 years' time as these latter will be the next generation, the children of those in the system in 5 years' time. [1]

The current relevance of models written and used in the past
The government/local authorities should have models which are still relevant even if they need parameters adjusting. [1]

The credibility of the data input
The data from the newspaper may be of doubtful credibility. It would be worth examining them in the light of past trends to see whether they fall within the range of past data. [1]

The credibility of the results output
This model will give a very crude answer which is pretty difficult to have much faith in. Again, it will be worth examining the output in the light of recently past trends to see whether they mark a break with the past. [1]

The dangers of spurious accuracy
There is no point calculating the number of children to many significant figures when the assumptions are so approximate and the size of individual schools so variable. [1]

The cost of buying or constructing, and of running the model
An advantage of the model is that it is very inexpensive. [1]

Ease of use and availability of suitable staff to use it
The model is very easy to use. [1]

The risk of the model being used incorrectly or with wrong inputs
This is low, as the model is so simple. [1]

The ease with which the model and its results can be communicated
Another advantage of the model is that it is very simple to communicate. [1]
Compliance with the relevant regulations
Regulations are unlikely to be applicable in this case. However changes in legislation concerning immigration might be an issue. [1]

The existence of clear documentation
It should be easy to produce clear documentation. [1]
[Max 6]
[Total 10]

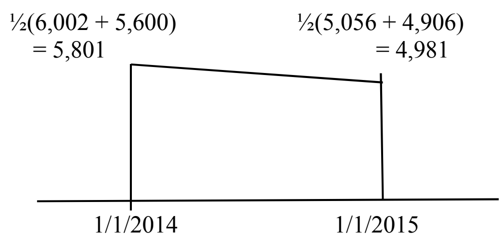
> Part (i) of this question was well answered, with many candidates scoring full marks. Only eight factors were required for full credit. The list above gives the complete range of factors that were awarded marks. Answers to part (ii) were more variable. The most able candidates produced thoughtful comments demonstrating engagement with the particular application described in the question, and a substantial minority of candidates scored close to full marks. Credit was given for sensible comments other than those listed above. On the other hand, some candidates wrote only general comments which added little to part (i).

Q8

(i) **Company *A***

Age 51 last = 0.5 * age 51 nearest + 0.5 * age 52 nearest. [½]

The Exposed-to-risk = 0.25 * (6,002 + 5,600 + 5,056 + 4,906) = 5,391. [1]



Company *B*

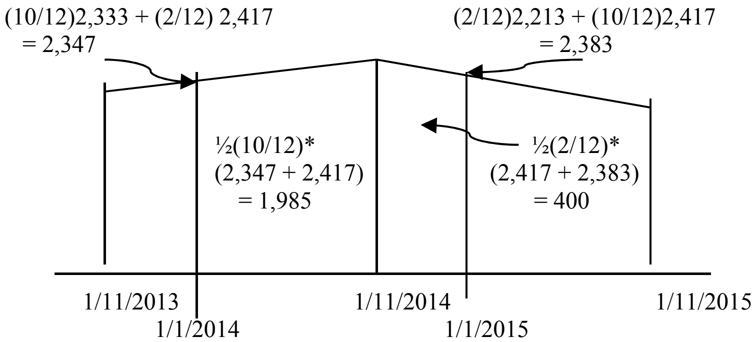
Population on 1/1/2014 is (2/12 * 2,417 + 10/12 * 2,333) = 2,347, [½]

so the contribution before 1/11/2014 is 0.5 * (2,347 + 2,417) * (10/12) = 1,985. [½]

The population on 1/1/2015 is (10/12 * 2,417 + 2/12 * 2,213) = 2,383, [½]

so the contribution after 1/11/2014 is 0.5 * (2,417 + 2,383) * 2/12 = 400. [½]

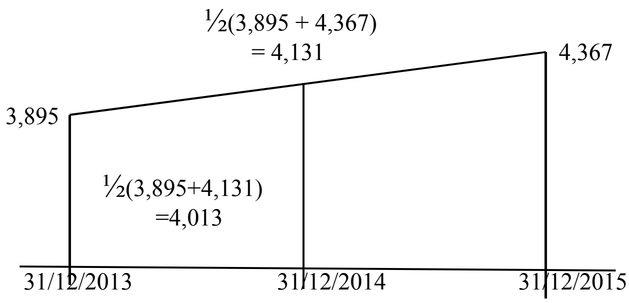
Therefore the overall Exposed-to-risk is 2,385. [½]



Company *C*

Population on 31/12/2014 is 0.5 * (3,895 + 4,367) = 4,131, [½]

so the Exposed-to-risk is 0.5 * (4,131 + 3,895) = 4,013 [½]



(ii) (a) Birthdays are evenly distributed across calendar years. [1]

(b) This is needed in order to average the data for 51 and 52 year olds or to adjust the Exposed-to-risk from age nearest to age last birthday for Company *A*. [1]

(a) 1 January data in year x can be taken as 31 December data in year $x - 1$. [1]

(b) This is needed in order to use start-2015 data as end-2014 data for Company *C*. [1]

(a) The population varies linearly between census dates. [1]

(b) This is needed in order to average between census dates (OR to apply the trapezium rule). [1]

[Total 11]

Answers to this question were variable. A minority of candidates did well, scoring full marks on part (i). Others made a range of errors. Some of these were simple, for example reading data for the wrong company from the examination paper. Others included averaging ages 50 and 51 for Company *A*, and simply averaging the estimated Exposed-to-risk figures for 1 January 2014 and 1 January 2015 for Company *B* (this is incorrect because we have additional information about the Exposed-to-risk on 1 November 2014 which we can use). In part (ii) there were a lot of marks available, and the Examiners were looking for accuracy and clarity. So, for example, for full credit candidates were required to state that the "population varies linearly *between census dates*". Many candidates failed to identify the second assumption, that we will assume 31 December data to be equivalent to 1 January in the following year.

Q9

(i) The maximum likelihood estimate of the transition intensity from state i to state j is the number of transitions from state i to state j divided by the total waiting time in state i. [1]

To estimate the transition intensities exactly we therefore need

the total time spent in each state
OR
entry and exit times for each individual for each state, [1]

and the total number of transitions of each type made. [1]

(ii) Define $p_{AA}(s,t)$ to be the probability of being in state Active at time $s+t$ if Active at time s.

Then EITHER

$$\frac{\partial}{\partial t} p_{AA}(s,t) = -p_{AA}(s,t)\mu$$ [½]

$$\frac{\partial}{\partial t} p_{AT}(s,t) = p_{AA}(s,t)\mu \ ,$$ [½]

OR

$$\frac{\partial}{\partial t} p(s,t) = p(s,t)M$$ [½]

where $M = \begin{pmatrix} -\mu & \mu \\ 0 & 0 \end{pmatrix}$ in order Active, Theft, [½]

OR

Integrated forward equations:

$$p_{AA}(s,t) = \exp\left(-\int_{u=s}^{t} \mu du\right)$$ [½]

$$p_{AT}(s,t) = \int_{u=0}^{t} p_{AA}(s,u).\mu.1du\,.$$ [½]

(iii) Measure from time zero i.e. $s = 0$ and drop s from notation.

EITHER

$$\frac{1}{p_{AA}(t)} \frac{\partial}{\partial t} p_{AA}(t) = -\mu\,.$$

$$\frac{\partial}{\partial t}(\ln(p_{AA}(t))) = -\mu\,,$$ [½]

hence $p_{AA}(t) = \exp(-\mu t + C)$.

As $p_{AA}(0) = 1$, $C = 0$, so

$$p_{AA}(t) = \exp(-\mu t)$$ [½]

A claim occurs with cost £C if moves to state "Theft Claim".

Hence the expected cost is $C(1-\exp(-\mu T))$ [1]

OR

Solving for p_{AT}, we have

$$\frac{\partial}{\partial t} p_{AT}(t) = p_{AA}(t)\mu = (1-p_{AT}(t))\mu$$ (as the model has only two states). [½]

Using an integrating factor, we can write

$$\frac{\partial}{\partial t}[\exp(\mu t)p_{AT}(t)) = \mu\exp(\mu t),$$ [½]

$$\exp(\mu t)p_{AT}(t) = \exp(\mu t) - 1,$$

$$p_{AT}(t) = 1 - \exp(-\mu t),$$

and hence the expected cost is $C(1-\exp(-\mu T))$. [1]

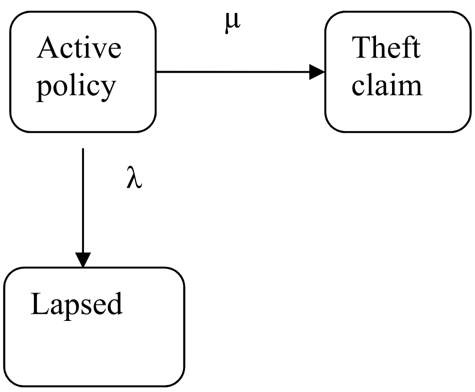
OR

Solving the integrated forward equation

$$P_{AT}(T) = \int_{s=0}^{T}\exp(-\mu s)\mu ds = [-\exp(\mu s)]_0^T = 1-\exp(-\mu T),$$ [1]

and hence the expected cost is $C(1-\exp(-\mu T))$. [1]

(iv)



[2]

(v) We now have $\frac{\partial}{\partial t}p_{AA}(t) = -p_{AA}(t)(\mu+\lambda)$. [½]

So $p_{AA}(t) = \exp(-(\mu+\lambda)t)$. [½]

We want $\frac{\partial}{\partial t}p_{AT}(t) = p_{AA}(t)\mu = \mu\exp(-(\mu+\lambda)t))$. [½]

Solving this produces $p_{AT}(t) = \left|\frac{-\mu}{(\mu+\lambda)}\exp(-(\mu+\lambda)t))\right|_0^T = \frac{\mu}{\mu+\lambda}(1-\exp(-(\mu+\lambda)T))$.

[1]

So claims become $\dfrac{\mu}{\mu+\lambda}C(1-\exp(-(\mu+\lambda)T))$. [½]

[Total 11]

> Answers to this question were often weak. In part (i) a substantial minority of candidates seemed not to have read the part of the question about a statement of the data required, and so lost marks. Parts (iii) and (v) were poorly answered, with part (v) not being attempted by many candidates.

Q10

(i) $\lambda(t{:}Z_i) = \lambda_0(t)\exp(0.065Z_1 - 0.035Z_2 - 0.06Z_3 + 0.085Z_4)$. [2]

Here

$\lambda(t{:}Z_i)$ is the hazard of being discharged at time t,

$\lambda_0(t)$ is the baseline hazard, [½]

and

Z_1 is the gender covariate = 1 if the patient is female and 0 if the patient is male.

Z_2 is the smoker covariate = 1 if the patient is a non-smoker and 0 if the patient is a smoker.

Z_3 is the non-drinker covariate = 1 if the patient does not drink and 0 if the patient drinks.

Z_4 is the heavy drinker covariate = 1 if the patient is a heavy drinker and 0 otherwise. [1½]

(ii) A male moderate drinker who does not smoke has hazard of leaving after 3 days of

$\lambda_0(3)\exp(0 - 0.035 + 0 + 0)$. [½]

So $0.6 = \lambda_0(3)\exp(-0.035)$. [½]

A female heavy drinker who smokes has a hazard of leaving of

$\lambda_0(3)\exp(0.065 + 0 + 0 + 0.085) = \lambda_0(3)\exp(0.15)$ [1]

So the probability that the female is discharged is $0.6\dfrac{\exp(0.15)}{\exp(-0.035)} = 0.7219$, [½] and the probability she is not discharged is $1 - 0.7219 = 0.2781$ or 28%. [½]

(iii) The colleague's null hypothesis is that the gender parameter is actually zero. [½]

From the data calculate the variance of the estimate of the parameter for gender. [½]

The 95% confidence interval is then the parameter ± 1.96 * standard deviation. [½]

If this range does not include 0, we can be 95% confident that the gender has a material impact and we can reject the null hypothesis. [½]

(iv) A suitable statistical test is the log likelihood test. [½]

The null hypothesis is that the coefficient of the marital status term is zero. [½]

We compare the model with and without the extra parameter. [½]

If the log-likelihoods for the two models are L_{with} and L_{without} respectively, then the test statistic is $-2(L_{\text{without}} - L_{\text{with}})$. [½]

This statistic has a chi-squared distribution with one degree of freedom (since the marital status term involves one parameter). [½]

If the test statistic is greater than 3.84 (the chi-squared critical value at 95% with 1 degree of freedom), we reject the null hypothesis [1]

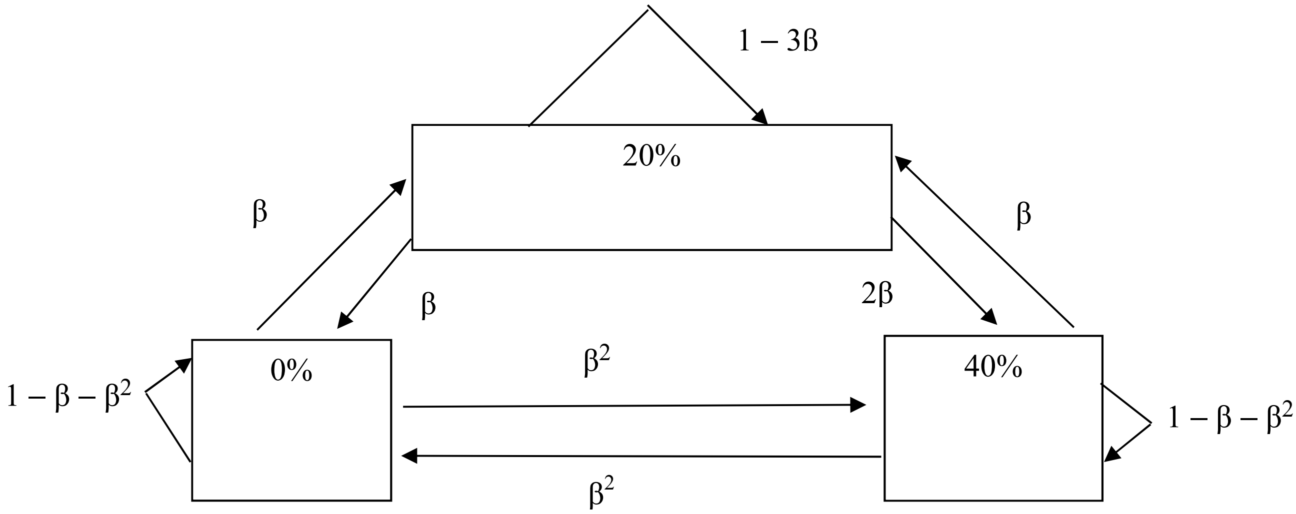
and conclude that the marital status term does improve the model. [½]

[Total 13]

In part (i) some candidates incorrectly defined the dummy variables Z_3 and Z_4 for drinking behaviour as "0 if the patient drinks and 1 if the patient is a moderate drinker" and "0 if the patient is a heavy drinker and 1 if the patient is a moderate drinker" respectively. This is incorrect as it leaves no category for the heavy drinkers and the non-drinkers respectively. Other candidates failed to mention the numerical values of the parameter estimates, which were required for full credit as the question said "this … model". In part (ii) many candidates interpreted the question as referring to a survival probability rather than a hazard. Credit was given to all reasonable interpretations of the question. In part (iii) it is, of course, possible to test the significance of the gender covariate using the likelihood ratio test, and full credit was given for this if fully explained.

Q11

(i)



[2]

(ii) We require each row of the transition matrix to sum to 1. [½]

Here this holds for all values of β. [½]

We require each of the following to lie between 0 and 1 inclusive:

$\beta, \beta^2, 2\beta, 1-3\beta, 1-\beta-\beta^2$.

The first two require that $0 \le \beta \le 1$. [½]

The third requires that $0 \le \beta \le 1/2$. [½]

The fourth that $\beta \le \frac{1}{3}, \beta \ge 0$. [½]

The fifth implies $\beta \le \frac{\sqrt{5}-1}{2}$ as the negative root is not viable. [1]

So overall $0 \le \beta \le \frac{1}{3}$. [½]

[Max 3]

(iii) If $\beta > 0$ then it can reach any other state (so it is irreducible) [1]

and it has a loop on each state (so it is aperiodic). [½]

However if $\beta = 0$ it can never leave its current state so it is reducible. [½]

(iv) In this case the matrix is $P = \begin{pmatrix} 0.89 & 0.1 & 0.01 \\ 0.1 & 0.7 & 0.2 \\ 0.01 & 0.1 & 0.89 \end{pmatrix}$ [½]

The stationary distribution satisfies $\pi = \pi P$. [½]

We have:

$$\pi_0 = 0.89\pi_0 + 0.1\pi_{20} + 0.01\pi_{40}$$
$$\pi_{20} = 0.1\pi_0 + 0.7\pi_{20} + 0.1\pi_{40}$$
$$\pi_{40} = 0.01\pi_0 + 0.2\pi_{20} + 0.89\pi_{40}$$

[½]

and

$$\pi_0 + \pi_{20} + \pi_{40} = 1.$$

[½]

The first and third equations give

$$0.11\pi_{40} - 0.22\pi_0 = 0.01\pi_0 - 0.02\pi_{40}$$
$$\pi_{40} = \frac{0.23}{0.13}\pi_0 = 1.769\pi_0$$
$$\pi_{20} = \frac{0.1 + 0.1769}{0.3}\pi_0 = 0.923\pi_0,$$

so $(1 + 0.923 + 1.769)\pi_0 = 1$, and hence [1]

$$\pi_0 = 0.271 = \frac{13}{48}$$
$$\pi_{20} = 0.25 = \frac{1}{4}$$
$$\pi_{40} = 0.479 = \frac{23}{48}$$

are the long term proportion of taxpayers at each marginal rate. [1]

(v) Looking for the rates two years' later, these are given by P^2, which is

$$\begin{pmatrix} 0.89 & 0.1 & 0.01 \\ 0.1 & 0.7 & 0.2 \\ 0.01 & 0.1 & 0.89 \end{pmatrix} . \begin{pmatrix} 0.89 & 0.1 & 0.01 \\ 0.1 & 0.7 & 0.2 \\ 0.01 & 0.1 & 0.89 \end{pmatrix} = \begin{pmatrix} 0.8022 & 0.16 & 0.0378 \\ 0.161 & 0.52 & 0.319 \\ 0.0278 & 0.16 & 0.8122 \end{pmatrix}$$

So the required probabilities are:

(a) 0.161
(b) 0.52
(c) 0.319. [2]

[Total 13]

> This question was generally well answered, except for part (ii). In part (ii) a large number of candidates considered that β could not take the values 0 and 1, even though it was a probability. In part (ii) candidates who gave the correct range of $0 \le \beta \le 1/3$ scored 2 marks. The other mark was for reasoning leading to the correct range. Candidates who gave an incorrect range could score partial credit for sensible reasoning. The most common error in part (iv) was to base the answer on P^3 rather than P^2.

Q12

(i) There are 11 age groups: [½]

EITHER
no parameters have been fitted and no table has been chosen
OR
we are not comparing our data with a set of graduated rates derived from our data [1]

so the number of degrees of freedom is 11. [½]

(ii) The null hypothesis is that the old rates are the true rates underlying the observed data. [½]

Age	*ETR*	*Deaths*	*Mortality*	*Expected Deaths*	*z*	z^2
55	5,842	150	0.0267	155.981	–0.479	0.229
56	5,630	132	0.0278	156.514	–1.959	3.840
57	4,281	126	0.0301	128.858	–0.252	0.063
58	3,955	98	0.0325	128.538	–2.694	7.255
59	3,879	142	0.0356	138.092	0.333	0.111
60	3,550	149	0.0387	137.385	0.991	0.982
61	4,006	162	0.0396	158.638	0.267	0.071
62	4,150	173	0.0410	170.150	0.218	0.048
63	3,520	158	0.0433	152.416	0.452	0.205
64	3,057	150	0.0458	140.011	0.844	0.713
65	3,666	200	0.0490	179.634	1.520	2.309
Total						15.825

The observed test statistic is 15.825. [3]

The critical value of the chi squared distribution at the 5% level with 11 degrees of freedom is 19.68. [½]

Since 15.85 < 19.68 [½]

we do not reject the null hypothesis. [½]

(iii) To test whether the shape of the mortality rates has changed over the age range we use the Grouping of Signs Test.

Under the null hypothesis that the old rates are the true rates underlying the observed data, [½]

G = Number of groups of positive deviations = 1,
m = number of deviations = 11,
n_1 = number of positive deviations = 7, and
n_2 = number of negative deviations = 4 [½]

EITHER

We want k^* the largest k such that

$$\sum_{t=1}^{k} \frac{\binom{n_1-1}{t-1}\binom{n_2+1}{t}}{\binom{m}{n_1}} < 0.05 .$$

The test fails at the 5% level if $G \leq k^*$. [½]

From the Gold Book the value of k^* is 1. [½]

Since, therefore, $G = k^*$ in this case [½]

we have sufficient evidence to reject the null hypothesis at the 5% level. [½]

OR

$$\Pr[t=1] = \frac{\binom{6}{0}\binom{5}{1}}{\binom{11}{7}} = \frac{5}{330} = 0.015152$$

$\Pr[t = 0] = 0$ [1]

So $\Pr[t <= 1] < 0.05$ [½]

Hence we have evidence to reject the null hypothesis at the 5% level. [½]

(iv) The Chi-Squared Test shows that overall the data are a good fit to the previous investigation. [½]

However, the Grouping of Signs Test shows that the shape of the mortality has changed, [½]

with the current investigation showing lower mortality at the lower end of the age range and higher mortality at the higher end. [½]

This change in shape was not picked up by the Chi-Squared test as it uses the square of the standard deviations, and therefore has no regard for the direction of the deviations. [½]

(v) Mortality may have improved/worsened due to changes in medical science, health care provision, the environment or the standard of living. [½]

This could have affects at all ages, for example when a new "cure all" drug is found, or over particular age ranges if a procedure or drug is discovered which most notably affects a certain age range of the population. [½]

The composition of the lives may have changed (e.g. a different mix of nationalities, or a different weighting between males and females) [1]

There may be more underwritten lives in one investigation than the other. [½]

One of the investigations could have included a period when an epidemic occurred. OR There could be an error in one of the investigations. [½]

There could be a general change in lifestyle or diet. [½]

Underwriting practices may have changed for example a new "preferred lives" category may have been created. [1]

[Max 2]

[Total 14]

Answers to part (i) of this question were the weakest of any part of the examination paper. The Core Reading clearly states that "[i]f we are comparing an experience with a standard table, then [the test statistic] can be assumed to have a χ^2 distribution with *m* degrees of freedom [where] *m* is just the number of age groups". It matters not how the previous table was constructed. In this case, indeed, there is no standard table as such, just the results of a previous investigation. Provided the current investigation is independent of the previous investigation, then we do not need to deduct any degrees of freedom for the "choice of standard table". In part (iii) many candidates incorrectly interpreted the question as referring to the "shape of the distribution of the mortality rates" rather than the shape of the rates themselves. Many candidates made sensible comments in part (v). Only a subset of the points mentioned above was required for full credit.

END OF EXAMINERS' REPORT