INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

September 2010 examinations

Subject CT8 — Financial Economics
Core Technical

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

T J Birse
Chairman of the Board of Examiners

December 2010

1 (i) $E[R] = 0.8 * 0\% + 0.2 * 30\% = 6\%$
$E[R^2] = 0.8 * 10\%^2 + 0.2*(30\%^2 + 10\%^2) = 0.028$

(ii) $\text{Var}(R) = 0.028 - 0.06^2 = 0.0244 = 0.1562^2$

$\text{Prob}(R < 0) = 0.8 * N(0; 0, 10\%) + 0.2 * N(0; 30\%, 10\%) = 0.8 * 0.5 + 0.2 * 0.00135 = 0.4 + 0.00027 = 0.40027$

$\text{Prob}(S < 0\%) = N(0; 6\%, 15.62\%) = 0.3504$

$\text{Prob}(R < -10\%) = 0.8 * N(-10\%; 0, 10\%) + 0.2 * N(-10\%; 30\%, 10\%) = 0.8 * 0.1587 + 0.2 * 0 = 0.1269$

$\text{Prob}(S < 10\%) = N(-10\%; 6\%, 15.62\%) = 0.1528$

(iii) Variance suggests risks are the same

Benchmark at 0% suggests *R* riskier than *S* – "weight" of probability around 0% with *R* makes *R* look riskier than *S*

Benchmark at −10% suggests *S* riskier than *R* – overall wider "spread" of *S* dominates at more extreme risk levels

(Note: candidate answers may differ slightly because approximations required in standard normal lookups from tables.)

2 (i) Excessive volatility is when the change in market value of stocks (observed volatility), cannot be justified by the news arriving. This is claimed to be evidence of market over-reaction which was not compatible with efficiency.

(ii) There are also well-documented examples of under-reaction to events (any two of these):

1. Stock prices continue to respond to earnings announcements up to a year after their announcement. An example of under-reaction to information which is slowly corrected.

2. Abnormal excess returns for both the parent and subsidiary firms following a de-merger. Another example of the market being slow to recognise the benefits of an event.

3. Abnormal negative returns following mergers (agreed takeovers leading to the poorest subsequent returns). The market appears to over-estimate the benefits from mergers, the stock price slowly reacts as its optimistic view is proved to be wrong.

3 One-factor models
All are arbitrage-free.
Vasicek: easy to implement but problem of possible negative interest rates

CIR: more tricky to implement but positive rates (for suitable choice of parameter values).

HW: more flexible as time-inhomogeneous, so better fit to market data (in particular option prices)., but negative rates are possible

Limitations:

1) historical data shows changes in the prices of bonds with different terms to maturity are not perfectly correlated

2) there have been sustained periods of both high and low interest rates with periods of both high and low volatility

3) we need more complex models to deal effectively with more complex derivative contracts e.g. any contract which makes reference to more than one interest rate should allow these rates to be less than perfectly correlated

Multiple-factor models: to capture more features of market data, better for pricing exotic derivatives.

There is no perfect model. A good model depends on the data available and the use of the model (basic assets, plain vanilla derivatives, more exotic derivatives, short or long maturities…).

Fit to historical data; realistic dynamics

4 (i) (a) A credit event is an event which will trigger the default of a bond and includes the following:

- failure to pay either capital or a coupon
- loss event
- bankruptcy
- rating downgrade of the bond by a rating agency such as Standard
- and Poor's or Moody's

(b) The outcome of a default may be that the contracted payment stream is:
- rescheduled
- cancelled by the payment of an amount which is less than the default-free value of the original contract
- continues but at a reduced rate
- totally wiped out

[any three of the above]

(c) In the event of a default, the fraction of the defaulted amount that can be recovered through bankruptcy proceedings or some other form of settlement is known as the recovery rate.

(ii) The model is in continuous time; it has two states N (not previously defaulted) and D (previously defaulted).

Under this simple model it is assumed that the default-free interest rate term structure is deterministic with $r(t) = r$ for all t.

If the transition intensity, under the real-world measure P, from N to D at time t is denoted by $\lambda(t)$, then if $X(t)$ is the state at time t:

$$\Pr P(X(t + dt) = N \mid X(t) = N) = 1 - \lambda(t)\, dt + o(dt) \text{ as } dt \to 0,$$

$$\Pr P(X(t + dt) = D \mid X(t) = N) = \lambda(t)\, dt + o(dt) \text{ as } dt \to 0.$$

(iii) The formula for the unit ZCB price is $e^{-rT}(1 - (1 - \delta)(1 - e^{-\lambda(i)T}))$, where δ is the recovery rate and $\lambda(i)$ is the (constant) default rate for bond i and T is the redemption time.

Thus

$1.6 = 2e^{-0.06}(1 - .4(1 - e^{-2\lambda(A)}))$ and
$2.2 = 3e^{-0.06}(1 - .4(1 - \mathrm{e}^{-2\lambda(B)}))$,

so $\lambda(A) = 0.2361$

and

$\lambda (B) = 0.4029$

(iv) (a) We seek a portfolio consisting of a units of £100 nominal of bond A, b units of £100 nominal of bond B, d units of the derivative, D, and c units of cash.

If this is to perfectly hedge the security then its value at time 2 should be zero unless both bonds default, in which case it should be 100.

At time 2 there are four possibilities: no defaults, bond A only has defaulted, bond B only has defaulted, both bonds have defaulted.

Equating the corresponding values of the portfolio and of the new security (at time 2) we obtain:

$100a + 100b + c = 0;$
$60a + 100b + 100d + c =0;$
$100a + 60b + 100d + \mathrm{c} = 0$
$60a + 60b + 100d + c = 100$

Solving gives $a = b = -2.5$, $c = £500$ and $d = -1$.

(b) Since this is a perfect hedge, the initial value of the hedging portfolio is the fair price for the new security, so the fair price is
$500e^{-0.06} - 250(1.6/2) - 250(2.2/3) - 52 = £34.55$

5 (i) The unique fair price is $V = E_P[e^{-rT}D]$, where P is the EMM

(ii) Standard interpolation using the Black-Scholes formula gives $r = 14.55\%$ or 14.5%

(iii) The price of the security is given in (i) so equals
$E_P[e^{-r} S_1^2] = E_P[S_0^2 e^{-r} \exp(2\sigma B_1 + 2r - \sigma^2)] = S_0^2 \exp(r + \sigma^2) = 5.5^2 \exp(.185) = £36.40$.

(iv) The amount of stock to hold in the hedging portfolio is Delta = $\partial f/\partial S$, where f is the price as a function of current stock price S. Thus the initial hedging portfolio holds $2S_0\exp(r + \sigma^2) = 13.235$ units of stock and is short $£S_0^2\exp(r + \sigma^2) = £36.40$.

6 (i) Set $g(x,t) = \exp(kx - (1/2)k^2t)$, then $\Lambda_t = g(W_t, t)$.

It follows from Ito's formula that
$d\Lambda_t = (\partial g/\partial t\,(W_t, t) + (1/2)\,\partial^2 g/\partial^2 x(W_t, t))dt + \partial g/\partial x(W_t, t)\,dW_t$
$= (-(1/2)k^2 g + (1/2)k^2 g)dt + kg\,dW_t = kg\,dW_t$.

It follows that Λ is a (local) martingale.

Hence $1 = \Lambda_0 = E[\Lambda_T] = E[\exp(kW_T - 1/2k^2T)$
$= E[\exp(kW_T]\exp(-1/2k^2T)$ so $E[\exp(kW_T]=\exp(1/2k^2T)$

(ii) (a) When

$$
\begin{aligned}
f(x) &= x,\ p_0 = E[e^{-rt} S_T \Lambda_t | F_0] \\
&= E[e^{-rt} S_0\exp(\sigma W_t + (\mu - 1/2\,\sigma^2)t)\Lambda_t | F_0] \\
&= E[e^{-rt} S_0\exp((\sigma + m) W_t + (\mu - 1/2\,\sigma^2 - 1/2\,m^2)t) | F_0] \\
&= e^{-rt} S_0\exp(1/2\,(\sigma + m)^2 t + (\mu - 1/2\,\sigma^2 - 1/2\,m^2)t) \\
&= S_0\exp((\sigma m + \mu - r)t)).
\end{aligned}
$$

(b) Now the price at time 0 of a unit of stock is S_0, so unless $p_0 = S_0$, which holds if and only if $m = (r - \mu)/\sigma$, there is an arbitrage opportunity.

7 (i) Denote the individual derivative by *f* and assume this is written on an underlying security *S*

$$\Delta = \frac{\partial f}{\partial s} \equiv \frac{\partial f}{\partial s}(t, S_t).$$

$$\Gamma = \frac{\partial^2 f}{\partial s^2}$$

$$\nu = \frac{\partial f}{\partial \sigma}$$

(Marks should also be awarded if these are defined in words.)

(ii) If the portfolio is Delta-hedged and has a high value of Γ then it will require more frequent rebalancing or larger trades than one with a low value of gamma. The need for rebalancing can, therefore, be minimised by keeping gamma close to zero.

The value of a portfolio with a low value of vega will be relatively insensitive to changes in volatility. Since σ is not directly observable, a low value of vega is important as a risk-management tool. Furthermore, it is recognised that σ can vary over time. Since many derivative pricing models assume that σ is constant through time the resulting approximation will be better if ν is small.

8 The proof of this result is an adaptation of that of the standard call-put parity. Two (self-financing) portfolios are considered:

- Portfolio A: buying the call and selling the put at time *t*. Its value at time *t* is $C_t - P_t$ and at time *T*, it is $S_T - K$ in all states of the universe.

- Portfolio B: buying the underlying asset for S_t and borrowing $K\exp\left(-r\left(T-t\right)\right)+D\exp\left(-r\left(T'-t\right)\right)$ at time *t*. Its value at time *t* is then: $-\left(K\exp\left(-r\left(T-t\right)\right)+D\exp\left(-r\left(T'-t\right)\right)-S_t\right)$. At time T', the dividend *D* is paid and added to the portfolio. Therefore the value at maturity of the portfolio is then $S_T - K$, taking into account the dividend payment.

Using the absence of arbitrage opportunity, both portfolios should have the same value at any intermediate time, in particular at time *t*. Hence:

$$C_t - P_t = S_t - K\exp\left(-r\left(T-t\right)\right) - D\exp\left(-r\left(T'-t\right)\right).$$

9 (i) (a) Key calculation in demonstrating no arbitrage is
$d = 0.8 < \exp(0.06) < u = 1.2$

(b) The binomial tree is:

		144
	120	
100		96
	80	
		64

(ii) To price the call option, we use the risk-neutral pricing formula. We use the following simplifying notation:

$$C_{uu} = \left(u^2 S_0 - K\right)^+ = 44 \ ; \ C_{ud} = \left(udS_0 - K\right)^+ = 0 \ ; \ C_{dd} = \left(d^2 S_0 - K\right)^+ = 0\,.$$

At time 1, we get in the upper state,

$C_1(u) = \exp(-r)\left[qC_{uu} + (1-q)C_{ud}\right] = 27.12$, and in the lower state
$C_1(d) = \exp\left[qC_{ud} + (1-q)C_{dd}\right] = 0$ where the risk-neutral probability of an upward move is $q = \dfrac{\exp(r) - d}{u - d} = 0.6545$.

At time 0, $C_0 = \exp(-r)\left[qC_1(u) + (1-q)C_1(d)\right]$.

Hence $C_0 = 16.72$. (this could be seen directly as $C = e^{-2}rp^{2}*44$)

(iii) (a) Only one path is relevant for this barrier option "up-up". Its probability of occurrence is q^2 and the associated payoff is $X_{uu} = 44$. Using the risk-neutral valuation formula, we get:

$$X_0 = \exp(-2r)\left(q^2 X_{uu}\right) = 16.72$$

(b) In practice this option "clearly" has less value than the option (ii) because it pays off in fewer cases. However it has the same price when calculated using the binomial tree approach – this reinforces the need for choosing binomial trees carefully when pricing derivatives.

END OF EXAMINERS' REPORT