INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

April 2013 examinations

Subject CT8 – Financial Economics
Core Technical

Introduction

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and using past papers as a revision aid and also those who have previously failed the subject.

The Examiners are charged by Council with examining the published syllabus. The Examiners have access to the Core Reading, which is designed to interpret the syllabus, and will generally base questions around it but are not required to examine the content of Core Reading specifically or exclusively.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report; other valid approaches are given appropriate credit. For essay-style questions, particularly the open-ended questions in the later subjects, the report may contain more points than the Examiners will expect from a solution that scores full marks.

The report is written based on the legislative and regulatory context pertaining to the date that the examination was set. Candidates should take into account the possibility that circumstances may have changed if using these reports for revision.

D C Bowie
Chairman of the Board of Examiners

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General comments on Subject CT8

Subject CT8 introduces the main concepts and principles of financial economics. These are developed in later subjects in the ST series of exams. This subject combines various types of skills. In particular, along with CT7, it is one of the first where candidates are expected to write lengthy passages of reasoned thought, rather than just complete calculations. This is a skill that will be new to many, and candidates are advised to pay particular attention to the answers to this type of question by studying many past papers.

Comments on the April 2013 paper

The general performance was good and better than on the previous session (September 2012). Candidates generally found this paper challenging, but well-prepared candidates scored well across the whole paper and the best candidates scored close to full marks. As in previous diets, questions that required an element of application of the core reading to situations that were not immediately familiar proved more challenging to most candidates. The comments that follow the questions concentrate on areas where candidates could have improved their performance.

1 **Variance of return**

Variance is mathematically tractable.

Variance fits neatly with a mean-variance portfolio construction framework.

Variance is a symmetric measure of risk. The problem of investors is really the downside part of the distribution.

Credit risky bonds have an asymmetric return distribution and as defaults are often co-dependent on economic downturns portfolios can have fat tails.

Neither skewness or kurtosis of returns is captured by a variance measure.

Downside semi-variance of return

Semi-variance is not easy to handle mathematically and it takes no account of variability above the mean.

Furthermore if returns on assets are symmetrically distributed semi-variance is proportional to variance.

As with variance of return, semi-variance does not capture skewness or kurtosis.

It takes into account the risk of lower returns.

It can be decomposed into systematic and non-systematic risk contributions.

Shortfall probability

The choice of benchmark level is arbitrary.

For a portfolio of bonds, the shortfall probability will not give any information on:

- upside returns above the benchmark level
- nor the potential downside of returns when the benchmark level is exceeded.

It gives an indication of the possibility of loss below a certain level.

It allows a manager to manage risk where returns are not normally distributed.

Value at Risk (VaR)

VaR generalises the likelihood of underperformance by providing a statistical measure of downside risk.

Portfolios exposed to credit risk, systematic bias or derivatives may exhibit non-normal distributions.

The usefulness of VaR in these situations depends on modelling skewed or fat-tailed distributions of returns.

The further one gets out into the "tails" of the distributions, the more lacking the data and, hence, the more arbitrary the choice of the underlying probability distribution becomes.

Tail Value at Risk (TailVaR)

Relative to VaR, TailVaR provides much more information on how bad returns can be when the benchmark level is exceeded.

It has the same modelling issues as VaR in terms of sparse data, but captures more information on tail of the non-normal distribution.

In general, and given that this was a straightforward question, this was surprisingly poorly answered with students losing marks for not knowing basic definitions.

2 Let the expected return on S_A be E_A and the variance of return be V_A. Then the expected return on S_B is $2E_A$ and the variance of return is $2V_A$.

(i) (a) The only zero risk portfolio can occur if the correlation is either 1 or −1. By considering diversification, the most efficient portfolio will occur when it is −1.

The overall portfolio variance is:

$$V = x_A^2 V_A + 2x_B^2 V_A + 2\sqrt{2}x_A x_B \rho V_A = V_A(x_A - \sqrt{2}x_B)^2$$
$$+2\sqrt{2}x_A x_B (1+\rho)V_A$$

Since $-1 \le \rho \le 1$ and $V_A > 0$
this can only be 0 if $\rho = -1$.

(b) Then, $V = 0 \Rightarrow\ x_A = \sqrt{2}x_B$ and the overall portfolio constrain $x_A + x_B = 1$ yields $x_A = \dfrac{\sqrt{2}}{\sqrt{2}+1}$ and $x_B = \dfrac{1}{\sqrt{2}+1}$.

(c) So the expected return on the overall portfolio $E = E_A . \dfrac{\sqrt{2}+2}{\sqrt{2}+1}$.

(ii) (a) In this case the maximum expected return is infinite (obtained by selling unlimited amounts of security S_A to purchase unlimited amounts of security S_B).

(b) In this case the maximum expected return is obtained by selling one unit of S_A to purchase two units of S_B. The maximum expected return is then $3\,E_A$.

(iii) In this case we have, using results from the core reading,

$$x_A = \frac{2V_A - 0.6V_A}{3V_A - 1.2V_A} = \frac{7}{9} = 0.7777 \text{ and so } x_B = \frac{2}{9} = 0.2222$$

And so the expected return is $\frac{11}{9}E_A = 1.2222\,E_A$.

This question was surprisingly poorly answered with most candidates missing the point of the question, which was to test their understanding of basic ideas about correlated assets.

3 We may not be justified in accepting a model simply because it passes the tests. Many of these tests (for example, tests of stationarity) have notoriously *low power,* and therefore may not reject incorrect models.

Indeed, even if the "true" model was not in the class of models being fitted, we would still end up with an apparently acceptable fit, because the rules say we keep generalising until we find one.

This process of generalisation tends to lead to *models which wrap themselves around the data, resulting in an understatement of future risk, and optimism regarding the accuracy of out-of-sample forecasts.*

For example, *Huber recently compared the out-of-sample forecasts of the Wilkie model to a naïve "same as last time" forecast over a 10 year period. The naïve forecasts proved more accurate.*

May candidates had not studied basic material covered in this question and answered poorly.

4 (i) The market portfolio is in proportion to the market capitalisation since every investor holds risky assets in proportion to that portfolio. Thus the market portfolio is $0.1A + 0.2B + 0.4C + 0.3D$ (asset E is the risk-free asset).

Asset	*A*	*B*	*C*	*D*	*E*	*Probability of being in state*
Annual return in						
State 1	3%	3%	3%	3%	3%	0.25
State 2	5%	7%	2%	8%	3%	0.5
State 3	7%	5%	8%	1%	3%	0.25
Market Capitalisation	10m	20m	40m	30m		

$E_A = 5\%$; $E_B = 5.5\%$; $E_C = 3.75\%$; $E_D = 5\%$

and so $E_M = (10\times5\%+20\times5.5\%+40\times3.75\%+30\times5\%)/100 = 4.6\%$

Now $\sigma^2_M = 0.25\times(3-4.6)^2+0.5\times(5.1-4.6)^2+0.25\times(5.2-4.6)^2 = 0.855\%$
and $\sigma_M = 0.92466\%$

(ii) market price of risk is $(E_M-r)/\sigma_M = (4.6 - 3)/0.92466 = 173\%$

(iii) $\beta_i = \mathrm{Cov}(R_i, R_M)/\mathrm{Var}(R_M)$.

Now $\mathrm{Cov}(R_A, R_M) = 0.25\times3\times3+0.5\times5\times5.1+0.25\times7\times5.2-5\times4.6 = 1.1\%$;
$\mathrm{Cov}(R_B, R_M) = 0.25\times3\times3+0.5\times7\times5.1+0.25\times5\times5.2-5.5\times4.6 = 1.3\%$;
$\mathrm{Cov}(R_C, R_M) = 0.25\times3\times3+0.5\times2\times5.1+0.25\times8\times5.2-3.75\times4.6 = 0.5\%$;
$\mathrm{Cov}(R_D, R_M) = 0.25\times3\times3+0.5\times8\times5.1+0.25\times1\times5.2-5\times4.6 = 0.95\%$

It follows that $\beta_A = 1.1/0.855=1.2865$, $\beta_B = 1.3/0.855 = 1.5205$,
$\beta_C = 0.5/0.855 = 0.5848$ and $\beta_D = 0.95/0.855 = 1.1111$.
OR
Assets all lie on the securities market line, so $E_i - r = \beta_i(E_M-r)$, so
$\beta_A = 2/1.6=1.25$, $\beta_B = 2.5/1.6 = 1.5625$
$\beta_C = 0.75/1.6 = 0.46875$ and $\beta_D = 2/1.6 = 1.25$.

(iv) Most of the assumptions of the basic model can be attacked as *unrealistic*. *Empirical studies do not provide strong support for the model.* There are basic *problems in testing the model* since, in theory, *account has to be taken of the entire investment universe open to investors,* not just capital markets.

Regrettably, there was an inconsistency with the CAPM in the question data.
Accordingly candidates could obtain full marks to part (iii) by giving either of the two answers above.
In general the question was answered well, with most candidates showing good familiarity with the CAPM.

5 (i) B_t has independent increments, i.e. $B_t - B_s$ is independent of $\{B_r, r \le s\}$ whenever $s < t$.

B_t has stationary increments, i.e. the distribution of $B_t - B_s$ depends only on $t - s$.

B_t has Gaussian increments, i.e. the distribution of $B_t - B_s$ is $N(0, t - s)$.

B_t has continuous sample paths $t \rightarrow B_t$.

$B_0 = 0$.

(ii) $$df\left(t, X_t\right) = \frac{\partial f}{\partial x} Y_t dB_t + \left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} A_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} Y_t^2\right] dt$$

(iii) Using Ito's lemma above we have:

$$df\left(t, X_t\right) = 2te^{2tX_t}\ Y_t dB_t + 2e^{2tX_t}\left[X_t + tA_t + t^2 Y_t^2\right] dt$$

This question was very well answered in general, with most candidates fully conversant with the basic properties of Brownian Motion and with Ito's Lemma.

6 (i) It means that a_t, b_t and c_t are known based on information up to but not including time *t*.

(ii) The instantaneous change in the value of the portfolio is given by:

$$dV_t = a_t dA_t + da_t A_t + da_t dA_t + b_t dB_t + db_t B_t + db_t dB_t + c_t dC_t + dc_t C_t + dc_t dC_t$$

(iii) The portfolio is self-financing if the instantaneous change in the value of the portfolio is equal to the pure investment gain.

In other words, $dV_t = a_t dA_t + b_t dB_t + c_t dC_t$

(iv) A replicating strategy is a self-financing strategy $\left(a_t, b_t, c_t\right)$ defined for $0 \le t < U$

(where *U* is the payment time for *X*) such that:

$$V_U = a_U A_U + b_U B_U + c_U C_U = X\ .$$

(v) An initial investment of $V_0 = a_0 A_0 + b_0 B_0 + c_0 C_0$ at time 0, if we follow the self-financing portfolio strategy $\left(a_t, b_t, c_t\right)$, will reproduce the derivative payment without risk. Hence, by no arbitrage the value of the derivative at time 0 must be V_0.

(vi) The market is complete if for any contingent claim *X* there is a replicating strategy $\left(a_t, b_t, c_t\right)$.

In contrast to question 5, the slightly more advanced knowledge about self-financing portfolios and simple stochastic calculus seemed beyond most candidates.

7 (i) First we calculate the risk neutral probability of an up-jump q:

$$q = \frac{1.01 - \frac{1}{1.2}}{1.2 - \frac{1}{1.2}} = 0.481818$$

Then the equation of value for the option price is,

$$50p = \frac{1}{1.01^2}\left(q^2 P_{uu} + 2q(1-q)P_{ud} + (1-q)^2 P_{dd}\right)$$

So

$$P_{uu} = 219.70826p - 2.15094P_{ud} - 1.15664P_{dd}\ .$$

(ii) P_{uu} represents the payoff from an option so cannot be negative. Likewise, it takes its maximum value when P_{ud} and P_{dd} are zero. So $0 < P_{uu} < 219.70826p$.

(iii) (a) If P_{uu} takes its maximum value then P_{ud} and P_{dd} are both zero.

If first stock price move is up then the new value of the option is:

$$\tilde{V} = \frac{qP_{uu}}{1.01} = 104.8113.$$

(b) As P_{ud} and P_{dd} are both zero if the first stock price move is down then the option will expire worthless.

Many candidates seemed uncomfortable with a basic binary tree calculation, despite these being well-explained in the Core Reading. Those with some familiarity scored very well.

8 (i)

- *Not perfect correlation across maturities.*

Firstly, if we look at historical interest rate data we can see that changes in the prices of bonds with different terms to maturity are not perfectly correlated as one would expect to see if a one-factor model was correct. Sometimes we even see, for example, that short-dated bonds fall in price while long-dated bonds go up.

Recent research has suggested that around three factors, rather than one, are required to capture most of the randomness in bonds of different durations.

- *Different volatility phases.*

Secondly, if we look at the long run of historical data we find that there have been sustained periods of both high and low interest rates with periods of both high and low volatility. Again these are features which are difficult to capture without introducing more random factors into a model.

This issue is especially important for two types of problem in insurance: the pricing and hedging of long-dated insurance contracts with interest-rate guarantees; and asset-liability modelling and long-term risk-management.

- *Pricing complex derivatives.*

Thirdly, we need more complex models to deal effectively with derivative contracts which are more complex than, say, standard European call options. For example, any contract which makes reference to more than one interest rate should allow these rates to be less than perfectly correlated in order to produce realistic pricing formulae.

(ii) This models two processes which satisfy the equations:

$$dr(t) = \alpha_r\left(m(t) - r(t)\right)dt + \sigma_{r1}dW_1(t) + \sigma_{r2}dW_2(t)$$
$$dm(t) = \alpha_m\left(\mu - m(t)\right)dt + \sigma_{m1}dW_1(t)$$

where $r(t)$ is the short rate, and $m(t)$, the local mean-reversion level for $r(t)$ and $W_1(t)$ and $W_2(t)$ are independent, standard Brownian motions under the risk-neutral measure Q.

Answers were mixed. Again, knowledge of basic Core Reading made all the difference to candidates scores on this question.

9 (i) $Q(\text{Max}_{t<1}S_t \geq 2)$

$= Q(\max_{t<1}\sigma B_t + (r - ½\sigma^2)t \geq \ln 2)$

$= Q(\max_{t<1}B_t + (r - ½\sigma^2)t/\sigma \geq \ln 2/\sigma)$

$= \Phi([-\ln 2 + (r - ½\,\sigma^2)]/\sigma) + \exp(2(r - ½\sigma^2)\ln 2/\sigma^2)\,\Phi(-[\ln 2 + (r - ½\sigma^2)]/\sigma)$

$= \Phi(-2.7776) + 0.9727 \times \Phi(-2.7676)$

$= 0.00274 + 0.9727 \times 0.00282$

$= 0.00548$

So $Q(\text{Max}_{t<1}S_t < 2) = 0.99452$.

(ii) (a) Denoting by Q the EMM and by F_1 the information available at time 1, the risk neutral pricing formula gives the following price C_1 for the option at time 1:

- if the event $\{\text{Max}_{t<1} S_t < 2\}$ occurs:

  $C_1 = C_1\,(\text{low}) = e^{-r} E_Q[100.\text{Max}(S_2/S_1 - 1;0)|F_1]$

- if the event $\{\text{Max}_{t<1} S_t \geq 2\}$ occurs:

  $C_1 = C_1\,(\text{up}) = 0$

(b) In the case $\{(\text{Max}_{t<1} S_t) < 2\}$, we need to compute the following conditional expectation: $C_1\,(\text{low}) = e^{-r} E_Q[100.\text{Max}(S_2/S_1 - 1;0)|F_1]$

Since S_2/S_1 is independent of the values of S up to time 1 under the EMM Q, the conditional expectation is a simple expectation:

$C_1\,(\text{low}) = e^{-r} E_Q[100.\text{Max}(S_2/S_1 - 1;0)] = e^{-r}.100.\,E_Q[\text{Max}(S_2/S_1 - 1;0)]$

But $S_2/S_1 = \exp((r - \sigma^2/2) + \sigma.(B_2 - B_1))$.

Hence, we simply need to compute the price of a standard European option with strike and initial stock price both equal to 1 and maturity 1 year.

After some simple calculations, we have, in the case $\{(\text{Max}_{t<1} S_t) < 2\}$
$C_1\,(\text{low}) = 100.[\Phi(d_1) - \exp(-3\%).\Phi(d_2)]$
with $d_2 = [r - \sigma^2/2]/\sigma$ and $d_1 = d_2 + \sigma$.

Hence, C_1 (low) = \$11.348

(c) Thus, the fair price at time 0 of the option is $C_0 = E[e^{-r}\, C_1 1_{\{\text{Max}_{t<1} S_t < 2\}}]$ where $1_{\{\text{Max}_{t<1} S_t < 2\}}$ is the indicator of the event $\{\text{Max}_{t<1} S_t < 2\}$, so takes the value 1 if $\{\text{Max}_{t<1} S_t < 2\}$ occurs and 0 otherwise.

So $C_0 = 0.99452 e^{-r}\, C_1(\text{low})$
C_0 = \$10.952

This question was very poorly answered, with most candidates unable to cope with path-dependent option, even though the steps to solution were laid out in the question. Familiarity with the actuarial tables would also have been helpful.

10 (i) In the two state model, the company defaults at time-dependent rate $\lambda(t)$ if it has not previously defaulted. Once it defaults it remains permanently in the default state. It is assumed that after default all bond payments will be reduced by a known factor $(1 - \delta)$, where δ is the recovery rate. Now we need to change to the risk neutral measure, which will change the default rate to $\lambda'(t)$. This rate is that implied by market prices.

(ii) The risk-neutral prices are given by

$$P_t = 100e^{-tR(t)} = 100e^{-rt}(1-(1-\delta)(1-\exp(-\int_0^t \lambda_s ds)))$$
$$= 100e^{-rt}(\delta + (1-\delta)Q(A_t)),$$

where $R(t)$ is the effective rate for a ZCB with redemption at t, λ_s is the risk neutral default rate at time s and A_t is the event that there has been no default by time t.

So, $P_1 = 100e^{-(.015+.04)} = \$94.6485$
and so $Q(A_1) = (e^r P_1/100 - \delta)/(1-\delta) = 0.90197$

And $P_2 = 100e^{-2(.015+.05)} = \$87.8095$

and so $Q(A_2) = (e^{2r} P_2/100 - \delta)/(1-\delta) = 0.76209$

(iii) Thus $\int_0^1 \lambda_s ds = \lambda_1 = -\ln[Q(A_1)] = 0.10317$

and $\int_0^2 \lambda_s ds = \lambda_1 + \lambda_2 = -\ln[Q(A_2)] = 0.27169$ and so $\lambda_2 = 0.16852$.

This was a difficult question and many candidates clearly didn't know the relevant material. A smaller number did and consequently performed well.

END OF EXAMINERS' REPORTS