Subject CT8 — Financial Economics
Core Technical

EXAMINERS' REPORT

September 2008

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

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1 (i) Asset 1 (random return X):

Mean = 4‰np = 4‰3‰½ = 6%;
Variance = 4^2‰npq = 16‰3‰½‰½ = 12%%;
Lower semi-variance = 3/8‰ $(6 - 4)^2$+1/8‰ $(6 - 0)^2$ = 6%%
(can just use symmetry of distribution around the mean);
Shortfall probability = $P(4B < 3) = P(B = 0) = 1/8$.

Asset 2 (random return Y):

Mean = 2‰µ = 2‰3 = 6%;
Variance = 2^2‰ µ =4‰3 = 12%%;
Lower semi-variance = e^{-3}(3^2/2‰ $(6 - 4)^2$ + 3/1‰ $(6 - 2)^2$ + 1.$(6 - 0)^2$)
= 5.078%%;
Shortfall probability = $P(2P < 3) = P(P = 0 \text{ or } 1) = e^{-3}\,(1 + 3)$ = .1991

(ii) Maximising the expected utility corresponds to minimising the lower semi-variance, hence choose asset 2.

2 (i) A multifactor model of security returns attempts to explain the observed historical return by an equation of the form

$$R_i = a_i + b_{i,1}\, I_1 + b_{i,2}\, I_2 + \dots + b_{i,L}\, I_L + c_i\ ,$$

where R_i is the return on security i,
a_i and c_i are the constant and random parts respectively of the component of return unique to security i,
$I_1 \dots I_L$ are the changes in a set of L factors which explain the variation of R_i about the expected return a_i,
$b_{i,k}$ is the sensitivity of security i to factor k.

(ii) **Macroeconomic factor models**

These use observable economic time series as the factors. They could include factors such as the annual rates of inflation and economic growth, short term interest rates, the yields on long term government bonds, and the yield margin on corporate bonds over government bonds.

Fundamental factor models

Fundamental factor models are closely related to macroeconomic models but instead of (or in addition to) macroeconomic variables the factors used are company specific variables. These may include such fundamental factors as:

- the level of gearing
- the price earnings ratio
- the level of R&D spending

- the industry group to which the company belongs

Statistical factor models

Statistical factor models do not rely on specifying the factors independently of the historical returns data. Instead a technique called principal components analysis can be used to determine a set of indices which explain as much as possible of the observed variance. However, these indices are unlikely to have any meaningful economic interpretation and may vary considerably between different data sets.

3 (i) (a) Beta of security i = Covar[R_i, R_M]/V_M

(b) Beta is useful because it allows the expected return of any security to be expressed as a linear function of that security's covariance with the market as a whole.

(ii) Using the formula in (i)(a), Expected Return = 6 + 0.50/0.70(12 - 6) = 10.29%

(iii) (a) Need expected return for each security N, variance of each security N, covariance between each pair of securities $N(N - 1)/2$.

(b) Just need Beta for each security, expected market return, and market variance. Total of $N + 2$.

4 (i) The claim of "excessive volatility" was first formulated into a testable proposition by Shiller in 1981. He considered a discounted cashflow model of equities going back to 1870. By using the actual dividends that were paid and some terminal value for the stock he was able to calculate the perfect foresight price, the "correct equity" price if market participants had been able to predict future dividends correctly. The difference between the perfect foresight price and the actual price arise from the forecast errors of future dividends. If market participants are rational we would expect no systematic forecast errors. Also if markets are efficient broad movements in the perfect foresight price should be correlated with moves in the actual price as both react to the same news.

Shiller found strong evidence that the observed level of volatility contradicted the EMH.

(ii) However, subsequent studies using different formulations of the problem found that the violation of the EMH only had borderline statistical significance. Numerous criticisms were subsequently made of Shiller's methodology, these criticisms covered

- the choice of terminal value for the stock price

- the use of a constant discount rate
- bias in estimates of the variances because of autocorrelation
- possible non-stationarity of the series, i.e. the series may have stochastic trends which invalidate the measurements obtained for the variance of the stock price

Although subsequent studies by many authors have attempted to overcome the shortcomings in Shiller's original work there still remains the problem that a model for dividends and distributional assumptions are required. Some equilibrium models now exist which calibrate both to observed price volatility and also observed dividend behaviour. However, the vast literature on volatility tests can at best be described as inconclusive.

5 Standard Brownian motion (also called the Wiener process) is a stochastic process $\{B_t, t \geq 0\}$ with state space $S = \mathbf{R}$ and the following defining properties:

- B_t has independent increments, i.e. $B_t - B_s$ is independent of $\{B_r, r \leq s\}$ whenever $s < t$.
- B_t has stationary increments, i.e. the distribution of $B_t - B_s$ depends only on $t - s$.
- B_t has Gaussian increments, i.e. the distribution of $B_t - B_s$ is $N(0, t - s)$.
- B_t has continuous sample paths $t \to B_t$.
- $B_0 = 0$.
- (Note that the stationarity property is not needed separately if the Gaussian property is set out in detail.)

6 The proof of this result is an adaptation of that of the standard spot-forward parity. Two (self-financing) portfolios are considered:

- Portfolio A: take a long position in the forward contract at time *t*. Its value at time *t* is 0 and at time *T*, it is $S_T - F_t^T$.
- Portfolio B: buying a fraction $\exp\left(-\delta\left(T-t\right)\right)$ of the underlying asset and borrowing $F_t^T \exp\left(-r\left(T-t\right)\right)$ at time t. Its value at time *t* is then – ($F_t^T \exp\left(-r\left(T-t\right)\right) - \exp\left(-\delta\left(T-t\right)\right) S_t$). Its value at maturity is $S_T - F_t^T$ (assuming reinvestment of dividends).

Using the absence of arbitrage opportunity, both portfolios should have the same value at any intermediate time, in particular at time t. Hence:

$$F_t^T = \exp\left(\left(r-\delta\right)\left(T-t\right)\right)S_t .$$

7 (i) There is no arbitrage in the market since $d = \frac{3}{4} < \exp\left(0.05\right) < u = \frac{3}{2}$.

(ii)

- First method: we construct a risk-neutral portfolio with 1 underlying asset and m call options. We choose the value of m such that this portfolio is risk neutral (its value in the upper state and in the lower state at time 1 should coincide). In this case, $m = -2$. Then, we use a no arbitrage argument: since the portfolio is risk-neutral, it should have the same rate of return as the risk-free asset. Hence, the initial value of the call:

  C_0 satisfies $S_0 - 2C_0 = 30e^{-r}$,

  so $C_0 = 5.732$.

- Second method: We use a replicating portfolio. This is a self-financing portfolio with φ_0 invested in the risk-free asset and φ_1 underlying asset at time 0. Its initial value is therefore $V_0 = \varphi_0 + \varphi_1 S_0$. At time 1, the portfolio should replicate the payoff of the call option. Therefore: $V_u = \varphi_0 \exp\left(r\right) + \varphi_1 S_u = C_u$ and $V_d = \varphi_0 \exp\left(r\right) + \varphi_1 S_d = C_d$. We can deduce the value of φ_0 and $\varphi_1 : \varphi_0 = -14.27$ and $\varphi_1 = 0.5$. By no arbitrage, the initial value of the portfolio and that of the call option should coincide.

  Hence $C_0 = 5.732$.

8 (i) $E_Q[e^{-r(T-t)}X|F_t]$, where X is the amount payable, Q is the risk-neutral measure and F_t is the sigma-algebra generated by the stock-price history up to time t.

(ii) From (a), the price is $E_Q[e^{-rT}1(K \le S_T)|F_0] = e^{-rT}Q(K \le S_T)|$.
Now, under Q, $S_T = S_0\exp(\sigma W_T + (r - ½\sigma^2)T)$, where W is a standard Brownian motion. Thus, $V_0 = e^{-rT}Q(W_T > (\ln(K/S_0) - (r - ½\sigma^2)T)/ \sigma)$
$= e^{-rT}(1 - \Phi((\ln(K/S_0) - (r - ½\sigma^2)T)/ \sigma\sqrt{T})) = e^{-rT}\Phi(d_2)$, where Φ is the standard normal distribution function and d_2 is as in the Black-Scholes formula in the tables.

(iii) If we are long one unit of this derivative and short K units of the derivative in part (ii) then we effectively hold a call option. Thus the value of this derivative must be the sum of the value of the call and KV_0 i.e $S_0\Phi(d_1)$.

(iv) If we go long 150,000 contracts of the type in part (iii) with a strike of 120p and short 150,000 such contracts with a strike of 150p then we have duplicated the contract. Thus the fair price is

$$1.1\times 150{,}000\times(\Phi(\ln(S_0/120) + (r + \tfrac{1}{2}\sigma^2)T)/\sigma\sqrt{T}) - \Phi(\ln(S_0/150) + (r + \tfrac{1}{2}\sigma^2)T)/\sigma\sqrt{T})))$$

$$=1.1\times 150{,}000\times(\Phi(.21184) - \Phi(-.51694)) = 165{,}000\times(.58389 - .30260)$$

$$= £46{,}413.$$

9 (i) The value of a portfolio with a low value of vega will be relatively insensitive to changes in volatility. Put another way: it is less important to have an accurate estimate of σ if vega is low. Since σ is not directly observable, a low value of vega is important as a risk-management tool. Furthermore, it is recognised that σ can vary over time. Since many derivative pricing models assume that σ is constant through time the resulting approximation will be better if ν is small.

(ii) Let f denote the price of a call option, then $f(s,T) = s\Phi(d_1) - Ke^{-rT}\Phi(d_2)$, where $d_1 = (\ln(S_0/K) + (r + \tfrac{1}{2}\sigma^2)T)/\sigma\sqrt{T}$ and $d_2 = d_1 - \sigma\sqrt{T}$. It follows (since $\Phi'(x) = \exp(-x^2/2)/\sqrt{2\pi}$) that $V = s\,(\exp(-d_1^2/2)/\sqrt{2\pi})\partial d_1/\partial\sigma - Ke^{-rT}$ $(\exp(-d_2^2/2)/\sqrt{2\pi})\partial d_2/\partial\sigma = s\,(\exp(-d_1^2/2)/\sqrt{2\pi})\partial(d_1 - d_2)/\partial\sigma$ $= s\,(\exp(-d_1^2/2)/\sqrt{2\pi})\sqrt{T}$.

(iii) Differentiating the Black-Scholes PDE in σ gives us
$\partial V/\partial t + rs\partial V/\partial s + \tfrac{1}{2}\sigma^2 s^2\partial^2 V/\partial s^2 + \sigma s^2\partial^2 f/\partial s^2 = rV$.

Then, since $\Gamma = \partial^2 f/\partial s^2$, we see that in the case where $\Gamma=0$ we have
$\partial V/\partial t + rs\partial V/\partial s + \tfrac{1}{2}\sigma^2 s^2\partial^2 V/\partial s^2 = rV$.

And so V satisfies the Black-Scholes PDE.

10 We will make use of the following notation:

$B(t,T)$	= Zero-coupon bond price
	= price at t for £1 payable at T
$r(t)$	= instantaneous risk-free rate of interest at t
$C(t)$	= unit price for investment at the risk-free rate
$F(t,T,S)$	= forward rate at t for delivery between T and S
$f(t,T)$	= instantaneous forward-rate curve
$R(t,T)$	= spot-rate (zero-coupon yield) curve

Zero-coupon bond prices are related to the spot-rate and forward-rate curves in the following way:

$$R(t,T) = \frac{-1}{T-t}\log B(t,T) \quad \text{for } t < T$$

or $$B(t,T) = \exp[-R(t,T)(T-t)]$$

$$F(t,T,S) = \frac{1}{S-T}\log\frac{B(t,T)}{B(t,S)} \quad \text{for } t < T < S$$

$$f(t,T) = \lim_{S \to T} F(t,T,S) = -\frac{\partial}{\partial T}\log B(t,T)$$

or $$B(t,T) = \exp\left[-\int_t^T f(t,u)du\right].$$

$$B(t,T) = E_Q\left[\exp\left(-\int_t^T r(u)du\right)\middle| r(t)\right]$$

for specific models.

It is important to remember that Q is an artificial computational tool. It is determined by combining (a) the model for $r(t)$ under the real world measure P and (b) the market price of risk established from knowledge of the dynamics of one bond.

11 (i) Merton's model assumes that a corporate entity has issued both equity and debt such that its total value at time t is of $F(t)$. $F(t)$ varies over time as a result of actions by the corporate entity which does not pay dividends on its equity or coupons on its bonds. Part of the corporate entity's value is zero-coupon debt with a promised repayment amount of L at a future time T. At time T the remainder of the value of the corporate entity will be distributed amongst the equity holders and the corporate entity will be wound up.

The corporate entity will default if the total value of its assets, $F(T)$ is less than the promised debt repayment at time T i.e. $F(T) < L$. In this situation, the bond holders will receive $F(T)$ instead of L and the equity holders will receive nothing. This can be regarded as treating the equity holders of the corporate entity as having a European call option on the assets of the company with maturity T and a strike price equal to the value of the debt.

The Merton model can be used to estimate either the risk-neutral probability that the company will default or the credit spread on the debt.

(ii) We assume the Merton model, so the value of the company is the value of a call on the assets. The underlying is the gross value and the strike is the debt.

Thus $S_0 = 10.009$, $\sigma = 0.2$, $T = 1$, $K = 8$, and 2.9428 is the value of the call (at time 0).

So, $2.9428 = 10.009\Phi((\ln(10.009/8) + .02 + r)/0.2) - 8e^{-r}\Phi((\ln(10.009/8) - .02 + r)/0.2) = 10.009\Phi(1.2202 + 5r) - 8e^{-r}\Phi(1.0202 + 5r)$. This is a differentiable and increasing function of r so interpolation should get a solution.

Setting $r = 10\%$, we get $10.009\Phi(1.2202 + 5r) - 8e^{-r}\Phi(1.0202 + 5r) = 10.009\Phi(1.7202) - 8e^{-.1}\Phi(1.5202) = 10.009 \times 0.95730 - 8e^{-.1} \times 0.93577 = 2.80786$, so we need to increase r.

Setting $r = 15\%$, we get $10.009\Phi(1.2202 + 5r) - 8e^{-r}\Phi(1.0202 + 5r) = 10.009\Phi(1.9702) - 8e^{-.1}\Phi(1.7702) = 10.009 \times 0.97559 - 8e^{-.15} \times 0.96166 = 3.14301$, so we need to decrease r.

Interpolating gives $r = 10 + 5X(2.9428 - 2.80786)/(3.14301 - 2.80786)\% = 12\%$.

If we try $r = 12\%$, we get $10.009\Phi(1.2202 + 5r) - 8e^{-r}\Phi(1.0202 + 5r) = 10.009\Phi(1.8202) - 8e^{-.12}\Phi(1.6202) = 10.009 \times 0.96564 - 8e^{-.12} \times 0.94740 = 2.9429$, so $r = 12\%$.

END OF EXAMINERS' REPORT