INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

April 2012 examinations

Subject CT8 – Financial Economics
Core Technical

Purpose of Examiners' Reports

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and who are using past papers as a revision aid, and also those who have previously failed the subject. The Examiners are charged by Council with examining the published syllabus. Although Examiners have access to the Core Reading, which is designed to interpret the syllabus, the Examiners are not required to examine the content of Core Reading. Notwithstanding that, the questions set, and the following comments, will generally be based on Core Reading.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report. Other valid approaches are always given appropriate credit; where there is a commonly used alternative approach, this is also noted in the report. For essay-style questions, and particularly the open-ended questions in the later subjects, this report contains all the points for which the Examiners awarded marks. This is much more than a model solution – it would be impossible to write down all the points in the report in the time allowed for the question.

T J Birse
Chairman of the Board of Examiners

July 2012

General comments on Subject CT8

Subject CT8 introduces the main concepts and principles of financial economics. These are developed in later subjects in the ST series of exams. This subject combines various types of skills. In particular, along with CT7, it is one of the first where candidates are expected to write lengthy passages of reasoned thought, rather than just complete calculations. This is a skill that will be new to many, and candidates are advised to pay particular attention to the answers to this type of question by studying many past papers.

Comments on the April 2012 paper

The general performance was good and better than on the previous session (September 2011). Candidates generally found this paper challenging, but well-prepared candidates scored well across the whole paper and the best candidates scored close to full marks. As in previous diets, questions that required an element of application of the core reading to situations that were not immediately familiar proved more challenging to most candidates. The comments that follow the questions concentrate on areas where candidates could have improved their performance. Candidates approaching the subject for the first time are advised to concentrate their revision in these areas and the ability to apply the core reading to similar situations.

1 (i) The assumptions underlying the Black-Scholes model are as follows:

1. The price of the underlying share follows a geometric Brownian motion.
2. There are no risk-free arbitrage opportunities.
3. The risk-free rate of interest is constant, the same for all maturities and the same for borrowing or lending.
4. Unlimited short selling (that is, negative holdings) is allowed.
5. There are no taxes or transaction costs.
6. The underlying asset can be traded continuously and in infinitesimally small numbers of units.

(ii) A Brownian Motion Z has the following properties

1. Z_t has independent increments, i.e. $Z_t - Zs$ is independent of $\{Zr\ , r \le s\}$ whenever $s < t$.
2. Z_s has stationary increments, i.e. the distribution of $Z_t - Zs$ depends only on $t - s$.
3. Z_s has Gaussian increments, i.e. the distribution of $Z_t - Zs$ is $N(0, t - s)$.
4. Z has continuous sample paths $t \to Z_t$ *(note that Property (4) is a consequence of (1)–(3)).*

(iii) The Black-Scholes formula describes option prices in terms of anticipated values of volatility over the term of the option. Given observed option prices in the market, it is possible to work backwards to the *implied volatility*, that is, the value of σ which is consistent with observed option. Examination of historic option prices suggests that volatility expectations fluctuate markedly over time.

The candidates who were familiar with the bookwork scored very well.

2 (i) $E_M = 9\%$

(ii)

Asset number	*1*	*2*	*3*	*4*	*5*
Expected return	6%	5%	8%	13%	11%
Market capitalisation (in \$)	2.6m	3.9m	5.2m	6.5m	1.3m
Beta	5/8	1/2	7/8	1.5	5/4

(iii) $\beta_P = 19/24$

(iv) *P* does not belong to the Capital Market Line because (except in degenerate cases) portfolios on the efficient frontier consist of linear combinations of the market portfolio and the risk-free asset.

Generally answered well by candidates. Most candidates were able to score full marks on parts (i) and (ii). The rest of the question proved to be a bit more difficult.

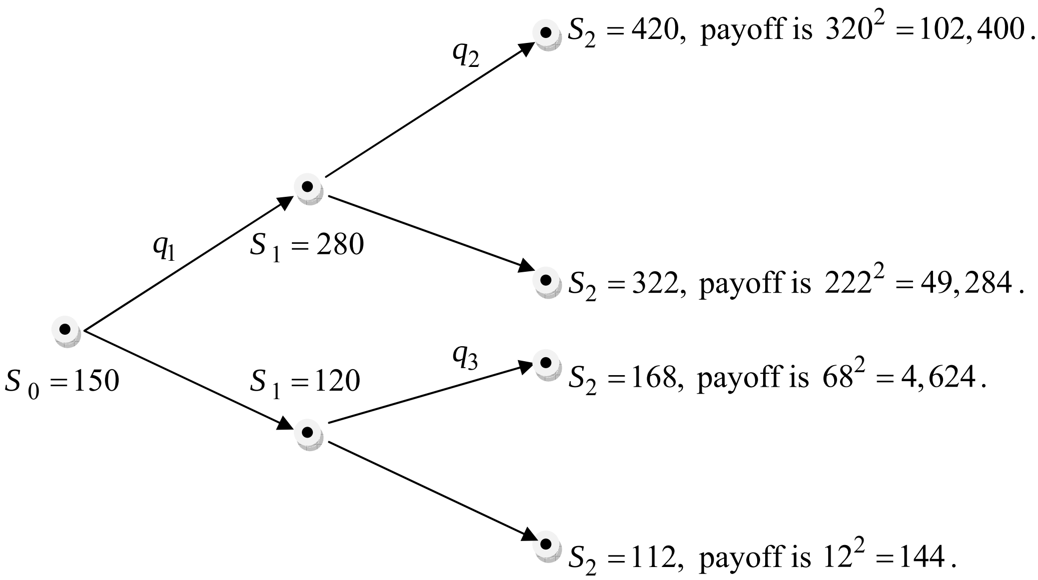
3 (i) The no arbitrage condition implies that $d < 1+r < u$.

At the current time, this implies that $0.8 < 1+r < 1.8667$.

After an *up* move we have $1.15 < 1+r < 1.5$.

After a *down* move we have $0.9333 < 1+r < 1.4$.
Since the rate of interest has to satisfy all of these inequalities, we obtain $15\% < r < 40\%$.

(ii) The model can be drawn as follows:



Then we can calculate

$$q_1 = \frac{180-120}{280-120} = 0.375\,,\ q_2 = \frac{336-322}{420-322} = 0.14286$$

and $$q_3 = \frac{144-112}{168-112} = 0.57143.$$

The value of the option is therefore

$$V = \frac{1}{(1+r)^2}\left[102{,}400 q_1 q_2 + 49{,}284 q_1 (1-q_2) + 4{,}624(1-q_1) q_3 + 144(1-q_1)(1-q_3)\right]$$

$$= 15{,}984.$$

Generally candidates scored well on this question.

4 (i) Consider a portfolio, A, consisting of a European call on the share and a sum of money equal to $\$30e^{-\frac{5\%}{3}} + \$0.50e^{-\frac{5\%}{6}}$.

After 4 months, portfolio A has a value which is equal to the value of the underlying share plus the dividend invested for two months, provided that the share value is above \$30. If the value of the share is below \$30, then the payoff from portfolio A is great than that from the share with the dividend reinvested. So

$$c + \$30e^{-\frac{5\%}{3}} + \$0.50e^{-\frac{5\%}{6}} \geq \$28$$

$$\Rightarrow c \geq -\$2$$

The call option gives the holder the right to buy the underlying share for \$30. So the payoff is always less than the value of the share after 4 months.

Therefore the value of the call option must be less than or equal to the value of the share:

$$c_0 \leq \$28.$$

(ii) Consider the following two portfolios:

A: one call option plus cash of $\$30e^{-\frac{5\%}{3}} + \$0.50e^{-\frac{5\%}{6}}$
B: one put option plus one share
After four months both portfolios have value

$$\max\{\$30, S_{4\,months}\} + \$0.50e^{-\frac{5\%}{6}}.$$

Therefore they should have the same value at any time before 4 months, so

$$c_0 + \$30e^{-\frac{5\%}{3}} + \$0.50e^{-\frac{5\%}{6}} = \$3 + \$28,$$

and so $c_0 = 1$.

Generally answered well by candidates. Most candidates were able to score full marks on part (i).

5 (i) $VaR_\alpha(X) = -t$ where $\mathbb{P}(X < t) = \alpha.$

(ii) Following the hint in the question:

$$\alpha = \mathbb{P}(X < -VaR_\alpha)$$

$$= \mathbb{P}\left(\frac{X-\mu}{\sigma} < \frac{-VaR_\alpha - \mu}{\sigma}\right)$$

$$= \mathbb{P}\left(Z < \frac{-VaR_\alpha - \mu}{\sigma}\right)$$

where Z is a standard Normal random variable.

Therefore $= \Phi\left(\frac{-VaR_\alpha - \mu}{\sigma}\right)$

$$\Rightarrow \Phi^{-1}(\alpha) = \frac{-VaR_\alpha - \mu}{\sigma},$$

and so $VaR_\alpha = -(\mu + \sigma\Phi^{-1}(\alpha)).$

(iii) As the loss distribution is continuous, we have

$$TailVaR_\alpha = \mathbb{E}\left(-X | X < -VaR_\alpha\right)$$

$$= -\frac{1}{\alpha}\int_{-\infty}^{-VaR_\alpha} x\varphi_{\mu,\sigma}\left(x\right)dx$$

$$= -\mu - \frac{\sigma}{\alpha}\int_{-\infty}^{-\Phi^{-1}(\alpha)} x\varphi_{0,1}\left(x\right)dx$$

$$= -\mu - \frac{\sigma}{\alpha}\left[-\varphi_{0,1}\left(x\right)\right]_{-\infty}^{-\Phi^{-1}(\alpha)}$$

$$= \frac{\sigma}{\alpha}\varphi\left(\Phi^{-1}\left(\alpha\right)\right) - \mu$$

(iv) $VaR = -£350m\left[10\% - 25\% \times 2.32635\right] = £168.56m$

$$TailVaR = -£350m\left[10\% - \frac{25\%}{1\%}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\times 2.32635^2}\right] = £198.21m$$

There were typographical errors in the question which should have defined $TailVaR_\alpha = \mathrm{E}(-X | X < -VaR_\alpha)$. *Generous consideration was given to all scripts containing any reasonable attempt in the marking of this question.*

6 (i) $\boldsymbol{dr\left(t\right) = \alpha\left(\mu - r\left(t\right)\right)dt + \sigma dW(t)}$ where $\boldsymbol{W}$ is a standard Brownian motion.

(ii) This process is an Ornstein-Uhlenbeck process.

(iii) Let $r\left(t\right) = s(t)e^{-\alpha t}$ so

$$ds\left(t\right) = d\left(r\left(t\right)e^{\alpha t}\right) = \alpha e^{\alpha t}r\left(t\right)dt + e^{\alpha t}dr\left(t\right)$$

$$= \alpha e^{\alpha t}r\left(t\right)dt + e^{\alpha t}\alpha\left(\mu - r\left(t\right)\right)dt + e^{\alpha t}\sigma dW\left(t\right)$$

$$= \alpha\mu e^{\alpha t}dt + \sigma e^{\alpha t}dW\left(t\right).$$

Thus $s(t) = s(0) + \mu\left(e^{\alpha t} - 1\right) + \sigma\int_0^t e^{\alpha s} dW(s)$,

and $r(t) = r(0)e^{-\alpha t} + \mu\left(1 - e^{-\alpha t}\right) + \sigma\int_0^t e^{\alpha(s-t)} dW(s)$.

(iv) For t given, $r(t)$ is normally distributed.

(v) For t given, the expected value of r(t) is given by:

$$E\left(r(t)\right) = E\left(r(0)e^{-\alpha t} + \mu\left(1 - e^{-\alpha t}\right) + \sigma\int_0^t e^{\alpha(s-t)} dW(s)\right)$$

Hence,

$$E\left(r(t)\right) = r(0)e^{-\alpha t} + \mu\left(1 - e^{-\alpha t}\right)$$

The second moment of r(t) is given by:

$$E\left(\left(r(t)\right)^2\right) = E\left(\left(r(0)e^{-\alpha t} + \mu\left(1 - e^{-\alpha t}\right) + \sigma\int_0^t e^{\alpha(s-t)} dW(s)\right)^2\right)$$

$$E\left((r(t)^2\right) = (r(0)e^{-\alpha t} + \mu\left(1 - e^{-\alpha t}\right))^2 + \sigma^2\int_0^t e^{2\alpha(s-t)} ds$$

(vi) The process may become negative which is undesirable in a nominal interest rate model

This was again standard material from the core reading and more successful candidates tended to score well, although this question proved to be generally challenging.

7 (i) The CEO essentially holds 5,000,000 call options on the stock with strike 100p and maturity 1 year.

(ii) Thus $C = S\Phi(d_1) - Ke^{-rT}\,\Phi(d_2)$,

with $S = 90$, $K = 100$, $d_1 = -.43978$, $d_2 = -.61978$, so $\Phi(d_1) = 0.33005$

and $\Phi(d_2) = .26770$

so that $C = 3.2009$p.

Then $\Delta_C = \partial C/\partial S = \Phi(d_1) = .33005$,
so the hedging portfolio is 5,000,000 ×.33005 = 1,650,250 shares
and 5,000,000 ×.032009 − 1,650,250 ×.9 = £1,325,180 short in cash.

(iii) (a) For 120 > S > 100, we need a and b to satisfy

$$0.6 \times a \times (S - 100) = 0.6 \times 5{,}000{,}000 \times (S - 100)$$

so $a = 5{,}000{,}000$;

For S>120, then we need

$$0.6 \times \{a(S - 100) - b(S - 120)\} = 600{,}000 + 0.2 \times 5{,}000{,}000(S - 120)$$

Equating coefficients of S, we must have $b = 3{,}333{,}333$.

We can then check that the constant terms agree in this equation, too.

(b) The amount saved is the cost of b call options with the higher strike.

We get $C' = S\Phi(d_1) - Ke^{-rT}\Phi(d_2)$, with $S = 90$, $K = 120$, $d_1 = -1.45268$, $d_2 = -1.63268$, $\Phi(d_1) = .07316$, $S\Phi(d_2) = 0.05127$

So $C' = .4932$p
The saving is 3,333,333 ×.004932 = £16,440.

Question 7 was generally found to be difficult. While part (i) was generally straightforward for most candidates, parts (ii) and (iii) proved to be very challenging and were only answered well by the best candidates.

8 (i) $D_0(a) = e^{-rT}E[I_a(S_T)]$

(ii) Since $C_0(K) = e^{-rT}E[\max(S_T - K,0)] = e^{-rT}E[\int_k^\infty I_a(S_T)da]$, we see that

$$C_0(K) = \int_k^\infty D_0(a)da.$$

(iii) It follows that

$$D_0(K) = -d/da\ C_0(K) = -d/dK(S\Phi(d_1) - Ke^{-rT}\Phi(d_2))$$
$$= e^{-rT}\Phi(d_2) - S\varphi(d_1)(dd_1/dK) - Ke^{-rT}\varphi(d_2)(dd_2/dK),$$

where φ is the standard normal density.

Now $(dd_1/dK) = -1/(K\sigma\sqrt{T}) = (dd_2/dK)$,
and $S\varphi(d_1) = (1/\sqrt{2\pi})\exp(\log S - d_1^2/2) = Ke^{-rT}\varphi(d_2)$
and so $D_0(K) = e^{-rT}\Phi(d_2)$.

(iv) We can decompose the payoff for this security as the sum of a call with strike 120 and 1 special option also with strike 120p.

Thus the price is

$$(S\Phi(d_1) - Ke^{-rT}\Phi(d_2) + 100e^{-rT}\Phi(d_2))$$
$$p = 110\Phi(d_1) - (120e^{-r} - 100e^{-r})\Phi(d_2)$$
$$= 110 * (.3877867) - (19.80100) * 0.3138046 = 36.443\text{p}$$

This question proved to be a bit challenging, despite being well within the syllabus..

9 (i) In the two state model, the company defaults at time-dependent rate $\lambda(t)$ if it has not previously defaulted. Once it defaults it remains permanently in the default state. It is assume d that after default all bond payments will be reduced by a known factor $(1 - \delta)$, where δ is the recovery rate. Now we need to change to the risk neutral measure, which will change the default rate to $\lambda'(t)$. This rate is that implied by market prices.

The Jarrow-Lando-Turnbull model generalises the two-state model to $n - 1$ credit ratings plus the default state with transitions possible between any pair of states except for the default state which is absorbing.

(ii) The risk-neutral prices are given by

$$82 = 100e^{-rt}(1 - (1 - \delta_A)(1\text{-}\exp(-\lambda_A t))),$$

and

$$79 = 100e^{-rt}(1 - (1 - \delta_B)(1\text{-}\exp(-\lambda_B t))),$$

so

$$\lambda_A = -\log[1\text{-}(1 - e^{.75r}\, 82/100) / (1 - \delta_A)] / .75 = .74204$$

and

$$\lambda_B = -\log[1\text{-}(1 - e^{.75r}\, 79/100) / (1 - \delta_B)] / .75 = .68583.$$

(iii) Let p denote the risk-neutral double-default probability, then if V is the price of the derivative security we have

$$V = 100{,}000\, e^{-.75r}p,$$

So

$$P = (7900/100000) * e^{.75r} = 0.07989,$$

the corresponding constant rate is $-\log(1 - p) / .75 = 0.111016$.

(iv) Now we get a double-default when both companies default, so this can't happen at rate faster than $\min(\lambda_A, \lambda_B) = \lambda_B = .68583$. This would give rise to a price for the derivative of $V' = 100000e^{-.75r}\, p'$, where $p' = 1 - \exp(-.75\,\lambda_B) = 0.40212$ so $V' = \$39{,}763$.

Alternatively, we can buy \$200,000 nominal of risk-free zero coupon bond and sell \$200,000 nominal of company B's bond. This will cost $200{,}000(e^{-.75r} - .79) = \$39{,}763$ and will pay \$100,000 in 9 months if and only if company B defaults.

Clearly, we would not be willing to pay more than this for the derivative.

Few candidates failed to score well on this exercise.

10 (i) APT requires that the returns on any stock be linearly related to a set of factor indices as shown below

$$R_i = a_i + b_{i,1}\, I_1 + b_{i,2}\, I_2 + \ldots + b_{i,L}\, I_L + c_i\,,$$

where R_i is the return on security i,

a_i and c_i are the constant and random parts respectively of the component of return unique to security i,

$I_1 \ldots I_L$ are the returns on a set of L indices,

$b_{i,k}$ is the sensitivity of security i to index k.

The more general result of APT, that all securities and portfolios have expected returns described by:

$$E_i = \lambda_0 + \lambda_1\, b_{i,1} + \lambda_2 b_{i,2} + \ldots + \lambda_L b_{i,L}\,.$$

The principal strength of the APT approach is that it is based on the no-arbitrage conditions.

(ii) Weaknesses:

(1) In order to apply APT, we need to define a suitable multi-index model.

(2) We also need to come up with the correct factor forecasts. The hard part is the factor forecasts: finding the amount of expected excess return to associate with each factor. The simplest approach is to calculate a history of factor returns and take their average. This implicitly assumes an element of stationarity in the market.

This was standard material from the core reading and more successful candidates tended to score well, although some struggled to get all points required for full marks.

END OF EXAMINERS' REPORT