INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

April 2011 examinations

Subject CT8 — Financial Economics
Core Technical

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

T J Birse
Chairman of the Board of Examiners

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Overall the paper was answered well and candidates' performance was satisfactory. The comments below each question indicate where candidates had the most difficulty.

1 (i) [Unit 13 pp1-2, Unit 8 p2] I

The assumptions underlying the Black-Scholes model are as follows:

1. The price of the underlying share follows a geometric Brownian motion.
2. There are no risk-free arbitrage opportunities.
3. The risk-free rate of interest is constant, the same for all maturities and the same for borrowing or lending.
4. Unlimited short selling (that is, negative holdings) is allowed.
5. There are no taxes or transaction costs.
6. The underlying asset can be traded continuously and in infinitesimally small numbers of units.

(ii) [Unit 8 p3 para 1]

A Brownian Motion Z has the following properties:

(1) Zt has independent increments, i.e. $Zt - Z_s$ is independent of $\{Z_r, r \leq s\}$ whenever $s < t$.

(2) Z has stationary increments, i.e. the distribution of $Zt - Z_s$ depends only on $t - s$.

(3) Z has Gaussian increments, i.e. the distribution of $Zt - Z_s$ is $N(0, t - s)$.

(4) Z has continuous sample paths $t \rightarrow Zt$

Candidates seemed to know this material well, and had no particular problems with this question.

2 (i)

- It is assumed that investors select their portfolios on the basis of the expected return and the variance of that return over a single time horizon.
- It is assumed that the expected returns, variance of returns and covariance of returns are known for all assets and pairs of assets.
- Investors are never satiated. At a given level of risk, they will always prefer a portfolio with a higher return to one with a lower return.

- Investors dislike risk. For a given level of return, they will always prefer a portfolio with lower variance to one with higher variance.

(ii) (a) Let the proportion invested in asset *i*, be *xi* , with expected return *Ei* , variance *Vi* and correlation ρ12. Let *E* be the return on the portfolio of the three assets and let λ and μ be Lagrange multipliers. Then, the Lagrangian function *W* satisfies:

$$W = \sum_{i=1}^{3} x_i^2 V_i + 2\rho_{12}\sigma_1\sigma_2 x_1 x_2 - \lambda(E_1x_1 + E_2x_2 + E_3x_3 - E) - \mu(x_1 + x_2 + x_3 - 1)$$

$$= 36x_1^2 + 144x_2^2 + 324x_3^2 + 108x_1x_2 - \lambda(4x_1 + 6x_2 + 8x_3 - E) - \mu(x_1 + x_2 + x_3 - 1)$$

(b) $\frac{\partial W}{\partial x_1} = 0 \Rightarrow 72x1 + 108x2 - 4\lambda - \mu = 0$

$\frac{\partial W}{\partial x_2} = 0 \Rightarrow 108x1 + 288x2 - 6\lambda - \mu = 0$

$\frac{\partial W}{\partial x_3} = 0 \Rightarrow 648x3 - 8\lambda - \mu = 0$

$\frac{\partial W}{\partial \lambda} = 0 \Rightarrow 4x1 + 6x2 + 8x3 = 5$

$\frac{\partial W}{\partial \mu} = 0 \Rightarrow x1 + x2 + x3 = 1$

Solving this set of simultaneous equations gives $x1 = 0.7125$, $x2 = 0.075$ and $x3 = 0.2125$.

$72x1 + 108x2 - 4\lambda - \mu = 0$ (1)
$108x1 + 288x2 - 6\lambda - \mu = 0$ (2)
$648x3 - 8\lambda - \mu = 0$ (3)
$4x1 + 6x2 + 8x3 = 5$ (4)
$x1 + x2 + x3 = 1$ (5)
(1) $\Rightarrow \mu = 72x1 + 108x2 - 4\lambda$ (6)
into (2) $36x1 + 180x2 - 2\lambda = 0 \Rightarrow \lambda = 18x1 + 90x2$ (7)
(4) and (5) into (3) $648x3 - 144x1 - 468x2 = 0$ (8)
(5) $\Rightarrow x3 = 1 - x1 - x2$ (9)
(7) into (4) $\Rightarrow 4x1 + 6x2 + 8 - 8x1 - 8x2 = 5$
$\Rightarrow 4x1 + 2x2 = 3$
$\Rightarrow x2 = 1.5 - 2x1$ (10)

(10) and (9) into (8) $\quad 648 - 792x1 - 1116x2 = 0$

$$648 - 792x1 - 1674 + 2232x1 = 0$$
$$1440\,x1 - 1026 = 0$$
$$\Rightarrow x1 = 0.7125$$
$$\Rightarrow x2 = 0.075$$
$$\Rightarrow x3 = 0.2125$$

Although most candidates could write down the Lagrangian, several missed the factor of 2 in front of the covariance term. The handling of the Lagrangian showed that many candidates could write down the partial differential equations for optimisation, but were unable to solve them simultaneously.

3 (i) The market price of risk is $(EM - r)/\sigma M$ so $\sigma M = (EM - r)/0.1 = .02/.1 = 20\%$

(ii) The market portfolio is in proportion to the market capitalisation since every investor holds risky assets in proportion to that portfolio. Thus the market portfolio is $.2A + .3B + .5C$ and so $EM = .2EA + .3EB + .5EC$ so $E_B = (.053 - .2\times.04 - .5\times.06)/.3 = 5\%$.

(iii) Assets all lie on the securities market line, so $Ei - r = \beta i(EM - r)$, where $\beta i = \text{Cov}(Ri, RM)/\text{Var}(RM)$.

It follows that $\beta A = .007/.02 = .35$, $\beta B = .017/.02 = .85$ and $\beta C = .027/.02 = 1.35$.

Then $\text{Var}(RM) = .04$ (from part (i)) so $\text{Cov}(RA, RM) = 0.014$, $\text{Cov}(RB, RM) = 0.034$ and $\text{Cov}(RC, RM) = 0.054$.

Generally well-answered by most candidates.

4 (i) Bookwork Unit

Strong form EMH: market prices incorporate all information, both publicly available and also that available only to insiders.

Semi-strong form EMH: market prices incorporate all publicly available information.

Weak form EMH: the market price of an investment incorporates all information contained in the price history of that investment.

(ii) Any reasonable comments:-the market was expecting more and reacted efficiently on the release of insider information. This does suggest that Strong form EMH doesn't hold. It doesn't seem to contradict weak or semi-strong EMH. However, the price fall could be an over-reaction which would contradict the semi-strong form.

Part (i) was well-answered by most candidates. In part (ii) the comments on the statement were disappointingly unclear.

5 (i) Consider an investment of x in cash and y in the stock at time t. Equating the value of this portfolio to the value of the derivative at time $t = 1$ we find the two simultaneous equations:

$xer + yu = \alpha$,
$xer + yd = \beta$.

Rearranging we find:

$$y = \frac{\alpha - \beta}{u - d}, \text{ and}$$

$$x = e^{-r}\frac{\beta u - \alpha d}{u - d}.$$

(ii) $x + y = e{-}r[q\alpha + (1 - q)\beta]$

where q is the risk-neutral probability we are seeking.

$$\text{So } q = \frac{(x + y)e^r - \beta}{\alpha - \beta}.$$

(iii) (a) For the call option we have:

$y = 55\tfrac{5}{9},\ x = -44\tfrac{4}{9},$ and so $x + y = 11\tfrac{1}{9}$.

For the put option we have:

$x = 55\tfrac{5}{9},\ y = -44\tfrac{4}{9},$ and so $x + y = 11\tfrac{1}{9}$.

(b) The strike price (for the at-the-money option) is just $St = 100$. Therefore, the put-call parity relation holds.

Several candidates misread the question and took y to denote the number of shares rather than their initial value. There were also a significant number of careless errors in the calculation.

6 (i) The lognormal model has independent, stationary normal increments for the log of the asset price. Thus, if S_u denotes the stock price at time u, then $\log(S_t/S_s) \sim N(\mu(t-s), \sigma^2(t-s))$ where μ is the drift and σ is the volatility parameter.

(ii) $p = P(S_1 > €2.20) = P(\log(S_1/S_0) > \log(2.2/1.83)) = P(N(0,1) > (\log(2.2/1.83) - .0428)/.12) = 1 - \Phi(1.17784) = 0.1194$

(iii) No, I would not buy the contract. Assuming the log normal model, we are in a Black-Scholes market and the fair price for the option is $f = E_Q[e^{-.02}C]$ where C is the contract value at expiry date, and Q is the EMM. Under the EMM, the discounted stock price will be a martingale i.e S will be lognormal with drift $.02 - ½\sigma^2 = .0128$ and volatility σ. Now $f = €1000e^{-.02}p'$, where $p' = Q(S_1 > €2.20)$, and since S has a smaller drift under Q than under the real-world measure, this will be a smaller price than I am being offered.

(iv) (a) μ' is $N(\mu, \sigma^2/10)$ so $\mu' - \mu \sim N(0, 0.00144)$.

(b) A priori, therefore, $P(\mu' - \mu > 0.03) = 1 - \Phi(.03/\sqrt{.00144})$
$= 1 - \Phi(.79057) = .21459$.

(v) (a) If the true value of μ is <0.0128 then p is smaller than p' and so the option is a bargain!

(b) The probability of this level of error in the estimate of μ is relatively large even though we have 10 years of data.

In fact, this shows the difficulty in estimating drifts in market models generally.

The poorer candidates answered this question in a way that is inconsistent with the Core Reading, taking the drift parameter to refer to the parameter in the Black Scholes model. This resulted in incorrect numerical answers.

7 According to the Core Reading the factors and the effect they would have are:

(1) The premium would decrease as the underlying share price increased.
(2) The premium would increase as the strike price increased.
(3) The premium would increase as the time to expiry increased.
(4) The premium would increase as the volatility of the underlying share increased.
(5) The premium would decrease as interest rates increased.

A very well answered question.

8 (i) The unique fair price is $V = E_Q[e{-}rTD]$, where Q is the EMM

(ii) (a) Standard interpolation using the Black-Scholes formula gives σ = 20% as follows:

using Black-Scholes, $C = S0\Phi(d1) - ke{-}rT\Phi\,(d2)$, with
$d1 = (rT + \tfrac{1}{2}\sigma 2T)/\sigma\sqrt{T} = (.01 + \tfrac{1}{2}\sigma 2)/\sigma$
and $d2 = (.01 + \tfrac{1}{2}\sigma 2)/\sigma$, $S0 = k = 118.57$ and $C = 10$.

Trying σ = 15% gives a value of $d1 = .14167$ and $d2 = -.00833$ which gives $\Phi(d1) = .55633$, $\Phi(d2) = .49668$, and thus a trial value for C of $118.57 \times (.55633 - e{-}.01 \times .49668) = 7.65868$.

Trying σ = 25% gives a value of $d1 = .165$ and $d2 = -.085$ which gives $\Phi(d1)$=.56553, $\Phi(d2)$=. 46613, and thus a trial value for C of $118.57 \times (.56553 - e{-}.01 \times .46613) = 12.33579$.

Interpolation gives a new trial value of σ of
$15 + (10 - 7.65868)/(12.33579 - 7.65868) \times 10 = 20\%$.

With this value for σ we get a value of
$d1 = 0.15$ and $d2 = -.05$ which gives
$\Phi(d1) = 0.5596$, $\Phi(d2) = 0.4801$, and thus a trial value for C of
$118.57 \times (0.5596 - e{-}.01 \times 0.4801) = 9.993$.

Thus σ = 20%.

(b) The call's Delta = $\Delta C = \partial f/\partial S = \Phi(d1)$, where
$d1 = (\log(S/K) + rT + \tfrac{1}{2}\sigma 2T) / \sigma\sqrt{T} = 0.15$ and
$\Phi(.15) = 0.55962$, so the hedge is $1000\Delta = 559.62$ units of stock and
£10,000 − 118.57 × 559.62 = − £56,354 in cash.

(c) Vega = $VC = \partial f/\partial\sigma = \partial/\partial\sigma(S\Phi(d1) - Ke^{-rT}\,\Phi(d_2))$
$= (S\varphi(d_1)\,\partial d1/\partial\sigma - Ke{-}rT\,\varphi(d2)\,\partial\,d2/\partial\sigma)$
$= (S\varphi(0.15)(\tfrac{1}{2} - r/\sigma 2) + Ke^{-rT}\,\varphi(\,-0.05)(\tfrac{1}{2} + r/\sigma 2))$
$= 118.57\times\,(0.25Xe{-}.01125 + 0.75{\times}e{-}.00125{\times}e{-}.01)/\sqrt{(2\pi)}$
$= 46.773$
[since $d2 = (\log(S/K) + r - \tfrac{1}{2}\sigma 2T)/\sigma\sqrt{T} = -0.05$ and
$\partial d2/\partial\sigma = -(\tfrac{1}{2} + (r + \log(S/K))/\sigma 2)$]

(iii) The put price is $p = Ke^{-rT}\,\Phi(-d2) - S\Phi(-d1)$, where
$d1 = (S/K) + r + \tfrac{1}{2}\sigma 2T)/\sigma\,\sqrt{T} = 0.52512$ and
$d2 = (\log(S/K) + r - \tfrac{1}{2}\sigma 2T)/\sigma\,\sqrt{T} = 0.32512$.
So the price is $110Xe{-}.01{\times}.37254 - 118.57{\times}.29975 = £5.0303$

(iv) If we have a portfolio of a shares, b puts and m cash we require to match the value, Delta and Vega of the option. This gives three equations:

(1) $aS + bp + m = 10$
(2) $a + b\,\Delta P = \Delta C$
(3) $bVP = VC$

Equation (3) gives $b = 1.18506$;
equation (2) then gives $a = 0.91484$;
equation (1) gives $m = -£104.43$.

This question was generally not well answered, with errors being made in simple calculations of hedging portfolios.

9 (i) (a) Under the Merton model, the value, Ft, of the firm follows a Geometric BM under the EMM. It follows that the terminal value of the debt is $\min(FT, L1)$, where Li is the tier i nominal debt (since FT is available to pay the senior debt).

(b) Subtracting this value from the value of the firm we see that the assets available to redeem the junior debt are $\max(FT - L1, 0)$. It follows that the terminal value of the junior debt is $\min(L2, \max(FT - L1, 0))$.

(ii) Using a Black-Scholes approach, the current value of the senior debt is $V1 = E[e{-}rT \min(FT, L1)] = E[e{-}rT\,(FT - \max(FT - L1, 0)] = F0 - C1$, where $C1$ is the initial value of a call on the value of the firm with strike $L1$. The current value of the junior debt is $V2 = E[e{-}rT\min(L2, \max(FT - L1, 0))]$.

We obtain immediately $V1 = 88.26*12{,}000 = 1059120$

Now the value of the junior debt is $C1 - C2 = F0 - V1 - C2$ – where
$C2 = E[e{-}rT\max(F_T - (L1 + L2), 0))]$.

Using Black-Scholes, $C2 = F_0\Phi(d1) - (L1 + L2)\, e{-}rT\, \Phi(d2)$, with
$d1 = (\ln(F0/L1 + L2) + rT + ½\sigma2T)/\sigma\sqrt{T}$
$= (\ln(1) + .12 + ½*.09*3)/.3*\sqrt{3} = 0.49075$
and $d2 = (\ln(F0/L1 + L2) + rT - ½\sigma2T)/\sigma\sqrt{T} = -.02887$.

$\Phi(d1) = 0.68819$ and $\Phi\,(d_2) = 0.48848$ and so
$C_2 = 3.2*.68819 - e{-}0.12*3.2*.48848 = 0.81583m = £815{,}830$. Thus the junior debt is worth $= C1 - C2 = 32000000 - 1059120 - 815830 = 1325050$.
This is the value of £2m nominal so the value of £100 nominal is £66.25.

With some notable exceptions, this question was generally very poorly answered. Candidates were unable to perform calculations related to the Merton model, and were unable to identify the payoffs from simple contingent contracts.

10 (i) Risk-neutral approach:

$$B(t,T) = \mathbb{E}_Q\left[\exp\left(-\int_t^T r_u du\right)|F_t\right]$$

State-price deflator approach:

$$B(t,T) = \frac{\mathbb{E}_{\mathbb{P}}\{A(T)|F_t\}}{A(t)}$$

Where $A(t)$ is the deflator.

(ii) The dynamics of the short rate *rt* under $\mathbb{Q}$ for the Vasicek model are:

drt = α(μ – *rt*)*dt* + σ*dZt*,

where Z is a $\mathbb{Q}$-Brownian motion.

This is an Ornstein-Uhlenbeck process.

(iii) Consider *st* = *e*α*trt*. Then

dst = α*e*α*trtdt* + *e*α*t drt*
= αμ*e*α*tdt* + σ*e*α*t dZt*

Thus *st* = *s*0 + μ(*e*α*t* – 1) + $\sigma\int_0^t e^{\alpha u} dZ_u$

and consequently

rt = *e*–α*tr*0 + μ(1 – *e*–α*t*) + $\sigma\int_0^t e^{\alpha(u-t)} dZ_u$

(iv) So *rt* has a Normal distribution and hence from (i), $B(t, T)$ has a lognormal distribution.

This question was largely from a section of the core reading with which some candidates seemed unfamiliar. Candidates need to study the sections relating to interest rate models more carefully. Candidates who knew the bookwork performed well.

END OF EXAMINERS' REPORT