INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

September 2014 examinations

Subject CT8 – Financial Economics
Core Technical

Introduction

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and using past papers as a revision aid and also those who have previously failed the subject.

The Examiners are charged by Council with examining the published syllabus. The Examiners have access to the Core Reading, which is designed to interpret the syllabus, and will generally base questions around it but are not required to examine the content of Core Reading specifically or exclusively.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report; other valid approaches are given appropriate credit. For essay-style questions, particularly the open-ended questions in the later subjects, the report may contain more points than the Examiners will expect from a solution that scores full marks.

The report is written based on the legislative and regulatory context at the date the examination was set. Candidates should take into account the possibility that circumstances may have changed if using these reports for revision.

F Layton
Chairman of the Board of Examiners

November 2014

General comments on Subject CT8

Subject CT8 introduces the main concepts and principles of financial economics. These are developed in later subjects in the ST series of exams. This subject combines various types of skills. In particular, along with CT7, it is one of the first where candidates are expected to write lengthy passages of reasoned thought, rather than just complete calculations. This is a skill that will be new to many, and candidates are advised to pay particular attention to the answers to this type of question by studying many past papers.

Comments on the September 2014 paper

The general performance was good. Candidates found some questions challenging, but well-prepared candidates scored consistently across the whole paper. As in previous diets, questions that required an element of application of the core reading to situations that were not immediately familiar proved more challenging to most candidates. A significant number of candidates failed to read some questions carefully enough to identify the relevant section of the course being examined.

1 The traditional theory of consumer choice has three main elements:

(i) the consumer's preferences
(ii) the budget constraint
(iii) how the consumer decides which consumption bundle to choose

(i) **The consumer's preferences**

Definitions

"Utility" is the satisfaction that a consumer obtains from a particular course of action.

The amount of one good that a consumer is prepared to swap for one extra unit of another good is known as the "marginal rate of substitution".

An "indifference curve" joins all the consumption bundles of equal utility. The slope of a consumer's indifference curves will depend on his or her individual preferences and is equal to the marginal rate of substitution.

A given combination of goods (e.g. two apples and five bananas) is called a "consumption bundle".

Assumptions and results

(a) A consumer can rank any two bundles.

A consumer can rank different bundles, and therefore can pick a set of consumption bundles that give the same utility.

(b) Consumers prefer more of a good to less of it.

Therefore indifference curves slope downwards from left to right and indifference curves further from the origin give higher utility.

(c) Consumer preferences exhibit diminishing marginal rates of substitution.

(ii) **The budget constraint**

The assumptions

(a) The prices of the goods are fixed.
(b) The consumer's income is fixed.

These two assumptions determine which consumption bundles are affordable.

The budget line joins all points that a consumer can afford, assuming that all income is spent.

(iii) **How consumers choose**

Economists assume that consumers' choices exhibit rational behaviour. A rational consumer will choose the consumption bundle that maximises his own utility. This is the concept of utility maximisation.

Implications

Combining the budget line with indifference curves, we can determine the consumption bundle which a consumer will choose. A rational consumer will choose a consumption bundle such that the marginal rate of substitution is equal to the slope of the budget line – that is, where the ratios of marginal utilities equal the ratios of prices.

[9]

This question was generally well answered by candidates who knew the bookwork on consumer choice theory. A significant minority of candidates wrote about behavioural finance (sometimes many pages) and scored few, if any, marks.

2 (i) S has a higher return and a lower variance so is preferable in a mean-variance framework. [2]

(ii) You can relax the assumption that investors solely select their portfolios on the basis of the expected return and variance of that return. [1]

(iii) $E[P] = 0.75 \times E[S] + 0.25 \times E[M] = 9.5\%$

$\text{Var}(P) = 0.75^2\ \text{Var}(S) + 0.25^2\ \text{Var}(M) + 2 \times 0.75 \times 0.25 \times 4 \times 5 \times 0.3$
$= 12.8125\%\%$

So standard deviation (P) = $\sqrt{(12.8125)}$ = 3.57945% [2]

(iv) The amount invested in S, x_S, will be,

$$x_S = \frac{V_M - C_{SM}}{V_S - 2C_{SM} + V_M} = \frac{25 - 4 \times 5 \times 0.3}{16 - 2 \times 4 \times 5 \times 0.3 + 25} = 0.655173$$

invested in S, and so 0.344828 invested in M. [2]

(v) The study suggests that the correlation between M and S will increase.

This means that portfolios containing positive amounts of M and S will have a higher variance.

If the correlation increases, then the minimum variance portfolio will contain relatively higher amounts of S and relatively lower amounts of M. [3]

[Total 10]

Generally well answered. A surprising number of candidates went about deriving the proportion of assets in the minimum variance portfolio from first principles. A number of candidates also weren't able to calculate the variance of a linear combination of correlated random variables. Part (v) required thinking beyond the core reading, and the better candidates scored here.

3 (i) One possible answer is as follows (other acceptable proofs could score full marks):

We need:

$W > 0$,
$W + X < 0 \Leftrightarrow X < 0$ and $\text{abs}(X) > \text{abs}(W)$

The probability of each of these inequalities is 0.5,
and they are all independent.

Therefore the overall probability is 1/8. [5]

(ii) $S_t = e^{A+\mu t+\sigma Z_t}$

Where A, μ and σ are constants and Z_t is the standard Brownian motion. [1]

(iii) However successful the Brownian motion model may be for describing the movement of market indices in the short run, it is useless in the long run, if only for the reason that a standard Brownian motion is certain to become negative eventually.

It could also be pointed out that the Brownian model predicts that daily movements of size 100 or more would occur just as frequently when the process is at level 100 as when it is at level 10,000. [2]

[Total 8]

Many candidates found part (i) challenging, and skipped to other questions without having a go at parts (ii) and (iii), which presented an opportunity for some relatively easy marks.

4 As the option is an American put, it may be optimal to exercise early and we have to test at each node on the binomial tree.

First let us calculate the value of the European put option at each node starting from expiry (the value of the American option is then the maximum of the value of the European option and the intrinsic value at any node).

The risk-neutral probability of an up-jump is:

$$q = \frac{1.04^2 - 0.8}{1.3 - 0.8} = 0.5632.$$

The value of the option payoff for the European options at t = 4 is given by:

Stock price	*Option payoff*
$S_{UU} = 65 \times 1.03^2 = 109.85$	0
$S_{UD} = S_{DU} = 65 \times 1.03 * 0.8 = 67.6$	\$2.40
$S_{DD} = 65 \times 0.8^2 = 41.6$	\$28.40

The value of the European option, and hence American options at time $t = 2$ are then:

Stock price	*American option payoff at t = 2*	*Value of European option at t = 2*
$S_U = 65 * 1.03 = 84.5$	0	\$0.97
$S_D = 65 * 0.8 = 52$	\$18	\$12.71

So if the first jump is down, the option should be exercised early.

Finally, the value at time zero is

$$\text{Max}(\$5, (0.5632 \times \$0.97 + (1 - 0.5632) \times \$18)/1.04^2) = \$7.77$$

[9]

Generally well answered. Most candidates realised that you can exercise an American option early, but few managed to adjust the option price appropriately to allow for this. Some candidates also slipped up over the two-year time steps.

5 (i) One possible answer is as follows (other acceptable proofs could score full marks):

Let K be the forward price. Now compare the setting up of the following portfolios at time 0:

A: one long forward contract.
B: borrow $K \exp(-rT)$ cash and buy one share at S_0.

If we hold both of these portfolios up to time T then both have a value of $S_T - K$ at T.

By the principle of no-arbitrage these portfolios must have the same value at all times before T.
In particular, at time 0, portfolio B has value $S_0 - K\exp(-rT)$ which must equal the value of the forward contract.

This can only be zero (the value of the forward contract at $t = 0$) if $K = S_0\exp(rT)$. [4]

(ii) Similarly consider two portfolios:

A: one long forward contract
B: long 1.05^{-4} units of the share and *short* $K * \exp(-0.06)$ in cash

Following similar logic to part (i), with the dividend being reinvested in the share at each dividend date we find that no arbitrage implies that the fair forward price $K = 500\exp(0.06)\times 1.05^{-4} = 436.79$. [4]

(iii) This does not provide an arbitrage opportunity since the dividend is not risk-free (and if the share price dropped significantly so would the dividend amount, even if the yield remained the same). [2]

[Total 10]

Part (i) was well answered. In part (ii) many candidates made a decent attempt at allowing for dividends, but few got to the right answer. A surprising number of candidates thought that a high dividend yield presented an arbitrage opportunity, failing to appreciate that dividends are not risk-free.

6 (i) We have started off with a process for $r(t)$ which is not a tradable asset. An arbitrage opportunity must relate to trading an asset, therefore arbitrage-free models must allow for trading. [2]

(ii) (a) The *market price of risk* is defined as:

$$\gamma(t,T_1) = \frac{m(t,T_1) - r(t)}{S(t,T_1)}.$$

(b) $\gamma(t,T_1)$ represents the excess expected return over the risk-free rate per unit of volatility in return for an investor taking on this volatility.

(c) With this identity we find that for all $t < T$

$$dB(t,T) = B(t,T)[m(t,T)dt + S(t,T)dW(t)]$$

$$= B(t,T)[(r(t) + \gamma(t)S(t,T))dt + S(t,T)dW(t)]$$

$$= B(t,T)[r(t)dt + S(t,T)(dW(t) + \gamma(t)dt)]$$

$$= B(t,T)\left[r(t)dt + S(t,T)d\tilde{W}(t)\right]$$

where $d\tilde{W}(t) = dW(t) + \gamma(t)dt$ is the standard Brownian motion under Q. [4]

(iii) For a one-factor model we have seen above the broad principal which transforms from P to Q. In order to say more about the basic price processes we must look at the effect of this transformation on $r(t)$.

Thus

$$dr(t) = a(t,r(t))dt + b(t,r(t))dW(t) \text{ under } P$$

$$= a(t,r(t))dt + b(t,r(t))\left(d\tilde{W}(t) - \gamma(t)dt\right)$$

$$= (a(t,r(t)) - \gamma(t)b(t,r(t)))dt + b(t,r(t))\,d\tilde{W}(t)$$

$$= \tilde{a}(t,r(t))dt + b(t,r(t))d\tilde{W}(t)$$

where $\tilde{a}(t,r(t)) = a(t,r(t)) - \gamma(t)b(t,r(t))$.

The final two lines give us the dynamics of $r(t)$ under the artificial measure Q.

We then use this to determine:

$$B(t,T) = E_Q\left[\exp\left(-\int_t^T r(u)du\right)\middle|\mathcal{F}_t\right]$$

for specific models. [3]

(iv) When modellers use this approach to pricing, from the practical point of view they normally start by specifying the dynamics of $r(t)$ under Q in order to calculate bond prices. Second, they specify the market price of risk as a component of the model, and this allows us to determine the dynamics of $r(t)$ under P. [2]

[Total 11]

Part (i) was generally well answered, but very few candidates managed parts (iii) and (iv) which were largely bookwork. Quite a few candidates tried solving the SDE for the log of the ZCB price with no mention of transforming to the risk neutral probability measure.

7 (i) Let $M = \max_{0 \le s \le t} (B_s + \mu s)$, then

$$P(M > y) = \Phi((\mu t - y)/\sqrt{t}) + e^{2\mu y}\,\Phi((-y - \mu t)/\sqrt{t}).$$

It follows that M has density f given by

$$f(y) = -\partial f/\partial y$$

$$= \varphi((\mu t - y)/\sqrt{t})/\sqrt{t} - 2\mu\, e^{2\mu y}\,\Phi((-y - \mu t)/\sqrt{t}) + e^{2\mu y}\,\varphi((-\mu t - y)/\sqrt{t})/\sqrt{t},$$

where φ is the standard normal density: it follows after a little algebra that

$$f(y) = 2\varphi((\mu t - y)/\sqrt{t})/\sqrt{t} - 2\mu\, e^{2\mu y}\,\Phi((-y - \mu t)/\sqrt{t}).$$ [3]

(ii) The fair price is $e^{-r(T-t)}\, E_Q\,[D_t\, \mathcal{F}_t]$,

where Q is the unique risk-neutral (or equivalent martingale) measure, r is the risk-free rate and $\mathcal{F}$ is the filtration. [3]

(iii) Under the EMM, Q, we have:

$$S_t/S_0 - \log N(r - 0.5\sigma^2)t, \sigma^2 t)$$

and $S_t/S_0 = \exp((r - 0.5\sigma^2)t + \sigma B_t)$

The value of the derivative is contingent on:

$$P\left[\max_{0 \le s \le t}(B_s + \mu s) > 1.44\right]$$

Comparing the $B_s + \mu s$ part of this expression with the expression inside the share price formula, we might take:

$$\mu = \frac{r - 0.5\sigma^2}{\sigma} = -0.5$$

(although the value of μ was not given in the question so we award full marks to any student who has derived the correct answer in terms of μ)

$$= 10e^{-0.06} P[\max_{0 \le s \le 2}(B_s + \mu s) > 1.44]$$

$$= 10e^{-0.06} \left\{\Phi\left(\frac{-1.44 - 2\mu}{\sqrt{2}}\right) + e^{2 \times 1.44\mu} \times \Phi\left(\frac{-1.44 + 2\mu}{\sqrt{2}}\right)\right\}$$

[4]
[Total 10]

Many candidates found this question tough and scored few marks. A number of candidates were seemingly put off by part (i) and didn't attempt the potentially easier marks available in later parts.

8 (i) Δ is the first partial derivative of the option price with respect to the underlying asset price. [1]

(ii) Using the formula for the Δ, we see that $\Phi(d_1) = 0.42074$ and hence $d_1 = -0.2$.

Thus $-0.2\,\sigma = -0.0600 + \frac{1}{2}\sigma^2$ or $\frac{1}{2}\sigma^2 + 0.2\sigma - 0.06 = 0$.

Solving the quadratic gives $\sigma = 20\%$ or -60% and rejecting the negative value gives $\sigma = 20\%$. [3]

(iii) When S_τ is at least K_1 then the holder is presented with a choice between \$100 now and the possibility of \$$c$ later. Clearly if $c \le 100$, the holder will always choose to exercise immediately. [2]

(iv) Just after τ, the optional element has expired and the holder is entitled to \$$c$ at time T if and only if $(S_T/S_\tau \ge K_2)$. And so the fair price after time τ is

$$ce^{-r(T-\tau)}\, Q(S_T/S_\tau \ge K_2 | F_\tau),$$

where Q is the EMM. [2]

(v) Since, under

$$Q,\ S_T/S_\tau = \exp(\sigma\,(B_T - B_\tau) + (r - \tfrac{1}{2}\sigma^2)(T - \tau)),$$

where B is a Brownian motion under Q, and since BM has independent increments, S_T/S_τ is independent of F_τ and so the value of the option just after time τ does not depend on S_τ. [2]

(vi) Inserting the parameter values, the value after time 1 is

$$200e^{-0.03}(1 - \Phi(-0.5)) = \$134.20.$$

Since this is greater than \$100, the holder will never exercise the option at time 1 and so

$$V_0 = e^{-0.03}V_1 = \$130.24.$$ [3]

[Total 13]

Reasonably well answered. Many candidates tried to find the volatility by trial and error, which was time-consuming compared to deriving and solving the quadratic equation. Some candidates derived the quadratic equation then still solved it by trial and error. Several candidates tried to determine the price of the exotic option as a call option for part (vi).

9 (i) $B(t,T) = e^{-r(T-t)}[1 - (1 - \delta)(1 - \exp - \int_t^T \lambda_s ds))]$,

where B is the bond price, λ is the risk-neutral default rate, δ is the recovery rate, and r is the risk-free rate. [2]

(ii) Using the formula,

$$0.9278 = e^{-0.03}[1 - 0.5(1 - \exp(-\int_0^1 \lambda_s ds)]$$
$$= e^{-0.03}[1 - 0.5p],$$

where p is the default probability.

Hence $p = 0.08789$. [3]

(iii) Under the Merton model, $Q(\text{default}) = Q(F_T < L)$, where L is the nominal loan amount, and F_t is the gross asset value of the company at time t and Q is the EMM.

Hence

$$Q(\text{default}) = Q(F_0 \exp(\sigma B_T + (r - ½\sigma^2)T) < L)$$

$$= Q(B_T < (\ln(L/F_0) - (r - ½\sigma^2)T)/\sigma)$$

$$= \Phi(\ln(L/F_0) - (r - ½\sigma^2)T)/\sigma\sqrt{T})$$ [3]

(iv) Thus

$$\Phi([\ln(L/F_0) - 0.02155]/0.13) = 0.08789$$

so

$$\ln(L/F_0) = 0.13\Phi^{-1}(0.08789) + 0.02155 = -0.154457$$

and so

$$L/F_0 = 0.85688.$$ [2]

(v) Thus,

$$L_0/F_0 = 0.9278*0.85688 = 0.79501$$

and so

$$E_0/F_0 = 1 - 0.79501 = 20.5\%,$$

where E_t is the net equity value at time t and L_t is the loan value at time t. [5]
[Total 15]

Candidates scored well on parts (i), (ii) and (iii). Well prepared candidates scored full marks but many did not attempt parts (iv) and (v). Many candidates thought that lambda was the probability of default (rather than the integral of lambda) and tried to find lambda without success.

10 (i) (a) The lognormal model has independent, stationary normal increments for the log of the asset price.

Thus, if S_u denotes the stock price at time u, then

$$\log(S_t/S_s) \sim N(\mu(t-s), \sigma^2(t-s))$$

where μ is the drift and σ is the volatility parameter.

(b) It follows that

$$E(S_t) = S_0\exp(\mu t + \tfrac{1}{2}\sigma^2 t)$$

and the variance is

$$\text{Var}(S_t) = S_0^2\exp(2\mu t + \sigma^2 t)(\exp(\sigma^2 t) - 1).$$

[*Alternative*: if students use Geometric BM as the foundation for the lognormal model then they will model

$$\log(S_t/S_s) \sim N(\mu(t-s) - \tfrac{1}{2}\sigma^2(t-s), \sigma^2(t-s)).$$

The formulae for mean and variance will then change to:

$$E(S_t) = S_0\exp(\mu t)$$

and

$$\text{Var}(S_t) = S_0^2\exp(2\mu t)(\exp(\sigma^2 t) - 1).]$$ [4]

(ii) From (i), we see that

$$\exp(\mu+\tfrac{1}{2}\sigma^2) = e^{0.4} \text{ and } \exp(2\mu + \sigma^2)(\exp(\sigma^2) - 1) = e^{-0.4},$$

so that $\exp(\sigma^2) - 1 = e^{-1.2}$.

It follows that

$$\sigma^2 = \log(1 + e^{-1.2}) = 0.2633$$

and

$$\mu = \log(e^{0.4}) - \tfrac{1}{2}\log(1 + e^{-1.2}) = 0.2684.$$

[*Alternative*: if students use Geometric BM then they should obtain

$$\exp(\mu) = e^{0.4} \text{ and } \exp(2\mu)(\exp(\sigma^2) - 1) = e^{-0.4},$$

so that $\exp(\sigma^2) - 1 = e^{-1.2}$.

It follows that

$$\sigma^2 = \log(1 + e^{-1.2}) = 0.2633, \text{ as before, and } \mu = \log(e^{0.4}) = 0.4.]$$

[3]
[Total 7]

Generally very well answered, with most students recalling and applying the bookwork correctly.

END OF EXAMINERS' REPORT