INSTITUTE AND FACULTY OF ACTUARIES

EXAMINATION

28 September 2011 (am)

Subject CT8 — Financial Economics
Core Technical

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*

2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*

3. *Mark allocations are shown in brackets.*

4. *Attempt all 11 questions, beginning your answer to each question on a separate sheet.*

5. *Candidates should show calculations where this is appropriate.*

Graph paper is NOT required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator from the approved list.

1 (i) Define an efficient portfolio in the context of modern portfolio theory. [1]

A market consists of two assets A and B. Annual returns on the two assets (R_A and R_B) have the following characteristics:

Asset	*Expected return %*	*Standard deviation %*
A	6	20
B	10	20

The correlation between the returns on the two assets is 0.25.

(ii) (a) Calculate the proportion that would be invested in each of the two assets in a minimum variance portfolio.

(b) Calculate the expected return of that portfolio.

[3]
[Total 4]

2 Consider the following three-factor model of security returns:

$$R_i = \alpha_i + \beta_{i1} I_1 + \beta_{i2} I_2 + \beta_{i3} I_3 + \varepsilon_i$$

Where:

- R_i is the return on security i
- α_i, β_{i1}, β_{i2} and β_{i3} are security-specific parameters
- I_1, I_2 and I_3 are the changes in the three factors on which the model is based; and
- ε_i are independent random normal variables, each with variance σ^2

(i) Describe three categories of model that could be used to help choose the factors I_1, I_2 and I_3. [6]

(ii) List examples of the variables that could be used for the factors I_1, I_2 and I_3, for two of these three categories of model. [2]
[Total 8]

3 An investor wishes to save for a retirement fund of £100,000 in 10 years' time. The instantaneous, constant continuously compounded risk-free rate of interest is 4% per annum. The investor can purchase shares on a non-dividend paying security with price S_t governed by the Stochastic Differential Equation (SDE):

$$dS_t = S_t(\mu dt + \sigma dZ_t)$$

where:

- Z_t is a standard Brownian motion
- $\mu = 12\%$
- $\sigma = 25\%$
- t is the time from now measured in years; and
- $S_0 = 1$

(i) (a) Derive the distribution of S_t.

(b) Calculate the amount, A, that the investor would need to invest in shares to give a 50:50 probability of building up a retirement fund of £100,000 in 10 year's time.

[4]

(ii) Calculate the following risk measures applied to the difference between the value of the fund and £100,000, if the investor invests A.

(a) Variance
(b) Shortfall probability relative to £90,000
(c) 99% Value at Risk

[6]

The investor decides that they do not need more than £100,000 so they write a call option giving up any upside return above £100,000. They also buy a put option to remove the downside risk of receiving less than £100,000.

(iii) Calculate the net cost at time zero of purchasing enough shares to give themselves a 50:50 chance of building up a retirement fund of £100,000, writing the call option on those shares and buying the put option on the shares.

[2]
[Total 12]

4 Assume that there is no arbitrage in the market. A forward contract is available on a physical asset. The continuously compounded costs of managing the asset are x% of its value, and it provides an income stream of £y per ton payable at six monthly intervals, a payment has just been made.

Let S_t be the spot price of one ton of the asset at time t and let r be the continuously compounded risk-free rate of interest per annum which is assumed to be constant.

Derive the current price of a forward contract written on one ton of the asset with maturity T years where (6 months < T < 1 year). [8]

5 (i) List the desirable characteristics of a model for the term structure of interest rates. [4]

(ii) Write down the stochastic differential equation for the short rate r_t under $\mathbb{Q}$ in the Hull-White model. [1]

(iii) Indicate whether or not the Hull-White model shows the characteristics listed in (i). [4]

[Total 9]

6 Under the real-world probability measure, $\mathbb{P}$, the price of a zero-coupon bond with maturity T is given by:

$$B(t, T) = \exp\left\{-(T-t)r_t + \frac{\sigma^2}{6}(T-t)^3\right\}$$

where r_t is the short rate of interest at time t and satisfies the following stochastic differential equation under the real-world measure $\mathbb{P}$:

$$dr_t = \mu r_t dt + \sigma dZ_t,$$

where $\mu > 0$ and Z_t is a standard Brownian motion under $\mathbb{P}$.

(i) Derive a formula for the instantaneous forward rate $f(t, T)$, based on this model. [2]

(ii) Derive an expression for the market price of risk. [4]

(iii) Deduce the stochastic differential equation for r_t under the risk-neutral measure $\mathbb{Q}$ defining all terms used. [2]

[Total 8]

7 A non-dividend-paying stock, S_t, has a current price of 200p. After 6 months the price of the stock could increase to 230p or decrease to 170p. After a further 6 months, the price could increase from 230p to 250p, or decrease from 230p to 200p. From 170p the price could increase to 200p or decrease to 150p. The semi-annually compounded risk-free rate of interest is 6% per annum and the real-world probability that the share price increases at any time step is 0.75. Adopt a binomial tree approach with semi-annual time-steps.

(i) Calculate the state-price deflator after one year. [5]

(ii) Calculate, using the state-price deflator from (i), the price of a non-standard option which pays out $\max\{0, \log(S_1 - 180)\}$ one year from now. [4]

(iii) State how the answer to (ii) would change if the real-world probability of a share price increase at each time step was 0.6. [1]

[Total 10]

8 A non-standard derivative is written on a stock with current price S_0 = \$2 and is exercisable at two dates, after exactly one year and at expiry, after exactly two years. If it is exercised at expiry it returns \$1000 if and only if the stock price is below \$2. If it is exercised after one year it returns \$500 if and only if the stock price is above \$2.

Assume the market is a Black-Scholes one with a continuously compounded risk-free rate of 2% per annum and a stock volatility of 30% per annum.

(i) (a) Explain how the option should be priced after $t = 1$ (assuming that it is not exercised at $t = 1$).

(b) Give an expression for the corresponding price, p_t.

[4]

(ii) Denoting the price just after 1 year by p_{1+}, explain why the fair price, p_1, at $t = 1$, is given by $p_1 = \max(p_{1+}, 500)$ if $S_1 <$ \$2 and by $p_1 = p_{1+}$ if $S_1 >$ \$2. [2]

(iii) (a) Show that a holder should exercise the option at $t = 1$ if $S_1 > k$ for a suitable value of k.

(b) Calculate the value of k.

[4]

[Total 10]

9 A European call option and a European put option on the same stock with the same strike price have an exercise date one year away and both are priced at 12p. The current stock price is 300p.

The continuously compounded risk free rate of interest is 2% per annum.

(i) Calculate the common strike price, quoting any results that you use. [3]

Assume the Black-Scholes model applies.

(ii) Calculate the implied volatility of the stock. [4]

(iii) Construct the corresponding hedging portfolio in shares and cash for 5000 of the call options. [2]
[Total 9]

10 (i) In the Wilkie model, the force of inflation, $I(t)$, is a mean-reverting AR(1) process.

(a) Explain what the statement above means.

(b) Show that the mean of $I(t)$ converges to m, using the formula:

$$I(t) = m + a(I(t-1) - m) + Z(t),$$

where the $Z(t)$'s are iid $N(0, \sigma^2)$ random variables and $0 < a < 1$.
[4]

(ii) Discuss the differences between and suitability of mean-reverting and random walk models for share prices, interest rates and inflation. [5]
[Total 9]

11 (i) Draw a diagram to illustrate the Jarrow-Lando-Turnbull model for credit default, defining any notation used. [4]

A model was proposed for a country's sovereign debt as follows:

The economy is in one of three states: 1 (good), 2 (bad) and 3 (default).
Transition intensities $\lambda_{i,j}$ are constant and as follows:

$$\lambda_{1,2} = 1; \lambda_{1,3} = 0; \lambda_{2,1} = 0.25, \lambda_{2,3} = 0.75; \lambda_{3j} = 0 \text{ for all } j \text{ and } \lambda_{1,1} = \lambda_{2,2} = -1.$$

It follows that if $p_i(t)$ is the probability that the economy is in state i at time t then:

$$\frac{dp_1(t)}{dt} = -p_1(t) + 0.25\,p_2(t)$$

and

$$\frac{dp_2(t)}{dt} = p_1(t) - p_2(t)\,.$$

Set $h(t) = 2p_1(t) - p_2(t)$.

(ii) (a) Show that $\dfrac{dh(t)}{dt} = -1.5h(t)$.

(b) Derive a similar equation for k defined by $k(t) = 2p_1(t) + p_2(t)$.

[2]

Suppose that this country's economy is in state 2 at time 0.

(iii) Find the probability that it is in default at time 2. [4]

Assume a continuously compounded risk-free interest rate of 2% per annum and a recovery rate of 60%.

(iv) (a) Deduce the price under this model for a zero-coupon bond in this country with a redemption value of 100 and a redemption date in two years' time.

(b) Calculate the credit spread.

[3]
[Total 13]

END OF PAPER