INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

September 2013 examinations

Subject CT8 – Financial Economics
Core Technical

Introduction

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and using past papers as a revision aid and also those who have previously failed the subject.

The Examiners are charged by Council with examining the published syllabus. The Examiners have access to the Core Reading, which is designed to interpret the syllabus, and will generally base questions around it but are not required to examine the content of Core Reading specifically or exclusively.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report; other valid approaches are given appropriate credit. For essay-style questions, particularly the open-ended questions in the later subjects, the report may contain more points than the Examiners will expect from a solution that scores full marks.

The report is written based on the legislative and regulatory context pertaining to the date that the examination was set. Candidates should take into account the possibility that circumstances may have changed if using these reports for revision.

D C Bowie
Chairman of the Board of Examiners

December 2013

General comments on Subject CT8

Subject CT8 introduces the main concepts and principles of financial economics. These are developed in later subjects in the ST series of exams. This subject combines various types of skills. In particular, along with CT7, it is one of the first where candidates are expected to write passages of reasoned thought, rather than just complete calculations. This is a skill that will be new to many, and candidates are advised to pay particular attention to the answers to this type of question by studying many past papers.

Comments on the September 2013 paper

The general performance was good and broadly in line with the previous session (April 2013). Candidates generally found this paper challenging, but well-prepared candidates scored well across the whole paper and the best candidates scored close to full marks. As in previous diets, questions that required an element of application of the core reading to situations that were not immediately familiar proved more challenging to most candidates. The comments that follow the questions concentrate on areas where candidates could have improved their performance. Candidates approaching the subject for the first time are advised to include in their revision these areas and the ability to apply the core reading to similar situations.

1 (i) (a) The expected utility theorem states that a function, $U(w)$ can be constructed representing an investor's utility of wealth, w, at some future date. Decisions are made on the basis of maximising the expected value of utility under the investor's particular beliefs about the probability of different outcomes.

(b) The expected utility theorem can be derived formally from the following four axioms.

1. *Comparability*

An investor can state a preference between all available certain outcomes.

2. *Transitivity*

If A is preferred to B and B is preferred to C, then A is preferred to C.

3. *Independence*

If an investor is indifferent between two certain outcomes, A and B, then he is also indifferent between the following two gambles:

(a) A with probability p and C with probability $(1 - p)$; and
(b) B with probability p and C with probability $(1 - p)$.

4. *Certainty equivalence*

Suppose that A is preferred to B and B is preferred to C. Then there is a unique probability, p, such that the investor is indifferent between B and a gamble giving A with probability p and C with probability $(1 - p)$.

B is known as the certainty equivalent of the above gamble.

(ii) It is usually assumed that people prefer more wealth to less. This is known as the principle of non-satiation and can be expressed as:
$U'(w)>0$ or U is strictly increasing.

Attitudes to risk can also be expressed in terms of the properties of utility functions.

A risk averse investor values an incremental increase in wealth less highly than an incremental decrease and will reject a fair gamble. The utility function condition is $U''(w) < 0$ or U is strictly concave.

(iii) The absolute risk aversion A is given by:

$$A(w) = \frac{-U''(w)}{U'(w)}.$$

Which for the utility function given can be calculated by taking derivatives as, $\frac{-2b}{1+2bw}$.

Now, given the condition $A(1) = 0.25$ yields $b = -0.1$.

Non-satiation means $U'(w) > 0 \iff 1+2bw > 0 \iff -\infty < w < 5$.

This bookwork question was largely well-answered although some candidates appeared to be unaware that non-satiation and absolute risk aversion do not have identical meanings.

2 (i) The single-index model expresses the return on a security as:

$$R_i = \alpha_i + \beta_i R_M + \varepsilon_i$$

where: R_i is the return on security i
α_i and β_i are constants
R_M is the return on the market

The ε_i are independent, zero-mean random variables, uncorrelated with R_M, representing the component of R_i not related to the market.

(ii) The expected return on security i is

$$E_i = \mathbb{E}(R_i) = \mathbb{E}(\alpha_i + \beta_i R_M + \varepsilon_i) = \alpha_i + \beta_i . E_M,$$

where E_M is the expected return on the market.

The variance of returns on security i is $V_i = Var(\alpha_i + \beta_i R_M + \varepsilon_i) = \beta_i^2 V_M + V_{\varepsilon_i}$, where V_M is the variance of returns on the market, V_{ε_i} is the variance of the random variable component of R_i not related to the market and the result holds because under the model ε_i is uncorrelated with R_M.

The covariance of returns between security i and security j is given by $C_{i,j} = Cov(R_i, R_j) = Cov(\alpha_i + \beta_i R_M + \varepsilon_i, \alpha_j + \beta_j R_M + \varepsilon_j) = \beta_i . \beta_j . V_M$, since under the model ε_i is uncorrelated with R_M and ε_i is independent of ε_j for all $i \neq j$.

(iii) Using the results from (ii), the variance of portfolio returns on a portfolio of N equally weighted securities is

$$V = \sum_{i,j=1}^{N} \frac{1}{N^2} Cov\left(R_i, R_j\right)$$

$$= \frac{1}{N^2}\sum_{i=1}^{N}(\beta_i^2 V_M + V_{\varepsilon_i}) + \frac{1}{N^2}\sum_{i,j=1, i\neq j}^{N} \beta i \ . \ \beta j \ .VM$$

$$= \left(\frac{1}{N}\sum_{i=1}^{N}\beta_i\right)^2 V_M \text{ plus terms which tend to zero as } N \to \infty.$$

In other words, the limiting portfolio variance depends on the average value of the β_i s and the variance of the market but not the specific risk of any individual security.

Alternative solution:

The single index model for a portfolio P of N assets held in proportions x_i, …, x_N is:

$$R_P = \alpha_P + \beta_P R_M + \varepsilon_P$$

where $\alpha_P = \sum_{i=1}^{N} x_i \alpha_i$, $\beta_P = \sum_{i=1}^{N} x_i \beta_i$ and $\varepsilon_P = \sum_{i=1}^{N} x_i \varepsilon_i$

So that (by the result in part (ii)):

$$V_P = \beta_P^2 V_M + V_{\varepsilon_P}$$

$$= \left(\sum_{i=1}^{N} x_i \beta_i\right)^2 V_M + \text{var}\left(\sum_{i=1}^{N} x_i \varepsilon_i\right)$$

If $x_i = \dfrac{1}{N}$ then:

$$V_P = \frac{1}{N^2}\left(\sum_{i=1}^{N}\beta_i\right)^2 V_M + \frac{1}{N^2}\left(\sum_{i=1}^{N} V_{\varepsilon_i}\right)$$

$$= \bar{\beta}^2 V_M + \frac{1}{N}\bar{V}$$

Where $\bar{\beta}$ is the average of the individual β_i's and $\bar{V}$ is the average of the V_{ε_i}'s.

As $N \to \infty$, the second component, which represents the specific risk, tends to 0.

(iv) More factors will always improve the fit of a regression to historic data, in other words reduce the residual errors in relation to the data fitted, although market correlation typically has the most explanatory power.

There is little evidence that multi-factor models are significantly better at forecasting the future correlation structure.

Again this was a largely well-answered question, although some candidates didn't define notation despite the explicit instruction to do so. Surprisingly few candidates used the results they had derived in part (ii) to prove part (iii). Some candidates confused this model with CAPM.

3 (i) **Strong form EMH**: market prices incorporate all information, both publicly available and also that available only to insiders.

Semi-strong form EMH: market prices incorporate all publicly available information.

Weak form EMH: the market price of an investment incorporates all information contained in the price history of that investment.

(ii) Scenario 1: The first event tells us nothing about the EMH-assuming this earthquake was not predictable, its happening could not have been discounted in market prices.

A quick adjustment of prices in response to a news announcement suggests evidence for the semi-strong form (and by implication the weak form) EMH.

However, although the price drop was quick, we have no idea how accurate it was. It is possible that the market has over or under reacted to the bad news and will correct itself later. If this is the case, then it suggests markets are not efficient.

Some earthquake specialists (insiders) may have known about the earthquake shortly in advance but there is no mention of price movements before the earthquake, perhaps this suggests the market is also strong form efficient.

Scenario 2: The second event strongly contradicts the strong-form EMH. Insiders are privy to all information about the merger talks and therefore there shouldn't be a sudden reaction.

Indeed, given the public nature of the negotiations, this seems even to contradict the semi-strong form (and by implication the strong form) of the EMH although perhaps markets were pricing in a significant probability of the merger failing or overreacting to the benefits and then correcting themselves.

This question was reasonably answered, although some candidates simply related the same form of EMH to both scenarios. Many candidates missed the fact that the merger had already been publicly negotiated and so wasn't new information.

4 (i) $I(2013) = 0.03 + 0.6(I(2012) - 0.03) + 0.005N = 0.0318 + 0.005N$,

where N is a standard normal r.v. It follows that a 95% confidence interval is

$(0.0318 \pm 0.005 \times 1.9600) = (0.0220, 0.0416)$

(ii) Not at all appropriate, since 1% does not lie in the 95% confidence interval for 2013 and it was 3.3% in 2012!

However there may be compensating assumptions which make this divergence unimportant, for example wage-inflation may also be underestimated.

The pension scheme may have a view that inflation will fall next year, e.g. due to a forecast recession. The Wilkie model parameters are estimated as averages over a historic time period (they are longitudinal estimates) and therefore may not reflect future conditions.

Candidates will be rewarded for making any other self-consistent and reasonable statement about this assumption.

A surprising number of candidates were unable to calculate a confidence interval, and there were many calculation slips. The majority of candidates struggled to interpret part (ii) of the question.

5 (i) (a) Divide by S_t to separate variables:

$$\frac{dS_t}{S_t} = 0.4dt + 0.5dB_t.$$

Use Itô's Lemma to calculate $d \log S_t$:

$$d \log S_t = \frac{dS_t}{S_t} - \frac{0.5}{S_t^2}(dS_t)^2 = 0.275dt + 0.5dB_t.$$

(b) Written in integral form, this reads:

$$\log S_t = \log S_0 + 0.275t + 0.5B_t.$$

or, finally,

$$S_t = S_0 e^{0.275t + 0.5B_t}.$$

So $\dfrac{St}{S0}$ has a lognormal distribution with parameters 0.275*t* and 0.25*t*, or equivalently St is lognormally distributed with parameters log $S0$ + 0.275*t* and 0.25*t*.

The properties of the lognormal distribution give us the expectation and variance of S_t:

$$\mathbb{E}\left(S_2\right) = 97e^{0.8} = 215.877p$$

$$\begin{aligned}\mathrm{Var}\left(S_2\right) &= 97^2(e^{0.55})^2\,\mathrm{Var}\left(e^{0.5B_2}\right)\\ &= 97^2(e^{0.55})^2\, e^{0.5}\left(e^{0.5}-1\right)\\ &= 30{,}232.41\mathrm{p}^2 = (173.87)^2 = (£173.87)^2 = £^2\ 3.0232\end{aligned}$$

(ii) (a) We wish to solve the stochastic differential equation:

$$dR_t = -0.4R_t dt + 0.5dB_t.$$

Consider $U_t = R_t e^{0.4t}$

$$\begin{aligned}dU_t &= 0.4R_t e^{0.4t}dt + e^{0.4t}dR_t\\ &= 0.5e^{0.4t}dB_t\end{aligned}$$

(b) so $U_t = U_0 + 0.5\int_0^t e^{0.4s}dB_s$

and hence

$$R_t = R_0 e^{-0.4t} + 0.5\int_0^t e^{0.4(s-t)}dB_s.$$

Now since

$$\mathbb{E}\left(0.5\int_0^t e^{0.4(s-t)}dB_s\right) = 0,$$

$$E\left(R_2\right) = 97e^{-0.8}p = 43.584p = £0.43584$$

and

$$Var\left(R_2\right) = Var\left(0.5\int_0^2 e^{0.4(s-2)} dB_s\right) = \frac{0.25}{0.8}\left(1-e^{-1.6}\right) = 0.2494\,p^2$$

$$= (0.499\,p)^2 = £^2 0.00002494 = (£0.00499)^2$$

Overall well-answered but some candidates did seem to struggle applying Ito's Lemma and with calculating the expectation and variance; Some candidates confused the Normal and LogNormal distributions, while others simply stated the answer rather than deriving it.

6 (i) First we calculate the risk-neutral probability of an upwards movement in the stock price from each state:

$$q(300) = \frac{1.02\times 300 - 270}{330-270} = 0.6$$

$$q(330) = \frac{1.02\times 330 - 300}{360-300} = 0.61$$

$$q(270) = \frac{1.02\times 270 - 240}{300-240} = 0.59.$$

Then the option price *V* is:

$$V = \frac{1}{1.02^2}\left[q(300)q(330)\times 70 + q(300)\left(1-q(330)\right)\times 10 + \left(1-q(300)\right)q(270)\times 10 + 0\right]$$

$= 29.143.$

Alternatively, if the candidates misinterpret the question and use 2% as a force of interest per quarter we get qs = (0.601007, 0.611107, 0.590906), and a value for V of 29.21234. 2 marks can be awarded for this or any other answer which has the right working but the wrong interpretation of the interest rate.

(ii) (a) Using the results from (i) we can calculate the values of the state-price deflator:

$$A(360) = \frac{q(300)q(330)}{\left(0.7\times 1.02\right)^2} = 0.71793$$

$$A(300) = \frac{q(300)\left(1-q(330)\right) + \left(1-q(300)\right)q(270)}{2\times 0.7\times 0.3\times 1.02^2} = 1.07559$$

$$A(240) = \frac{(1-q(300))(1-q(270))}{(0.3\times 1.02)^2} = 1.75146.$$

Alternatively candidates may have calculated four deflators, one for each path:

$A(uu) = 0.71793$, $A(ud) = 1.071017$, $A(du) = 1.080171$,
$A(dd) = 1.75146$.

If candidates have used 2% as a force of interest:

$A(uu) = 0.72016$, $A(ud) = 1.069345$, $A(du) = 1.078681$,
$A(dd) = 1.742506$.

Or if using three nodes the middle node is 1.074013.

(b) Then the option premium V can be calculated as:

$$\begin{aligned} V &= E_P(A_2V_2) \\ &= \left(0.7^2 A(360)\times 70\right) + \left(2\times 0.7\times 0.3\times A(300)\times 10\right) \\ &\qquad + \left(0.3^2 \times A(240)\times 0\right) \\ &= 29.143. \end{aligned}$$

(c) This is the same answer as under part (i) as expected – under a given model the option price should not vary depending on how we evaluate the model.

(iii) (a) $A(360)$ would rise as the denominator decreases; $A(240)$ and $A(300)$ would shrink as the denominator rises.

(b) Overall the option price would remain unchanged as it does not depend on real-world probabilities.

Generally reasonably answered, although some candidates only calculated one risk-neutral probability instead of three and many struggled to calculate correct state price deflators or more surprisingly confused real-world and risk-neutral probabilities.

7 (i) Let K be the forward price and denote the stock price at time t by S_t. Now compare the setting-up of the following two portfolios at time 0:

- A: one long forward contract.
- B: borrow Ke^{-rT} in cash and buy e^{-qT} units of the stock priced at S_0. Invest dividends in the stock.

At time T portfolio A is worth $S_T - K$.
At time T portfolio B is worth $S_T - K$.

By the principle of no-arbitrage these portfolios must have the same value at all times before T.

In particular, at time 0, portfolio B has value $e^{-qT}S_0 - Ke^{-rT}$ which must equal the value of the forward contract (which must be zero at time 0).

So $K = e^{(r-q)T}S_0$.

(ii) Consider a portfolio consisting of e^{-qT} units of stock, a European put option and short a European call option both of which expire at time T at strike price K, whose prices at time t are denoted by S_t, p_t and c_t respectively.

At expiry:

If $S_T > K$, the put option expires worthless, the call option will be exercised (or be worthless if $S_T = K$) and the stock will be delivered in return for K. I.e. the value of the portfolio will be K.

If $S_T < K$, the call option will not be exercised, and the stock can be sold via the put option for K, so the value of the portfolio will be K.

Since the portfolio will be worth a known, fixed amount at time T, by the principle of no-arbitrage it must earn the risk free rate up to time T.

This question differentiated between the stronger and weaker candidates. Candidates who knew how to adjust the portfolio construction arguments to forward pricing scored well.

8 (i) The stochastic differential equation for the short rate r is:

$$dr_t = \alpha(\mu - r_t)\,dt + \sigma dB_t.$$

where B is a standard Brownian motion, σ is the volatility, and α and μ are the drift parameters. [1]

(ii) **Desirable:**

Arbitrage-free
Instantaneous and other rates mean reverting
Ease of computation/pricing of derivatives and bonds

Undesirable:

Short rate not necessarily positive
Does not generate realistic dynamics/yield curves
e.g. bonds of all durations perfectly correlated
e.g. constant volatility of the short rate
Does not provide good historical fit (even with suitable parameter values)
Is not easy to calibrate
Is not sufficiently flexible – e.g. cannot price derivatives whose value depends on more than one interest rate

(iii) (a) We wish to solve the stochastic differential equation:

$$dr_t = \alpha(\mu - r_t)\,dt + \sigma dB_t.$$

Consider $u_t = r_t e^{\alpha t}$

$$du_t = \alpha r_t e^{\alpha t} dt + e^{\alpha t} dr_t$$

$$= \alpha\mu e^{\alpha t} \mathrm{d}t + \sigma e^{\alpha t} dB_t.$$

so $u_T = u_t + \mu(e^{\alpha \mathrm{T}} - e^{\alpha \mathrm{t}}) + \sigma\int_t^T e^{\alpha s} dB_s$

and hence $r_T = r_t e^{-\alpha(\mathrm{T}-t)} + \mu\left(1 - e^{-\alpha(\mathrm{T}-t)}\right) + \sigma\int_t^T e^{-\alpha(\mathrm{T}-\mathrm{s})} dB_s.$

(b) From this we see that under the risk-neutral measure r_T follows a Gaussian distribution

with mean

$$r_t e^{-\alpha(T-t)} + \mu\left(1 - e^{-\alpha(T-t)}\right)$$

and variance $\dfrac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha(T-t)}\right)$.

Largely, well-answered bookwork question, although the candidates found the later sections of this question progressively more difficult.

9 (i) The Δ of the call holding must be minus the Δ of the shareholding, which, by definition is – 18673, so the Δ of a call is $\Delta_C = 0.18673$.

(ii) Δ_C for a call is $\Phi(d_1)$, where $d_1 = (\ln(S_0/k) + r + ½\sigma^2))/\sigma = (\ln(1.1798/1.5) + 0.02 + ½\sigma^2))/\sigma = -0.22/\sigma + ½\sigma$.

Now $\Phi(d_1) = 0.18673$ so $d_1 = -0.89$

which implies that

$-0.22 + 0.89\,\sigma + ½\,\sigma^2 = 0$ so $\sigma = -0.89 \pm (0.89^2 + 0.44)^{½}$.
Rejecting the negative root gives a value of $\sigma = 22\%$.

(iii) $d_2 = d_1 - \sigma\sqrt{T} = -1.11$. Thus $P = Ke^{-rT}\,\Phi(-d_2) - S_0\,\Phi(-d_1)$
$= 150e^{-r}\,\Phi(-d_2) - 117.98\Phi(-d_1) = 147.0298\,\Phi(-d_2) - 117.98\Phi(-d_1)$
$= 147.0298 \times 0.8665 - 117.98 \times 0.81327 = \$31.4517$

(iv) Using C to denote the call option, P the put option and S the stock we know that:

$$\Delta_C - \Delta_P = \Delta_S = 1$$
$$\Gamma_C = \Gamma_P \text{ and } \Gamma_S = 0$$

So since we hold 100,000 call options, we must be short 100,000 put options and 100,000 shares to get a gamma and delta neutral portfolio.

Alternative calculation approaches were awarded full marks if candidates reached the right conclusions.

Candidates found this question difficult, especially the latter part which only the strongest candidates answered well.

10 (i)

- Merton's model assumes that a corporate entity has issued both *equity and debt such that its total value at time t is F(t).* $F(t)$ varies over time as a result of actions by the corporate entity which does not pay dividends on its equity or coupons on its bonds.

- Part of the corporate entity's value is *zero-coupon debt with a promised repayment amount of L at a future time T.* At time T the remainder of the value of the corporate entity will be distributed amongst the equity holders and the corporate entity will be wound up.

- The corporate entity will *default if the total value of its assets, F(T) is less than the promised debt repayment at time T i.e. F(T) < L.* In this situation, the bond holders will receive $F(T)$ instead of L and the equity holders will receive nothing.

- This can be regarded as treating the *equity holders of the corporate entity as having a European call option on the assets* of the company with maturity T and a strike price equal to L.

(ii) (a) Under the Merton model, the value at redemption is $\min(F(4),120) = 120 - \max(120 - F(4),0) = F(4) - \max(F(4) - 120, 0)$, where $F(t)$ is the gross value of the company at time t.

Thus the value at time 0 is
$e^{-4r}E[\min(F(4),120)] = e^{-4r}E[F(4) - \max(F(4) - 120,0)]$,

[Alternative expressions are fine, as per the first part of (ii) (a).]

where the expectation is under the risk-neutral measure, so equals $F(0) - C$, where C is a call option on the gross value with strike 120.

[Alternatively = $120e^{-4r} - P$, where P is a call option on the gross value with strike 120.]

(iii) The bond price is $120 \times e^{-4(r+.045)}$ = \$92.5262m.

(iv) The call price is \$87.474 with $T = 4$, $r = 0.02$, $S_0 = 180$, $K = 120$.

This leads to an estimated volatility of 40%.

Try a volatility of 20%. This gives an option price of \$72.266.

A volatility of 50% gives a price of \$92.293.

Interpolating gives a volatility of:

$$20 + (87.276 - 72.266 / (96.06 - 72.266) \times 30 = 39\%.$$

This gives a price of \$86.413.

Interpolating again gives a volatility of 40%.

For reference when marking:

volatility = 30%, c = \$78.985, vol = 40%, c = \$87.275, vol = 45%, c = \$91.645

(v) $Q(\text{default}) = Q(F(4) < 120)$
$= 1 - \Phi(d_2) = 1 - \Phi(0.206831)$
$= 1 - 0.58192 = 0.41808 = 42\%$

Candidates struggled to gain many marks on this question, and many seemed to be short of time reflecting the importance of time management in these exams.

END OF EXAMINERS' REPORT