INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

September 2015

Subject CT8 – Financial Economics
Core Technical

Introduction

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and using past papers as a revision aid and also those who have previously failed the subject.

The Examiners are charged by Council with examining the published syllabus. The Examiners have access to the Core Reading, which is designed to interpret the syllabus, and will generally base questions around it but are not required to examine the content of Core Reading specifically or exclusively.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report; other valid approaches are given appropriate credit. For essay-style questions, particularly the open-ended questions in the later subjects, the report may contain more points than the Examiners will expect from a solution that scores full marks.

The report is written based on the legislative and regulatory context pertaining to the date that the examination was set. Candidates should take into account the possibility that circumstances may have changed if using these reports for revision.

F Layton
Chairman of the Board of Examiners
December 2015

A. General comments on the *aims of this subject and how it is marked*

1. The aim of the Financial Economics subject is to develop the necessary skills to construct asset liability models and to value financial derivatives. These skills are also required to communicate with other financial professionals and to critically evaluate modern financial theories.

2. The marking approach for CT8 is flexible in the sense that different answers to those shown in the solution can earn marks if they are relevant and appropriate. Marks for the methodology are also awarded.

B. General comments on *student performance in this diet of the examination*

The general performance was good. Candidates found some questions challenging, but well-prepared candidates scored consistently across the whole paper. As in previous diets, questions that required an element of application of the core reading to situations that were not immediately familiar proved more challenging to most candidates. A significant number of candidates failed to read some questions carefully enough to identify the relevant section of the course being examined.

C. Comparative pass rates for the past 3 years for this diet of examination

Year	*%*
September 2015	50
April 2015	60
September 2014	56
April 2014	55
September 2013	51
April 2013	51

Reasons for any significant change in pass rates in current diet to those in the past:

The pass rates are comparable with standard pass rates for CT8, even if slightly lower than recent diets. The paper was standard but candidates struggled with some questions.

Solutions

Q1 From the core reading:

1.1 The consumer's preferences

1.1.1 Definitions

"Utility" is the satisfaction that a consumer obtains from a particular course of action. The amount of one good that a consumer is prepared to swap for one extra unit of another good is known as the "marginal rate of substitution".

An "indifference curve" joins all the consumption bundles of equal utility. The slope of a consumer's indifference curves will depend on his or her individual preferences and is equal to the marginal rate of substitution.

A given combination of goods (e.g. two apples and five bananas) is called a "consumption bundle".

1.1.2 Assumptions and results

(i) A consumer can rank any two bundles.

A consumer can rank different bundles, and therefore can pick a set of consumption bundles that give the same utility.

(ii) Consumers prefer more of a good to less of it.

Therefore indifference curves slope downwards from left to right and indifference curves further from the origin give higher utility.

(iii) Consumer preferences exhibit diminishing marginal rates of substitution.

This means that if it takes, say, *n* extra apples to persuade a consumer to give up one banana, it will take more than another *n* extra apples to persuade her to give up yet another banana. Indifference curves are "convex to the origin".

1.2 The budget constraint

1.2.1 The assumptions

(i) The prices of the goods are fixed.

(ii) The consumer's income is fixed.

These two assumptions determine which consumption bundles are affordable. The budget line joins all points that a consumer can afford, assuming that all income is spent.

1.3 How consumers choose

Economists assume that consumers' choices exhibit rational behaviour. A rational consumer will choose the consumption bundle that maximises his own utility. This is the concept of utility maximisation.

1.4 Implications

Combining the budget line with indifference curves, we can determine the consumption bundle which a consumer will choose. A rational consumer will choose a consumption bundle such that the marginal rate of substitution is equal to the slope of the budget line – that is, where the ratios of marginal utilities equal the ratios of prices.

> Well prepared students scored well on this bookwork question. Some students confused consumer choice theory with expected utility theory but still managed to score some marks. Several students wrote about efficient portfolios and scored nothing.

Q2 (i) $A(w) = -U''(w) / U'(w) = -2c / (b+2cw)$
$R(w) = -wU''(w) / U'(w) = -2cw/(b + 2cw)$

(ii) For a gamble with an equal size gain or loss, the requirement that $p \geq 0.55$ implies that the investor is risk averse. (Alternatively, they have increasing absolute and relative risk aversion.)

(iii) With w =100, the (certain) utility if the gamble is rejected is:

(1) $610 = a + 100b + 10{,}000c$

whereas the expected utility if the gamble is accepted with $p = 0.55$ is:

$U(100) = 0.55 * U(120) + 0.45 * U(80)$
$\Rightarrow 610 = 0.55 * (a + 120b + 14{,}400c) + 0.45 * (a + 80b + 6{,}400c)$
(2) $\Rightarrow 610 = a + 102b + 10{,}800c$

With w =120, the (certain) utility if the gamble is rejected is:

(3) $U(120) = a + 120b + 14{,}400c$

whereas the expected utility if the gamble is accepted with $p = 0.5625$ is:

$U(120) = 0.5625 * U(140) + 0.4375 * U(100)$
$\Rightarrow U(120) = 0.5625 * (a + 140b + 19{,}600c) + 0.4375 * (a + 100b + 10{,}000c)$
(4) $\Rightarrow U(120) = 0.4375 * U(100) + 0.5625 * U(140) = a + 122.5b + 15{,}400c$

Solving these gives:

$a = (17{,}080 - 25 * U(120)) / 3$
$b = (U(120) - 610) / 9$
$c = -(U(120) - 610) / 3{,}600$

(iv) $U'(w) = b + 2cw$
$U'(w) = 0 \Rightarrow w = -b / 2c$
$b = -400c$ (from above) $\Rightarrow w = £200$

> Early parts of this question were largely completed well, though some students used the incorrect formulae despite them appearing in the tables. Many students identified the simultaneous equations to solve, but only the best students proceeded to solve them.

Q3 (i) A portfolio is efficient if the investor cannot find a better one in the sense that it has a higher expected return with the same variance, or a lower variance with the same expected return.

(ii) The assumptions are:

Investors are never satiated.

Investors dislike risk.

Investors select assets based on mean and variance of return only.

Mean return, variance (or standard deviation) and co-variances are known for all assets.

(iii) $E = \sum_i x_i E_i$ where E_i is the expected return on security i.

(iv) $V = \sum_i \sum_j x_i x_j C_{ij}$ where C_{ij} is the covariance of the returns on securities i and j and we write $C_{ii} = V_i$.

(v) With only two securities the variance is

$$V = x_A{}^2 V_A + (1 - x_A)^2 V_B + 2x_A(1 - x_A)C_{AB}$$

Differentiating wrt x_A gives

$$\frac{dV}{dx_A} = 2x_A V_A - 2(V_B - x_A V_B) + 2(1 - 2x_A)C_{AB}$$

$$= (2V_A + 2V_B - 4C_{AB})x_A - 2V_B + 2C_{AB}$$

Setting this to zero gives

$$x_A = \frac{V_B - C_{AB}}{V_A + V_B - 2C_{AB}}$$

Checking the second derivative shows that this is a minimum:

$$\frac{d^2V}{dx_A^{\ 2}} = 2V_A + 2V_B - 4C_{AB} \geq 0$$

Generally well answered, though not all students stated the requirements for an efficient portfolio in both directions.

Q4 (i) Arbitrage pricing theory (APT) is an equilibrium market model that does not rely on the strong assumptions of the capital asset pricing model (CAPM).

APT requires that the returns on any stock be linearly related to a set of factor indices as shown below

$R_i = a_i + b_{i,1} I_1 + b_{i,2} I_2 + ... + b_{i,L} I_L + c_i$ (*)

where R_i is the return on security i, a_i and c_i are the constant and random parts respectively of the component of return unique to security i, $I_1 ... I_L$ are the returns on a set of L indices,

$b_{i,k}$ is the sensitivity of security i to index k.

We have

$E[c_i] = 0$,
$E[c_i c_j] = 0$ for all i, j where $i \neq j$,
and $\text{Cov}(c_i, I) = 0$ for all stocks and indices.

This is exactly the same as the multi-index model for returns on individual securities. The contribution of APT is to describe how we can go from a multi-index model for individual security returns to a equilibrium market model. Non-mathematically, the argument can be made as follows. Consider a two index model. The return on the ith security is given by
$R_i = a_i + b_{i,1} I_1 + b_{i,2} I_2 + c_i$.

For investors who hold well-diversified portfolios the specific risk of each security, represented by c_i can be diversified away so an investor need only be concerned with expected return, $b_{i,1}$ and $b_{i,2}$ in choosing his portfolio.

Suppose we hypothesize the existence of three widely diversified portfolios, represented by the points $(E_i, b_{i,1}, b_{i,2})$ in $E - b_1 - b_2$ space where $i = 1, 2, 3$. These three portfolios define a plane in $E - b_1 - b_2$ space with equation
$E[R_i] = \lambda_0 + \lambda_1 b_{i,1} + \lambda_2 b_{i,2}$.

A portfolio having any combination of b_1 and b_2 can be formed by combining portfolios 1, 2 and 3 in the correct proportions. For example the portfolio P, obtained by taking one third each of each of 1, 2 and 3 would have:

$b_{P,1} = (b_{1,1} + b_{2,1} + b_{3,1})/3$,
$b_{P,2} = (b_{1,2} + b_{2,2} + b_{3,2})/3$,
and $E[R_P] = \lambda_0 + \lambda_1 b_{P,1} + \lambda_2 b_{P,2}$. (**)

Now, consider what would happen if another portfolio Q existed, with exactly the same values of b_1 and b_2 but a higher expected return. Both portfolios would have the same degree of systematic risk but Q would have a higher expected return than P. Rational investors would therefore sell P and buy Q, and this would continue until the forces of supply and demand had brought portfolio Q onto the same plane as portfolios 1, 2 and 3.

Thus, in equilibrium, all securities and portfolios must lie on a plane in $E - b_1 - b_2$ space.

(ii) Since I_M is a traded index it must satisfy the formula (**). But the portfolio consisting of just the index has $b_{M,M} = 1$ and $b_{M,C} = 0$
and has expected return λ_M
so we must have $\lambda_0 = 0$.

(iii) We must have

$R = b_{i,M} I_M + b_{i,C} I_C + c_i$, where c_i is independent of I_M and I_C.

So,

$\text{Cov}(R, I_M) = b_{i,M}\text{Var}(I_M) + b_{i,C}\text{Cov}(I_M, I_C) = 0.04 b_{i,M} - 0.4 * 0.01\, b_{i,C}$
$= 0.02$,

while $E_i = b_{i,M}\,\lambda_M + b_{i,C}\,\lambda_C = 0.07\,b_{i,M} + 0.02b_{i,C} = 0.09$

so $b_{i,M} = 0.8235$ and $b_{i,C} = 1.6176$.

(iv) $\text{Cov}(R, I_C) = b_{i,M}\,\text{Cov}(I_M, I_C) + b_{i,C}\,\text{Var}(I_C) = 0.8235 * -0.008 + 1.6176 * 0.01 = 0.0096$.

> Well prepared students scored well here. Many made mistakes in the calculations or tried to apply formulae for the single-index case to the two-factor model.

Q5 (i) Using Ito's Lemma:

$d\log S_t = 1/S_t dS_t - 1/(2 * S_t^2)(dS_t)^2 = (\mu - \sigma^2/2)dt + \sigma dZ_t$

Integrating both sides gives

$\log S_t = \log S_0 + (\mu - \sigma^2/2)t + \sigma Z_t$
$\Rightarrow S_t = S_0 \exp((\mu - \sigma^2/2)t + \sigma Z_t)$

So S_t is lognormal with parameters

$(\mu - \sigma^2/2)t = 0.08t$ and $\sigma^2 t = 0.04t$

(ii) To find the initial investment we need the 60th percentile of $\log S_t$, which is:

$P((\log S_t - 0.4) / \sqrt{(0.2)} < X) = 0.6$
$\Leftrightarrow X = 0.253$
$\Leftrightarrow \log S_t = 0.253 * \sqrt{(0.2)} + 0.4 = 0.51315$
$\Leftrightarrow S_t = 1.6705$

So:

$A = €20{,}000 / 1.6705 = €11{,}972$

(iii) (a) $\text{Var}(S_t) = \exp(2\mu t)(\exp(\sigma^2 t) - 1)$
$\Rightarrow \text{Var}(10{,}000S_t) = (10{,}000)^2\exp(2\mu t)(\exp(\sigma^2 t) - 1)$
$= (10{,}000)^2 * \exp(2 * 0.5)(\exp(0.2) - 1) = €^2\ 60{,}183{,}509$
$\Rightarrow \text{SD} = €7{,}758$

(b) We need the 5th percentile of $\log S_t$, which is:

$P((\log S_t - 0.4) / \sqrt{(0.2)} < X) = 0.05$
$\Leftrightarrow X = -1.645$
$\Leftrightarrow \log S_t = -1.645*\sqrt{(0.2)} + 0.4 = -0.335666$
$\Leftrightarrow S_t = 0.71486$
So the VaR is €20,000 – (€10,000*0.71486) = € 12,851
Credit was also given for calculation of the 95th percentile of $\log S_t$, which is:

$P((\log S_t - 0.4) / \sqrt{(0.2)} < X) = 0.95$
$\Leftrightarrow X = 1.645$
$\Leftrightarrow \log S_t = 1.645*\sqrt{(0.2)} + 0.4 = 1.135666$
$\Leftrightarrow S_t = 3.11325$
So the VaR is €20,000 – (€10,000*3.11325) = –€ 11,132

This question was surprisingly poorly answered. Many students derived the correct distribution but few calculated the parameter values using the numbers in the question. Many students struggled to calculate the Value at Risk correctly.

Q6 (i) Let K be the forward price. Now compare the setting up of the following portfolios at time 0:

A: one long forward contract.
B: borrow Ke^{-rT} cash and buy one share at S_0.

If we hold both of these portfolios up to time T then both have a value of $S_T - K$ at T.

By the principle of no arbitrage these portfolios must have the same value at all times before T.

In particular, at time 0, portfolio B has value $S_0 - Ke^{-rT}$ which must equal the value of the forward contract.

This can only be zero (the value of the forward contract at $t = 0$) if $K = S_0e^{rT}$.

(ii) $K = £5*e^{0.03+0.05+2*0.02} = £5.64$

(iii) Consider at time $t = 1$ portfolio A = the forward and $5.64e^{-0.09}$ cash, portfolio B = one share.

These have equal value at $t = 4$, so must be equal at $t = 1$ by the principle of no arbitrage.

So value of existing contract = $6 - 5.64e^{-0.09}$ = £0.85

> Part (i) was a standard proof that was largely well answered. In part (iii) some students applied the risk-free rate to the share price or simply calculated the value of a forward contract at time 1 using the £6 share price.

Q7 (i) Lower bound = $\max\{0, S_0 - Ke^{-rT}\} = 10 - 12e^{-0.04*5} = 0.175$ = \$0.18
Upper bound = S_0 = \$10

(ii) Trial and error gives volatility of 16%.

Sample values:

Volatility	*Option value*
10%	\$0.97
15%	\$1.41
20%	\$1.84
25%	\$2.27
30%	\$2.69
35%	\$3.11
40%	\$3.51

(iii) $\Phi(d_1) = 0.59$ so the hedge is 100 * 0.59 = 59 shares
and 100 * 1.5 – 59 * 10 = \$440 short in cash.

> Largely well-answered, though some students produced a negative lower bound for the option price. Most students attempted to find sigma through trial and interpolation, but some were let down by failing to interpolate correctly.

Q8 (i) $P(Y \le y) = P(F(X) \le y) = P(X \le F^{-1}(y)) = F(F^{-1}(y)) = y$ whenever $0 \le y \le 1$.

(ii) $S_t = S_0 \exp(\sigma B_t + (r - \frac{1}{2}\sigma^2)t)$, where B is a standard Brownian motion under Q.

Hence B_t has a $N(0,t)$ distribution under Q and so $\ln(S_t)$ has a $N(\ln S_0 + (r - \frac{1}{2}\sigma^2)t, \sigma^2 t)$ distribution under Q.

It follows that, using the values for the parameters given, $\ln(S_2)$ has a $N(0.09596, 0.0968)$ distribution under Q
and so $a = 0.09596$, $b = 0.31113$.

(iii) The fair price is $V_0 := E_Q[e^{-2r}D_2]$
$= E[100e^{-.04}U^2]$
where U is a $U(0,1)$ random variable. Thus $V_0 = 100e^{-.04} \int_0^1 u^2 du$
$= 100e^{-.04}/3 = £32.03$.

(iv) Now the fair price is $V_1 := E_Q[e^{-2r}D_2 1_{(U>0.5)}]$
$= E[100e^{-.04}U^2 1_{(U>0.5)}]$
where U is a $U(0,1)$ random variable. Thus $V_1 = 100e^{-.04} \int_{0.5}^1 u^2 du$
$= 100e^{-.04}(1 - 0.125) / 3 = £28.02$.

Few students managed to score more than a few marks here, and some didn't attempt the question or scored zero. Many students managed part (ii).

Q9 (i) The Merton model is a structural model for credit risk.

It assumes that the shareholders are entitled to net assets of the company after redemption of the loan.

Gross assets are modelled as the share price in a Black-Scholes market

Thus, if L_t is the loan value at time t, F_t is the gross asset value, E_t is the equity value at time t and the loan matures at time T, then $L_T = \min(L, F_T)$, where L is the nominal amount of loan.

It follows that E_t is the value of a call option on the gross assets with strike L and $F_t = E_t + L_t$.

(ii) We know from (i) that $E_0 = E_Q e^{-rT}(F_T - L)^+$, where Q is the equivalent martingale measure.

(iii) We know E_0, L, r, T and σ so we only lack F_0, the initial price in the Black-Scholes formula for a call: $E_0 = F_0\Phi(d_1) - Le^{-rT}\Phi(d_2)$.

Answers will vary depending on the initial trial values chosen.

Trying $F_0 = 15$ gives $E_0 = 6.595$

Trying $F_0 = 30$ gives $E_0 = 20.619$

Interpolating gives a value for F_0 of 21.3258 which gives $E_0 = 12.272$

Eventually we get $F_0 = 21.58$

(iv) From (iii) we get that $L_0 = F_0 - E_0 = 9.0696$.

This implies a yield of ln(12.3/9.0696)/5 = 6.09%, which gives a credit spread of 1.09%

> Standard bookwork which was largely well answered. Some students confused the Merton and Jarrow-Lando-Turnbull models. Many students struggled to apply the bookwork to part (iii). The most common mistake was to confuse the value of E_0 and F_0. Again some students failed to interpolate between two values correctly.

Q10 (i)

t	*F(t – 1,t)*	*B(0,t)*	*R(0,t)*	*C(t)*
0	-	-	-	£100.00
1	2%	£98.02	2.0%	£102.02
2	4%	£94.18	3.0%	£106.18
3	3%	£91.39	3.0%	£109.42
4	5%	£86.94	3.5%	£115.03

i.e. (a) = 5%
(b) = £98.02
(c) = 3.0%
(d) = £109.42

(ii)

t	*F(t – 1,t)*	*B(0,t)*
0	-	-
1	-	-
2	5%	95.12
3	4%	91.39
4	6%	86.07

The investor bought 10 bonds maturing at $t = 2$ and 20 bonds maturing at $t = 4$, at a total cost of 10 * 94.18 + 20 * 86.94 = £2,680.60.

The bonds are now worth 10 * 95.12 + 20 * 86.07 = £2,672.60.
Profit is 2,672.60 – 2,680.60 = –£8.00 (i.e. a loss of £8.00).

(iii) Any portfolio consisting only of risk-free assets will return the risk-free rate of interest if rates remain unchanged.

The investor would therefore need to invest in other risky assets, or assets linked to another interest rate, in order to recoup her loss.

> Well-prepared students scored full marks here, but a surprising number failed to find all four values in part (i). Some students assumed interest was compounded annually, which cannot be possible given the values in the table.

END OF EXAMINERS' REPORT