Faculty of Actuaries **Institute of Actuaries**

EXAMINATIONS

September 2006

Subject CT8 — Financial Economics

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

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Chairman of the Board of Examiners

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Comments

No comments are given

1 (a) $\text{Var}(R) = 100{,}000^2\ \text{Var}(N) = 10^{10}$

(b) Downside semi-variance of $R = 10^{10} \times$ upside semi-variance of N; the upside semi-variance of N is ½, so downside semi-variance of R is 5×10^9

(c) $P(R < 50{,}000) = P(N > 2) = 1 - \Phi(1) = 1 - .8413 = .1587$

(d) If $\text{Va}R_{5\%}(R) = t$ then $P(R \leq - t) = 0.05$, so $P(250{,}000 - 100{,}000N \leq -t) =$ $P(N > 2.5 + (t/100{,}000)) = 5\%$, hence (since $N - 1$ is a standard normal r.v) $\Phi(1.5 + (t/100{,}000) = .95$, so $t = 100{,}000(1.645\text{-}1.5) = £14{,}500$.

2 (i) The pricing measure **Q** must satisfy:

$$\mathbf{E_Q}\left[\frac{1}{1+r}S_{t+1}|\mathbf{F}_t\right] = S_t;$$

so, if we set

$$q_t = \mathbf{Q}(S_{t+1} = 1.25S_t\,|\mathbf{F}_t),$$

then

$$1.04 = 1.25q_t + 0.8(1 - q_t) \Leftrightarrow q_t = q = 8/15$$

Thus the unique pricing measure makes S a multiplicative random walk with up-jump probability of 8/15.

(ii) The price of the derivative is $P = \mathbf{E}_Q[X/(1 + r)^2]$, where X is the terminal value of the derivative.

Thus,

$$\begin{aligned} P &= 1{,}000Q(S_2 \neq 800)/1.04^2 \\ &= 1{,}000 \times ((8/15)^2 + (7/15)^2)/1.04^2 \\ &= 464.33\text{p} \end{aligned}$$

3 (i) Suppose that at time t we hold the portfolio (φ_t, ψ_t) where φ_t represents the number of units of S_t held at time t and ψ_t is the number of units of the cash bond held at time t.

We denote the value of the portfolio at time t by $V(t)$.

The portfolio strategy is described as self-financing if $dV(t)$ is equal to $\varphi_t\, dSt + \psi_t\, dB_t$: that is, at time $t + dt$ there is no inflow or outflow of money necessary to make the value of the portfolio back up to $V(t + dt)$.

(ii) Let F_t be the discounted value of a derivative (priced using the EMM) then since it's martingale, there is (by martingale representation) a φ_t such that $dF_t = \varphi_t dD_t$, where D is the discounted price of the underlying. This φ_t is the derivative's delta.

(iii) It follows from the above that if we hold φ_t in the underlying asset and $\psi_t = F_t - \varphi_t D_t$ in the bond, then the discounted value of our holding is F_t.

The holding is self-financing, since $dV(t) = d(e^{rt}F_t) = re^{rt}\, F_t dt + e^{rt}\, dF_t$
$= re^{rt}F_t dt + e^{rt}\, \varphi_t dS_t + re^{rt}\,(F_t - \varphi_t D_t)\, dt = \varphi_t dS_t + \psi_t\, dB_t$.
The final discounted value of our holding is F_T, and so we have hedged the derivative with terminal value of V_T.

4 (i) In the Wilkie model, the force of inflation, $I(t)$, over the period $t - 1$ to t is an autoregressive model of order 1, $AR1$: $I_{t+1} = (1 - \alpha)\, m + \alpha I_t + e_t$, where the e_t are iid normal errors.

It follows that it is mean-reverting and the longitudinal distribution from the 1,000 year simulation will converge to stationarity.

Consequently we will get the unconstrained s.d., whereas the s.d. from repeated one year simulations (cross-sectional) will depend strongly on initial conditions.

(ii) In a pure random walk environment, the force of inflation would be independent across the years and (as for any model) across simulations. As a result, cross-sectional and longitudinal quantities would coincide. This happens if $\alpha = 0$.

(iii) In a statistical model, the model structure is derived from past time series, together with some intuition regarding what model formulae look reasonable. However, these statistical models can produce some odd results. It can be useful to impose additional economic constraints on model behaviour. The advantage of using more economic theory is that it gives us a more concrete way of interpreting model output. For example, if we model a market which is broadly governed by rational pricing rules, we can apply those same pricing rules to simulated output from a model. This gives us a market-based way of comparing strategies, and deciding which strategy is most valuable. The difficulty with this approach is that the model's optimal strategy may not be the strategy that managers wish to follow. In this context, a more flexible judgmental approach may better meet the client's needs.

5 (i) Arbitrage free
Positive rates
Instantaneous and other rates mean reverting
Ease of computation/pricing of derivatives and bonds
Realistic dynamics/yield curves
Historical fit (with suitable parameter values)
Ease of calibration
Flexibility (to cope with range of derivatives)

(ii) The stochastic differential equation for the short rate r is:

$$dr_t = \sigma dB_t + \alpha(\mu - r_t)\, dt.$$

(iii) Arbitrage free - yes
Positive rates - no
Instantaneous and other rates mean reverting - yes
Ease of computation/pricing of derivatives and bonds - yes
Realistic dynamics/yield curves - no
Historical fit (with suitable parameter values) - no
Ease of calibration - no
Flexibility (to cope with range of derivatives) - no
-not very good as a model.

6 (i) Since the efficient frontier consists of pairs of points (in (s.d., return) coordinates) such that no higher return is available for the same or lower s.d. we see that to get a return of r, greater than or equal to .05, we need a portfolio of $((r - .05)/.05,\ 1 - ((r - .05)/.05) = (20r - 1,\ 2 - 20r)$, this portfolio has a standard deviation $0.2\,(20r - 1)$, hence the efficient frontier is the straight line $(4r - 0.2,\ r)$ which does indeed pass through the two specified points.

(ii) A portfolio with x invested in A and $(1 - x)$ invested in C has an expected return of $.06 + .04x$ and s.d. of $\sqrt{(.04x^2 + .01(1 - x)^2)}$. Thus we seek x to maximize $(.01 + .04x) / \sqrt{(.04x^2 + .01(1 - x)^2)}$. Taking logs and differentiating we see (after a lot of algebra) that the optimal x is 5/9, so the optimal portfolio is (5/9, 4/9).

(iii) The efficient frontier in the presence of a risk free asset is the tangent to the efficient frontier (without a risk free asset) which passes through the point in (s.d., return)-space corresponding to the risk free asset.

Clearly this is the line through (0, .05) with maximal gradient which passes through some point of the efficient frontier.

Consider the point corresponding to the portfolio in part (ii): it is on the efficient frontier for the pair A and C, and the line from (0, .05) to it has gradient
$(.01 + .04x) / \sqrt{(.04x^2 + .01(1 - x)^2)}$

Hence the new efficient frontier is a straight line which passes through (0, .05) and $(\sqrt{(.04(5/9)^2 + .01(1 - 5/9)^2)}, .06 + .04 \times 5/9)$.

This is the line $y = .05 + .2692x$, which clearly passes through (.1, .076926).

7 (i) The three types are:

Macroeconomic factor models

These use observable economic time series as the factors. They could include factors such as the annual rates of inflation and economic growth, short term interest rates, the yields on long term government bonds, and the yield margin on corporate bonds over government bonds. Once the set of factors has been decided on, a time series regression is performed to determine the sensitivities for each security in the sample.

Fundamental factor models

Fundamental factory models are closely related to macroeconomic models but instead of (or in addition to) macroeconomic variables the factors used are company specific variables. These may include such fundamental factors as:

- the level of gearing
- the price earnings ratio
- the level of R&D spending
- the industry group to which the company belongs

Again, the models are constructed using regression techniques.

Statistical factor models

Statistical factor models do not rely on specifying the factors independently of the historical returns data. Instead a technique called principal components analysis can be used to determine a set of indices which explain as much as possible of the observed variance.

(ii) Denoting the changes in the two indices by I_t and J_t, let $K_t = J_t - c\, I_t$, where $c = \mathrm{Cov}(I_t, J_t)/\mathrm{Var}(I_t)$, then the two factors I and K are orthogonal. We can check: $\mathrm{Cov}(I_t, K_t) = \mathrm{Cov}(I_t, J_t) - c\,\mathrm{Var}(I_t) = 0$. Alternatively, we may regress index J on index I to obtain $J = a + bI + d_2$, and set $K = d_2$, where a is a constant and d_2 is uncorrelated with I.

(iii) Suppose that $R_i = \alpha_i + \beta_{i,1}I + \beta_{i,2}K + \varepsilon_i$, then $\mathrm{Var}(R_i) = \beta_{i,1}^2\,\mathrm{Var}(I) + \beta_{i,2}^2\,\mathrm{Var}(K) + \sigma_i^2$.

(iv) The interpretation is (as in principal components analysis) that we have a decomposition of the variance into the portion explained by the behaviour of the first index, that explained by the second and the residual or unexplained error or variance.

8 (i) Under the Black Scholes assumptions, the unique risk-neutral measure is Q, where, under Q,

$$dS_t = rS_t\,dt + \sigma S_t\,dB_t,$$

with B a standard Brownian motion.

(ii) The unique fair price for a derivative security which pays C at time T is

$$V_0 = E_Q\,[e^{-rT}C].$$

(iii) For the special option, $C = 1$ if S_T is in $[a, b]$, otherwise 0, so

$$\begin{aligned} V_0 &= E_Q\,[e^{-rT}1_{[a,b]}(S_T)] \\ &= e^{-rT}Q(S_T \text{ in } [a, b]) \end{aligned}$$

Because B is a Brownian motion $\ln\dfrac{S_T}{S_0}$ is normally distributed (under Q) with mean $\left(r - \dfrac{\sigma^2}{2}\right)T$ and standard deviation $\sqrt{T}\sigma$.

Hence $Q(S_T < x) = \Phi(d(x))$ where $d(x) = \dfrac{\ln\dfrac{x}{S_0} - \left(r - \dfrac{\sigma^2}{2}\right)T}{\sqrt{T}\sigma}$

Hence $V_0 = e^{-rT}\,[\Phi(d(b)) - \Phi(d(a))]$

(iv) The value is .5% of the holding, so is $.005S_0$.

(v) Payoff = $.001S_1 + .004(S_1 - S_0)^+ + .005(S_1 - U)^+ + .004S_0 1_{(S_1 - S_0)} + .005U1_{(S_1 > U)}$

Denoting the prices of the four options in the decomposition immediately above as $c_1, c_2, c_3,$ and c_4:

$$c_1 = S_0\Phi(d_1) - S_0 e^{-r}\Phi(d_1 - \sigma) \text{ where } d_1 = \frac{r + ½\sigma^2}{\sigma} = \frac{r}{\sigma} + ½\sigma$$

$$= 100\Phi(0.325) - 100e^{-0.05}\Phi(0.075)$$

$$= 12.33599$$

$$c_2 = S_0\Phi(d_3) - Ue^{-0.05}\,\Phi(d_3 - \sigma) \text{ where } d_3 = \frac{\ln\left(\frac{S_0}{U}\right) + (r + ½\sigma^2)}{\sigma}$$

$$c_3 = 100[e^{-0.05}(1 - \Phi(d_4))] \text{ where } d_4 = \frac{-\left(r - \frac{\sigma^2}{2}\right)}{\sigma} = \frac{-r}{\sigma} + \frac{\sigma}{2}$$

$$= 100[0.5040495] = 50.40495$$

$$c_4 = e^{-0.05}(1 - \Phi(d_5)) \text{ where } d_5 = \frac{\ln\left(\frac{U}{S_0}\right) - \left(r - \frac{\sigma^2}{2}\right)}{\sigma}$$

$$= \frac{-\ln\left(\frac{S_0}{U}\right) - \left(r - \frac{\sigma^2}{2}\right)}{\sigma}$$

$$= -(d_3 - \sigma)$$

Value = 0.1

$+\ 12.33599 \times .004$

$+\ [100\Phi(d_3) - Ue^{-0.05}\,\Phi(d_3 - \sigma)] \times 0.005$

$+\ 50.40495 \times .004$

$+\ U \times e^{-0.05}(1 - \Phi\{-(d_3 - \sigma)\}) \times .005$

$$\text{Value} = 0.1 + 12.33599 \times 0.04 + 50.40495 \times .004$$

$$+ 0.5\Phi(d_3) - Ue^{-0.05}\,\Phi(d_3 - \sigma)\,0.005$$

$$+ Ue^{-0.05} \times 0.005 \times (1 - (1 - \Phi(d_3 - \sigma)))$$

$$= 0.35096376 + 0.5\Phi(d_3) = 0.5$$

$$\Rightarrow d_3 = \Phi^{-1}\left(1 - \frac{0.35096376}{0.5}\right)$$

$$= -0.52995$$

$$\Rightarrow \ln\frac{100}{U} = -0.52995 \times 0.25 - 0.05 - ½0.25^2$$

$$\Rightarrow U = 123.83$$

END OF EXAMINERS' REPORT