INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

September 2012 examinations

Subject CT8 – Financial Economics
Core Technical

Introduction

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and using past papers as a revision aid and also those who have previously failed the subject.

The Examiners are charged by Council with examining the published syllabus. The Examiners have access to the Core Reading, which is designed to interpret the syllabus, and will generally base questions around it but are not required to examine the content of Core Reading specifically or exclusively.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report; other valid approaches are given appropriate credit. For essay-style questions, particularly the open-ended questions in the later subjects, the report may contain more points than the Examiners will expect from a solution that scores full marks.

D C Bowie
Chairman of the Board of Examiners

December 2012

General comments on Subject CT8

Subject CT8 introduces the main concepts and principles of financial economics. These are developed in later subjects in the ST series of exams. This subject combines various types of skills. In particular, along with CT7, it is one of the first where candidates are expected to write lengthy passages of reasoned thought, rather than just complete calculations. This is a skill that will be new to many and candidates are advised to pay particular attention to the answers to this type of question by studying many past papers.

Comments on the September 2012 paper

The general performance was good and better than on the previous session (April 2012). Candidates generally found this paper challenging, but well-prepared candidates scored well across the whole paper and the best candidates scored close to full marks. As in previous diets, questions that required an element of application of the core reading to situations that were not immediately familiar proved more challenging to most candidates. The comments that follow the questions concentrate on areas where candidates could have improved their performance.

1 (i) Let the risk neutral default probability for AA be p_{AA}. Consider the equation of value for a £100 investment in AA:

$$100 = (1 - p_{AA}) \times \frac{£106}{1.04} + p_{AA} \times 0 \Rightarrow p_{AA} = 1.8868\%,$$

and similarly

$$100 = (1 - p_{BB}) \times \frac{£108}{1.04} + p_{BB} \times 0 \Rightarrow p_{BB} = 3.7037\%.$$

(ii) (a) The 95% VaR is zero. The 95% TailVar is

$$\frac{£106\, p_{AA}}{p_{AA}} = £106$$

(b) The 95% VaR is zero. The 95% TailVar is:

$$\frac{£108\, p_{BB}}{p_{BB}} = £108$$

(c) The distribution of returns is:

£107 with probability $(1 - p_{AA})(1 - p_{BB}) = 0.94479$
£54 with probability $p_{AA}(1 - p_{BB}) = 0.01817$
£53 with probability $p_{BB}(1 - p_{AA}) = 0.03634$
£0 with probability $p_{AA} p_{BB} = 0.00070$

So the 95% VaR is $£107 - £54 = £53$.

The 95% TailVaR is

$$\frac{£107\, p_{AA} p_{BB} + £54(1 - p_{AA}) p_{BB}}{p_{BB}} = £55$$

(iii) Investing in diversified (i.e. not perfectly correlated) assets generally leads to a lower dispersion of returns and hence lower risk.

Portfolio (c) is diversified compared to (a) and (b). However, the 95% VaR for portfolio (c) is higher than for either (a) or (b) where it is zero. So an increase in VaR could, in this circumstance, correspond to a decrease in risk.

Zero VaR does not necessarily mean zero risk.

The 95% TailVaR for portfolio (c) is lower than (a) and (b).

The question was framed sufficiently openly that candidates could quote values at risk relative to the maximum return, the expected return or the initial investment. Full marks were available for any approach if it was followed through correctly although the below sets out answers relative to the maximum return.

In general, this was poorly answered with candidates struggling to gain more than a few marks. Some candidates calculated the transition rates in part (i) instead of the probability or calculated the probabilities assuming a continuous time model. In part (ii), many candidates calculated VaR and TailVaR using a continuous model instead of the discrete model in the question. Some candidates confused VaR and variance.

2 The risk-neutral probability of an up jump at any time is:

$$q = \frac{e^{6\%} - 1/1.2}{1.2 - 1/1.2} = 0.62319 \qquad [1]$$

There are eight possible paths the option could take. The paths, probabilities of those paths, final stock prices and option payoffs are shown in the following table.

Path	Probability of path	Final stock price	Option payoff
Up up up	$q^3 = 0.24203$	691.2	Nil
Up up down	$q^2(1-q) = 0.14634$	480	21.91
Up down up	$q^2(1-q) = 0.14634$	480	21.91
Up down down	$(1-q)^2 q = 0.08848$	333.33	18.26
Down up up	$q^2(1-q) = 0.14634$	480	Nil
Down up down	$(1-q)^2 q = 0.08848$	333.33	18.26
Down down up	$(1-q)^2 q = 0.08848$	333.33	Nil
Down down down	$(1-q)^3 = 0.05350$	231.48	Nil

The price of the option is then

$$V = e^{-3\times 6\%} \sum_{\text{paths}} \text{probability of path} \times \text{option payoff} = 8.05\text{p}$$

If candidates worked in units of £s rather than p, they will have found an answer of £0.805 (or 80.5p) and full marks were available.

Also, if candidates took a down movement to mean 0.8S_t rather than S_t/1.2 then full marks were available in this case.

Largely well answered. Some candidates didn't calculate the price correctly because they miscalculated the number of paths to the nodes with non-zero payoffs. Some candidates miscalculated the probability by using e^{-r} rather than e^{r}. A few candidates calculated the probability according to classical probability theory (favourable outcomes / possible outcomes) rather than risk-neutral.

3 (i) The key assumptions are:

(a) That investors select their portfolios on the basis of the expect return and the variance of that return over a single time horizon.

(b) Investors are never satiated. At a given level of risk, they will always prefer a portfolio with a higher return to one with a lower return.

(c) Investors dislike risk. For a given level of return they will always prefer a portfolio with lower variance to one with higher variance.

(ii) Suppose an investor can invest an any of N securities, $i = 1, \ldots, N$.
A proportion x_i is invested in security S_i. The return on the portfolio R_P is

$$R_P = \sum_{i=1}^{N} x_i R_i,$$

where R_i is the return on security i.

The expected return on the portfolio E is

$$E = \mathbb{E}\left[R_P\right] = \sum_{i=1}^{N} x_i E_i,$$

where E_i is the expected return on security i.

The variance is

$$V = Var\left[R_P\right] = \sum_{i,j=1}^{N} x_i x_j C_{ij},$$

where C_{ij} is the covariance of the returns on securities i and j and we write $C_{ii} = V_i$.

(iii) A portfolio is efficient if the investor cannot find a better one in the sense that it has both a higher expected return and a lower variance.

When there are N securities the aim is to choose x_i to minimise V subject to the constraints

$$\Sigma_i x_i = 1$$

and

$$E = E_P, \text{ say,}$$

in order to plot the minimum variance curve.

One way of solving such a minimisation problem is the method of Lagrangian multipliers.

The Lagrangian function is

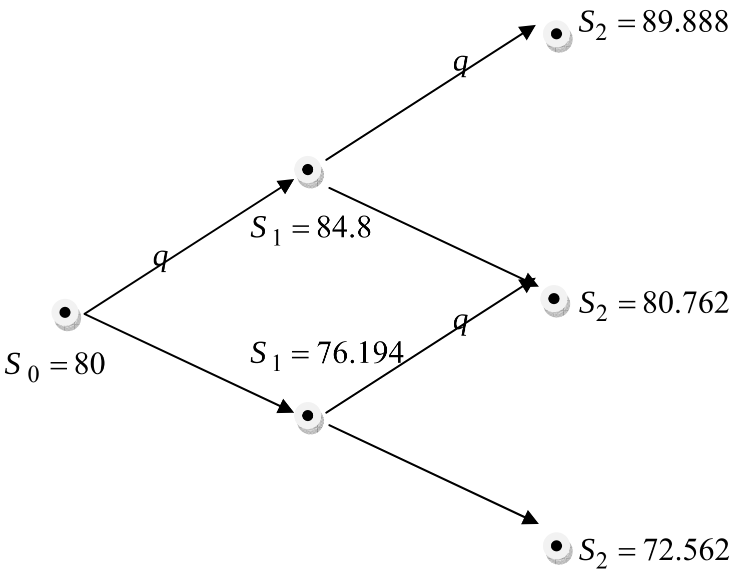
$$W = V - \lambda(E - E_P) - \mu(\Sigma_i x_i - 1).$$

To find the minimum we set the partial derivatives of W with respect to all the x_i and λ and μ equal to zero. The result is a set of linear equations that can be solved.

The usual way of representing the results of the above calculations is by plotting the minimum standard deviation for each value of E_P as a curve in expected return – standard deviation ($E - \sigma$) space. In this space, with expected return on the vertical axis, the efficient frontier is the part of the curve lying above the point of the global minimum of standard deviation.

Largely well answered. Some candidates forgot the "single time period" assumption in part (i) or included simplifying assumptions such as "no transaction charges etc." as major assumptions. Some candidates confused efficient portfolios with optimal portfolios.

4 (i) The model can be drawn as follows:



Payoff	
Call at 82	Put at 82
7.888	0
0	1.238
0	9.438

Now $q = \dfrac{e^{5\%/4} - \dfrac{1}{1.05}}{1.06 - \dfrac{1}{1.05}} = 0.55936.$

So, $c_0 = e^{-5\%/2}\left[q^2 \times 7.888 + 0\right] = 2.40707$

(ii) (a) Similarly, $p_0 = e^{-\frac{5\%}{2}}\left[(1-q)^2 \times 9.438 + 2q(1-q) \times 1.238\right] = 2.38248$

(b) The put-call parity entails $c_0 + 82e^{-\frac{5\%}{2}} = p_0 + 80$

Using the value for the call found in question (i), we get p_0 = 2.38248.

(iii) Early exercise would happen at time zero or after three months.

At time zero, the value of the American put option is at least as great as the European put option, i.e. greater than 2.38248. The intrinsic value of the option is 2. Therefore early exercise is not optimal.

After three months, if the first move is up the option is out of the money, so early exercise is not optimal.

After three months, if the first move is down, the intrinsic value of the option is 82 – 76.19048 = 5.8095. The value of holding on to the option until 6 months is given by

$$e^{-\frac{5\%}{4}}\left(q\times 1.238+\left(1-\text{q}\right)\times 9.438\right)=4.7909.$$

So, it would be optimal to exercise the option early if the first move was down.

Candidates who calculated a down move as multiplying by 0.95 rather than dividing by 1.05 were awarded full marks.

Largely well answered with most candidates earning full marks in parts (i) to (iii). The majority of candidates discussed whether it was optimal to exercise American put options in general in part (iv), or even American call options, rather than the particular option in the question.

5 The model should be arbitrage free.

Interest rates should be positive.

The short rate and other interest rates should exhibit some form of mean-reverting behaviour.

It should be straightforward to calculate the prices of bonds and certain derivative contracts.

The model should produce realistic dynamics.

The model should be able to be calibrated easily to current market data.

The model should be flexible enough to cope properly with a range of derivative contracts.

The model should provide a satisfactory fit to historical data.

Generally well done as straightforward book work. Some candidates answered this with a series of questions such as "is it easy to calculate? does it fit historical data?... In this situation marks were awarded according to the extent that the candidates identified the key points set out below. Some candidates did write other assumptions like "constant volatility" or "the share follows geometric Brownian motion".

6 (i) Under the risk neutral measure $\mathbb{Q}$ the short rate under the Vasicek model has the dynamics

$$dr(t) = \alpha(\mu - r(t))dt + \sigma dW(t)$$

The short rate under the Cox-Ingersoll-Ross model has the dynamics

$$dr(t) = \alpha(\mu - r(t))dt + \sigma\sqrt{r(t)}dW(t)$$

(ii) So, if the short-rate changes, the volatility of the process is unchanged in the Vasicek model, but it will change in the CIR model (an increase in the short rate will lead to an increase in the volatility).

Most candidates gained full marks in part (i). Some candidates wrote about how a change in volatility could affect the short-term rate rather than vice versa. One or two candidates only included the sigma in the volatility term of the CIR model. Many candidates gave generic statements about the interest rate models in answer to part (ii), rather than responding to the question.

7 (i) Standard interpolation gives a volatility of $\sigma = 436\%$

(ii) Under the risk free measure, the stock price $S_{0.25} = S_0 \exp(\sigma Z_{0.25} - 0.5\sigma^2(0.25) + 0.25r)$

While the stock price at time 6 months is

$$\begin{aligned} S_{0.5} &= S_{0.25} \exp(2\sigma(Z_{0.5} - Z_{0.25}) - 0.5(2\sigma)^2 (0.25) + 0.25r) \\ &= S_0 \exp(2\sigma(Z_{0.5} - Z_{0.25}) + \sigma Z_{0.25} - 0.5\sigma^2(1.25) + 0.5r) \end{aligned}$$

Full marks were available if candidates provided the formulae under the real world probability measure.

(iii) Since Z has stationary independent increments, $S_{0.5}$ has the same distribution as

$$S_0 \exp(\sqrt{(2.5)}\sigma Z_{0.5} - 0.5\sigma^2(1.25) + 0.5r),$$

which corresponds to the stock price at 6 months with volatility $\sqrt{(2.5)}\sigma$.

Now, using the Black-Scholes formula, the put price is

$$p = K^{-rT} \Phi(-d_2) - S_0 \Phi(-d_1).$$
$$d_1 = 2.4378$$
$$d_2 = -2.4368,$$

so

$$p = 120(e^{-.5r}\,\Phi(-d_2) - \Phi(-d_1)) = \$1.179$$

In general, this was poorly answered. There was evidence of candidates spending a significant amount of time in part (i) but many then proceeded with an assumed value for the volatility and were awarded full marks where they completed parts (ii) and (iii) in a self-consistent way. Most candidates forgot the volatility doubled between time three and six. Several candidates managed to calculate a value in part (iii) using their assumed values from part (i).

8 (i) Merton's model assumes that a corporate entity has issued both equity and debt such that its total value at time t is of $F(t)$.

It is an example of a structural credit risk model.

$F(t)$ varies over time as a result of actions by the corporate entity which does not pay dividends on its equity or coupons on its bonds. Part of the corporate entity's value is zero-coupon debt with a promised repayment amount of L at a future time T. At time T the remainder of the value of the corporate entity will be distributed amongst the equity holders and the corporate entity will be wound up.

The corporate entity will default if the total value of its assets, $F(T)$ is less than the promised debt repayment at time T, i.e. $F(T)<L$. In this situation, the bond holders will receive $F(T)$ instead of L and the equity holders will receive nothing.

This can be regarded as treating the equity holders of the corporate entity as having a European call option on the assets of the company with maturity T and a strike price equal to the value of the debt.

The Merton model can be used to estimate either the risk-neutral probability that the company will default or the credit spread on the debt.

(ii) Under the Merton model, the value at redemption is min($F(T)$, £3,200m), where $F(t)$ is the gross value of the company at time t.

Thus the value at time 0 is

$$e^{-3r}E[\min(F(3),3200)] = e^{-3r}E[F(3) - \max(F(3) - 3200,0)],$$

where the expectation is under the risk-neutral measure, so equals $F(0) - C$, where C is a call option on the gross value with strike £3,200m.

(iii) The market value of the debt is £3,200 × £92.603/£100 = £2,963.3m

The market value of the equity (i.e. the call option on the company's assets is then £6,979m – £2,963.3m = £4,015.7m.
We can calculate the implied volatility of the company's assets as 29.8%

The risk neutral price for the insurance (ignoring credit risk of the insurer themselves) is then:

$$£1\text{m} \times e^{-6\%} \times (1 - \Phi(d_2)) = £1\text{m} \times e^{-6\%} \times 0.085518 = £80{,}538.2$$

Whether or not this represents an arbitrate opportunity depends on whether there is a market (e.g. credit default swaps) where you can trade these contracts/go short in relation to Risky plc.

Candidates had no major problems in part (i) describing the Merton model. Some candidates did confuse the facts that shareholders had a call option while bondholders had a put, although given put-call parity there are various ways to value these options. Only some candidates managed to answer part (ii) and some did answer it by reference to valuing a put option rather than the call in the marking schedule for which full marks were awarded.

9 (i) Since equal market capitalisation: $w_A = 0.5$ and $w_B = 0.5$.

(ii) Let r_M denote the return of the market portfolio, r_A (resp. r_B) denote the return of asset A (resp. asset B).

Then, $V(r_M) = V(0.5r_A + 0.5r_B) = 0.5^2 * V(r_A) + 0.5^2 * V(r_B) + 2 * 0.5^2 \text{cov}(r_A, r_B)$.

$\text{Beta}_A = \text{cov}(r_A, r_M)/V(r_M)$
$= (0.5 * V(r_A) + 0.5 * \text{cov}(r_A, r_B))/ 0.5^2 * V(r_A) + 0.5^2 * V(r_B) + 2 * 0.5^2 \text{cov}(r_A, r_B)$

As $\text{Cov}(r_A, r_M) = \text{cov}(r_A, 0.5r_A + 0.5r_B) = 0.5 * V(r_A) + 0.5 * \text{cov}(r_A, r_B)$

Similarly, $\text{Beta}_B = (0.5 * V(r_B) + 0.5 * \text{cov}(r_A, r_B)) / 0.5^2 * V(r_A) + 0.5^2 * V(r_B) + 2 * 0.5^2 \text{cov}(r_A, r_B)$

(iii) The equation of the Security Market line gives:

$\underline{r}_i = r_f + \text{Beta}_i (\underline{r}_M - r_f)$ where $\underline{r}_i$ is the expected return of asset i (for $i = A,B$).

Hence, using the numerical values, we get

$\underline{r}_A = 0.2$ and $\underline{r}_B = 0.16$

(iv) Using the separation theorem, we have:

$\underline{r}_P = w_0\ rf + w_M\, \underline{r}_M$

where w_0 is the weight of the risk-free asset in the portfolio P and w_M is the weight of the market portfolio in the portfolio P.

Moreover, there is the constraint $w_0 + w_M = 1$

Solving the system leads to:

$w_0 = -0.25$ and $w_M = 1.25$

(v) The Capital Market Line equation is:

$\underline{r}_P = r_f + \text{sigma}_P * ((\underline{r}_M - r_f) / \text{sigma}_M)$

where sigma_P (resp. sigma_M) is the standard deviation of the portfolio P (resp. the market portfolio.

So, we get $\text{sigma}_P = 17.6\%$

This question posed little difficulty to well-prepared candidates. Some included the risk-free asset in their answer to part (i). In part (ii), a lot of candidates defined beta in terms of the market portfolio rather than the risky assets as asked for in the question. Part (iii) posed little problem for the majority of candidates although some struggled to calculate a numerical value for beta. Only a handful of candidates included the risk-free asset in part (iv). The majority only included the risky assets.

10 (i) $U'(w) > 0$.

(ii) This means that the probability of portfolio B producing a return below a certain value is never less than the probability of portfolio A producing a return below the same value and exceeds it for at least some value of x.

Alternative answer:

First order stochastic dominance holds if:

$F_{\text{A}}(x) \le F_{\text{B}}(x)$, for all x, and

$F_{\text{A}}(x) < F_{\text{B}}(x)$, for some value of x.

(iii) The expected utility of A is

$$E[U_{\text{A}}] = \int_a^b U(w)\,\mathrm{d}F_{\text{A}}(w),$$

and the expected utility of an investment in portfolio B is

$$E[U_B] = \int_a^b U(w)\,\mathrm{d}F_B(w).$$

Thus, if A is preferred to B

$$\int_a^b U(w)\,\mathrm{d}F_\mathrm{A}(w) - \int_a^b U(w)\,\mathrm{d}F_\mathrm{B}(w) > 0.$$

Now, the left hand side can be written as

$$\int_a^b U(w)[\mathrm{d}F_\mathrm{A}(w) - dF_\mathrm{B}(w)]$$

and integrating by parts yields

$$\left[U(w)\left(F_\mathrm{A}(w) - F_\mathrm{B}(w)\right)\right]_a^b - \int_a^b U'(w)[F_\mathrm{A}(w) - F_\mathrm{B}(w)]dw.$$

Now, $F_\mathrm{A}(a) = F_\mathrm{B}(a) = 0$ by definition, and $F_\mathrm{A}(b) = F_\mathrm{B}(b) = 1$ by definition so for the expression to be positive we require the value of the integral to be negative.

$U'(w) > 0$ by assumption, so for the integral to be negative, no matter what the exact form of $U'(w), F_\mathrm{A}(w) - F_\mathrm{B}(w)$ must be less than or equal to zero for all values of w with $F_\mathrm{A} < F_\mathrm{B}$ for at least one value of w if the value is not to be zero.

Largely well-answered. However, some candidates appeared to confuse inequality signs. In part (i), this meant defining non-satiation as having a decreasing utility function while in part (ii) this meant the distribution function of A was greater than that of B. Fewer candidates than expected scored well in part (iii) for what appeared to be a textbook proof.

END OF EXAMINERS' REPORT