Subject CT8 — Financial Economics

EXAMINERS' REPORT

April 2008

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

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Chairman of the Board of Examiners

June 2008

1 *Gamma*

$$\Gamma = \frac{\partial^2 f}{\partial s^2}$$

Vega

$$\nu = \frac{\partial f}{\partial \sigma}$$

f is the price of the derivative; s is the price of the underlying asset; σ is the volatility of the stochastic process of the price of the underlying

2 Consider the stochastic differential equation

$$dS_t = \alpha S_t\, dt + \sigma S_t\, dB_t\ .$$

$$\log S_t = \log S_0 + \left(\alpha - ½\sigma^2\right) t + \sigma B_t$$

or, finally,

$$S_t = S_0 \exp\left[\left(\alpha - ½\sigma^2\right) t + \sigma B_t\right].$$

3 (i) There is no arbitrage in the market since

$$d < \exp(r) < u \qquad \text{with } r = 8\%.$$

(ii) To price the call option, we use the risk-neutral pricing formula. The risk-neutral probability of an upward move is

$$q = \frac{\exp(r) - d}{u - d} = 0.9164.$$

The price of the call is determined by a backward procedure:

$$\begin{cases} Cuu = (u^2 S_0 - K)^+ = 21 \\ Cud = (udS_0 - K)^+ = 0 \\ Cdd = (d^2 S_0 - K)^+ = 0 \end{cases} \Rightarrow \begin{cases} C_1(u) = \exp(-r)(qCuu + (1-q)Cud) = 17.7655 \\ C_1(d) = \exp(-r)(qCud + (1-q)Cdd) = 0 \end{cases}$$

$$\Rightarrow \quad C_0 = \exp(-r)(qC_1(u) + (1-q)(C_1(d)) = 15.0292$$

4 (i) **The two-state model for credit ratings with a constant transition intensity**.

A model can be set up, in continuous time, with two states N (not previously defaulted) and D (previously defaulted). Under this simple model it is assumed that the default-free interest rate term structure is deterministic with $r(t) = r$ for all t. If the transition intensity, under the real-world measure P, from N to D at time t is denoted by $\lambda(t)$, this model can be represented as:

$$\boxed{\text{No default, } N} \overset{\lambda(t)}{\text{―――――}} \boxed{\text{Default, } D}$$

and D is an absorbing state.

(ii) $B(t,T) = e^{-r(T-t)}\,[1 - (1 - \delta)(1 - \exp(-\int_t^T \tilde{\lambda}(s)ds))]$

5 (i) **Mean Return**

<u>Asset 1</u> $-1 \times 8^1/_3\% + 11 \times 91^2/_3\% = 10\%$

<u>Asset 2</u> $0 \times 50\% + 20 \times 50\% = 10\%$

Variance of Return

<u>Asset 1</u> $(10 - (-1))^2 \times 8^1/_3\% + (10 - 11)^2 \times 91^2/_3\% = 11\%\%$

<u>Asset 2</u> $(10 - 0)^2 \times 50\% + (10 - 20)^2 \times 50\% = 100\%\%$

Semi-Variance of Return

<u>Asset 1</u> $(10 - (-1))^2 \times 8^1/_3\% = 10.08333\%\%$

<u>Asset 2</u> $(10 - 0)^2 \times 50\% = 50\%\%$

Shortfall Probability

<u>Asset 1</u> $8^1/_3\%$

<u>Asset 2</u> 0%

(ii) Both have same expected return. The variance is appropriate risk measure in this case.

=> Choose Asset 1

(iii)

- Mathematically tractable.
- Leads to elegant solutions for optimal portfolios.

- Often a good approximation to the other possible methodologies.
- Gives optimum portfolios if returns are normally distributed or investors have quadratic utility functions

6 (i) Investors select their portfolios on the basis of the expected return and the variance of the return over a single time horizon.

Investors are never satiated. At a given level of risk, they will always prefer a portfolio with a higher return to one with a lower return.

Investors dislike risk. For a given level of return they will always prefer a portfolio with lower variance to one with higher variance.

(ii) (a) 100% in X – expected return of 12%

(b) Proportion in $X = (V_Y + C_{XY})/(V_X + V_Y + C_{XY})$

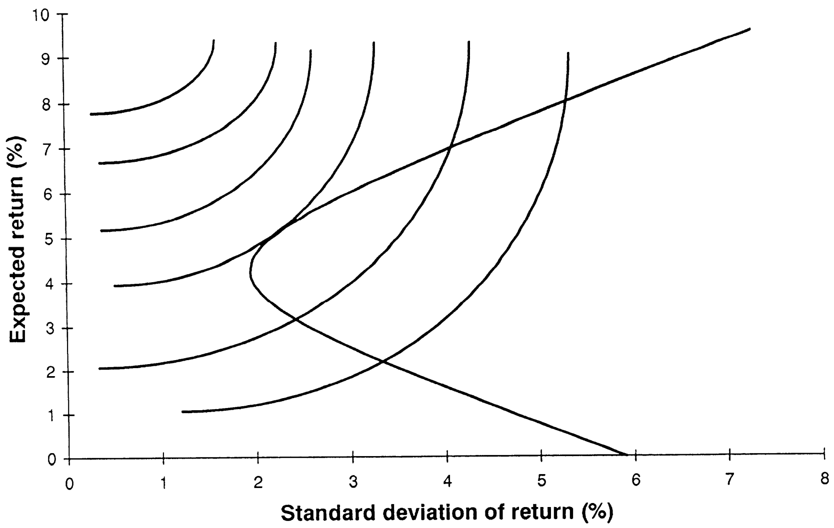
$= (15\%\% - 0.5 \times (30\%\% - 15\%\%)^{0.5})/(30\%\% + 15\%\% + 0.5 \times (30\%\% - 15\%\%)^{0.5})$

$= 18.47\%$

(iii) Plot indifference curves in return-standard deviation space.

Utility is maximised by choosing the portfolio on the efficient frontier where the frontier is at a tangent to the indifference curve.

Graphically candidates should reproduce a diagram similar to Figure 3 from the core reading.



7 (i) The extra assumptions of CAPM are:

- All investors have the same one-period horizon.

- All investors can borrow or lend unlimited amounts at the same risk-free rate.

- The markets for risky assets are perfect. Information is freely and instantly available to all investors and no investor believes that they can affect the price of a security by their own actions.

- Investors have the same estimates of the expected returns, standard deviations and covariances of securities over the one-period horizon.

- All investors measure in the same "currency" e.g. pounds or dollars or in "real" or "money" terms.

(ii) If investors have homogeneous expectations, then they are all faced by the same efficient frontier of risky securities. If in addition they are all subject to the same risk-free rate of interest, the efficient frontier collapses to the straight line in $E - \sigma$ space which passes through the risk-free rate of return on the E-axis and is tangential to the efficient frontier for risky securities.

All rational investors will hold a combination of the risk-free asset and the portfolio of risky assets at the point where the straight line through the risk-free return touches the original efficient frontier. Because this is the portfolio held in different quantities by all investors it must consist of all risk assets in proportion to their market capitalisation. It is commonly called the "market portfolio". The proportion of a particular investor's portfolio consisting of the market portfolio will be determined by their risk-return preference.

(iii) The market price of risk is $(E_m - r)/\sigma_m$, where

$$E_m = (30{,}000 \times (5\% \times 0.4 + 8\% \times 0.1 + 3\% \times 0.5) + 50{,}000 \times (6\% \times 0.4 + 2\% \times 0.1 + 5\% \times 0.5) + 30{,}000 \times (7\% \times 0.4 + 1\% \times 0.1 + 4\% \times 0.5)) \div 110{,}000$$

$$= 4.8273\%$$

$$\sigma_{m\cdot} = [(30{,}000 \times 5\% + 50{,}000 \times 6\% + 30{,}000 \times 7\%) \div 110{,}000 - 4.8273\%]^2 \times 0.4 + [(30{,}000 \text{ x } 8\% + 50{,}000 \times 2\% + 30{,}000 \times 1\%) \div 110{,}000 - 4.8273\%]^2 \times 0.1 + [(30{,}000 \times 3\% + 50{,}000 \times 5\% + 30{,}000 \times 4\%) \div 110{,}000 - 4.8273\%]^2 \times 0.5$$

$$= 9.7264 \times 10^{-5} = 0.9862\%^2$$

Thus the market price of risk is (4.8273% – 4%)/0.9862%

= 83.89%

8 (i) (a) $\log(S_u) - \log(S_t) \sim N[\mu(u - t), \sigma^2(u - t)]$

(b) $E[S_u] = S_t \exp(\mu(u - t) + \tfrac{1}{2}\sigma^2(u - t))$

(c) $\mathrm{Var}[S_u] = (S_t)^2 \exp(2\mu(u - t) + \sigma^2(u - t)).\ [\exp(\sigma^2(u - t)) - 1]$

(ii) As the model incorporates independent returns over disjoint intervals, it is impossible to use past history to deduce that prices are cheap or dear at any time.

(iii) Technical analysis does not lead to excess performance.

Estimates of σ vary widely according to what time period is considered, and how frequently the samples are taken.

Examination of historic option prices suggests that implied volatility based on the Black Scholes model fluctuate markedly over time.

There appears to be some evidence for some mean reversion in markets, but the evidence rests heavily on the aftermath of a small number of dramatic crashes. Furthermore, there also appears to be some evidence of momentum effects, which imply that a rise one day is more likely to be followed by another rise the next day.

In particular, market crashes appear more often than one would expect from a normal distribution.

While the random walk produces continuous price paths, jumps or discontinuities seem to be an important feature of real markets. Furthermore, days with no change, or very small change, also happen more often than the normal distribution suggests.

One measure of these non-normal features is the *Hausdorff fractal dimension* of the price process. A pure jump process (such as a Poisson process) has a fractal dimension of 1. Random walks have a fractal dimension of 1½. Empirical investigations of market returns often reveal a fractal dimension around 1.4.

9 The idea is to assume that the converse inequality holds true and show that this leads to an arbitrage opportunity. More precisely, let us assume that

$$\forall\lambda \in [0,1] \quad \lambda C(K_1)+(1-\lambda)C(K_2) < C(\lambda K_1+(1-\lambda)K_2).$$

Then we construct the following (self-financing) portfolio:

- At time 0: we sell one call with strike price $K_3 = \lambda K_1 + (1-\lambda)K_2$ and buy λ calls with strike K_1 and $(1-\lambda)$ calls with strike K_2. We lend the difference. The total value of the portfolio at time 0 is equal to 0.

At time 0	
sell $C(K_3)$	$C(K_3)$
buy $\lambda C(K_1)$	$-\lambda C(K_1)$
buy $(1-\lambda)C(K_2)$	$-(1-\lambda)C(K_2)$
lend the difference	$M \equiv \lambda C(K_1) + (1-\lambda)C(K_2) - C(K_3)$
Total	0

- At time T: we look at the various possibilities depending on the value S_T of the underlying asset at that time. In all situations, the terminal value of the portfolio is either >0 or ≥ 0.

	$S_T < K_1$	$K_1 < S_T < K_3$	$K_3 < S_T < K_2$	$K_2 < S_T$
$\lambda C(K_1)$	0	$\lambda(S_T - K_1)$	$\lambda(S_T - K_1)$	$\lambda(S_T - K_1)$
$(1-\lambda)C(K_2)$	0	0	0	$(1-\lambda)(S_T - K_2)$
$C(K_3)$	0	0	$-(S_T - K_3)$	$-(S_T - K_3)$
lending	$M\exp(rT)$	$M\exp(rT)$	$M\exp(rT)$	$M\exp(rT)$
Total	$M\exp(rT) > 0$	$M\exp(rT) + \lambda(S_T - K_1) > 0$	$M\exp(rT) + (1-\lambda)(K_2 - S_T) > 0$	$M\exp(rT) > 0$

This is an arbitrage opportunity. Hence the result.

10 (i) This is an Ornstein-Uhlenbeck process. The solution is given by:

$$r(t) = r_0\exp(-at) + b(1-\exp(-at)) + \sigma\exp(-at)\int_0^t \exp(as)\,dW_s.$$

(ii) Using similar arguments, we can get for $u \geq t$:

$$r(u) = r(t)\exp(-a(u-t)) + b(1-\exp(-a(u-t))) + \sigma\exp(-au)\int_t^u \exp(as)\,dW_s.$$

Hence

$$\int_t^T r(u)du = r(t)\int_t^T \exp(-a(u-t))du + b\int_t^T \left(1-\exp(-a(u-t))\right)du + \sigma\int_t^T \exp(-au)\int_t^u \exp(as)\,dW_s du.$$

After some computation:

$$\int_t^T r(u)du = b(T-t) + (r(t)-b)\frac{1-\exp(-a(T-t))}{a} + \frac{\sigma}{a}\int_t^T \left(1-\exp(-a(T-s))\right)dW_s.$$

(iii) Hence, $\int_t^T r(u)du$ is also a Gaussian random variable.

(iv) Since the bond market is complete, the price of a zero-coupon bond can be written as

$$B(t,T) = E\left[\exp\left(-\int_t^T r(s)ds\right)\middle| F_t\right].$$

Since $\int_t^T r(u)du$ is a Gaussian random variable, we can compute explicitly the price of the zero-coupon bond in terms of the expected value and variance (conditional) of $\int_t^T r(u)du$:

$$B(t,T) = \exp\left[-E\left[\int_t^T r(s)ds \middle| F_t\right] + \frac{1}{2}V\left[\int_t^T r(s)ds \middle| F_t\right]\right].$$

11 (i) Let f denote the price of a call option, then

$$f(s,T) = s\Phi(d_1) - Ke^{-rT}\Phi(d_2),$$

where

$$d_1 = (\ln(S_0/K) + (r + ½\sigma^2)T)/\sigma\sqrt{T} \text{ and } d_2 = d_1 - \sigma\sqrt{T}.$$

It follows (since $\Phi'(x) = \exp(-x^2/2)/\sqrt{2\pi}$) that

$$\Delta = \partial f/\partial s = \Phi(d_1) + s\exp(-d_1^2/2)/\sqrt{2\pi}\,\partial d_1/\partial s - Ke^{-rT}\exp(-d_2^2/2)/\sqrt{2\pi}\,\partial d_2/\partial s.$$

If we now notice that

$$\partial d_1/\partial s = \partial d_2/\partial s$$

and

$$d_2^2 = d_1^2 - (2r + \sigma^2)T - 2\ \ln(s/K) + \sigma^2 T = d_1^2 - 2rT - 2\ln(s/K)$$

we see that the last two terms in the expression for Δ cancel and we are just left with $\Delta = \Phi\ (d_1)$.

In this case, we must have $100{,}000\Delta = 75{,}000$ and so $\Delta = 0.75$.

(ii) $\Delta = .75$ and so $d_1 = 0.6745$. It follows (rearranging the expression for d_1) that $(.02469 + .07 + 0.5\sigma^2) = 0.6745\sigma$. Solving the quadratic we obtain (choosing the root less than 1) $\sigma = 0.6745 \pm \sqrt{0.26557} = 0.159165 = 15.9\%$.

(iii) We need to calculate:

$$Ke^{-rT}\Phi(d_2) = 8e^{-r}\Phi(d_1 - \sigma\sqrt{T}\,) = 8e^{-r}\ 0.696825 = £5.19772$$

Clearly, the value of the loan is = £519,772 and the option price is

$$100{,}000 * 8.2 * 0.75 - 519{,}772 = £95{,}228.$$

(iv) Use put-call parity. This merely assumes that borrowing is allowed and the market is arbitrage free.

$$p_0 = c_0 + Ke^{-rT} - S_0 = .95228 + 8\ e^{-rT} - 8.2 = .21143$$

END OF EXAMINERS' REPORT