Subject CT8 — Financial Economics. Core Technical

September 2009 Examinations

EXAMINERS REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

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Chairman of the Board of Examiners

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Comments for individual questions are given with the solutions that follow.

1

(i) $R_i = a_i + b_{i,1} I_1 + b_{i,2} I_2 + \dots\dots. + b_{i,L} I_L + c_i$

Where

R_i is the return on security i.

a_i and c_i are the constant and random parts respectively of the component of return unique to security i.

$I_1 \dots\dots I_L$ are the changes in a set of L factors which explain the variation in R_i about the expected return.

$b_{i,k}$ is the sensitivity of security i to factor k.

(ii) **Macroeconomic factor models**

These use observable economic time series as the factors.

Examples: rate of inflation, economic growth, short term interest rates, yields on long-term government bonds, yield margin on corporate bonds over government bonds.

Fundamental factor models

These use company specific variables as the factors.

Examples: level of gearing, price earnings ratio, the level of R & D spending, the industry group to which the company belongs

Statistical factor models

Principal components analysis is used to determine a set of indices which explain as much as possible of the observed variance.

These indices are unlikely to have any meaningful economic interpretation and may vary considerably between different data sets.

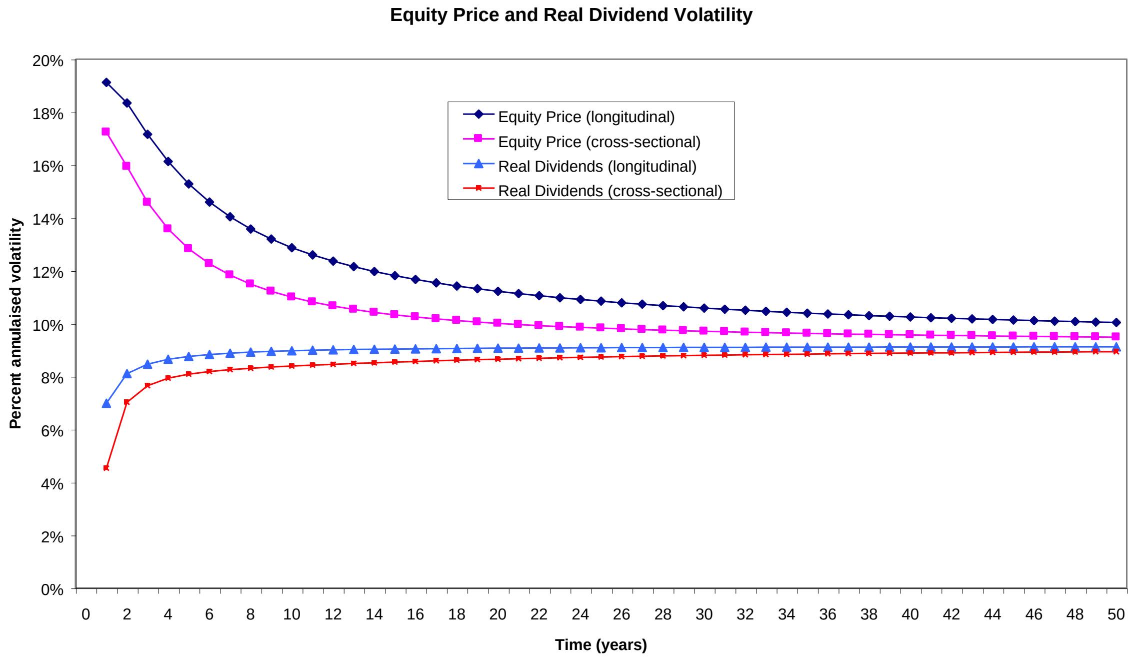
2

(i) Cross-sectional property fixes a time horizon and looks at the distribution over all the simulations. E.g. what will inflation be next year? The estimates are implicitly conditional on past information. They can be deduced from prices of options and other derivatives.

Longitudinal property looks at the distribution over a long period of time. E.g. what will the distribution of inflation be over the next 1000 years? Unlike cross sectional properties does not reflect market conditions at a particular time.

(ii)

a. Estimates are the same — random walk returns are independent across years.
b. The main point is longitudinal volatilities are higher. Longitudinal volatilities represent unconditional values whilst cross-sectional volatilities depend on the information set. The difference between the two shows the value of extra information. Over long horizons the two values converge to the same point. Students might draw a graph similar to that on Unit 6 page 13.



3

(i) **Investment A**

Expected return = $E[0.1 + N] = 0.1 + 1 = 1.1$
Variance = 1

Investment B

Expected return = $1.5 \times 0.99 - 5.0 \times 0.01 = 1.435$
Variance = $(1.435 - 1.5)^2 \times 0.99 + (1.435 - (-5))^2 \times 0.01 = 0.418275$

Investment B has both higher expected return and lower variance so would be preferred on this basis. However there is an issue with the possibility of very bad returns. Also there might be an issue with the estimated probabilities of investment B being somewhat unreliable as they are probably derived from the fat tail part of a distribution. Thus it might be wise to have a margin of error regarding this calculation in particular.

(ii)

a. **Investment A**

Semivariance = 0.5

Investment B

Semivariance = $(1.435 -(-5))^2 \times 0.01 = 0.41409$

b. **Investment A**

Shortfall probability of return below 0.
This is probability of the return from $N(1,1)$ being below $-0.1 = 0.13567$.

Investment B

Shortfall probability of return below 0 is 0.01.

c. **Investment A**

Shortfall probability of return below –2.
This is probability of the return from $N(1,1)$ being below $-2.1 = 0.00097$.

Investment B

Shortfall probability of return below –2 is 0.01.

(iii) Definitions of VAR, Tail VaR and Expected Shortfall. (Unit 1, page 3)
Clearly will show high but unlikely risk in the tail of Investment B.

(iv) Give points for sensible discussion:

Might not always be best to optimise on basis of expected return and variance.

No one "correct" measure of risk — different definitions give different orderings in this case.

Which investment is preferable depends somewhat on solvency of company — can they afford large, albeit unlikely, losses (analogies with "credit crunch").

Consider the rest of portfolio.

4

(i) Expected Return = $\Sigma_i x_i E_i$

where E_i is expected return on security i.

Variance is $\Sigma_i \Sigma_j x_i x_j C_{ij}$

where C_{ij} is the covariance of the returns on securities i and j and $C_{ii} = V_i$

where V_i is variance of security i.

(Unit 2 page 2)

(ii) Proportion in A = $(V_B - C_{AB}) / (V_A + V_B - 2C_{AB})$

From (Unit 2 page 3) or can fairly easily be calculated from first principles

$C_{AB} = 1 \times sd_A \times sd_B = 1 \times 4\% \times 2\% = 8\%\%$

Thus Proportion in A = $(4\%\% - 8\%\%) / (16\%\% + 4\%\% - 2 \times 8\%\%) = -1$
Proportion in B = 2
i.e. Short sell a unit of A and buy 2 of B.

(iii) Expected Return of portfolio in (ii) is $-1 \times 4 + 2 \times 3 = 2\%$

Variance = $1^2 \times 16\%\% + 2^2 \times 4\%\% + 2 \times 2 \times -1 \times 2\% \times 4\% = 0$
(i.e. risk free)

Now if we borrow at 1% p.a. can invest in the portfolio in (ii) make a return of 2% pay back the loan and will have make 1% over the year.

5

(i) The formula states that

$P(\max_{0 \le s \le t} B_s + \mu\, s > y) = \Phi((-y + \mu t) / \sqrt{t}) + e^{2\mu y}\Phi(-(y + \mu t) / \sqrt{t})$.

Thus the density of $\max_{0 \le s \le t} B_s + \mu\, s$ is

$-d/dy(\Phi((-y + \mu t) / \sqrt{t}) + e^{2\mu y}\Phi(-(y + \mu t) / \sqrt{t}))$

$= 2\mu e^{2\mu y}\Phi(-(y + \mu t) / \sqrt{t})) + e^{2\mu y} \exp(-(y + \mu t)^2 / t) / \sqrt{(2\pi t)})$
$\quad + \exp(-(-y + \mu t)^2/t) / \sqrt{(2\pi t)})$

$= 2\mu e^{2\mu y}\Phi(-(y + \mu t) / \sqrt{t})) + 2 \exp(-(-y + \mu t)^2 / t) / \sqrt{(2\pi t)}).$ [4]

(ii) We need to price the derivative under the risk neutral measure. Under this measure,

$\max_{0\le s\le T} S_s = S_0 \exp(\max_{0\le s\le T} \sigma B_s + s(r - 1/2\ \sigma^2))$
$= S_0 \exp(\sigma(\max_{0\le s\le T} B_s + \mu\ s)),$

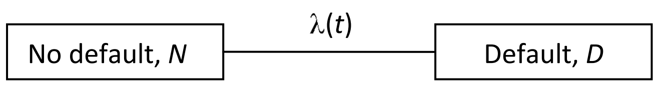
with $\mu = (r - 1/2\ \sigma^2) / \sigma$, and B a standard Brownian motion. Thus the price is

$E[e^{-rT} S_0 \exp(\sigma(\max_{0\le s\le T} B_s + \mu\ s))]$
$= e^{-rT} S_0 \int [2\ e^{-(-y + \mu T)^2/2T} / \sqrt{(2\pi T)} - 2\mu e^{2\mu y}\Phi(-(y + \mu T)/\sqrt{T})] \exp(\sigma y) dy$ [4]

[Total 8]

6

(i) The model is a continuous time Markov with two states: N (not previously defaulted) and D (previously defaulted). Under this simple model it is assumed that the default-free interest rate term structure is deterministic with $r(t) = r$ for all t. If the transition intensity, under the real-world measure P, from N to D at time t is denoted by $\lambda(t)$, this model can be represented as:



and D is an absorbing state.

If $X(t)$ is the state at time t, the transition intensity, $\lambda(t)$, can be interpreted as:

$\Pr_P(X(t + dt) = N \mid X(t) = N) = 1 - \lambda(t)\, dt + o(dt) \quad$ as $dt \to 0$,
$\Pr_P(X(t + dt) = D \mid X(t) = N) = \lambda(t)\, dt + o(dt) \quad$ as $dt \to 0$.

(ii) We need to show that under P_λ, $E[e^{-rt} D_t | F_s] = e^{-rs} D_s$.
If the default time $\tau \le s$ then $e^{-rs} D_s = e^{-rs}\, \delta\, e^{-r(2-s)} = e^{-rt} D_t$.
If $s \le \tau$, then $e^{-rt} D_t = \delta e^{-2r}$, if default has occurred prior to time t

$= e^{-2r}(\delta(1 - e^{-\lambda(2-t)}) + e^{-\lambda(2-t)})$ otherwise.
Thus,
$E[e^{-rt} D_t | F_s] = \delta e^{-2r}(1 - e^{-\lambda(t-s)}) + e^{-2r}(\delta(1 - e^{-\lambda(2-t)}) + e^{-\lambda(2-t)})\, e^{-\lambda(t-s)}$

$= e^{-2r}((\delta(1 - e^{-\lambda(2-s)}) + e^{-\lambda(2-s)})$ which is $e^{-rs} D_s$ in this case.

(iii) Seeking a portfolio of the form $aD + bB$, we need the value to be 1000 at time 2 if default has occurred, so we need $a\delta + b = 1000$. Similarly we need the value to be zero at time 2 if default has not occurred, so we need $a + b = 0$. Hence $b = -a$ and $b = 1000/(1 - \delta)$.
The fair price for the derivative must be the set-up cost for this portfolio which is $be^{-2r} + aD_0 = 1000/(1 - \delta)(e^{-2r} - e^{-2r}(\delta(1 - e^{-2\lambda}) + e^{-2\lambda})) = 1000\, e^{-2r}(1 - e^{-2\lambda})$

(iv) We need to check that the initial value of this portfolio is $E[e^{-2r}V]$ under P_λ : under P_λ , $E[e^{-2r}V] = 1000e^{-2r}P(default) = 1000e^{-2r}(1 - e^{-2\lambda})$ as required (V is the final value of the derivative).

This then accords with the fact that if we can hedge without arbitrage then the price is that given by the EMM

7

(i) The assumptions underlying the Black-Scholes model are as follows:

1. The price of the underlying share follows a geometric Brownian motion.
2. There are no risk-free arbitrage opportunities.
3. The risk-free rate of interest is constant, the same for all maturities and the same for borrowing or lending.
4. Unlimited short selling (that is, negative holdings) is allowed.
5. There are no taxes or transaction costs.
6. The underlying asset can be traded continuously and in infinitesimally small numbers of units.

(ii) It is clear that each of these assumptions is unrealistic to some degree, for example:

- Share prices can jump. This invalidates assumption 1. since geometric Brownian motion has continuous sample paths. It also invalidates assumption 2. However, hedging strategies can still be constructed which substantially reduce the level of risk.
- The risk-free rate of interest does vary and in a unpredictable way. However, over the short term of a typical derivative the assumption of a constant risk-free rate of interest is not far from reality. (More specifically the model can be adapted in a simple way to allow for a stochastic risk-free rate, provided this is a predictable process.)
- Unlimited short selling may not be allowed except perhaps at penal rates of interest. These problems can be mitigated by holding mixtures of derivatives which reduce the need for short selling. This is part of a suitable risk management strategy as discussed in Section 2 below.
- Shares can normally only be dealt in integer multiples of one unit, not continuously and dealings attract transaction costs: invalidating assumptions 2., 5., 6. and 7. Again we are still able to construct suitable hedging strategies which substantially reduce risk.

- Distributions of share returns tend to have fatter tails than suggested by the log-normal model, invalidating assumption 1.

8

(i) A corollary to the Cameron-Martin-Girsanov theorem states that there exists a process η_t such that, for any F_T-measurable derivative payoff X at time T,

$$E_Q[X \mid F_t] = E_P\left[\frac{\eta_T}{\eta_t} X \mid F_t\right].$$

Define $A_t = e^{-rt}\eta_t$

The process A_t is called a state-price deflator (also *deflator*; *state-price density*; *pricing kernel*; or *stochastic discount factor*). [3]

(ii) If we define $\eta_t = \exp(-\gamma Z_t - \frac{1}{2}\gamma^2 t)$, where $\gamma = (\mu - r)/\sigma$, then the state price deflator is $A_t = \eta_t e^{-rt}$. [3]

(iii) If a contract has terminal value V then its price at time t is $E_P[A_T V/A_t|F_t]$. So in this case we obtain

$$p_t = E_P[\exp(\gamma Z_1)\, 1_{(Z1>1)}\exp(-\gamma Z_1 - \tfrac{1}{2}\gamma^2)\, e^{-r} / (\exp(-\gamma Z_t - \tfrac{1}{2}\gamma^2 t)\, e^{-rt})|F_t]$$
$$= P[Z_1 > 1|F_t]\exp(\gamma Z_t - (\tfrac{1}{2}\gamma^2 + r)(1 - t))$$
$$= (1 - \Phi((1 - Z_t) / \sqrt{(1 - t)}))\exp(\gamma Z_t - (\tfrac{1}{2}\gamma^2 + r)(1 - t))$$ [5]

9

The real-world probability measure P can be interpreted in the following way. Let A be some event contained in F (for example, suppose that A is the event that S_1 is greater than or equal to 100). Then $P(A)$ is the *actual* probability that the event A will occur. On a more intuitive level with m independent realisations of the future instead of one we would find that the event A occurs on approximately a proportion $P(A)$ occasions (with the approximation getting better as m gets larger and larger).

Two measures P and Q which apply to the same sigma-algebra F are said to be *equivalent* if for any event E in F: $P(E) > 0$ if and only if $Q(E) > 0$, where $P(E)$ and $Q(E)$ are the probabilities of E under P and Q respectively.

In the context of the binomial model and using the above definition of equivalence the only constraint on the real-world measure P is that at any point in the binomial tree the probability of an up move lies strictly between 0 and 1. The only constraint on Q is the same but this can be equated to the requirement that the risk-free return must lie strictly between the return on a down move and the return on an up move. This gives us considerable flexibility in the range of possible equivalent measures.

10

Consider an investor holding an American call. She needs some cash at time $t < T$. Two strategies are available for her:

(i) She sells the options on the market and gets the price of it in exchange C_t^A,

(ii) She exercises the option and obtains the intrinsic value $S_t - K$.

But, we know that $C_t^A \geq C_t^E \geq \max\left(0; S_t - K\exp\left(-r\left(T-t\right)\right)\right)$.

Hence $C_t^A \geq \max\left(0; S_t - K\exp\left(-r\left(T-t\right)\right)\right) \geq \left(S_t - Ke^{-r(T-t)}\right) \geq S_t - K$.

As a consequence, the first strategy is better and it is never optimal for the agent to exercise her option early.

11

(i) $B(t,T)$ = Zero-coupon bond price
= price at t for £1 payable at T
$r(t)$ = instantaneous risk-free rate of interest at t

Take a specific bond with maturity at T_1. Suppose its SDE under the real-world measure P is

$$dB(t,T_1) = B(t,T_1)\left[m(t,T_1)dt + S(t,T_1)dW(t)\right]$$

where, besides $S(t, T_1)$, $m(t, T_1)$ might be stochastic. The *market price of risk* is defined as

$$\gamma(t,T_1) = \frac{m(t,T_1) - r(t)}{S(t,T_1)}.$$

(ii) Define C_t = cash account at time t

Portfolio A: a_t units of $B(t, T_2)$ and b_t units of C_t

Portfolio *B*: 1 unit of $B(t, T_1)$

Self financing implies:

$$a_t B(t, T_2) + b_t C_t = B(t, T_1)$$

and

$$a_t\, dB(t, T_2) + b_t\, dC_t = dB(t, T_1)$$

So

$$a_t\, B(t, T_2)\, [m(t, T_2)\, dt + S(t, T_2)\, dW_t] + b_t r_t C_t\, dt = B(t, T_1)\, [m(t, T_1)\, dt + S(t, T_1)\, dW_t]$$

equating the coefficients of dt and dW_t we obtain

$$a_t = \frac{S(t,T_1)B(t,T_1)}{S(t,T_2)B(t,T_2)}$$

and $b_t = \dfrac{1}{r_t C_t}\left[m(t,T_1)B(t,T_1) - \dfrac{m(t,T_2)B(t,T_1)S(t_1T_1)}{S(t,T_2)} \right]$

Substituting these back:

$$\frac{S(t_1T_1)B(t,T_1)}{S(t,T_2)B(t,T_2)} B(t,T_2) +$$

$$B_t \frac{1}{r_t B_t}\left[m(t,T_1)B(t,T_1) - \frac{m(t,T_2)B(t,T_1)S(t,T_1)}{S(t,T_2)} \right]$$

$$= B(t,T_1)$$

Simplifying:

$$\frac{S(t,T_1)}{S(t,T_2)} + \frac{1}{r_t}\left[m(t,T_1) - \frac{m(t,T_2)S(t_1T_1)}{S(t,T_2)} \right] = 1$$

$$\Rightarrow \quad \frac{m(t,T_1) - r_t}{S(t,T_1)} = \frac{m(t,T_2) - r_t}{S(t,T_2)}$$

END OF EXAMINERS' REPORT