INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

April 2015 examinations

Subject CT8 – Financial Economics
Core Technical

Introduction

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and using past papers as a revision aid and also those who have previously failed the subject.

The Examiners are charged by Council with examining the published syllabus. The Examiners have access to the Core Reading, which is designed to interpret the syllabus, and will generally base questions around it but are not required to examine the content of Core Reading specifically or exclusively.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report; other valid approaches are given appropriate credit. For essay-style questions, particularly the open-ended questions in the later subjects, the report may contain more points than the Examiners will expect from a solution that scores full marks.

The report is written based on the legislative and regulatory context at the date the examination was set. Candidates should take into account the possibility that circumstances may have changed if using these reports for revision.

F Layton
Chairman of the Board of Examiners

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1 (i) 1. **Comparability**

An investor can state a preference between all available certain outcomes.

2. **Transitivity**

If A is preferred to B and B is preferred to C, then A is preferred to C.

3. **Independence**

If an investor is indifferent between two certain outcomes, A and B, then he is also indifferent between the following two gambles:

(i) A with probability p and C with probability $(1 - p)$; and
(ii) B with probability p and C with probability $(1 - p)$.

4. **Certainty equivalence**

Suppose that A is preferred to B and B is preferred to C. Then there is a unique probability, p, such that the investor is indifferent between B and a gamble giving A with probability p and C with probability $(1 - p)$. B is known as the certainty equivalent of the above gamble.

(ii) Non-satiated: $U'(w) > 0$
Risk neutral: $U''(w) = 0$

(iii) $R(w) = -wU''(w)/U'(w)$
$= 1 - \gamma$

(iv) $R'(w) = 0$ so the relative risk aversion is constant; "iso-elastic" is also acceptable.

This question was generally well-answered.

2 (i) VaR $= -t$ where $P(X < t) = 5\%$

(ii) Expected shortfall $= E[\max(2\% - X, 0)]$

$$= \int_{-\infty}^{2\%} (2\% - x) f(x) dx$$

(iii) $P(X < t) = 5\%$ where $x \sim N(5, 100)$
$\Leftrightarrow P(Z \le (t - 5) / 10) = 5\%$
$\Leftrightarrow (t - 5) / 10 = -1.645$
$\Leftrightarrow t = -11.45\%$

Therefore the 5% VaR is $-t = 11.45\%$.

(iv) VaR does not illustrate the size of the loss in the tail of the distribution, only the likelihood.

The usefulness of VaR may be limited by a lack of data to determine the tail of the distribution.

This question was relatively poorly answered for such a standard. Many candidates confused the 5th and 95th percentiles. In part (iv) some students mentioned that VaR calculations often assume a normal distribution, which was not relevant in the context of downside risk and gained no marks.

3 (i) (a) Let f denote the price of a put option,
then $d_1 = (\ln(S_0/K) + (r + ½\sigma^2)T)/ \sigma\sqrt{T}$
and then $\Delta = -\Phi\,(-\,d_1) = \Phi\,(d_1) - 1$.

(b) In this case, we must have $100{,}000\Delta = -24{,}830$ and so $\Delta = -0.25$

(ii) $\Delta = -.2483$ and so $d_1 = 0.68$. It follows (rearranging the expression for d_1) that $(.01575 + .03 + 0.5\sigma^2) = 0.68\sigma$. Solving the quadratic equation we obtain $\sigma = 0.68 \pm \sqrt{0.3709} = 0.07098 = 7.1\%$ (choosing the root less than 1).

(iii) We need to calculate $K\,e^{-rT}\Phi(-d_2) = e^{-r}\Phi(-d_1 + \sigma\sqrt{T}\,)$
$= 630e^{-0.03}\,\Phi(-0.609)\text{p} = 630e^{-0.03} * 0.2712 = 165.806\text{p}$.

Clearly the option price is 165.806 – 24830 * 640/100,000 = 6.894p.
and the value of the cash holding is 100,000 * 165.806p = £165,806

This question was relatively poorly answered for such a standard question. Few candidates scored on part (i)(a) or seemed to realise that the delta of a put must be negative.

4 *Regrettably, there was a mistake on the paper, giving an incorrect formula for the variance of X_t. Candidates were not penalised for this as they were credited with the greater of their actual score on this question and their average mark for the other questions on the paper.*

The calculations are given both for the incorrect formula and the correct one.

The formula for the distribution of X_{t+s} should be:

$N(0.75^t X_s, (0.75^{2(t-1)} + 0.75^{2(t-2)} \ldots +1).25^2)$

Since this is in the question the answers are given following through the mistake and the correct answers afterwards in italics (except for (i)(b) itself).

(i) (a) Conditional on X_t, X_{t+1} is normal, and has mean $0.75X_t$ and variance $0.25^2\text{Var}(e_{t+1}) = 0.25^2$

(b) *CORRECT ANSWER: By induction: true for t = 1, then if the statement is true for t = n (and all s) then X_{n+1+s} is $N(0.75^n X_{s+1}, (0.75^{2(n-1)} + 0.75^{2(n-2)} \ldots + 1).25^2)$, conditional on X_{s+1} and X_{s+1} is $N(0.75X_s, 0.25^2)$ conditional on X_s, so X_{n+1+s} is $N(0.75^{n+1}X_s, 0.25^2 + 0.75^2 (0.75^{2(n-1)} + 0.75^{2(n-2)} \ldots + 1).25^2)$ conditional on X_s, establishing the inductive step.*

(ii) (a) Taking the answer from part (i)(b), we see that the mean converges and so does the variance, so the distribution converges to a Normal.

(b) The mean tends to 0 and the variance tends to $(.25)^2/(1 - (.25)^2) = 1/15$, so the limiting distribution is $N(0,1/15)$.

CORRECT ANSWER … the variance tends to $(.25)^2/(1 - (.75)^2) = 1/7$, so the limiting distribution is $N(0,1/7)$

(iii) Conditional on Y_0, $\log Y_3$ is $N(0.75^3 \log Y_0, 0.25^2 + \ldots + 0.25^{2*3}) = N(0.08042, .06665)$

So $E[\log Y_3 | Y_0 = 1.21] = 0.75^3 \log 1.21 = 0.08042$ and $\text{Var}(\log Y_3 | Y_0 = 1.21) = .06665$.

Thus $E[Y_3 | Y_0 = 1.21] = \exp(0.08042 + ½ * 0.06665) = 1.12047$.

CORRECT ANSWER
Conditional on Y_0, $\log Y_3$ is $N(0.75^3 \log Y_0, (0.75^4+0.75^2+1).25^2)) = N(0.08042, 0.11743)$
So $E[\log Y_3 | Y_0 = 1.21] = 0.25^3 \log 1.21 = 0.08042$ and $\text{Var}(\log Y_3 | Y_0 = 1.21) = 0.11743$.
Thus $E[Y_3 | Y_0 = 1.21] = \exp(0.08042 + ½ * 0.11743) = 1.14928$.

(iv) The long-run distribution is $\text{logNormal}(0, 0.25^2/(1 - 0.25^2)) = \text{logNormal}(0, 0.06667)$, so the long-run mean annual increase is $e^{½*0.06667} = 1.033895$ or 3.3895%.

CORRECT ANSWER
The long-run distribution is $\text{logNormal}(0, 0.25^2/(1 - 0.75^2)) = \text{logNormal}(0, 0.14285)$, so the long-run mean annual increase is $e^{½*142851} = 1.07404$ or 7.404%.

5 (i)

Write down Ito's formula for $f(t,X_t)$ when $dX_t = \mu_t dt + \sigma_t dW_t$

$$
\begin{aligned}
df(t,X_t) &= \frac{\partial f(t,X_t)}{\partial t}dt + \frac{\partial f(t,X_t)}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 f(t,X_t)}{\partial x^2}\left(dX_t\right)^2 \\
&= \frac{\partial f(t,X_t)}{\partial t}dt + \frac{\partial f(t,X_t)}{\partial x}\left(\mu_t dt + \sigma_t dW_t\right) + \frac{1}{2}\frac{\partial^2 f(t,X_t)}{\partial x^2}\left(\sigma_t^2 dt\right) \\
&= \left(\frac{\partial f(t,X_t)}{\partial t} + \frac{\partial f(t,X_t)}{\partial x}\mu_t + \frac{1}{2}\frac{\partial^2 f(t,X_t)}{\partial x^2}\sigma_t^2\right)dt + \frac{\partial f(t,X_t)}{\partial x}\sigma_t dW_t
\end{aligned}
$$

This is actually from Taylor's formula and the last line alone would have scored full marks. Versions involving the integrals would also have scored full marks.

(ii) Consider $X_t = U_t e^{\lambda t}$.

Then $dU_t = d(e^{-\lambda t}X_t)$

$$= -\lambda e^{-\lambda t}X_t dt + e^{-\lambda t}dX_t$$

$$= -\lambda e^{-\lambda t}X_t dt + e^{-\lambda t}(\lambda X_t dt + \sigma dW_t) = \sigma e^{-\lambda t}dW_t$$

So $U_t = U_0 + \sigma\int_0^t e^{-\lambda s}dW_s$

So $X_t = e^{\lambda t}U_t = e^{\lambda t}X_0 + \sigma\int_0^t e^{\lambda(t-s)}dW_s$

This question was generally well-answered. However many candidates showed an eccentric use of stochastic calculus in part (ii).

6 (i) Consider two portfolios:

A: one call plus cash of $Ke^{-r(T-t)}$.
B: one put plus one share.

Both portfolios have a payoff at the time of expiry of the options of $\max\{K, S_T\}$.

Since they have the same value at expiry and since the options cannot be exercised before then they should have the same value at any time $t < T$, by no-arbitrage: that is

$$c_t + Ke^{-r(T-t)} = p_t + S_t$$

(ii) Put call parity implies a security price of $12.80.

(iii) Trial and error yields a volatility of 26%.

Sample values:

Volatility	*Call value*
10%	$3.44
15%	$3.50
20%	$3.64
25%	$3.83
30%	$4.05
35%	$4.29
40%	$4.54

This question was generally well-answered. However it is a cause for concern that many candidates were unable to calculate the implied volatility.

7 (i) Denote the individual derivative by f and assume this is written on an underlying security S

Delta = $\partial f/\partial S$
Gamma = $\partial^2 f/\partial S^2$
Vega = $\partial f/\partial \sigma$

(ii) Delta = 0.801

(iii) The hedge is delta = 0.801 shares = and 17.91 – 0.801 * 60 = $30.15 short in cash.

(iv) Using the approximation $f(S, \sigma + \delta) \approx f(S, \sigma) + \delta df/d\sigma$, we obtain an option price ≈ 17.91 + 29.00 * 0.02 = $18.49.

This question was generally well-answered. Very few candidates seemed to be able to use the Vega to perform the approximation in part (iv) and instead opted to recalculate the option price using the Black-Scholes formula. This still scored full marks if done correctly, but was time-consuming and unnecessary.

8 (i) Investors select their portfolios on the basis of the expected return and the variance of that return over a single time horizon.

The expected returns, variance of returns and covariance of returns are known for all assets and pairs of assets.

Investors are never satiated. At a given level of risk, they will always prefer a portfolio with a higher return to one with a lower return.

(ii) Let the proportion invested in asset i, be x_i, with expected return E_i, variance V_i and correlation ρ12. Let E be the return on the portfolio of the three assets and let λ and μ be Lagrange multipliers.

Then, the Lagrangian function W satisfies:

$$W = \sum_{i=1}^{3} x_i{}^2 V_i + 2\rho_{13}\sigma_1\sigma_3 x_1 x_3 - \lambda(E_1x_1 + E_2x_2 + E_3x_3 - E) - \mu(x_1 + x_2 + x_3 - 1)$$

$$= 16x_1{}^2 + 144x_2{}^2 + 64x_3{}^2 + 48x_1x_3 - \lambda(2x_1 + 4x_2 + 3x_3 - E) - \mu(x_1 + x_2 + x_3 - 1)$$

(iii) $\dfrac{\partial W}{\partial x_1} = 32x_1 + 48x_3 - 2\lambda - \mu = 0$ $\dfrac{\partial W}{\partial x_2} = 288x_2 - 4\lambda - \mu = 0$

$$\frac{\partial W}{\partial x_3} = 128x_3 + 48x_1 - 3\lambda - \mu = 0$$

Substituting the values given for x_i, we obtain three equations for λ and μ, solving these gives $\lambda = 64.8$ and $\mu = -100.8$ and we can check that these values satisfy the constraints.

(iv) Without short selling, the only way to get an expected return of 4% is to invest wholly in asset 2.

This question was generally answered quite well. Well answered on part (i) but many students listed all assumptions rather than the main ones. Part (ii) was well answered but some students did not know the formula for the Lagrangian function or the variance for the portfolio of three assets. Many students took the "show" instruction in part (iii) to mean prove by solving the equations rather than "verify" and spent a considerable amount of time. Few scored on part (iv) with many making vague comments about the variance changing.

9 (i) The three types of credit risk model are:

- structural models: these are explicit models of a corporate entity issuing both debt and equity. They aim to link default events explicitly to the fortunes of the issuer.

- reduced-form models: these are statistical models which use market statistics (such as credit ratings) rather than specific data relating to the issuer, and give statistical models for their movement.

- intensity-based models: these model the factors influencing the credit events which lead to default and typically do not consider what triggers these events.

(ii) In the Merton model, the company is modelled as having a fixed debt, L and variable assets F_t. This means the equity holders can be regarded as holding a European call on the assets with a strike of L. It follows from the Black-Scholes model that we can deduce the (risk-neutral) default probability from the share price.

(A correct quantitative answer was also rewarded.)

(iii) In the two state model for credit rating with deterministic transition intensity, the formula for the zero coupon bond price is

$$B(t,T) = e^{-r(T-t)}\,(1 - (1 - \delta)\ (1 - \exp(-\textstyle\int_t^T \lambda(s)ds))).$$

(iv) It follows that the risk-neutral default intensity is given by $\lambda(s) = s^2/2$.

(v) The fair price is

$$1.2\exp(-2r)\,(1 - \exp(-\textstyle\int_0^1 \lambda(s)ds)) = 1.2e^{-.04}(1 - e^{-1/6}) = £176{,}998.$$

There has been an average performance on this question. Many candidates seem unfamiliar with this standard material. Many candidates did not know the formula for the bond price with deterministic, but varying, transition intensity. Some candidates only listed the three forms of credit model and others struggled with part (iv). Some also ended up with answers for (v) in terms of delta, and/or (T – t), demonstrating a lack of competence with simple differentiation.

10 (i)

We then have the following formula for bond prices:

$$B(t,T) = e^{a(T-t)-b(T-t)r(t)}$$

where $b(\tau) = \dfrac{1-e^{-\alpha\tau}}{\alpha}$

$$a(\tau) = (b(\tau)-\tau)\left(\mu - \frac{\sigma^2}{2\alpha^2}\right) - \frac{\sigma^2}{4\alpha} b(\tau)^2$$

and $\tau = T - t$.

(ii) τ = 1

$b(\tau)$ = (1 – exp(–0.5 * 1)) / 0.5 = 0.7869

$a(\tau)$ = (0.7869 – 1) * (0.04 – 0.1^2/(2 * 0.5^2)) – 0.1^2/ (4 * 0.5) * 0.7869^2
= –0.0074

$B(0,1)$ = 100 * exp(–0.0074 – 0.7869 * 0.02) = 100 * 0.9772 = \$97.72

(iii) $R(t,T) = \dfrac{-1}{T-t} \ln B(t,T)$ for $t < T$

$R(0,3)$ = –1/(3 – 0) * ln(0.9) = 3.51%

(iv) $F(t,T,S) = \dfrac{1}{S-T} \ln \dfrac{B(t,T)}{B(t,S)}$ for $t < T < s$

$F(0,1,3)$ = 1/(3 – 1) * ln(97.72/90) = 4.11%

Candidates performed very poorly indeed on this question. Many people tried to derive the formula from first principles for part (i) producing many pages of working for no credit. Since there is a version of the formula in the tables, this is even more wasteful. A reasonable number of candidates scored the easy marks in part (iii). Less realised the opportunity in part (iv).

END OF EXAMINERS' REPORT