EXAMINATION

April 2007

Subject CT8 — Financial Economics
Core Technical

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

M A Stocker
Chairman of the Board of Examiners

June 2007

Comments

No comments given.

1 (i) $\Theta = \dfrac{\partial f}{\partial t}$

From the Black-Scholes PDE we have

$$\frac{\partial f}{\partial t} + rs\frac{\partial f}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 f}{\partial s^2} = rf$$

or $\Theta + rs\Delta + \dfrac{1}{2}\sigma^2 s^2 \Gamma = rf$

(ii) Deep out of the money, delta and gamma will be close to zero which implies that theta will equal the risk free rate of return.

2 (i) The expression for the variance of the portfolio can be rewritten as:

$$V = \Sigma_i \; x_i^2 \, V_i + \sum_{i \neq j} x_i . x_j . C_{ij}$$

(ii) If we assume that equal amounts are invested in each asset, then with N assets the proportion invested in each is $1/N$. Thus:

$$V = \Sigma_i \; (1/N)^2 \, V_i + \sum_{i \neq j} \; (1/N)(1/N).C_{ij}$$

Factoring out $1/N$ from the first summation and $(N - 1)/N$ from the second yields:

$$V = 1/N \, \Sigma_i \, V_i \,/N + (N - 1)/N \sum_{i \neq j} \; C_{ij}/N(N - 1)$$

Replacing the summation by averages we have

$$V = 1/N \, V_i + (N - 1)/N . \overline{C}_{ij}$$

The contribution to the portfolio variance of the variances of the individual securities goes to zero as N gets very large. However, the contribution of the covariance terms approaches the average covariance as N gets large. The individual risk of securities can be diversified away, but the contribution to the total risk caused by the covariance terms cannot be diversified away.

3 (i) Derivative pays off after two down moves

Derivative price = 0.1448 = $A(2,dd) \times (1 - p)^2 \times 1$
$A(2,dd) = A(1,d)^2 = 2.316$
Hence $p = 0.75$

(ii) $A(1,u) = 0.7610 = A(0) \times \exp(-0.05) \times q/p$
Knowing p, we can get $q = 0.6$

The alternative approach shown below is possible for (i) and (ii), students were given full credit for either approach.

(i) and (ii) p and q, the risk neutral probability measure, can be obtained by solving the equation for the state price deflator (Unit 11 Page 9 of Core Reading)

$$A_1 = e^{-r}q/p \qquad \text{if } S_1 = S_0u$$
$$\quad = e^{-r}(1 - q)/(1 - p) \qquad \text{if } S_1 = S_0d$$

This gives $p = 0.75$ and $q = 0.6$.

Note if this approach is used it is not necessary to know that the price of the derivative is 0.1448.

(iii) Price = $0.6^2 \times \exp(-.05 \times 2) \times 1 = 0.3257$

The solution to (iii) given above assumes that $ud = 1$, students who worked on this basis were given full credit as this is a common presumption in this type of work. However, some students realized that the strict definition of a recombining model is that $ud = du$. In this case

If $ud = 1$ the solution given above holds i.e. price = 0.3257 (only upper node pays off)

There is another possible case where $ud > 1$ (upper and middle nodes both pay off)

In this case price $= 0.6^2e^{-0.05\times2} + 2 \times 0.6 \times 0.4e^{-0.05\times2}$
$= 0.3257 + 0.4344$
$= 0.7601$

4 (a) SV = Downside semi-variance = $\int_{0.5}^{\mu}(t-\mu)^2 f(t)dt$

$$\mu = \int_{0.5}^{\infty} tf(t)dt = -\frac{c}{2t^2}\Bigg]_{0.5}^{\infty} = 0.75$$

$$SV = c\left\{\int_{0.5}^{0.75} t^{-2}dt - 2\mu\int_{0.5}^{0.75} t^{-3}dt + \mu^2\int_{0.5}^{0.75} t^{-4}dt\right\}$$

$$= c\left\{-t\Big]_{0.5}^{0.75} - 1.5\left[-\frac{t^{-2}}{2}\right]_{0.5}^{0.75} + 0.75^2\left[-\frac{t^{-3}}{3}\right]_{0.5}^{0.75}\right\}$$

$= 0.02083$

(in units of $(£m)^2$)

<u>OR</u>

$$SV = \int_{0.5}^{\infty}\frac{(t-\mu)^2}{t^4}dt - \int_{0.75}^{\infty}\frac{(t-\mu)^2}{t^4}dt$$

then, using the same integration steps as above,

$SV = 0.1875 - 0.16666$

$= 0.02083$

(b) $P(R > x) = \int_{x}^{\infty} f(t)dt = \frac{c}{3}x^{-3}$

$Pr[S \le -t] = Pr[R \le 0.7 - t]$

$= 1 - Pr[R > 0.7 - t]$

$= 1 - \frac{c}{3}(0.7-t)^{-3}$

For 5% VaR:

$Pr[S \le -t] = 1 - \frac{c}{3}(0.7 - t)^{-3} = 0.05$

$\Rightarrow t = 0.1914$

5 (i) $$C_{ij} = \sum_k \sum_l \beta_{ik}\beta_{jl}\text{Cov}(I_k, I_l)$$

$$+\sum_k \beta_{ik}\text{Cov}(I_k, \varepsilon_j)$$

$$+\sum_k \beta_{jk}\text{Cov}(I_k, \varepsilon_i)$$

$= \sum_k \beta_{ik}\beta_{jk}\text{Var}(I_k)$ because of independence of all the other terms.

(ii) low covariance if betas are low, i.e. pick stocks with different sensitivities to the factors

(iii) This is exactly the same as the multi-index model for returns on individual securities. The contribution of APT is to describe how we can go from a multi-index model for individual security returns to a equilibrium market model. Non-mathematically, the argument can be made as follows. Consider a two index model. The return on the *i*th security is given by

$$R_i = a_i + b_{i,1} I_1 + b_{i,2} I_2 + c_i .$$

For investors who hold well diversified portfolios the specific risk of each security, represented by c_i can be diversified away so an investor need only be concerned with expected return, $b_{i,1}$ and $b_{i,2}$ in choosing his portfolio. Suppose we hypothesize the existence of three widely diversified portfolios, represented by the points $(E_i, b_{i,1}, b_{i,2})$ in $E - b_1 - b_2$ space where $i = 1, 2, 3$. These three portfolios define a plane in $E - b_1 - b_2$ space with equation

$$E[R_i] = \lambda_0 + \lambda_1 b_{i,1} + \lambda_2 b_{i,2} .$$

A portfolio having any combination of b_1 and b_2 can be formed by combining portfolios 1, 2 and 3 in the correct proportions. For example the portfolio P, obtained by taking one-third each of each of 1, 2 and 3 would have

$$b_{P,1} = (b_{1,1} + b_{2,1} + b_{3,1})/3 ,$$

$$b_{P,2} = (b_{1,2} + b_{2,2} + b_{3,2})/3 ,$$

and $$E[R_P] = \lambda_0 + \lambda_1 b_{P,1} + \lambda_2 b_{P,2} .$$

Now, consider what would happen if another portfolio Q existed, with exactly the same values of b_1 and b_2 but a higher expected return. Both portfolios would have the same degree of systematic risk but Q would have a higher expected return than P. Rational investors would therefore sell P and buy Q, and this would continue until the forces of supply and demand had brought portfolio Q onto the same plane as portfolios 1, 2 and 3. Thus, in equilibrium, all securities and portfolios must lie on a plane in $E - b_1 - b_2$ space.

The more general result of APT, that all securities and portfolios have expected returns described by the L-dimensional hyperplane

$$E_i = \lambda_0 + \lambda_1 b_{i,1} + \lambda_2 b_{i,2} + \ldots + \lambda_L b_{i,L} ,$$

can be derived by a more rigorous mathematical argument.

6 (i) The Hull & White (HW) model does this by extending the Vasicek model in a simple way. We define the SDE for $r(t)$ under Q as follows

$$dr(t) = \alpha(\mu(t) - r(t))dt + \sigma d\tilde{W}(t)$$

where $\mu(t)$ is a deterministic function of t. $\mu(t)$ has the natural interpretation of being the local mean-reversion level for $r(t)$.

A simple example of a multifactor model is the 2-factor Vasicek model. This models two processes: $r(t)$, as before, and $m(t)$, the local mean-reversion level for $r(t)$. Thus

$$dr(t) = \alpha_r(m(t) - r(t))dt + \sigma_{r1} d\tilde{W}_1(t) + \sigma_{r2} d\tilde{W}_2(t)$$

$$dm(t) = \alpha_m(\mu - m(t))dt + \sigma_{m1} d\tilde{W}_1(t)$$

where $\tilde{W}_1(t)$ and $\tilde{W}_2(t)$ are independent, standard Brownian motions under the risk-neutral measure Q. This looks superficially like the Hull & White model, but the HW model has a deterministic mean-reversion level, whereas here $m(t)$ is stochastic.

(ii) We will now look at a simple extension of the Vasicek model. Recall the SDEs for both the Vasicek and CIR models gave us time-homogeneous models. This means that bond prices at t depend only on $r(t)$ and on the term to maturity. This results in a lack of flexibility when it comes to pricing related contracts. For example, on any given date theoretical bond prices will probably not match exactly observed market prices. We can re-estimate $r(t)$ to improve the match and even re-estimate the constant parameters α, μ and σ but we will still, normally, be unable to get a precise match.

A simple way to get theoretical prices to match observed market prices is to introduce some elements of *time-inhomogeneity* into the model. The Hull & White (HW) model does this by extending the Vasicek model in a simple way.

(iii) One factor models have certain limitations which it is important to be familiar with. First, if we look at historical interest rate data we can see that changes in the prices of bonds with different terms to maturity are not perfectly correlated as one would expect to see if a one-factor model was correct. Sometimes we even see, for example, that short-dated bonds fall in price while long-dated bonds go up. Recent research has suggested that around three factors, rather than one, are required to capture most of the randomness in bonds of different durations.

Second, if we look at the long run of historical data we find that there have been sustained periods of both high and low interest rates with periods of both high and low volatility. Again these are features which are difficult to capture without introducing more random factors into a model. This issue is especially important for two types of problem in insurance: the pricing and hedging of long-dated insurance contracts with interest-rate guarantees; and asset-liability modelling and long-term risk-management.

Third, we need more complex models to deal effectively with derivative contracts which are more complex than, say, standard European call options. For example, any contract which makes reference to more than one interest rate should allow these rates to be less than perfectly correlated.

7 (i) If we were dealing with an ordinary differential equation, integration would lead to the expression $\mu t + \sigma Z_t$ for $\log(S_t/S_0)$ and thus to $S_0 \exp(\mu t + \sigma Z_t)$ for S_t. To solve the problem within stochastic calculus, use Itô's Lemma to calculate $d \log S_t$:

$$d \log S_t = \frac{1}{S_t} dS_t + \tfrac{1}{2}\left(-\frac{1}{S_t^2}\right)(dS_t)^2$$

$$= \left(\mu - \tfrac{1}{2}\sigma^2\right) dt + \sigma\, dZ_t .$$

Written in integral form, this reads

$$\log S_t = \log S_0 + \left(\mu - \tfrac{1}{2}\sigma^2\right) t + \sigma Z_t$$

or, finally,

$$S_t = S_0 \exp\left[\left(\mu - \tfrac{1}{2}\sigma^2\right) t + \sigma Z_t\right].$$

We see that the process S satisfying the equation above is a geometric Brownian motion with parameter $\mu - \tfrac{1}{2}\sigma^2$. Since $\log S_t$ is normally distributed, it follows that S_t has a lognormal distribution with parameters $\left(\mu - \tfrac{1}{2}\sigma^2\right) t$ and $\sigma^2 t$.

Should insert the parameters given in the question, i.e. $\mu = 0.1$ and $\sigma = 0.2$.

(ii) The properties of the lognormal distribution give us the expectation and variance of S_t:

$$E(S_t) = \exp\left(\left(\mu - \tfrac{1}{2}\sigma^2\right)t + \tfrac{1}{2}\sigma^2 t\right) = e^{\mu t},$$

$$\mathrm{Var}(S_t) = e^{2\mu t}\left(e^{\sigma^2 t} - 1\right)$$

Need to look at

$X = 9000\ S_1 - 10{,}000$

which will also have a lognormal distribution

(a) $0.049846 \times 9{,}000^2 = 4{,}037{,}526$

(b) $\Pr[X < 0] = \Pr[S_1 < 10/9] = \Pr[\log S_1 < \log 10/9]$, then use normal distribution to get 0.55

(iii) 1021.42 based on Black Scholes

$S = 9{,}000$, $K = 10{,}000$, other parameters as in question

$d_1 = -0.1768$, $d_2 = 0.3768$, use Black Scholes to get price of call option as 509.12.

Use put call parity (or Black Scholes formula for put option directly) to get price of put = price of call + 10,000*exp(-0.05) - 9,000 = 1021.42.

8 (i) Attempts to explain this phenomenon gave rise to the *efficient markets hypothesis*, which claims that market prices already incorporate the relevant information. The market price mechanism is such that the trading pattern of a small number of informed analysts can have a large impact on the market price. Lazy (or cost conscious) investors can then take a free ride, in the knowledge that the research of others is keeping the market efficient.

If we assume that there are no arbitrage opportunities in a market, then it follows that any two securities or combinations of securities that give exactly the same payments must have the same price. This is sometimes called the "Law of One Price".

Arbitrage-free markets can be inefficient.

(ii) One measure of these non-normal features is the *Hausdorff fractal dimension* of the price process. A pure jump process (such as a Poisson process) has a fractal dimension of 1. Random walks have a fractal dimension of 1½. Empirical investigations of market returns often reveal a fractal dimension around 1.4.

(iii) Even mean reversion can be consistent with efficient markets. After a crash, many investors may have lost a significant proportion of their total wealth; it is not irrational for them to be more averse to the risk of losing what remains. As a result, the prospective equity risk premium could be expected to rise.

(iv) Several observers have commented that stock prices are "excessively volatile". By this they mean that the change in market value of stocks (observed volatility), could not be justified by the news arriving. This was claimed to be evidence of market over-reaction which was not compatible with efficiency.

The claim of "excessive volatility" was first formulated into a testable proposition by Shiller in 1981. He considered a discounted cashflow model of equities going back to 1870. By using the actual dividends that were paid and some terminal value for the stock he was able to calculate the perfect foresight price, the "correct equity" price if market participants had been able to predict future dividends correctly. The difference between the perfect foresight price and the actual price arise from the forecast errors of future dividends. If market participants are rational we would expect no systematic forecast errors. Also if markets are efficient broad movements in the perfect foresight price should be correlated with moves in the actual price as both react to the same news.

Shiller found strong evidence that the observed level of volatility contradicted the EMH. However, subsequent studies using different formulations of the problem found that the violation of the EMH only had borderline statistical significance. Numerous criticisms were subsequently made of Shiller's methodology, these criticisms covered

- the choice of terminal value for the stock price
- the use of a constant discount rate
- bias in estimates of the variances because of autocorrelation
- possible non-stationarity of the series, i.e. the series may have stochastic trends which invalidate the measurements obtained for the variance of the stock price

Although subsequent studies by many authors have attempted to overcome the shortcomings in Shiller's original work there still remains the problem that a model for dividends and distributional assumptions are required. Some equilibrium models now exist which calibrate both to observed price volatility and also observed dividend behaviour. However, the vast literature on volatility tests can at best be described as inconclusive.

9 (i) The three types of credit risk model are:

structural models: these are explicit models of a corporate entity issuing both debt and equity. They aim to link default events explicitly to the fortunes of the issuer.

reduced-form models: these are statistical models which use market statistics (such as credit ratings) rather than specific data relating to the issuer, a nd give statistical models for their movement.

intensity-based models: these model the factors influencing the credit events which lead to default and typically do not consider what triggers these events.

(ii) In the Merton model, the company is modelled as having a fixed debt, 40 with term 10 years and variable assets S_t. The equity holders can be regarded as holding a European call on the assets with a strike of 40.

In the current question the value of the option is 20.

Using Black Scholes formula, with $(T - t) = 10$, $K = 40$, $S_0 = 60$, $r = 0.05$, solve for σ, the implied volatility.
[Candidates need not actually do this calculation]

The assets of the company therefore follow a geometric Brownian motion under the risk neutral measure with drift $r = 0.05$ and volatility σ.

Therefore $\log(S_{10}/ S_0)$ follows a normal distribution with mean $10*(0.05 - \sigma_2/2)$ and variance $10* \sigma^2$.

The risk neutral probability of default is obtained by calculating the probability that $\log(S_{10}/ S_0)$ is less than log(40/60).

(iii) In the two state model for credit rating with deterministic transition intensity, the formula for the Zero Coupon Bond price is

$$B(t, T) = e^{-r(T-t)} (1 - (1 - \delta) (1 - e^{-\int_t^T \tilde{\lambda}(s)ds})).$$

It follows that the risk-neutral default intensity is given by

$$\tilde{\lambda}(s) = s/2.$$

END OF EXAMINERS' REPORT