EXAMINATIONS

September 2007

Subject CT8 — Financial Economics

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

M A Stocker
Chairman of the Board of Examiners

December 2007

1 (i) The Greeks are the derivatives of the price of a derivative security with respect to the different parameters needed to calculate the price. Thus:

$$\Delta = \frac{\partial f}{\partial s}; \Gamma = \frac{\partial^2 f}{\partial s^2}; \theta = \frac{\partial f}{\partial t}; \lambda = \frac{\partial f}{\partial q}; \rho = \frac{\partial f}{\partial r}; \nu = \frac{\partial f}{\partial \sigma};$$

where f is the value of the derivative, s is the price of the underlying security, q is the continuous dividend yield on the security, σ is the volatility, r is the interest rate and t is time. In each case the relevant Greek measures sensitivity (rate of change) of the option price to change in that variable.

(ii) Suppose we hold one European call option with strike k and maturity T and are short one put with the same strike and maturity. Our payoff at maturity is exactly the same as if we held one share of the underlying and were short k zero coupon bonds with maturity T. The value at time 0 of such a payoff, under the assumption of no arbitrage, is $S_0 - ke^{-rT}$, where S_t is the price of the stock and r is the risk-free interest rate. Thus is P and C are the price at time 0 of the put and call, we have:

$$C = P + S_0 - ke^{-rT}$$

Hence $\Delta_C = \Delta_P + 1$ and $\Gamma_C = \Gamma_P$.

(iii) Since the original portfolio is delta-hedged, and the delta of a share is 1, we must have

$$-.212n + 1M = 0 \Rightarrow n = 4716981.$$

(iv) Using the formulae in part (ii), the delta and gamma of the call are .788 and .377 respectively. The Γ of the original portfolio is

$$1M \times .377 = 377{,}000,$$

so we need:

$$.222d + .788c = 0$$

and

$$.111d + .377c + .377 \times 4716981 = 0.$$

Solving these two simultaneous equations, we get

$$c = 104,\ 605,\ 990,\ d = -371,\ 304,\ 146.$$

2 (i) We need to equate the expected average return on the stock and the return on a bond: so we solve solve $1.25p + 0.8q = 1.1 \Rightarrow p = \frac{2}{3}$. Thus, under Q, the risk-neutral measure, S is a multiplicative random walk with up-jump probability $\frac{2}{3}$.

(ii) The fair price at time t for a claim of X payable at time T is $E_Q[e^{-r(T-t)}X \mid \mathcal{F}_t]$, so, if $S_1 = 125$,

$$D_1 = (p \times 1 + q \times 2)/1.1 = 4/3.3 = 1.2121,$$

while if $S_1 = 80$ the fair price

$$D_1 = (p \times 2 + q \times 0)/1.1 = 4/3.3 = 1.2121.$$

Hence, the value $D_0 = 1.2121/1.1 = 1.1019$.

(iii) to hedge at time 1 if $S_1 = 125$ we let the amount invested in the stock be ϕ and the amount invested in cash be ψ and solve:

$$1.25\phi + 1.1\psi = 1$$

$$0.8\phi + 1.1\psi = 2$$

which gives

$\phi = \dfrac{20}{9} = -2.22222$ (the equivalent shareholding is
$-2.2222/125 = -.017778$)
and $\psi = \dfrac{340}{9} = 3.4343$.

If $S_1 = 80$ then we solve

$$1.25\phi + 1.1\psi = 2$$

$$0.8\phi + 1.1\psi = 0$$

which gives

$\phi = \dfrac{40}{9} = 4.4444$ (the equivalent shareholding is
$4.4444/80 = .0556$),
and $\psi = -\dfrac{320}{99} = -3.2323$.

Finally, we solve

$$1.25\phi + 1.1\psi = 1.2121$$

$$0.8\phi + 1.1\psi = 1.2121$$

which gives

$$\phi = 0 \text{ and } \psi = \frac{1.2121}{11} = 1.1019.$$

(iv) This last is obvious since the value of D_1 doesn't depend on S_1 and hence we must hedge using the risk-free asset only.

3 (i) No frictions; short-selling permitted; small investor (i.e. does not "move market"); market is arbitrage-free; stock price is given by $dS_t = \mu_t S_t dt + \sigma S_t dB_t$ for some process μ_t, where B is a Brownian motion.

All are, in some sense, implausible. Friction (spreads and commission) is present; short-selling is available but on very different terms; "small investor" not true for an investment bank; stock-market returns are not compatible with normality (fat tails); arbitrages occur (for short periods).

(ii) (a) Since

$$\tilde{S}_t = \cosh(\sigma Z_t)\exp(-\sigma^2 t),$$

$\tilde{S}$ can be written as

$$\tilde{S}_t = f(Z_t, t),$$

where

$$f(x, t) = \cosh(\sigma x)e^{-\sigma^2 t}.$$

Applying Ito's lemma we get that

$$d\tilde{S}_t = f_x dZ_t + (f_t + \frac{1}{2}f_{xx})dt$$

$$= \sigma\sinh(\sigma Z_t)e^{-\sigma^2 t}dZ_t + (\frac{1}{2}\sigma^2\cosh(\sigma Z_t) - \sigma^2\cosh(\sigma Z_t))e^{-\sigma^2 t}dt$$

$$= \sigma e^{-\sigma^2 t}\sinh(\sigma Z_t)dZ_t - \frac{1}{2}\sigma^2 e^{-\sigma^2 t}\cosh(\sigma Z_t)dt$$

$$= \sigma\sqrt{e^{-2\sigma^2 t} - \tilde{S}_t^2}\,dZ_t - \frac{1}{2}\sigma^2 \tilde{S}_t dt.$$

(b) The drift (the *dt*) term is not 0.

(c) The drift term would be zero under the risk-neutral measure: the *dZ* term would be unchanged, therefore we'd get:

$$d\tilde{S}_t = \sigma\sqrt{e^{-2\sigma^2 t} - \tilde{S}_t^2}\,d\tilde{Z}_t,$$

where $\tilde{Z}$ is a Brownian Motion under the risk-neutral measure.

4 (i) The general (zero coupon) bond pricing formula is

$$B(t, T) = E[\exp(-\int_t^T r_s ds) \mid \mathcal{F}_t],$$

where $B(t, T)$ is the price at time t of the ZCB with maturity T, and r is the random short rate.

(ii) Under the Vasicek model we have

$$dr_t = \alpha(\mu - r_t)\,dt + \sigma dZ_t,$$

where Z is Brownian Motion under the risk-neutral measure, Q, and α, μ and σ are positive constants.

Under the Cox-Ingersoll-Ross (CIR) model,

$$dr_t = \alpha(\mu - r_t)\,dt + \sigma\sqrt{r_t}\,dZ_t.$$

Finally, in the Hull and White model,

$$dr_t = \alpha(\mu_t - r_t)\,dt + \sigma dZ_t,$$

for a deterministic function of t, μ_t.

(iii) The corresponding bond-pricing formula for the Vasicek model is

$$B(t, T) = \exp(a(T - t) - b(T - t)r(t)),$$

where $b(\tau) = \frac{1-e^{-\alpha\tau}}{\alpha}$ and $a(\tau) = (b(\tau) - \tau)\,(\mu - \frac{\sigma^2}{2\alpha^2}) - \frac{\sigma^2}{4\alpha} b(\tau)^2$.

(iv) Historical data suggests that three factors are necessary;

historically, there have been sustained periods of both high and low interest rates with periods of both high and low volatility — features which are difficult to capture without more random factors;

we need more complex models to deal with more complex derivative contracts, for example those which depend on more than one interest rate should allow for these rates to be less than perfectly correlated.

5 Assume no other debt, frictionless markets, perfect information, black scholes type model for the movement of the assets

Use black scholes to calculate the value of a call option based on the value of the assets exceeding a strike of 10 after 10 years. Equal to equity of firm = E = 15.07631

Value of bond is $B = 20 - E = 4.923688$

Solve $10^*\exp(-10\, rb) = \mathrm{B} \Rightarrow \mathrm{rb} = -\frac{1}{10}\ln\frac{8}{10} = 7.085\%$

rb – 5% = credit spread in continuously compounded form = 2.085%

6 (i) $Var(R_A) = Var(R_B) = .2^2 = .04$, $Var(R_C) = .1^2 = .01$.
$Cov(R_A, R_C) = Cov(R_B, R_C) = -.5 \times .2 \times .1 = -.01$,
$Cov(R_A, R_B) = -.25 \times .2 \times .2 = -.01$.

(ii) An efficient portfolio is one with minimum variance of return for the given expected return, or maximum expected return for the given variance.

(iii) (a) The variance is given by

$$\left(\frac{2}{9}+4c\right)^2 \times .04 + \left(\frac{2}{9}+c\right)^2 \times .04 + \left(\frac{5}{9}-5c\right)^2 \times .01$$

$$-2\times.01\left(\frac{2}{9}+4c\right)\left(\frac{2}{9}+c\right) - 2\times.01\left(\frac{2}{9}+c\right)\left(\frac{5}{9}-5c\right) - 2\times.01\left(\frac{2}{9}+4c\right)\left(\frac{5}{9}-5c\right)$$

$= .01/9 + 1.35c^2 = .001111 + 1.35c^2$

(b) Thus the minimum variance portfolio is when $c = 0$ with variance .001111 and portfolio $(\frac{2}{9}, \frac{2}{9}, \frac{5}{9})^T$.

(iv) It follows from previous answers that the efficient frontier consists of points of the form (σ_c, r_c) with $c \geq 0$, $r_c = .05 + .27c$ and $\sigma_c = \sqrt{.001111+1.35c^2}$. Now the tangent at the point parametrised by c has gradient

$$g_c = \frac{r_c'}{\sigma_c'} = \frac{.27\sqrt{.001111+1.35c^2}}{1.35c}$$

and this intercepts the vertical axis at the point

$$p = r_c - g_c\sigma_c = .05 + .27c - \frac{.27(.001111+1.35c^2)}{1.35c}.$$

Setting $p = 0.04$ we obtain

$$.04 \times 1.35c = (.05 + .27c)\ 1.35c - .27(.001111 + 1.35c^2).$$

Rearranging this we get

$$c = .27 \times .001111/.0135 = .02222 = \frac{2}{90}.$$

The corresponding portfolio is $(\frac{14}{45}, \frac{11}{45}, \frac{4}{9})^T$. If the CAPM holds then these should be the proportions corresponding to the market capitalisation of the companies so they should be (56bn,44bn,80bn) respectively.

7 (i) Investor 1:

(a) variance = 1000^2*0.04*0.96 = 38,400
(b) value at risk = 0 as there is a less than 5% chance of any loss
(c) value at risk = 0
(d) prob of shortfall = 4%

Investor 2:

(a) variance = 1000*.04*.96 = 38.4

(b) normal approximation has mean 0.04*1000 = 40 and std deviation of 6.20 (sqrt of above).

(c) VaR = 40 + 1.645*6.20 = 50.2

(d) VaR = 40 + 1.282*6.20 = 47.9

(e) probability of shortfall = *c*.100% (mean loss is over 6 standard deviations from 0)

(ii) Investor 2 is clearly holding a more diversified portfolio, but two of four measures of risk would suggest the diversified portfolio was riskier.

Value at risk is highly sensitive to the confidence level chosen with 90% level suggesting investor 2 is riskier than investor 1 and 95% level vice versa.

8 (i) The three forms are:

Strong — stock prices reflect all current information relevant to the stock, including information which is not public.

Semi-strong — stock prices reflect all current, publicly available information relevant to the stock.

Weak — stock prices reflect all information available in the past history of the stock price.

(ii) Tests need to make assumptions (which may be invalid) such as normality of returns or stationarity.

Transaction costs may prevent the exploitation of anomalies, so that the EMH might hold net of transaction costs.

Allowance for risk: the EMH does not preclude higher returns as a reward for risk; however the EMH does not tell us how to price such risks.

END OF EXAMINERS' REPORT