EXAMINERS' REPORT

April 2010 examinations

Subject CT8 — Financial Economics
Core Technical

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

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Chairman of the Board of Examiners

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1 (i) Write down Ito's formula for $f(t,X_t)$ when $dX_t = \mu_t dt + \sigma_t dW_t$

$$\begin{aligned} df(t,X_t) &= \frac{\partial f(t,X_t)}{\partial t}dt + \frac{\partial f(t,X_t)}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 f(t,X_t)}{\partial x^2}\left(dX_t\right)^2 \\ &= \frac{\partial f(t,X_t)}{\partial t}dt + \frac{\partial f(t,X_t)}{\partial x}\left(\mu_t dt + \sigma_t dW_t\right) + \frac{1}{2}\frac{\partial^2 f(t,X_t)}{\partial x^2}\left(\sigma_t^2 dt\right) \\ &= \left(\frac{\partial f(t,X_t)}{\partial t} + \frac{\partial f(t,X_t)}{\partial x}\mu_t + \frac{1}{2}\frac{\partial^2 f(t,X_t)}{\partial x^2}\sigma_t^2\right)dt + \frac{\partial f(t,X_t)}{\partial x}\sigma_t dW_t \end{aligned}$$

(ii) Consider $X_t = U_t e^{-\gamma t}$.

Then

$$\begin{aligned} dU_t &= d(e^{\gamma t} X_t) = \gamma e^{\gamma t} X_t\, dt + e^{\gamma t}\, dX_t \\ &= \gamma e^{\gamma t} X_t\, dt + e^{\gamma t}(-\gamma X_t\, dt + \sigma dW_t) = \sigma e^{\gamma t}\, dW_t\,. \end{aligned}$$

Thus

$$U_t = U_0 + \sigma \int_0^t e^{\gamma s} dW_s$$

and consequently

$$X_t = e^{-\gamma t}\, U_t = X_0 e^{-\gamma t} + \sigma \int_0^t e^{\gamma(s-t)} dW_s$$

2 The proof of this result is an adaptation of that of the standard call-put parity. Two (self-financing) portfolios are considered:

- Portfolio A: buying the call and selling the put at time t. Its value at time t is $P_t - C_t$ and at time T, it is $S_T - K$ in all states of the universe.

- Portfolio B: buying a fraction $\exp\left(-\delta\left(T-t\right)\right)$ of the underlying asset for $S_t \exp\left(-\delta\left(T-t\right)\right)$ and borrowing $K\exp\left(-r\left(T-t\right)\right)$ at time t. Its value at time t is then $K\exp\left(-r\left(T-t\right)\right) - S_t \exp\left(-\delta\left(T-t\right)\right)$. Its value at maturity is then $S_T - K$ by taking into account the dividends which are paid continuously at rate δ.

Using the absence of arbitrage opportunity, both portfolios should have the same value at any intermediate time, in particular at time *t*. Hence:

$$C_t - P_t = S_t \exp\left(-\delta\left(T-t\right)\right) - K\exp\left(-r\left(T-t\right)\right).$$

Another proof can include the following portfolios:

Portfolio A: At time t, buying a call option and lending $K\exp\left(-r\left(T-t\right)\right)$

Portfolio B: At time t, buying the put option and buying one share.

3 (i) There is no arbitrage in the market since $d = 0.8 < \exp(0.05) < u = 1.2$.

(ii) To price the call option, we use the risk-neutral pricing formula. We use the following simplifying notation:

$$C_{uu} = \left(u^2 S_0 - K\right)^+ = 44;$$

$$C_{ud} = \left(udS_0 - K\right)^+ = 0;$$

$$C_{dd} = \left(d^2 S_0 - K\right)^+ = 0.$$

At time 1, we get in the upper state,

$$C_1\left(u\right) = \exp\left(-r\right)\left[qC_{uu} + \left(1-q\right)C_{ud}\right] = 26.29,$$

and in the lower state

$$C_1\left(d\right) = \exp(-r)\left[qC_{ud} + \left(1-q\right)C_{dd}\right] = 0$$

where the risk-neutral probability of an upward move is

$$q = \frac{\exp\left(r\right)-d}{u-d} = 0.628.$$

At time 0,

$$C_0 = \exp\left(-r\right)\left[qC_1(u) + \left(1-q\right)C_1(d)\right].$$

Hence

$$C_0 = 15.71.$$

(iii) For the special option, we need to compute the average for the different possible trajectories, the probability of each path and the associated payoff:

trajectory	*average*	*probability*	*payoff of the option*
$up - up$	$X_{uu} = 121.33$	$q^2 = 0.394$	$(X_{uu} - K)^+ = 21.33$
$up - down$	$X_{ud} = 105.33$	$q(1-q) = 0.234$	$(X_{ud} - K)^+ = 5.33$
$down - up$	$X_{du} = 92$	$(1-q)q = 0.234$	$(X_{du} - K)^+ = 0$
$down - down$	$X_{dd} = 81.33$	$(1-q)^2 = 0.138$	$(X_{dd} - K)^+ = 0$

The price of the option is obtained as

$$X_0 = \exp(-2r)\left(q^2 (X_{uu} - K)^+ + q(1-q)(X_{ud} - K)^+\right) = 8.744 .$$

4 (i) The price of a zero-coupon bond can be written as

$$B(t,T) = E\left[\exp\left(-\int_t^T r(s)ds\right)|F_t\right].$$

Since $\int_t^T r(u)du$ is a Gaussian random variable, we can compute explicitly the price of the zero-coupon bond in terms of the expected value and variance (conditional) of $\int_t^T r(u)du$:

$$B(t,T) = \exp\left[-E\left[\int_t^T r(s)ds|F_t\right] + \frac{1}{2}V\left[\int_t^T r(s)ds|F_t\right]\right]$$

where $E\left[\int_t^T r(s)ds|F_t\right] = b(T-t) + (r(t) - b)\left(\frac{1-\exp(-a(T-t))}{a}\right)$ and

$$V\left[\int_t^T r(s)ds|F_t\right] = \frac{\sigma^2}{a^2}(T-t) - \frac{\sigma^2}{2a^3}\left(\exp(-2a(T-t)) - 1\right) + \frac{2\sigma^2}{a^3}\left(\exp(-a(T-t)) - 1\right).$$

(ii) Main issue: possibility to have negative interest rates when using the Vasicek model. An alternative is the CIR model:

$$dr(t) = a(b - r(t))dt + \sigma\sqrt{r(t)}dW_t .$$

5 (i) Try $\sigma = 10\%$. Black Sholes formula gives a price of $p_{10} = 44.05$.
Try $\sigma = 40\%$. Black Sholes formula gives a price of $p_{40} = 76.05$.

Interpolating gives a trial value of
(76.05 – 52.73) / (76.05 – 44.05) * 10 + (52.73 – 44.05) / (76.05 – 44.05) * 40
= 20.2%.

Evaluating gives $p_{20.2} = 52.96$.

Interpolation with p_{40} give
σ = ((76.05 – 52.73) * 20.2 + (52.73 – 52.96) * 40) / (76.05 – 52.96) = 21.9%
(to the nearest .5%)

$p_{21.9} = 54.75$.

Actual answer is 20%.

(ii) The payoff is
$100\min(1, \max(S_T - 320,0)) = 100(\max(S_T - 320,0) - \max(S_T - 321,0))$
so is 100 times the difference between two call options with the corresponding strikes.

Using the Black-Scholes formula, the price of the second call option is 52.06p

and hence the value of the derivative is p = 100 * (52.73–52.06) = 67p.

(iii) The option essentially pays £1 if the final security price is greater than 320p.

Thus its price is approximately $e^{-r}P(S_1 > 320)$ (where P is the EMM).

So

$$P(S_1 > 320) = e^{.04} * p = 0.70.$$

Other answers are also possible. In particular, using the distribution of
$\ln\left(\frac{S_1}{S_0}\right)$ and use it to calculate the probability directly.

6 (i) A model can be set up, in continuous time, with two states N (not previously defaulted) and D (previously defaulted). Under this simple model it is assumed that the default-free interest rate term structure is deterministic with $r(t) = r$ for all t. If the transition intensity, under the real-world measure P, from N to D at time t is denoted by $\lambda(t)$, this model can be represented as:

$$\boxed{\text{No default, } N} \xrightarrow{\lambda(t)} \boxed{\text{Default, } D}$$

and D is an absorbing state.

If $X(t)$ is the state at time t. The transition intensity, $\lambda(t)$, can be interpreted as:

$$\Pr_P(X(t+dt) = N \mid X(t) = N) = 1 - \lambda(t)\,dt + o(dt) \qquad \text{as } dt \to 0,$$
$$\Pr_P(X(t+dt) = D \mid X(t) = N) = \lambda(t)\,dt + o(dt) \qquad \text{as } dt \to 0.$$

Another correct answer will be:

$$\Pr_P(X(T) = N \mid X(t) = N) = \exp\left(-\int_t^T \lambda_s ds\right)$$

$$\Pr_P(X(T) = \mathrm{D} \mid X(t) = N) = 1 - \exp\left(-\int_t^T \lambda_s ds\right)$$

If τ is a stopping time defined as:

$$\tau = \inf\{t : X(t) = D\} \text{ (with } \inf \emptyset = \infty)$$

and if $N(t)$ is a counting process defined as:

$$N(t) = \begin{cases} 0 & \text{if } \tau > t, \\ 1 & \text{if } \tau \le t. \end{cases}$$

Then τ can be interpreted as the time of default and $N(t)$ can be interpreted is the number of defaults up to and including time t.

It is assumed that if the corporate entity defaults all bond payments will be reduced by a known, deterministic factor $(1 - \delta)$ where δ is the recovery rate, i.e. for a zero-coupon bond which is due to pay 1 at time T, the actual payment at time T will be 1 if $\tau > T$ and δ if $\tau \le T$.

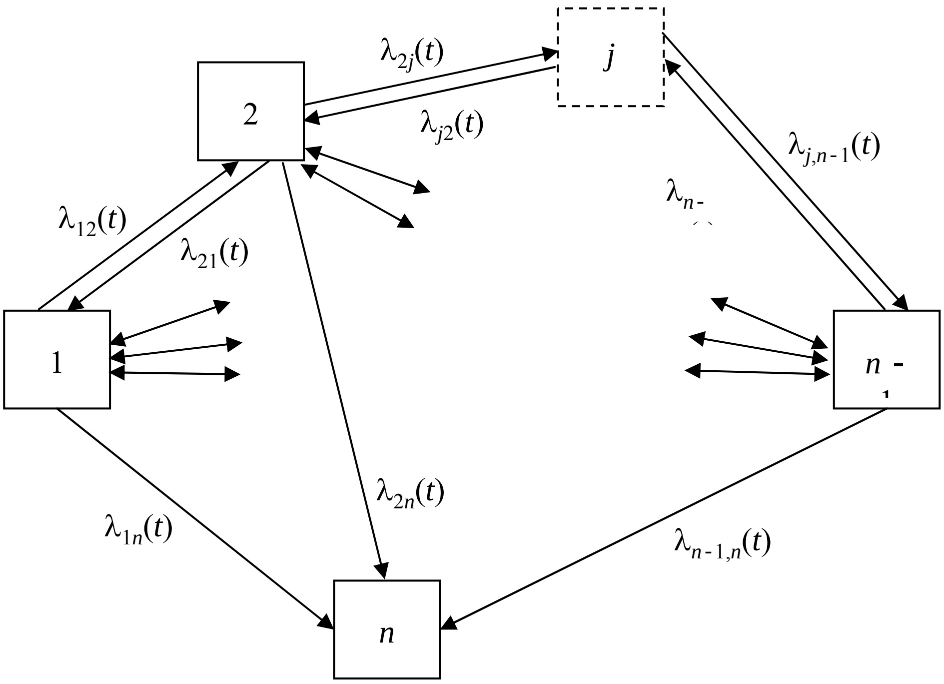
The formula for the zero-coupon bond price is

$$P(t,T) = B(t,T)\left(1-(1-\delta)\left(1-\exp\left(-\int_t^T \lambda_s ds\right)\right)\right)$$

Where $P(t,T)$ is the price at time t of a risky zero-coupon bond and $B(t,T)$ is the price at time t of a risk-free zero-coupon bond.

(ii) A more general and more realistic model with multiple credit ratings rather than the simple default/no default model, used above was developed by Jarrow, Lando and Turnbull. In this model there are $n - 1$ credit ratings plus default.

If the transition intensities, under the real-world measure P, from state i to state j at time t are denoted by $\lambda_{ij}(t)$ where the $\lambda_{ij}(t)$ are assumed to be deterministic then this model for default risk can be represented by the following diagram:



In this n-state model transfer is possible between all states except for the default state n, which is absorbing.

7 (i) (The Cameron-Martin-Girsanov theorem)

Suppose that Z_t is a standard Brownian motion under P. Furthermore suppose that γ_t is a previsible process. Then there exists a measure Q equivalent to P and where $\overline{Z}_t = Z_t + \int_0^t \gamma_s ds$ is a standard Brownian motion under Q.

Conversely, if Z_t is a standard Brownian motion under P and if Q is equivalent to P then there exists a previsible process γ_t such that $\overline{Z}_t = Z_t + \int_0^t \gamma_s ds$ is a Brownian motion under Q.

(ii) The answer is $a = \frac{1}{2}\sigma^2$. This can be proved using two different approaches: for showing all working correctly using one of the approaches below.

(1) Write the martingale condition and consider the expected value of the process at time *t*, conditional on the filtration up to an earlier time *s*.

(2) Write Ito's formula for the function $f(t,B_t)$ = exp(sigma $B_t - a_t$), and set the drift term equal to 0.

(iii) We know that $e^{-rt} S_t$ is a martingale under the EMM and so is $e^{-rt}D_t$. So, setting $W_t = B_t + ct$ we can write $e^{-rt} S_t = S_0\exp(0.2W_t - (r + 0.2c - 0.2)t)$ and we require $r + 0.2c - 0.2 = 1/2(0.2)^2 = 0.02$.

Similarly, we can write $e^{-rt}D_t = 2\exp(0.6W_t - (r + 0.6c - 0.39)t)$ and we then require $r + 0.6c - 0.39 = 1/2(0.6)^2 = 0.18$.

Eliminating r from these two equations gives
$0.4c - 0.19 = 0.16$, or $0.4c = 0.35$ so $c = 0.875$.

Substituting in the first equation gives $r + 0.175 - 0.2 = 0.02$ so $r = 4.5\%$.

8

- EMH states that market fully reflects all available information and the implication is therefore that investors are not able to make "excess" returns (rather than any returns at all!).

- 3 forms of EMH defining what type of information is available: weak for historical price information, semi-strong for all public information and strong for all information.

- Although illegal, insider information appears to enable investors to make money. Reasonable to conclude the other way round as studies of directors' share dealings suggest that, even with inside information, it is difficult to out-perform.

- Difficult to define publicly available information – might be that some very difficult-to-obtain information enables profits but at a high cost of obtaining the information.

- Investors taking higher risks may earn higher returns – this does not contradict the EMH.

- EMH does not specify how information is priced, so very difficult to test.

- Conflicting empirical evidence from supporters and detractors.

- Difficult to determine when, precisely, information arrives.

9 (i) Let J be the return to the investor in the junior loan.

$J = 54$, with probability 0.75

$= 0$ with probability $0.25 * 0.5$

$= 50 * U$ with probability $0.25 * 0.5$, where U is uniform over (0,1)

$E[J] = 0.75 * 54 + 0.25 * 0.5 * 0 + 0.25 * 0.5 * 0.5 * 50 = 43.625$

$E[J^2] = 0.75 * 54^2 + 0 + 0.25 * 0.5 * 50^2 * E[U^2]$
$= 0.75 * 54^2 + 312.5 * (0.25 + 0.083) = 2291$

$\mathrm{Var}[J] = 2291 - 43.625^2 = 388$

S = return to investor in senior loan

$S = 53$ with prob 0.75

$= 50$ with prob $0.25 * 0.5$

$= 50 * U$ with prob $0.25 * 0.5$

$E[S] = 0.75 * 53 + 0.125 * 50 + 0.125 * 50 * 0.5 = 49.125$

$E[S^2] = 0.75 * 53^2 + 0.125 * 50^2 + 0.125 * 50^2/3 = 2523$
$\mathrm{Var}[S] = 2523 - 49.125^2 = 110$

Alternative answers:

The word "return" can be interpreted in different ways, leading to several possible answers.

In the detailed solution above, it is total return.

If using percentage return, as a percentage, then

$J = 0.08$ with probability 0.75, -1 with probability 0.25 * 0.5 and $U - 1$ with probability 0.25 * 0.5 with U uniformly distributed over [0,1]

The expected value is then $E(J) = -0.1275$ and the variance is $V(J) = 0.1552$

$S = 0.06$ with probability 0.75, 0 with probability 0.25 * 0.5 and $U - 1$ with probability 0.25 * 0.5 with U uniformly distributed over [0,1].

The expected value is then $E(S) = -0.0025$ and the variance is $V(S) = 0.0441$.

(ii) $\Pr(J < 50) = 0.25$
$\Pr(S < 50) = 0.125$

END OF EXAMINERS' REPORT