INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

September 2011 examinations

Subject CT8 — Financial Economics
Core Technical

Purpose of Examiners' Reports

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and who are using past papers as a revision aid, and also those who have previously failed the subject. The Examiners are charged by Council with examining the published syllabus. Although Examiners have access to the Core Reading, which is designed to interpret the syllabus, the Examiners are not required to examine the content of Core Reading. Notwithstanding that, the questions set, and the following comments, will generally be based on Core Reading.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report. Other valid approaches are always given appropriate credit; where there is a commonly used alternative approach, this is also noted in the report. For essay-style questions, and particularly the open-ended questions in the later subjects, this report contains all the points for which the Examiners awarded marks. This is much more than a model solution – it would be impossible to write down all the points in the report in the time allowed for the question.

T J Birse
Chairman of the Board of Examiners

December 2011

General comments on Subject CT8

Subject CT8 introduces the main concepts and principles of financial economics. These are developed in later subjects in the ST series of exams. This subject combines various types of skills. In particular, along with CT7, it is one of the first where candidates are expected to write lengthy passages of reasoned thought, rather than just complete calculations. This is a skill that will be new to many, and candidates are advised to pay particular attention to the answers to this type of question by studying many past papers.

Comments on the September 2011 paper

The general performance was slightly worse than in April 2011 and candidates found this paper more challenging, but well-prepared candidates scored well across the whole paper and the best candidates scored close to full marks. As in previous diets, questions that required an element of application of the core reading to situations that were not immediately familiar proved very challenging to most candidates. The comments that follow the questions concentrate on areas where candidates could have improved their performance. Candidates approaching the subject for the first time are advised to concentrate their revision in these areas and the ability to apply the core reading to similar situations.

1 (i) A portfolio is efficient if the investor cannot find a better one in the sense that it has the same expected return and a lower variance, or the same variance and a higher expected return.

(ii) We have:

$$V = x_A^2 V_A + x_B^2 V_B + 2 x_A x_B C_{AB}$$

Which is a minimum at

$$x_A = \frac{V_B - C_{AB}}{V_A - 2C_{AB} + V_B}$$

= 0.5

So $x_B = 0.5$

And the expected return on the portfolio is 8%.

Generally candidates scored well on this question. Some students struggled to calculate the weighting in each asset class or failed to distinguish between the correlation and the correlation coefficient.

2 (i) **Macroeconomic factor models**

These use observable economic time series as the factors. They could include factors such as the annual rates of inflation and economic growth, short term interest rates, the yield on long term government bonds, and the yield margin on corporate bonds over government bonds. A related call of model uses a market index plus a set of industry indices as the factors.

Fundamental factor models

These are closely related to macroeconomic models but instead of (or, in addition to) macroeconomic variables the factors used are company specific variables. These may include such fundamental factors as:

- the level of gearing;
- the price earnings ratio;
- the level of R&D spending; or
- the industry group to which the company belongs.

Commercial fundamental factor models are available which use many tens of factors. They are used for risk control by comparing the sensitivity of a portfolio to one of the factors with the sensitivity of a benchmark portfolio.

Statistical factor models

These do not rely on specifying the factors independently of the historical returns data. Instead a technique called principal components analysis can be used to determine a set of indices which explain as much as possible of the observed variance. However, these indices are unlikely to have any meaningful economic interpretation and may vary considerably between different data sets.

(ii) There are many acceptable answers, but for example:

Macroeconomic factor model

I_1 = annual inflation; I_2 = annual GDP; I_3 = equity dividend yield

Fundamental factor model

I_1 = quick ratio; I_2 = book value; I_2 = industry group to which the company belongs

The candidates who were familiar with the bookwork scored very well. Some candidates were able to score some marks using economic knowledge from subject CT7.

3 (i) Using Ito's Lemma:

$$d\log S_t = \frac{1}{S_t}dS_t + \frac{-1}{2S_t^2}(dS_t)^2$$

$$= \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dZ_t$$

Written in integral form, this reads

$$\log S_t = \log S_0 + \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma Z_t \,.$$

Or, finally, $S_t = S_0 \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma Z_t\right\}$.

So, S_t has a lognormal distribution with parameters $\left(\mu - \frac{\sigma^2}{2}\right)t = 0.08875t$ and $\sigma^2 t = 0.0625t$.

The initial investment (based on a 50:50 chance) can be calculated by choosing the 50th percentile point of $Z_t = 0$, i.e. the initial investment is:

$$\frac{£100{,}000}{\exp(0.08875\times 10 + 0.25\times 0)} = £41{,}168 = \text{A}$$

(ii) (a) We know that:

$$\text{Var}(S_t) = e^{2\mu t}(e^{\sigma^2 t} - 1) \text{ or equivalently,}$$
$$\text{Var}(AS_t) = 100{,}000^2 e^{\sigma^2 t}\left(e^{\sigma^2 t} - 1\right)$$

So the variance of the investment is:

$£41{,}168^2 \text{Var}(S_{10}) = £41{,}168^2\, e^{2.4}(e^{0.625} - 1)$
$= £41{,}168^2 \times 9.571$
$= 16{,}220{,}971{,}227.90$

(b) As S_t has a lognormal distribution

$\mathbf{P}(£41{,}168\ S_{10} < £90{,}000) = \mathbf{P}\ (S_{10} < 2.1862) = \mathbf{P}\ (\log S_{10} < 0.7821)$
$= \mathbf{P}((\log S_{10} - 0.8875)/\ \sqrt{0.625} <- 0.1333) = 0.4470$

(c) The 99th percentile of the Normal distribution is given by $Z_t = 2.3263$. So the 99th percentile worst outcome for the investment is:

$S_{10} = £41{,}168\ \ e^{0.8875-\sqrt{0.625}\times 2.3263} = £15{,}896.$
So the VaR relative to A is £25,272 and relative to £100,000 is £84,104.

(iii) In this case the investor has removed all risk, so by the principle of no arbitrage the portfolio will earn the risk free rate. Therefore, the amount they need to invest at time 0 is:

$$\frac{£100{,}000}{e^{10\times 4\%}} = £67{,}032.$$

Many candidates scored well on part (i) which was a fairly standard proof using Ito's lemma.
Many struggled with manipulating the log-normal distribution and calculating risk metrics relating to it.

4 The proof of this result is an adaptation of that of the standard no arbitrage approach to pricing forward contracts. For ease of exposition we use 100x% rather than x% in the calculations.

Two self-financing portfolios are considered at time zero:

Portfolio A: entering into the forward contract to receive one ton of the asset at time T. Its value at time zero is zero, and at time T it is $S_T - F_0^T$.

Portfolio B: buying e^{xT} units of the underlying asset and borrowing $F_0^T e^{-rT} + ye^{x(T-\frac{1}{2})-\frac{r}{2}}$ at time zero. Its value at maturity is $S_T - F_0^T$ by taking account of the storage costs and the income stream.

Using the absence of arbitrage opportunity, both portfolios should have the same value at any intermediate time, in particular at time 0. Hence:

$$F_0^T = S_0 e^{(x+r)T} - ye^{(x+r)(T-\frac{1}{2})}.$$

Many candidates struggled with the concept of creating two portfolios using the principle of no arbitrage. They were unable to apply the core reading to a related situation. The question was challenging overall, with many candidates struggling to score well.

5 (i) Arbitrage free.
Positive interest rates.
Mean reversion of rates.
Ease of calculation of bonds and certain derivative contracts.
Realistic dynamics.
Goodness of fit to historical data.
Ease of calibration to current market data.
Flexible enough to cope with a range of derivative contracts.

(ii) $dr_t = \alpha(\mu_t - r_t)dt + \sigma dZ_t$ or alternatively $dr = \left[\theta(t) - ar\right]dt + \sigma dz$
Where in both cases Z is a Brownian motion under $\mathbb{Q}$.

(iii) Arbitrage free. Yes
Positive interest rates. No
Mean reversion of rates. Yes
Ease of calculation of bonds and certain derivative contracts. Yes
Realistic dynamics. No
Goodness of fit to historical data. Yes.
Ease of calibration to current market data. Yes
Flexible enough to cope with a range of derivative contracts. No.

This was standard material from the core reading and more successful candidates tended to score well, although some struggled to get all points required for full marks.

6 (i) $f(t, T) = -\frac{\partial}{\partial T}\log B(t,T)$

$$= [r_t - \frac{\sigma^2}{2}(T-t)^2]$$

(ii) The market price of risk, γ_t, is defined as:

$$\gamma_t = \frac{m(t,T) - r_t}{S(t,T)},$$

where

$$dB(t, T) = B(t, T)[m(t, T)\,dt + S(t, T)\,dZ_t].$$

Now,

$$\frac{\partial B(t,T)}{\partial t} = B(t,T)\left[r_t - \frac{\sigma^2}{2}(T-t)^2\right]$$

$$\frac{\partial B(t,T)}{\partial r_t} = B(t,T)[-(T-t)]$$

$$\frac{\partial^2 B(t,T)}{\partial r_t^2} = B(t,T)(T-t)^2$$

So, using Itô's lemma, we have

$$dB(t, T) = B(t, T)\{[-\mu(T-t)\,r_t + r_t]\,dt - \sigma(T-t)\,dZ_t\}$$

and so

$$\gamma_t = \frac{\mu r_t}{\sigma}.$$

(iii) The stochastic differential equation for r_t under the risk-neutral measure $\mathbb{Q}$ is given by

$$dr_t = \sigma d\tilde{Z}$$

where $\tilde{Z}$ is a standard Brownian motion under $\mathbb{Q}$

$$dr_t = \mu r_t dt + \sigma(d\tilde{Z} - \gamma_t dt)$$

$$= \mu r_t dt + \sigma\left(d\tilde{Z} - \frac{\mu r_t dt}{\sigma}\right)$$

$$= \mu r_t\, dt - \mu r_t dt + \sigma d\tilde{Z}$$

$$= \sigma d\tilde{Z}\ .$$

Question 6 was generally challenging. While part (i) was generally straightforward for most candidates, part (ii) where application of first principles was necessary was only answered well by the best candidates.

7 (i) First we calculate the risk-neutral probability of an upwards movement in the share price from each state:

$$q(200) = \frac{1.03 \times 200 - 170}{230 - 170} = 0.6$$

$$q(230) = \frac{1.03 \times 230 - 200}{250 - 200} = 0.738$$

$$q(170) = \frac{1.03 \times 170 - 150}{200 - 150} = 0.502$$

We can use these to calculate the state-price deflators:

$$A_2(250) = \frac{q(200)q(230)}{(0.75 \times 1.03)^2} = 0.742$$

$$A_2(200) = \frac{q(200)[1 - q(230)] + [1 - q(200)]q(170)}{2 \times 0.75 \times 0.25 \times 1.03^2} = 0.900$$

$$A_2(150) = \frac{[1 - q(200)][1 - q(170)]}{(0.25 \times 1.03)^2} = 3.004$$

(ii) The option premium, V, can be calculated as

$$V = E_{\mathbb{P}}(A_2 V_2)$$

$$= p^2 A_2(250) \log(70) + 2p(1 - p) A_2(200) \log(20) + (1 - p)^2 A_2(150) \times 0$$

$$= 2.784$$

(iii) It would not change at all.

This question was overall well answered, showing that many candidates have understood the broad concept of state price deflators. Well- prepared candidates were able to score near full marks on all three parts of the question.

Some candidates lost marks through ignoring the semi-annual interest rate. Part (iii) was designed to test the understanding of the candidates on how option pricing theory works in practice, but disappointingly many candidates got this part wrong.

8 (i) If the first exercise date has passed then the owner now has a derivative contract which pays \$1000 at time 2 years if and only if the stock price $S_2 < 2$.

The derivative should then be priced using the formula

$p_t = E_Q[e^{-r(2-t)}C|F_t]$,

where C is the claim value at $t = 2$ and Q is the risk-neutral probability measure.

This gives a value of p_t of

$1000\, e^{-r(2-t)}\, Q(S_2 < 2|F_t)$
$= 1000\, e^{-r(2-t)}\, Q(S_2/S_t < 2/S_t)$
$= 1000\, e^{-r(2-t)}\, Q(\log(S_2/S_t) < \log(2/S_t))$
$= 1000\, e^{-r(2-t)}\, \Phi(\{\log(2/S_t) - (.02- 0.045)(2- t)\}/ (.3\sqrt{(2 - t)}))$.

(ii) At t=1 the holder can choose between the value of the residual contract: p_{1+} and the current immediate exercise reward of \$500 if $S_1 > 2$ (and 0 otherwise). A rational holder will maximise value by choosing whichever has a greater current value.

There was a typo in the question where the inequalities were the wrong way around, full credit will be given to students who assumed this part was correct and had the inequalities the other way around.

In other words, an acceptable answer would be: At t=1 the holder can choose between the value of the residual contract: p_{1+} and the current immediate exercise reward of \$500 if $S_1 < 2$ (and 0 otherwise).

A rational holder will maximise value by choosing whichever has a greater current value.

(iii) (a) Since the current exercise value increases with S_1 and the value of p_{1+} decreases with S_1, the holder will choose to exercise the option at $t = 1$ if and only if the stock price is greater than some critical value k.

(b) At the critical value the holder should be indifferent i.e. we should have $p_{1+} = \$500$. So we seek k such that

$1000\, e^{-r}\, \Phi(\{\log(2/k) + .025\}/.3) = 500$
so $\Phi(\{\log(2/k) + .025\}/.3) = 0.51010$

so $\{\log(2/k) + .025\}/.3 = .02531$
so $k = 2.0343$

Performance on this question was very variable.
Part (i) was generally well-answered. A number of candidates highlighted that the inequality was the wrong way around in part (ii). Part (iii) was generally poorly answered.

9 (i) Using put-call parity, $0 = S - Ke^{-rT}$, so $K = Se^{rT} = 306.06$p

(ii) $d_1 = (\log(S/K) + r + ½\sigma^2 T) / \sigma\sqrt{T} = ½\sigma$,
while $d_2 = (\log(S/K) + r - ½\sigma^2 T)/\sigma\sqrt{T} = -½\sigma$.

Thus $C = S\Phi(d_1) - Ke^{-rT}\,\Phi(d_2) = S(\Phi(½\sigma) - \Phi(-½\sigma)) = 300(2\Phi(½\sigma) - 1)$
so $\Phi(½\sigma) = 0.52$ so $\sigma = .1003 = 10.0\%$.

(iii) $\Phi(d_1) = 0.52$ so the hedge is $5000 \times 0.52 = 2600$ shares
and $600 - 2600 \times 3 = £7200$ short in cash.

Generally answered well by candidates. Most candidates were able to score full marks on part (i) and proceed to score well on parts (ii) and part (ii).

10 (i) (a) Mean reversion means that the force of inflation will tend to move towards its average value.

An AR(1) process is a linear auto-regressive model of order one (i.e. the impulse at time t depends on the process one step before) whose formula is of the form of the equation given in the question.

(b) Denote by $i(t)$ the mean value of $I(t)$, then taking expectations in the formula, we see that $i(t) = m + a(i(t-1) - m)$
or $i(t) - m = a(i(t-1) - m)$. It follows that $i(t) - m$ tends to zero at a geometric rate.

(ii) A random walk process can be expected to grow arbitrarily large with time.

If share prices follow a random walk, with positive drift, then those share prices would be expected to tend to infinity for large time horizons.

However, there are many quantities which should not behave like this. For example, we do not expect interest rates to jump off to infinity, or to collapse back to zero.

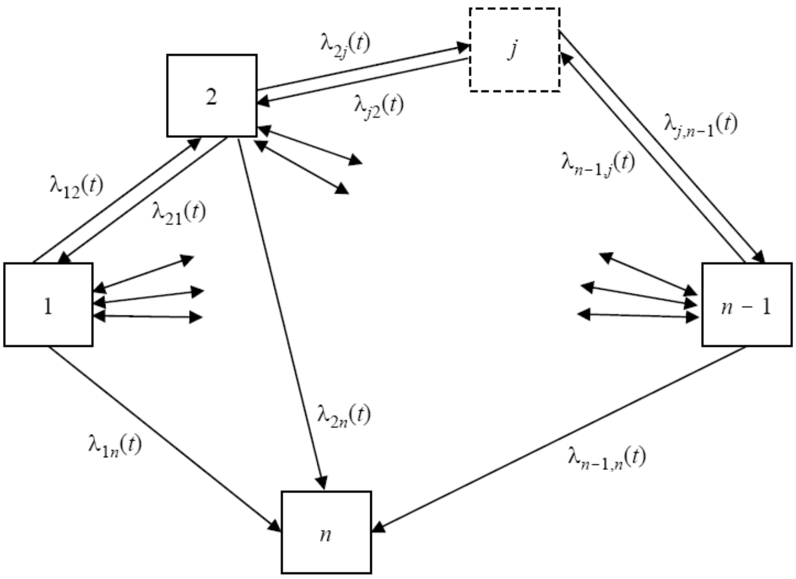
Instead, we would expect some mean reverting force to pull interest rates back to some normal range. In the same way, while inflation can change substantially over time, we would expect them, in the long run, to form some stationary distribution, and not run off to infinity. Similar considerations apply to the annual rate of growth in share prices.

In each case, these quantities are not independent from one year to the next; times of high interest rates or high inflation tend to bunch together i.e. the models are auto-regressive.

One method of modelling this is to consider a vector of mean reverting processes. These processes might include (log) yields, or the instantaneous growth rate of income streams. The reason for the log transformation is to prevent negative yields.

The question was straightforward bookwork. Candidates struggled to score full marks on part (ii) but were generally able to describe the basic concepts of the two models. Unfortunately many candidates chose to write extensive details about share price models and their characteristics rather than focus on the question about the random walk versus mean reverting models.

11 (i)



The n states represent $n-1$ credit ratings plus default.

$\lambda_{ij}(t)$ are the deterministic transition intensities from state i to state j at time t under the real world measure P.

(ii) (a) $h'(t) = 2p_1'(t) - p_2'(t) = -3p_1(t) + 3/2p_2(t) = -3/2h(t)$.

Similarly

$k'(t) = 2p_1'(t) + p_2'(t) = -p_1(t) - ½p_2(t) = -½k(t)$.

(b) Solving these linear differential equations with initial conditions $h(0) = -1$ and $k(0) = 1$ we get $h(t) = -e^{-3/2t}$ and $k(t) = e^{-½t}$.

It follows that $p_1(t) = \frac{1}{4}(h(t) + k(t)) = \frac{1}{4}(e^{-\frac{1}{2}t} - e^{-3/2t})$
while $p_2(t) = \frac{1}{2}(k(t) - h(t)) = \frac{1}{2}(e^{-\frac{1}{2}t} + e^{-3/2t})$.

Now, since $p_3(t) = 1 - p_1(t) - p_2(t)$,
we obtain $p_3(t) = 1 - 3/4e^{-\frac{1}{2}t} - 1/4\, e^{-3/2t}$.

And so $p_3(2) = .71164$.

(iv) (a) The bond price is thus $e^{-.04}\,(1 - p_3(2))£100 + p_3(2)£60) = £68.729$.

(b) The equivalent no-default interest rate is ½log(100/68.729) = 18.75%.
Thus the credit spread is 16.75%.

Few candidates failed to score well on parts (i) and (ii). In contrast, very few students were able to apply the results to part (iii) where scores were disappointing and often nil. Candidates did not understand the relevance of h(t) and k(t) and they may have gotten further if they had worked with them. Marks were picked up in question (iv) where candidates continued with the question.

END OF EXAMINERS' REPORT