INSTITUTE AND FACULTY OF ACTUARIES

EXAMINATION

23 September 2019 (pm)

Subject SP6 – Financial Derivatives
Specialist Principles

Time allowed: Three hours and fifteen minutes

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*

2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*

3. *Mark allocations are shown in brackets.*

4. *Attempt all questions, begin your answer to each question on a new page.*

5. *Candidates should show calculations where this is appropriate.*

Graph paper is required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator from the approved list.

1 The price of an asset, x, follows the process:

$$dx = xmdt + sdz$$

where m and s are constants dependent only on x and t and z is standard Brownian motion. Let f_1 and f_2 be the prices of two derivatives dependent only on x and t that follow the processes:

$$df_1 = \mu_1 f_1 dt + \sigma_1 f_1 dz$$
$$df_2 = \mu_2 f_2 dt + \sigma_2 f_2 dz$$

where μ_1, μ_2, σ_1 and σ_2 are functions of x and t.

(i) Write down an expression for the market price of risk λ_1 of f_1, defining any additional notation used. [1]

(ii) Write down the stochastic process followed by the value of a money market asset g which earns the instantaneous risk-free rate r at any given time. [1]

(iii) Show that a portfolio of $\sigma_2 f_2$ units of f_1 and $-\sigma_1 f_1$ units of f_2 earns the risk-free rate. [3]

(iv) Show that the market price of risk for f_1 and f_2 are the same, using your answers to (ii) and (iii). [2]

[Total 7]

2 The Black-Scholes differential equation for the price of a derivative $c(t,x)$ at time t and with current underlying price $x = S(t)$ is:

$$\frac{\partial c}{\partial t} + rx\frac{\partial c}{\partial x} + 0.5\sigma^2 x^2 \frac{\partial^2 c}{\partial x^2} = rc$$

where r and σ are constants.

(i) Show that $c(x,t) = Ax + Be^{rt}$ is a solution of the differential equation for any constants A and B. [2]

(ii) Describe what $c(x,t) = Ax + Be^{rt}$ represents in financial terms. [2]

Suppose $f(x,t)$ is a real, continuous and twice differentiable function in x, a real, continuous and differentiable function in t and $X(t)$ is a stochastic process satisfying: $dX(t) = u(t)dt + s(t)dW(t)$, where $u(t)$ and $s(t)$ are functions of t and $W(t)$ is standard Brownian motion. In the context of this question, you may consider a martingale to be a stochastic process with zero drift.

(iii) Derive, using Ito's lemma, a partial differential equation satisfied by f such that $f(X(t),t)$ is a martingale. [2]

(iv) Derive conditions for $c(X(t),t) = AX(t) + Be^{rt}$ from part (i) to be a martingale. [4]

[Total 10]

3 A large listed company is launching employee stock options (ESOs) for all its employees. The current share price of the company is \$100. Employees will be offered call options in the listed shares of the company with the following key terms:

- Strike price in \$ = K
- Term to expiry in years = T.

(i) State why the company may want to issue ESOs. [2]

(ii) Propose, giving reasons, values for the parameters K and T based on your answer to (i). [3]

(iii) Explain why other investors in the company's shares may have concerns with the launch of ESOs. [2]

Shortly after the launch of the ESOs, the company's share price has appreciated significantly and is now twice the value of the strike price, K.

(iv) Discuss the market risk of the ESOs from an employee's perspective following the rise in the share price. [2]

(v) Suggest reasons why employees should be particularly concerned with the market risk outlined in part (iv). [2]

[Total 11]

4 (i) Describe how the bootstrapping method can be used to derive a zero-coupon yield curve from the market price of conventional bonds. [2]

A bond trader is using the bootstrapping technique to calculate the term structure of the continuously compounded zero-coupon spot yields, $R_{Bootstrap}(t)$. The table below shows for each year $t = 1$ to $t = 10$, the price of a t-year bond of £100 notional paying coupons of 5% annually in arrears.

t	*Bond price (£100 nominal, 5% coupon, annually in arears)*	*Zero coupon spot yield, continuously compounded,* $R_{Bootstrap}(t)$
0	N/A	5.00%
1	100.88	4.00%
2	103.69	3.00%
3	106.20	2.75%
4	109.12	X
5	N/A – no bonds available	Y
6	105.49	4.00%
7	103.45	4.50%
8	106.72	4.00%
9	N/A – no bonds available	3.75%
10	111.66	3.50%

(ii) Calculate $R_{Bootstrap}(t = 4)$ (X in the table above) using the bootstrapping method. [3]

(iii) Estimate $R_{Bootstrap}(t = 5)$ (Y in the table above). [1]

The bond trader is modelling the instantaneous short rate r_t to investigate the impact of future scenarios on the value of their bond portfolio. The bond portfolio incorporates long and short positions in bonds of different terms. The bond trader uses the Vasicek model and has set the parameters below to ensure the model replicates $R_{Bootstrap}(t)$ at $t = 0$ and $t = 10$. The bond trader assumes:

- $dr_t = a(b - r_t)dt + \sigma dz_t$
- $a = 0.27$, $b = 0.03$, $\sigma = 0.02$
- $r_{t=0} = 5\%$

Let $R_{Model}(t)$ be the zero-coupon spot yield derived from the bond trader's model.

(iv) State why the Vasicek model is an equilibrium model. [2]

(v) Sketch a chart showing $R_{Bootstrap}(t)$ versus $R_{Model}(t)$ for $0 \leq t \leq 10$. [4]

(vi) Evaluate whether the above model will be suitable for the bond trader's intended use. [3]

(vii) Explain how the bond trader switching to the Hull and White model can improve the closeness of fit to $R_{Bootstrap}(t)$. [2]

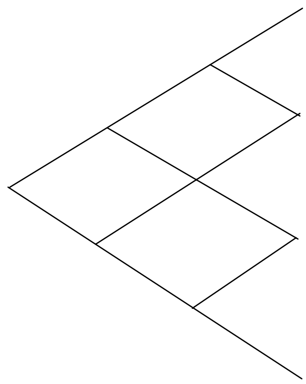
[Total 17]

5 (i) Sketch a three-dimensional graph of the implied volatility surface against the strike price and the time to maturity of call options written on the same underlying asset, as assumed in the Black-Scholes model. [2]

In a country there is a deep market in equity options. The current market implied volatilities for all equity options with a short time to maturity are at a historic low.

(ii) (a) Describe the likely shape of the implied volatility surface for equity options.
(b) Explain the reasons for the shape of this implied volatility surface.
[3]

An asset manager in this country is considering using a binomial tree to value an equity option. The risk-neutral binomial tree for the underlying equity price has the following shape with equally spaced time periods and constant volatility:



The asset manager decides to use an equity price model to value this equity option which has the equity volatility as a deterministic function of time and equity price. At each time step or equity price the value of the volatility changes.

(iii) Explain separately for (a) and (b) how the shape of the above binomial tree will change if it is assumed that the equity volatility is:

(a) a strictly decreasing function of time with the same starting volatility and equity price as in the diagram above, and no dependency on equity price; and
(b) a strictly increasing function of share price with the same starting volatility and equity price as in the diagram above, and no dependency on time.
[4]

(iv) Sketch separately for (a) and (b) in part (iii) the resulting risk-neutral binomial trees. [2]

(v) Suggest possible reasons why this equity price model may be difficult to implement. [2]

The chief risk officer of the asset manager is considering an alternative model for valuing equity options. This is a Black-Scholes model but with the volatility modelled stochastically. The stochastic terms of the equity price model and volatility model are correlated.

(vi) Assess the difficulties in modelling volatility stochastically. [4]

[Total 17]

6 (i) Show that if A and B are two events, then $P(A) = P(A|B)P(B) + P(A|\overline{B})P(\overline{B})$, where P is a probability measure and $\overline{B}$ is the event that B does not occur. [2]

Let W_t be standard Brownian motion for a given time t, x a positive real number and let T_x be the first time that W_t has a value of x.

(ii) Explain why $P(W_t \geq x | T_x > t) = 0$. [2]

(iii) Show that $P(W_t \geq x | T_x \leq t) = 0.5$, stating any assumptions made. [3]

(iv) Show that $P(T_x \leq t) = \frac{2}{\sqrt{2\pi}} \int_{\frac{x}{\sqrt{t}}}^{\infty} exp\left(\frac{-y^2}{2}\right) dy$. [3]

Let x be a fixed value.

(v) Prove that Brownian motion attains this value x in a finite amount of time with probability 1. [3]

(vi) Assess the importance of the distribution of T_x in pricing exotic options. [2]

One form of exotic options are barrier options.

(vii) Suggest possible reasons for investors using barrier options rather than vanilla options. [3]

[Total 18]

7 A bank runs a large arbitrage desk that trades interest rate derivatives including swaptions and interest rate swaps. Let:

- P be a swaption to pay a fixed rate of s_K and receive LIBOR between times T_1 and T_2.
- R be a swaption to receive a fixed rate of s_K and pay LIBOR between times T_1 and T_2.
- *SwapR* be a forward swap to receive a fixed rate of s_K and pay LIBOR between times T_1 and T_2.
- V_P, V_R and V_{SwapR} be the value of P, R and *SwapR* respectively.
- $v(P)$ and $v(R)$ be the vega of P and R respectively.

(i) Define what is meant by arbitrage. [1]

(ii) Show that $V_P + V_{SwapR} = V_R$, stating any assumptions made. [3]

(iii) Determine the value of the fixed rate s_K for which $V_P = V_R$. [2]

(iv) Show that $v(P) = v(R)$. [2]

The bank defines Delta as the impact of a 0.01% parallel decrease in interest rates on the value of a contract. The bank is assessing an arbitrage opportunity on P, R and *SwapR* when $s_K = 4\%$.

(v) Write down the Black formula for determining V_R, defining all terms used. [3]

(vi) Explain how the equation in part (v) can be used to determine the Delta of R. [1]

(vii) Sketch a chart showing how the Delta of a long position in each of P, R and *SwapR* varies with respect to the forward swap rate between T_1 and T_2 when $s_K = 4\%$. Use a single chart and assume the same notional in each instrument. [3]

The bank's traders have now undertaken a wide range of arbitrage trades in swaptions and swaps. The risk department is reviewing all the positions to ensure that the overall market risk is minimised.

(viii) Propose information that the risk department should request from the traders so it can assess whether market risk is minimised. [4]

(ix) Suggest a reason why an arbitrage trade in P, R and *SwapR* could introduce liquidity risk. [1]

[Total 20]

END OF PAPER