INSTITUTE AND FACULTY OF ACTUARIES

EXAMINATION

18 April 2012 (pm)

Subject ST6 – Finance and Investment Specialist Technical B

Time allowed: Three hours

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*
2. *You have 15 minutes before the start of the examination in which to read the questions. You are strongly encouraged to use this time for reading only, but notes may be made. You then have three hours to complete the paper.*
3. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*
4. *Mark allocations are shown in brackets.*
5. *Attempt all eight questions, beginning your answer to each question on a separate sheet.*
6. *Candidates should show calculations where this is appropriate.*

Graph paper is required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

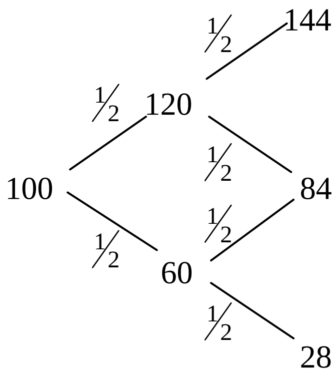
In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator from the approved list.

NOTE: In this examination, you are never required to prove the use of an arbitrage-free methodology unless clearly stated in the question.

1 Consider a stochastic process that evolves in discrete time t according to a recombining binomial tree. The set of possible values of the process are shown below for $t = 0$, 1 and 2, together with two different probability measures, **P** and **Q**, that apply to the (same) process.

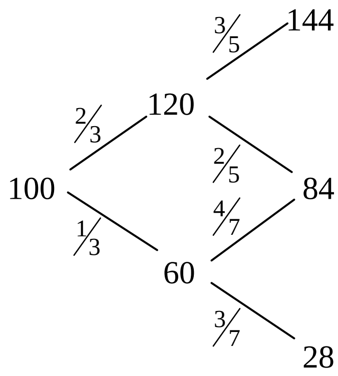
Tree of values under measure P

$t = 0$ $t = 1$ $t = 2$



Tree of values under measure Q

$t = 0$ $t = 1$ $t = 2$



(i) Demonstrate that:

(a) **P** and **Q** are equivalent measures.

(b) The process is a martingale under measure **Q** but not under measure **P**.

[4]

(ii) (a) Calculate the values of the discrete Radon-Nikodym derivative $\frac{d\mathbf{Q}}{d\mathbf{P}}$ for all nodes in the tree.

(b) Evaluate the process $\zeta_t = \mathrm{E}_{\mathbf{P}}\left[\frac{d\mathbf{Q}}{d\mathbf{P}} \middle| F_t\right]$ at time $t = 1$ for both up and down nodes, where F_t is the filtration (history) up to time t.

[4]

Consider a claim X that pays a fixed amount of 20 at time $t = 2$ if the tree value at that time is more than 75, otherwise zero.

(iii) Demonstrate by direct calculation that $\mathrm{E}_{\mathbf{Q}}[X] = \mathrm{E}_{\mathbf{P}}\left[\frac{d\mathbf{Q}}{d\mathbf{P}} X\right]$ at time $t = 0$. [3]

[Total 11]

2 A bank has written a five-year European Call option on a dividend bearing market index. The option has a strike of 2,000 and a notional value of £1,000 per index point.

Market conditions are currently as follows:

- market index M = 1,500
- risk-free rate r = 4% per annum, continuously compounded
- dividend yield q on the index = 2% per annum, continuously compounded
- market implied volatility (for options such as this one) σ = 25% per annum

There is a liquid market in a one-year futures contract on the index, with one contract worth £10 per index point.

The bank wishes to delta hedge its position using the assumption that implied Black-Scholes volatilities are σ_{fixed} = 25% for all strikes.

(i) Determine the number of futures that the bank should trade to delta hedge the position, stating whether a long or short position should be created. [4]

A trader within the bank has suggested that the delta hedging calculation be adjusted for skew. Skew is defined as the dependence of market implied volatilities on the amount by which an option is in- or out-of-the-money. She suggests that the constant volatility assumption σ_{fixed} within the Black-Scholes formula be replaced by a market adjusted value σ_{skew} as follows:

$$\sigma_{\text{skew}} = \sigma_{\text{fixed}} + 0.01\left[\frac{M - 1{,}500}{100}\right]$$

(ii) (a) Show that the delta of a European Call option on a stock S with market adjusted volatility $\sigma = aS + b$ is:

unadjusted delta + a × unadjusted vega

(b) Calculate the number of futures that the bank would need to transact if the trader's proposal above were adopted.

[Hint: You may use the result that the vega of a Call option is $Se^{-qT}\phi(d_1)\sqrt{T}$ *using the usual Black-Scholes notation and* $\phi(x) = \frac{1}{\sqrt{2\pi}}\exp(-\tfrac{1}{2}x^2)$ *.]*

[4]

(iii) Demonstrate graphically the result in part (ii)(b), explaining the impact of the trader's proposal. [3]

[Total 11]

3 In an economy where interest rates are deterministic and constant for all time t, consider a non-dividend paying stock, price S_t at time t, which evolves in the real world according to the stochastic process:

$$dS = \mu S dt + \sigma S dW_t$$

where μ and σ are constants, and W_t is a standard Brownian motion.

(i) Demonstrate that:

$$S_t = S_0 \exp[(\mu - \tfrac{1}{2}\sigma^2)t + \sigma\sqrt{t}Z]$$

where Z is a sample from the standard normal distribution. [2]

Consider also a risk-free bond, price B_t at time t.

(ii) (a) Explain what is meant by a change of numeraire in the context of the martingale approach to pricing options.

(b) Show that $\frac{B_t}{S_t}$, the bond price expressed in terms of the stock price, is a martingale under the measure where $\mu = r + \sigma^2$.

[4]

Consider now a European Call option on the stock with strike K and expiry time T, which has price C_t at time t.

(iii) (a) Explain why $\frac{C_t}{S_t}$, the Call price expressed in terms of the stock price, is a martingale under the same measure as in part (ii)(b).

(b) Show that the payoff of the Call can be written as:

$$S_T I_{S_t \geq K} - K I_{S_t \geq K}$$

where $I_{S_t \geq K} = 1$ for $S_T \geq K$ and 0 otherwise.

(c) Hence derive the first term of the Black-Scholes formula for the price of the Call at time 0, i.e. with the usual definition of d_1,

$$C_0 = S_0 \Phi(d_1) + \ \ldots$$

where $\Phi(\ldots)$ is the cumulative standard normal distribution.

[5]

[Total 11]

4 A proprietary derivative trading firm deals in swaps, options, futures and forwards on a variety of instruments and currencies. Option products are based on several underlyings: equity indices, interest rates, commodities and currency pairs (FX). In addition, there are a few exotic options that have complex payoffs. The maturities of the contracts extend to 50 years, although the average maturity is just under 10 years.

(i) Outline how the firm would prepare a consolidated report of market risks based on sensitivities and Value at Risk. [3]

The firm uses proprietary in-house models to price its contracts, built by the trading team and calibrated to mid-market parameters which they update throughout the day. The Finance department records profit and loss based on these prices. The models also calculate sensitivities and Value at Risk but, since the models have been written only relatively recently, the Market Risk department has not yet been able to use them to run stress tests on the firm's exposures.

A recent audit report of the Market Risk and Finance departments has highlighted a number of deficiencies around model validation, prudent valuations and stress testing. The management of these departments is also planning to review the models that price certain exotic options, as it has been found through the credit collateralisation process that the firm's prices are often out of line with the rest of the market.

(ii) (a) Describe the adjustments which might be made to the current valuation methodology to give a more realistic picture of the profit and loss.

(b) Describe how stress testing might supplement the existing risk management process.

(c) Suggest some conclusions that the management review might reach regarding exotic option pricing.

[8]
[Total 11]

5 Consider a standard Brownian motion process W_t.

(i) Demonstrate that $\frac{1}{\sqrt{c}} W_{ct}$ is also a standard Brownian motion for any constant $c > 0$ (the "scaling property" of Brownian motion). [2]

(ii) For sequential times $t_1 < t_2 < t_3$, compute the following expectations:

(a) $E[(W_{t_3} - W_{t_2})(W_{t_2} - W_{t_1})]$

(b) $E[(W_{t_3} - W_{t_1})(W_{t_2} - W_{t_1}) \mid W_{t_1} = 1]$

(c) $E[W_{t_3} W_{t_2}]$

[4]

Consider two independent standard Brownian motions, V_t and W_t.

(iii) Show that the process $X_t = V_t^2 W_t - \int_0^t W_u du$ is a martingale. [4]

[Total 10]

6 (i) Describe the following in the context of an exchange-traded futures contract:

(a) contract specifications
(b) open interest
(c) initial and variation margin
(d) physical vs cash settlement

[7]

(ii) Describe the purpose and effect of using an ISDA agreement with a Credit Support Annexe for over-the-counter (OTC) derivative trades. [3]

Following concerns about systemic risk in OTC derivative trades, global regulators have proposed the introduction of central clearing houses to act in a similar capacity for these OTC trades as futures exchanges do for exchange-traded futures.

(iii) Discuss the proposed regulatory changes by:

(a) explaining the impact they will have on credit risk and systemic risk.
(b) describing any problems and new risks that might be encountered.

[5]

[Total 15]

7 A wealthy investor has just died. At the reading of his will, it is announced that his portfolio of securities and derivatives, all of which mature in five years' time, is to be divided among his three sons according to the following conditions:

- The proceeds of the portfolio are only to be distributed after five years.

- At that time, one of the three sons will inherit the entire portfolio. It will be:

  - the youngest son if the FTSE is then below X
  - the middle son if the FTSE is then between X and Y
  - the oldest son if the FTSE is then above Y

- X and Y are to be set so that the market values of the three sons' shares in the portfolio are equal today.

The portfolio consists of:

- a five-year European Put option on the FTSE with a strike of 3,000
- a five year European Call option on the FTSE with a strike of 10,000
- a five-year risk-free zero coupon bond paying 10,000 after five years

Market conditions today are as follows:

- the FTSE is at 5,000
- the five-year risk-free rate is 5% per annum, continuously compounded
- the dividend yield on the FTSE is 2% per annum, continuously compounded
- FTSE option annualised implied volatilities are 20% per annum for all maturities and strikes

(i) Verify either that the market value of the Put is 44.03 or that the market value of the Call is 142.13. [3]

(ii) Derive values for X and Y that will meet the investor's requirements.

[*Hint: Consider the "payoff" structure to which each son is entitled in five years' time. You may assume that 3,000 < X < 5,000 < Y < 10,000.*]

[7]

(iii) Describe how X and Y would change from the result in part (ii) in each of the following three scenarios:

(a) The dividend yield on the FTSE is higher.

(b) Market implied volatility is higher.

(c) Market implied volatilities are skewed higher for options with low strikes and lower for options with high strikes.

[6]

[Total 16]

8 (i) Describe the features of:

(a) UK gilt STRIPS
(b) vanilla interest rate swaps

[6]

(ii) State the basic equations that would need to be satisfied for the pricing of gilt STRIPS and vanilla interest rate swaps so as not to present arbitrage opportunities. [3]

(iii) Discuss the usefulness of gilt STRIPS and vanilla interest rate swaps as hedging instruments. [3]

An investment bank transacts a wide range of simple and exotic swaps and swaptions. At regular points during the day, the bank determines a portfolio of swap transactions that would minimise the sensitivity of its net positions to yield curve movements.

(iv) Describe how this portfolio of swap transactions is likely to be determined.

[3]
[Total 15]

END OF PAPER