

# INSTITUTE AND FACULTY OF ACTUARIES



## EXAMINATION

29 September 2016 (pm)

### Subject ST6 – Finance and Investment Specialist Technical B

*Time allowed: Three hours*

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*
2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*
3. *You have 15 minutes of planning and reading time before the start of this examination. You may make separate notes or write on the exam paper but not in your answer booklet. Calculators are not to be used during the reading time. You will then have three hours to complete the paper.*
4. *Mark allocations are shown in brackets.*
5. *Attempt all seven questions, beginning your answer to each question on a new page.*
6. *Candidates should show calculations where this is appropriate.*

***Graph paper is required for this paper.***

*AT THE END OF THE EXAMINATION*

*Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.*

*In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator from the approved list.*

- 1 (i) Define a tradable asset. [1]

The price of a tradable dividend-paying stock follows a geometric Brownian motion.

- (ii) Write down an expression for the market price of risk for this stock, defining all notation used. [2]

In a securities market there are two tradable stocks,  $S$  and  $R$ , whose price processes are:

$$dS(t) = S(t)\mu dt + 0.1S(t)dW(t)$$
$$d[\ln R(t)] = 0.04dt + 0.25dW(t)$$

where  $\mu$  is a constant and  $W$  is standard Brownian motion.

$S$  is a non-dividend paying stock and  $R$  pays dividends of amount  $0.00125R(t)dt$  between the times  $t$  and  $t + dt$ , where  $t$  is measured in years.

The continuously compounded risk-free rate of interest is 5% per annum.

- (iii) Determine  $\mu$ , stating any assumptions made. [6]  
[Total 9]

- 2 A pension scheme provides increases to all benefit payments to members on 1 January each year. The rate of increase is equal to inflation over the previous year, as measured by the increase in a defined index, subject to a minimum of 0% and a maximum of 5%.

The benefit liabilities are closely matched using standard index-linked bonds, which are linked to the same index as the benefit increases. The benefit liabilities are valued by assuming that they increase each year in line with the same inflation index. The pension scheme balance sheet is perfectly inflation delta hedged, based on the current inflation expectations of 2.5% per annum at all terms.

The pension scheme has assets and liabilities each of \$100m (i.e. zero current surplus) and currently values all cash flows using a deterministic inflation assumption in line with inflation expectations.

- (i) Describe:
- index-linked bonds.
  - Limited Price Indexation (LPI) bonds.
  - LPI swaps.
- [3]

The pension scheme now wishes to model the impact on both assets and liabilities of inflation expectations immediately changing from 2.5% per annum to  $x\%$  per annum at all terms, where  $-5 \leq x \leq 10$ . You should assume that nominal interest rates do not change.

- (ii) Sketch three separate graphs showing the value of the assets, liabilities and surplus for all possible values of  $x$ . [4]
- (iii) (a) Explain why the scheme's liability valuation approach does not reflect inflation vega risk.
- (b) Identify for which values of  $x$  this valuation approach may overstate the surplus in the scheme. [4]

A bank offers an LPI instrument under which the scheme would receive the following payment on expiration of the instrument at time  $T$ , and receive no other payments:

$$\text{Payment}(T) = \text{Notional} \times \max \left[ \min \left[ \frac{I_T}{I_0}, (1 + 5\%)^T \right], 1 \right]$$

where  $I_0$  and  $I_T$  are the inflation indices at times 0 and  $T$  respectively.

The scheme would sell all of the index-linked bonds to purchase a series of these instruments with maturities  $T = 1, 2, 3, \dots$ , with the notional amount for each maturity  $T$  being chosen to match the expected liability cash flows as appropriate.

- (iv) Assess how effective this LPI instrument would be at matching the scheme's inflation risk under the following scenarios:
- sustained periods of negative inflation
  - sustained periods of inflation above 5% per annum
  - volatile periods of high annual inflation and negative annual inflation.

[4]

[Total 15]

**3** Consider a consumption asset with:

- spot price  $S_0$ .
- forward price  $F_0$  for delivery at time  $T$  years.
- storage costs  $u$  (expressed as a proportion of the spot price, continuously compounded per annum).

The constant continuously compounded risk-free rate per annum is  $r$ .

- (i) (a) Show that  $F_0 \leq S_0 e^{(r+u)T}$ .
- (b) Write down the equivalent of the above expression if the convenience yield is incorporated. [3]

An airline based in the European Union is proposing to hedge its exposure to fuel prices using one-year oil forwards and rolling the hedge each year.

- (ii) Explain the weaknesses of structuring the hedge in this way. [5]
- (iii) Describe the impact that the European Market Infrastructure Regulation (EMIR) may have on the airline's hedging approach. [3]
- [Total 11]

4 Consider a European call option on a non-dividend paying stock with price  $S(t)$  at time  $t$ . The option grants the holder the right to buy the stock at a fixed price  $K$  at a fixed time  $T$  in the future. The current time is  $t = 0$  and time is measured in years. The continuously compounded fixed risk-free interest rate is  $r$  per annum.

(i) Write down an expression for the payoff to the option holder at time  $T$ . [1]

(ii) Write down an expression for the expectation at time  $t = 0$  of the value of this payoff, stating any assumptions made and defining any additional notation used. [2]

(iii) Set out an algorithm for estimating this expectation using sampling, stating any assumptions made.

[Hint: Use the fact that if a sequence of independent random numbers  $X_1, X_2, \dots, X_n$  is sampled from the same distribution with mean  $\mu$  then  $S_n = (X_1 + X_2 + \dots + X_n) / n$  tends towards  $\mu$  with probability 1 as  $n$  gets larger.] [3]

Let  $\mathbb{P}$  be a probability measure on  $\mathbb{R}$  (the real numbers), with associated probability density function  $p$ .

(iv) Show that  $\mathbb{E}_{\mathbb{P}}[e^{-rT} \max\{0, S - K\}] = e^{-rT} \int_K^{\infty} (S - K)p(S)dS$ , where  $p(S)$  is the probability density function of  $S(T)$ . [1]

Consider the following integral:

$$\mathbb{E}_{\mathbb{P}}[h(X)] = \int_{\mathbb{R}} h(x)f(x)dx,$$

where  $X$  is a distribution on  $\mathbb{R}$  with probability density function  $f$  and  $h$  is a function from  $\mathbb{R}$  to  $\mathbb{R}$ .

Let  $\mathbb{Q}$  be another probability measure and  $g$  another probability density function on  $\mathbb{R}$ .

(v) Show that this expectation can be represented as  $\mathbb{E}_{\mathbb{Q}}[h(X)f(X)/g(X)]$ , stating any assumptions made. [3]

(vi) Explain the change in representation from  $\mathbb{E}_{\mathbb{P}}[h(X)]$  to  $\mathbb{E}_{\mathbb{Q}}[h(X)f(X)/g(X)]$ . [1]

(vii) Express  $\mathbb{E}_{\mathbb{Q}}[(h(X)f(X)/g(X))^2]$  in terms of  $\mathbb{E}_{\mathbb{P}}$ . [1]

(viii) Explain why representing  $\mathbb{E}_{\mathbb{P}}[h(X)]$  as  $\mathbb{E}_{\mathbb{Q}}[h(X)f(X)/g(X)]$  may be useful in financial mathematics. [3]

[Total 15]

- 5 (i) Describe the setting process for LIBOR. [4]
- (ii) Assess the likelihood of one individual bank being able to manipulate the LIBOR rate for its own gain using the setting process. [2]

A credit-risky floating-rate note (FRN) of term  $T$  (years) pays quarterly floating coupons in arrears of  $x$  per annum above three-month LIBOR. LIBOR interest rates are  $i$  per annum for all periods.

The FRN is redeemed at a par value of \$100m and trades in the market at a discount spread of  $s$  per annum above LIBOR (i.e. a total yield to redemption of  $(i + s)$  per annum).  $i$ ,  $x$  and  $s$  are quarterly compounded.

- (iii) (a) Show that the value of the FRN is given by:

$$100 \left[ \frac{(i+x)}{(i+s)} (1-v^{4T}) + v^{4T} \right]$$

where  $v$  is the quarterly discount factor which you should define as part of your answer.

- (b) Calculate the value of the FRN when  $T = 10$ ,  $x = 0.02$ ,  $i = 0.03$  and  $s = 0.01$ . [3]
- (iv) (a) Determine, to the nearest \$1m, the market value impact of a 1% per annum addition to the LIBOR rate,  $i$ , with all other parameters remaining as in part (iii)(b).
- (b) Determine, to the nearest \$1m, the market value impact of a 1% per annum addition to the discount spread,  $s$ , with all other parameters remaining as in part (iii)(b). [3]
- (v) Explain how the answers to parts (iv)(a) and (iv)(b) would change as  $x$  increases. [3]

The FRN is now modified to include a downgrade “step-up” clause. This means that the quarterly coupon amount over LIBOR will increase by an addition of 1% per annum for every rating that the current credit rating is below that at issue. All credit ratings are sourced from a single credit rating agency.

- (vi) Explain why this feature may be attractive to investors in the FRN. [1]
- (vii) Suggest limitations of using the credit rating to determine when the coupon should “step-up”. [3]

[Total 19]

- 6 A bank is issuing a two year term, \$100m notional, second-to-default basket swap referencing two global supermarkets: Chain A and Chain B. The bank will receive the premium up-front and will settle any payments following a relevant default at the end of the second year.

The bank is now trying to price the up-front premium required to cover its expected loss under the following model and assumptions:

- Loss given default = 70%
  - Risk-free rates are 3% per annum continuously compounded at all terms.
  - Probability of default of Chain A each year is 0.40, conditional on Chain A not defaulting in a prior year.
  - Probability of default of Chain B each year is 0.20, conditional on Chain B not defaulting in a prior year.
  - Probability of both Chain A and B defaulting each year is  $X$ , conditional on both Chain A and Chain B not defaulting in a prior year.
- (i) State the minimum and maximum values of  $X$  that the bank could assume when pricing the basket swap. [1]
- (ii) Calculate the up-front premium required, assuming that  $X = 0.15$ . [5]
- (iii) Sketch a graph showing approximately how the up-front premium depends on  $X$ . [2]
- (iv) Describe how the bank could use a Gaussian copula model to calculate the up-front premium more accurately. [4]
- (v) Suggest reasons why a financial institution may wish to purchase the basket swap. [2]

In order to mitigate counterparty risk, the bank is proposing to collateralise the full market value of the basket swap on a daily basis, using the equity of another global supermarket, Chain C.

- (vi) Assess the suitability of using the equity of Chain C to collateralise the derivative. [4]
- [Total 18]

- 7 Let the stochastic process  $\{W_t\}_{t \geq 0}$  be a standard  $\mathbb{P}$ -Brownian motion, with respect to a probability measure  $\mathbb{P}$ .

Let  $\text{cov}[W_s, W_t]$  be the covariance between the Brownian motions at times  $s$  and  $t$ , for  $s > 0$  and  $t > 0$ .

- (i) Show that  $\text{cov}[W_s, W_t] = \min\{s, t\}$ . [3]

Let  $\{W_t\}_{t \geq 0}$  and  $\{B_t\}_{t \geq 0}$  be independent standard  $\mathbb{P}$ -Brownian motions. Let  $-1 \leq \rho \leq 1$  be a given constant.

For  $0 \leq t \leq T$  define a new process  $Z_t = \rho W_t + \sqrt{1 - \rho^2} B_t$ . It can be assumed that  $Z_t$  is a standard  $\mathbb{P}$ -Brownian motion and that the correlation between  $Z_t$  and  $W_t$  is  $\rho$ .

Let  $V(S_t, R_t, t)$  be the price of a derivative, with  $S_t$  and  $R_t$  being the prices of the two underlying assets at time  $t$ . Assume that  $V$  is an infinitely differentiable function of  $S_t$ ,  $R_t$  and  $t$ .

$S_t$  and  $R_t$  follow the processes:

$$\begin{aligned} dS_t &= S_t \mu dt + S_t \sigma dW_t \\ dR_t &= R_t \omega dt + R_t \alpha dZ_t \end{aligned}$$

where  $W_t$  and  $Z_t$  are as described above and  $\mu, \omega, \sigma$  and  $\alpha$  are constants.

- (ii) Show, by ignoring second order terms and higher and using the result  $dW_t dB_t = 0$ , that:

$$\begin{aligned} dV &= \sigma S_t \frac{\partial V}{\partial S_t} dW_t + \alpha R_t \frac{\partial V}{\partial R_t} dZ_t + \\ &\left[ \frac{\partial V}{\partial t} + \mu S_t \frac{\partial V}{\partial S_t} + \omega R_t \frac{\partial V}{\partial R_t} + 0.5 \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} + 0.5 \alpha^2 R_t^2 \frac{\partial^2 V}{\partial R_t^2} + \rho \sigma \alpha S_t R_t \frac{\partial^2 V}{\partial S_t \partial R_t} \right] dt. \end{aligned}$$

[Hint: Use Taylor's theorem.] [6]

- (iii) Comment on how this equation relates to Itô's lemma. [1]

- (iv) Propose ways in which the construction of  $Z_t$  outlined above and the equation given in part (ii) may be used in derivative pricing. [3]

[Total 13]

**END OF PAPER**