INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

September 2014 examinations

Subject ST6 – Finance and Investment
Specialist Technical B

Introduction

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and using past papers as a revision aid and also those who have previously failed the subject.

The Examiners are charged by Council with examining the published syllabus. The Examiners have access to the Core Reading, which is designed to interpret the syllabus, and will generally base questions around it but are not required to examine the content of Core Reading specifically or exclusively.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report; other valid approaches are given appropriate credit. For essay-style questions, particularly the open-ended questions in the later subjects, the report may contain more points than the Examiners will expect from a solution that scores full marks.

The report is written based on the legislative and regulatory context at the date the examination was set. Candidates should take into account the possibility that circumstances may have changed if using these reports for revision.

F Layton
Chairman of the Board of Examiners

December 2014

General comments on Subject ST6

This subject deals with mathematical techniques for the valuation and risk management of derivatives, together with some aspects of their practical application.

Different numerical answers may be obtained to those shown in these solutions, depending on whether figures were obtained from tables or from calculators. Allowance was made for these minor differences. However, candidates may have been penalised for using excessive rounding or showing insufficient working.

Derivative theory is an interesting but very exacting subject which needs to be tackled in a precise and disciplined manner. Questions that appear unfamiliar can, with some clear thinking at the start of the exam, be found to be straightforward examples of well known techniques. Thorough preparation of the course material is essential, however. This subject also requires quite a varied approach in writing solutions. Algebraic and numerical content needs to be supplemented with well reasoned but brief arguments where the question demands. Candidates should always try to provide several distinct relevant points for the discursive questions, not lengthy paragraphs developing a single idea.

Comments on the September 2014 Paper

The overall performance of students was not as good as in recent sessions. It is expected that there will be some variation in performance between sessions, but it was disappointing not to be able to award the same level of passes when the overall difficulty of the paper itself was felt by the examiners to be similar to that of recent sessions. However, despite many students apparently finding this a challenging paper, it was reassuring that well-prepared candidates scored well above the pass mark.

A common theme amongst students was the lack of breadth: both in knowledge of the core reading and in being able to generate ideas to questions which required an application of core reading.

Question 3 in particular was perceived by many students as a difficult and technical question. Although the LIBOR Market Model is one of the more demanding areas of the course, the principles being tested in this question are covered in some detail in the core reading and there are recent questions from 2011 and 2012 which provide students with examples of the type of question which can be asked around this area of the syllabus. Students who had learned and understood this part of the core reading and had practised these questions were able to do well here, but many others seemed to struggle or simply did not attempt the question.

The comments that follow the questions concentrate on areas where candidates could have improved their marks, as an attempt to help future candidates to revise accordingly and to develop their ability to apply the core reading to related situations. This skill is likely to be valuable both in exams and in using this theory in their professional careers.

1 (i) Let $\mathcal{T}$ be a binomial tree and let $\mathbb{P}$ and $\mathbb{Q}$ be equivalent measures on $\mathcal{T}$.

Let X be a real valued random variable defined on the nodes of $\mathcal{T}$ with X_t

the value of the random variable at time *t* in the binomial tree.

Define $\frac{d\mathbb{Q}}{d\mathbb{P}}$ to be a random variable defined on paths of the tree which are measurable under $\mathbb{P}$ and

taking positive real values such that:

- $\mathbb{E}_{\mathbb{Q}}\left(X_T\right) = \mathbb{E}_{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}} X_T\right)$; and

- $\mathbb{E}_{\mathbb{Q}}\left(X_t \mid \mathcal{F}_s\right) = \zeta_s^{-1}\mathbb{E}_{\mathbb{P}}\left(\zeta_t X_t \mid \mathcal{F}_s\right)$ $(s \le t \le T)$,

  where $\zeta_t = \mathbb{E}_{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}} \mid \mathcal{F}_t\right)$ is the Radon-Nikodym process and $\mathcal{F}_t$ is the filtration on the tree

Alternatively

In words, the Radon-Nikodym derivative is a way of encoding the differences between $\mathbb{P}$ and $\mathbb{Q}$ based on a ratio of probabilities.

The Radon-Nikodym derivative is the random variable of the mapping of paths of the tree to the ratio described above.

A suitable illustration illustrating this concept, for example on pages 64, 65 or 67 of Financial Calculus by Baxter and Rennie.

[4]

(ii) The Radon-Nikodym derivative is an efficient method of coding information; in particular all the information about the measure $\mathbb{Q}$ can be extracted from $\mathbb{P}$ and $\frac{d\mathbb{Q}}{d\mathbb{P}}$.

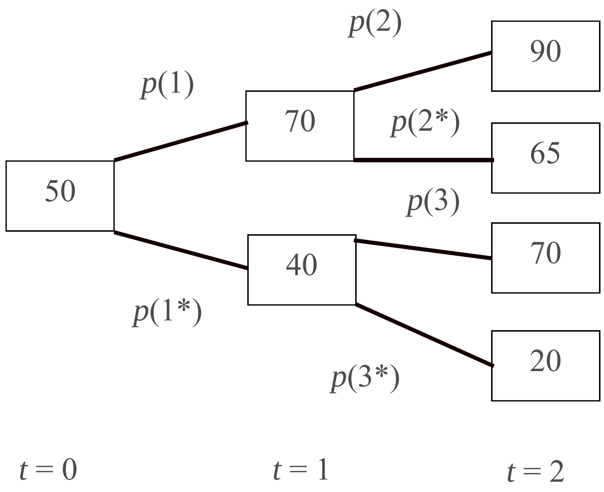
It therefore is useful in explaining how to change measure.

In particular, it can be used to change from the actual measure to the risk-neutral measure.

In order to price derivatives the conditional expectations with respect to the risk neutral measure are required. These are provided by the Radon-Nikodym process.

[2]

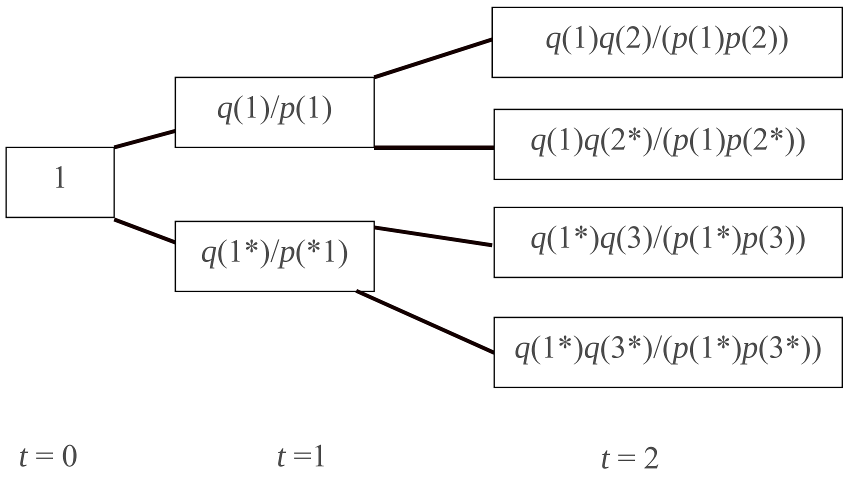
(iii) Denote the probability measure $\mathbb{P}$ symbolically as follows:



where $p(x^*) = 1 - p(x)$.

Similarly, for the probability measure $\mathbb{Q}$.

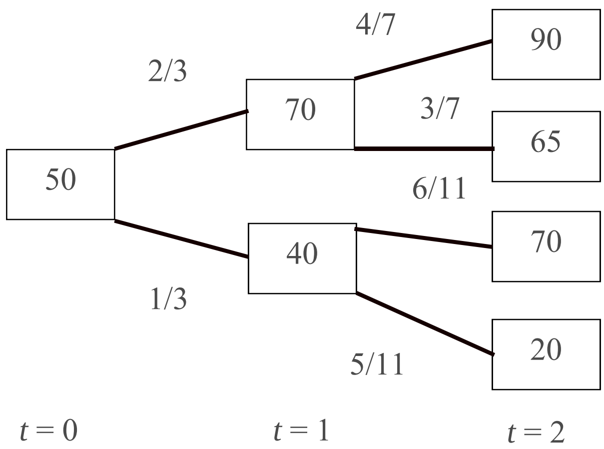
From the definition of the Radon-Nikodym process the process is illustrated as follows:



These six equations can be solved by using the probability measure $\mathbb{P}$ and by beginning with the t =1 equations.

For example: $q(1)/p(1) = q(1)/(3/5) = 10/9$, which gives a value of $q(1) = 2/3$.

The required values of $q(.)$ are:



[4]
[Total 10]

Part (i) was generally not well answered, and in particular students did not include sufficient detail given the 4 mark allocation. Students generally defined the Radon-Nikodym in terms of a ratio of probabilities. The full definition requires defining it as a random process using expectations, as on page 68 of Baxter and Rennie.

Parts (ii) and (iii) were generally well answered.

2 (i) (a) An interest rate floor is a derivative that pays out the shortfall in interest of a specified floating rate relative to a specified floor rate (whenever this difference is positive) on a specified notional amount over a series of dates. [So payouts would be:
notional × max(floor rate(t_i) − floating rate(t_i), 0) × ($t_{i+1} - t_i$)
at times t_{i+1} for $i = 1, 2, \ldots, n - 1$.]

(b) An interest rate floorlet is an interest rate floor over a single timestep. It pays:
notional × max(floor rate(t) − floating rate(t), 0) × ($t' - t$) at time t'

(c) A plain vanilla swap is an agreement between two parties where one party pays the other floating interest (e.g. LIBOR) on a fixed notional amount in return for interest at a fixed rate (the swap rate) on the same notional amount. [The payments are typically due every three or six months and netted off rather than both taking place.]

(d) A European receiver swaption gives the holder the option to enter, on the specified exercise date, into a swap of a specified term and on a specified notional, where the holder receives fixed interest at a strike rate specified at outset in return for paying LIBOR. [Often the swaption is settled for cash at the exercise date.]

(e) A forward rate agreement is an arrangement where one counterparty agrees to deposit a notional amount with another on a fixed date, in return for receiving back the notional with a specified amount of interest on a fixed date further into the future.

[5]

(ii) (a) The floor is an overhedge because....

....if the guarantee is not exercised, any payoffs provided by the floor are windfalls to the financial institution in that it does not have offsetting liability cashflows, whereas....

...if the guarantee is exercised, then:

- In months where the floor pays off, the payoff is exactly what the financial institution requires
- In months where there is no floor payoff, the financial institution profits by paying 3% while receiving LIBOR, which is higher.

(b) This is identical to (a).

(c) The swaption is the perfect hedging instrument – it exactly matches the guarantee that the bank has written. Hedging using the swaption would eliminate all future market risk associated with the guarantee.

(d) This is similar to (c), since the payoffs after five years are likely to be equal in value.

However, the redemption amount of the bond is not guaranteed.

This is due to the market conditions at redemption …

… and the credit risk due to the issuer of the bond.

(e) The forward rate agreement (FRA) will ultimately result in a positive or negative cashflow to the financial institution at time 5.

If it results in a positive cashflow, this exactly matches the cost of the guarantee (and the conditions under which the cashflow is positive will be those under which the guarantee will be exercised).

But it can result in a negative cashflow to the financial institution (if forward rates exceed the guaranteed rate), with no corresponding positive cashflow.

So the FRA would simply exchange one contingent claim for another.

Put-call parity can be used to demonstrate that the combination of a shorted payer swaption and an FRA is equivalent to a shorted receiver swaption.

[7]
[Total 12]

Part (i) was standard core reading and the majority of students scored highly here. However, some students did not appear to understand these terms at all or were not able to provide sufficient accurate detail to score highly.

Many students struggled with part (ii). Even though they had given the correct definitions of the terms, they were unable to apply these with much coherency to the given problems. One approach that could be taken was to consider the cashflows of the guarantee and the cashflows of the potential hedging instruments, in order to evaluate the effectiveness.

3 (i) A world is forward risk-neutral with respect to a certain asset when the market price of risk equals the volatility of that asset.

The rolling forward risk-neutral world is always forward risk-neutral with respect to a bond maturing at the next reset date. In this world we can discount from time $k + 1$ to time k using the zero rate observed at time k for a maturity $k + 1$.

[1]

(ii) The approach required is to look at the dz terms, hence less detail is needed in the workings for the drift terms.

Start from

$$dP(t,i) = \text{drift } dt + v_i(t)P(t,i)\, dz$$

Using Ito, and

$$d\ln P/dP = 1/P \text{ and } d^2\ln(P)/dP^2 = -1/P^2$$

$$d\ln P(t,i) = [(1/P)\text{drift} - 0.5\,(1/P^2)\, v_i(t)^2P^2]\, dt + (1/P)\, P\, v_i(t)\, dz$$

and a similar expression with i replaced with $i + 1$

And starting from

$$dF_i(t) = \text{drift } dt + \zeta_i(t)F_i(t)dz \text{ (as given in question)}$$

Using Ito and

$$d\ln(1 + F)/dF = 1/(1 + F) \text{ and } d^2\ln(1 + F)/dF^2 = -1/(1 + F)^2$$

$$d\ln(1 + F_k(t)) = [(1/(1 + F))\text{ drift} - 0.5(1/(1 + F)^2)\, \zeta_i(t)^2 F^2]\, dt + (1/1 + F)\, \zeta_i(t)\, F dz$$

Then equate the dz terms to get

$$\nu_i(t) - \nu_{i+1}(t) = (1/1 + F_i(t))\, \zeta_i(t)\, F_i(t) = F_i(t)\, \zeta_i(t) / (1 + F_i(t))$$

as required. [4]

(iii) We already have

$$dF_k(t) = \zeta_k(t)[\nu_{\text{int}(t)}(t) - \nu_{k+1}(t)]\, F_k(t)dt + \zeta_k(t)\, F_k(t)dz$$

The dz term is already in the required format. We just need to concentrate on the $\nu_{\text{int}(t)}(t) - \nu_{k+1}(t)$ within the drift term.

$\nu_{\text{int}(t)}(t) - \nu_{k+1}(t)$ can be broken down into

$$(\nu_{\text{int}(t)}(t) - \nu_{\text{int}(t)+1}(t)) + (\nu_{\text{int}(t)+1}(t) - \nu_{\text{int}(t)+2}(t)) + \ldots\ldots + (\nu_k(t) - \nu_{k+1}(t))$$

Which, using the result in (ii), is equal to

$$F_{\text{int}(t)}(t)\, \zeta_{\text{int}(t)}(t)/(1 + F_{\text{int}(t)}(t)) + F_{\text{int}(t)+1}(t)\, \zeta_{\text{int}(t)+1}(t)/(1 + F_{\text{int}(t)+1}(t)) + \ldots + F_k(t)\, \zeta_k(t)/(1 + F_k(t))$$

$$= \sum_{j=\text{int}(t)}^{k} F_j(t)\, \zeta_j(t)/(1 + F_j(t))$$

So $dF_k(t) = \zeta_k(t) F_k(t) \displaystyle\sum_{j=\text{int}(t)}^{k} F_j(t)\, \zeta_j(t)/(1 + F_j(t))\, dt + \zeta_k(t) F_k(t) dz$

[5]

(iv) The Λs will be related to the Black volatilities σ_k via

$$k\sigma_k^2 = \sum_{i=1}^{k} \Lambda_{k-i}^2$$

So we can work through iteratively:

$$\Lambda_1^2 = \sigma_1^2$$

$$\Lambda_2^2 = 2\sigma_2^2 - \Lambda_1^2$$

$$\Lambda_3^2 = 3\sigma_3^2 - \Lambda_1^2 - \Lambda_2^2$$

etc.

Results:

Term	*1*	*2*	*3*	*4*	*5*
Black volatility	15%	18%	18%	18%	17%
Lambda	15.00%	20.57%	18.00%	18.00%	12.21%

[4]
[Total 14]

Many students did not even attempt this question and it can only be assumed that they had not covered the LIBOR Market Model in sufficient detail in their studies. However, those who were well-prepared scored close to full marks. This emphasises the need to have studied all areas of the syllabus.

This question was not as difficult as it may have seemed, as hopefully the solution demonstrates. This question, and previous questions on the LIBOR Market Model, should be studied to help students become more familiar with this interest rate model.

4 (i) If S = index level, then the bond payoff is

$$\text{Max}(P, P(1 + xR)) = P + \max(0, xRP)$$

$$= P + \max(0, xP(S(5) - S(0))/S(0))$$

$$= P + xP/S(0) \max(0, S(5) - S(0))$$

Therefore the bond can be decomposed into:

- A zero-coupon bond paying P at time $t = 5$
- $xP/S(0)$ European calls on S …
- … with term of five years
- … and exercise price $S(0)$

[3]

(ii) First note that the zero-coupon bond would be worth

$$P \times \exp(-5 \times 0.08) = 0.6703P$$

Then calculate the price of one of the calls.

Price of a call option is

$$c = Se^{-qt}N(d_1) - Xe^{-rt}N(d_2)$$

Where

$$d_1 = [\ln(S/X) + (r - q + \sigma^2/2)T] / \sigma\sqrt{T}$$

And

$$d_2 = d_1 - \sigma\sqrt{T}$$

Here:

- S is spot, i.e. $S(0)$
- X is strike, i.e. $S(0)$
- r is risk-free rate, 8%
- q is dividend income, 3%
- σ is market implied volatility, 22%
- T is term, 5 years

So:

$$d_1 = [\ln(1) + (0.08 - 0.03 + 0.22^2/2) \times 5] / [\sqrt{5} \times 0.22] = 0.75416$$

And

$$d_2 = 0.75416 - \sqrt{5} \times 0.22 = 0.26223$$

$$N(d_1) = 0.7746$$

$$N(d_2) = 0.6034$$

$$\begin{aligned} c &= S(0) \times \exp(-0.03 \times 5) \times 0.7746 - S(0) \times \exp(-0.08 \times 5) \times 0.6034 \\ &= 0.2622 \times S(0) \end{aligned}$$

So the price of $xP/S(0)$ call options is $xP/S(0)c = 0.2622xP$.

(a) Now considering the profit/expense requirement and using the values from above, of every premium P:

- $0.05P$ will go towards profit and expenses
- $0.6703P$ will go towards the zero-coupon bond
- $0.2622xP$ will go towards the option

So we require $1 = 0.05 + 0.6703 + 0.2622x$

Which means $x = 0.2797/0.2622 = 107\%$

(b) In this case, profit/expense loading y is the unknown and we have

$$1 = y + 0.6703 + 0.2622 \times 1.15$$

So $y = 2.81\%$

[8]

(iii) Changes in risk-free rates

A fall in risk-free rates would reduce *x*.

It will make the zero-coupon bond more expensive.

It will also reduce the value of the call option…

… but by a lesser amount than the increase in the value of the zero-coupon bond.

This is one reason why these bonds are no longer common today.

Changes in market implied volatilities

An increase in market implied volatilities would reduce *x*.

This is because it would make the call option more expensive.

Changes in index dividends

A reduction in dividends would reduce *x*.

This is because it would make the call option more expensive.

[4]

(iv) First note that the hedging instruments would now be holdings of a zero-coupon bond and a call option where payoff is a function of the average of the index on a number of dates.

An option where payoff is a function of the average of the index on a number of dates is known as an Asian option.

Asian calls, being based on averages, will naturally have lower implied volatilities than their European equivalents.

This makes the call option cheaper.

This means that the bond issuer can offer a higher *x*.

Alternatively, if *x* is not changed, the financial institution could take a higher profit margin. [2]

[Total 17]

Students generally made good attempts at parts (i) and (ii) of this question, but there were often many basic numerical errors in the solutions. Although credit is given for the correct methodology, ignoring numerical errors carried forward, care should be taken in setting out solutions clearly.

Parts (iii) and (iv) were good differentiators of students. There seemed to be a degree of guessing for the answer to part (iii), which often led to contradictory answers. A good approach was to consider the components of the portfolio separately and to examine the effects of the various changes, before recombining them to look at the overall effects.

Part (iv) required the student to spot that the call in the resulting portfolio is an Asian option. Basic knowledge of Asian options could then be used to answer the question.

This question is a good example of a question which is simply looking for basic applications of derivatives theory to unusual situations.

5 *For reference, the methodology she is considering is termed CrashMetrics.*

(i) (a) VaR is the expected loss from an adverse market movement which will not be exceeded with a specified probability over a particular period of time.

(b) As an example of VaR: over the next two weeks, there is a 95% chance that the portfolio will lose no more than $50m. *[Any sensible example]*
[2]

(ii) There are a variety of acceptable answers; the following captures some of the fundamental points:

- a sudden fall in a significant percentage of stocks on a stock market;
- the decline may continue for months or years;
- often double digit percentage falls in the stock market index over several days;
- panic selling;
- a significant loss in paper wealth;
- it usually occurs following a prolonged period of rising stock prices and economic optimism;
- a reduction in market makers;
- increased volatility;
- a higher correlation of assets;
- reduced liquidity;
- cost of protection increases sharply;

- flight to cash and gold; and
- it can reduce the savings of investors in the short term (with potentially longer-term effects).

[3]

(iii) (a) Using a Taylor expansion of P:

$$P(S+\delta S)=P(S)+\delta S\times\frac{\partial P}{\partial S}+0.5\times\delta S^2\times\frac{\partial^2 P}{\partial S^2}+O\left(\delta S^3\right).$$

Let $\delta P=P(S+\delta S)-P(S)$.

Let $\Delta=\frac{\partial P}{\partial S}$ (the delta of the whole portfolio) and $\Gamma=\frac{\partial^2 P}{\partial S^2}$ (the gamma of the whole portfolio).

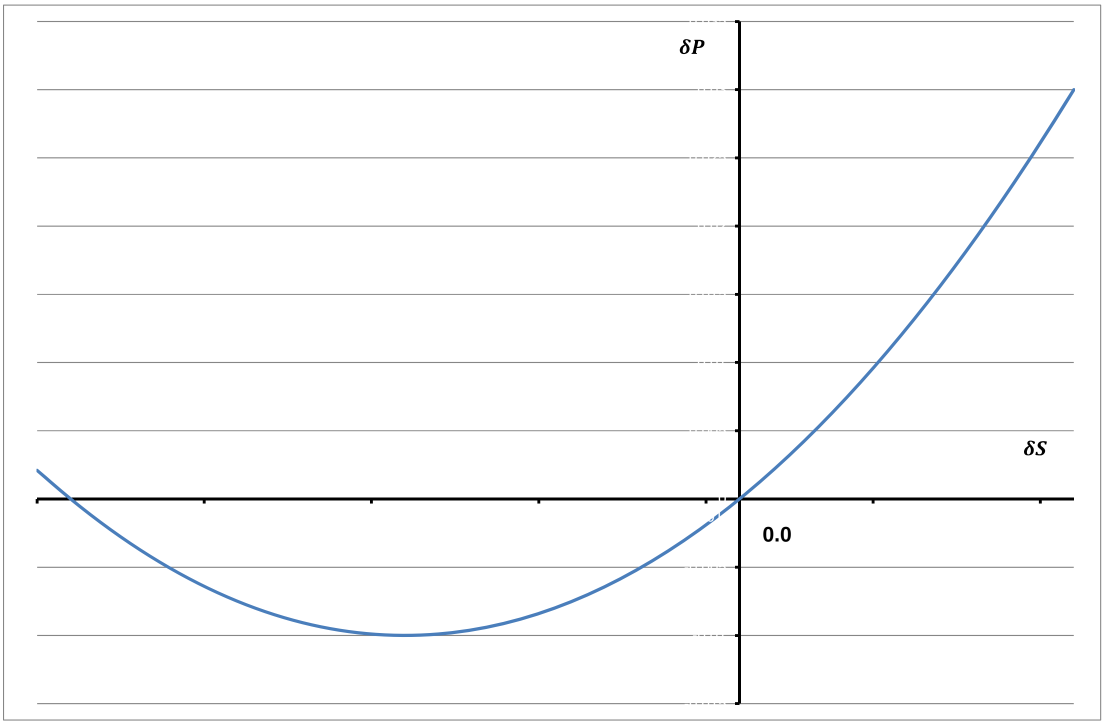
Then, ignoring δS^3 terms the result follows:

$$\delta P=\delta S\times\Delta+0.5\times\delta S^2\times\Gamma.$$

(b) Since the portfolio comprises long call options, gamma must be positive.

Delta must also be positive.

The key features of the graph are therefore that it is quadratic in δS showing a minimum (due to positive gamma) with the curve passing through the origin (with a positive gradient at that point due to positive delta, i.e. the minimum occurs for some $\delta S < 0$):



(c) The largest negative instantaneous change to the portfolio occurs at the minimum of the function

i.e. no time derivative needs to be considered.

This occurs when:

$$\delta^2 P = \Delta + \delta S \times \Gamma = 0$$

i.e. $\delta S = -\Delta / \Gamma$.

The portfolio change in this worst case scenario is:

$$\delta P_{\text{worst}} = -\Delta^2 / 2\Gamma.$$

(d) In these circumstances, the quadratic in δS is now approximately a linear function …

in delta only, e.g. $\delta P = \delta S \times \Delta$.

(e) Since delta is positive, the worst case scenario will be an immediate fall of the price of stock S to zero.

This is fairly unlikely, as even in the worst market crashes relatively few stocks reach a zero valuation.

For this reason the change in the underlying stock will probably need to be constrained (e.g. –20%) to restrict the fall in the portfolio value to more realistic levels.

It should also be noted that the use of the Taylor series is only an approximation; it does not make any assumption about the pricing of the options in the portfolio…

… and it may not remain reasonable for this substantial move in the underlying stock S.

[8]

(iv) The total change to the hedged portfolio is a linear combination of:

- the original change in the portfolio P;
- the change in the value of the hedging contracts; and
- the guaranteed loss due to the bid-offer spread.

This can be expressed as:

$$\delta P^* = \delta S \times \left(\Delta + \lambda\Delta^*\right) + 0.5 \times \delta S^2 \times \left(\Gamma + \lambda\Gamma^*\right) - |\lambda| C,$$

Alternatively, the guaranteed loss can be ignored (if this has been assumed to have been incurred in setting-up the hedge).

This can be expressed as:

$$\delta P^* = \delta S \times \left(\Delta + \lambda\Delta^*\right) + 0.5 \times \delta S^2 \times \left(\Gamma + \lambda\Gamma^*\right).$$

[2]

(v) The optimal value of λ is chosen so that:

- the value of the Greeks in the (new) portfolio under the worst case scenario are as low as possible; and

- the value of the (new) portfolio under the worst case scenario is as high as possible.

A guaranteed loss (due to the bid-offer spread) is being exchanged for a reduced worst case loss.

The optimisation could be extended to considering different types of contracts with the same underlying asset as the portfolio.

The process can be extended to look at:

- longer time horizons;
- non-stock assets; and
- multiple asset portfolios

[2]
[Total 17]

Parts (i) and (ii) were relatively straightforward and most students were close to full marks, as it was basic core reading.

The first parts of (iii) were well generally well answered, but students often struggled with the later parts. The hint in part (a) led many candidates on the correct path but many failed to spot that the question was worded so that only an expansion due to the change in stock price was required (i.e. not the full set of Greeks). This led to problems in being able to answer the rest of the part (iii). Those who had correctly identified the change in portfolio value to be a quadratic in the change in stock price scored highly.

Parts (iv) and (v) were not answered by many students. Part (iv) should have been attemptable as the new portfolio was a linear combination of two assets. The only care had to be taken with the bid–offer spread, but this could be dealt with in a couple of ways. Part (v) was difficult but still an application of some basic techniques.

6 (i) One way of summarising the differences between actual market prices of derivatives and those generated from a pricing model (e.g. Black-Scholes) is to plot the implied volatility parameter as a function of the strike price and time to expiry.

Volatility Smile

The volatility smile is the graph observed by plotting the implied volatility against the strike price…
… for a set of options with the same maturity and underlying asset.

For foreign currency options the shape is often in the shape of a smile.

Volatilities are higher for options further out-of- and in-the- money than for those that are closer to or at-the-money.

For equity options the shape is often a convex monotonic decreasing function (called a volatility smirk)…

… i.e. lower strikes have higher volatility than higher strikes.

Volatility Term Structure

The volatility term structure is the graph of implied volatilities against term to maturity …

… for a set of options with the same strike price and underlying asset.

Volatility Surface

The volatility surface is a three-dimensional graph of the implied volatilities against the strike price and against the term to maturity for a set of options on the same underlying asset.

Uses

These are used in pricing options more accurately (i.e. to allow for the non-lognormality of the underlying asset process).

There is a close correspondence between the shape of the implied volatility smile and the shape of the implied risk-neutral density function. In particular they can be used to derive the underlying asset's price distribution.

The volatility surface is used to predict the future volatility.

The volatility surface can be used as an interpolation tool for pricing other options.

[5]

(ii) A strategy is based on the theoretical advantage of buying and selling options on the same underlying with the same time to expiry. The volatility skew will generally imply that these options will have different implied volatility projections for the same underlying.

The expectation is that the implied volatilities involved in the options converge to a single value at or before expiry.

Therefore, one option is over-priced relative to the other.

The trader will need to determine which option is over-priced, which is equivalent to a view on the single implied volatility at expiry.

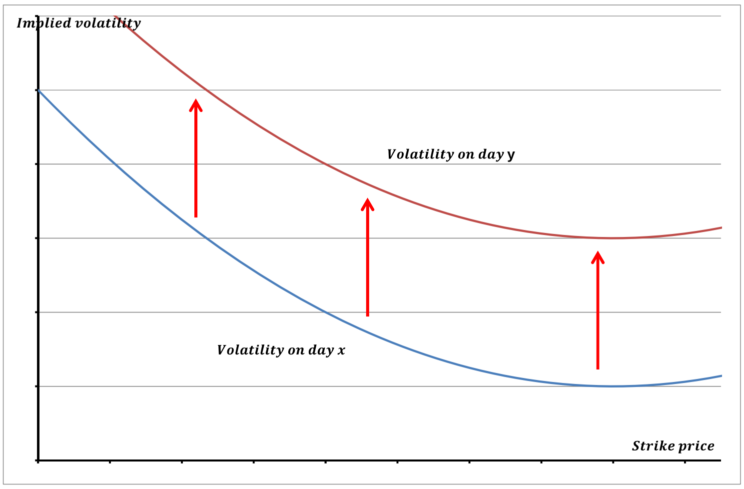
The trader should set up a spread by buying the cheaper options and selling the more expensive options.

There is a risk that a spread is affected by the term structure of volatility and liquidity and availability of such options.

[2]

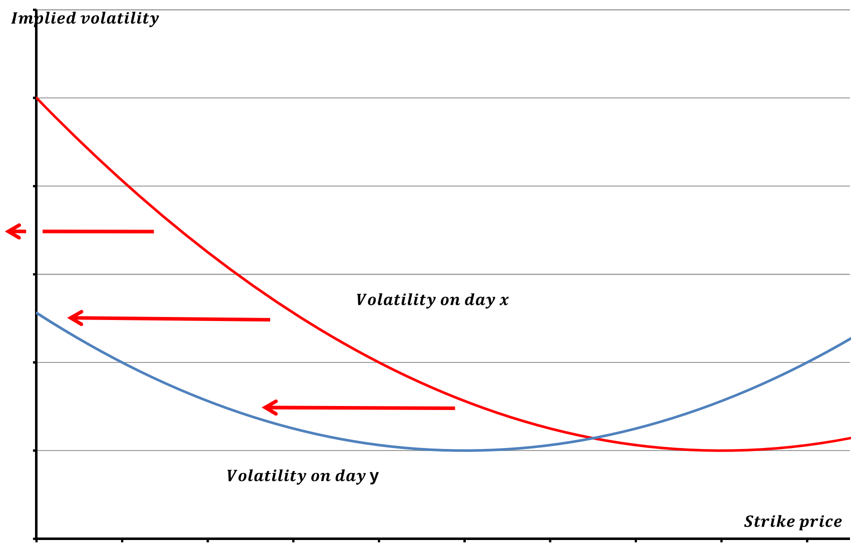
(iii) The volatility smile could shift vertically.

This is illustrated by the chart below:



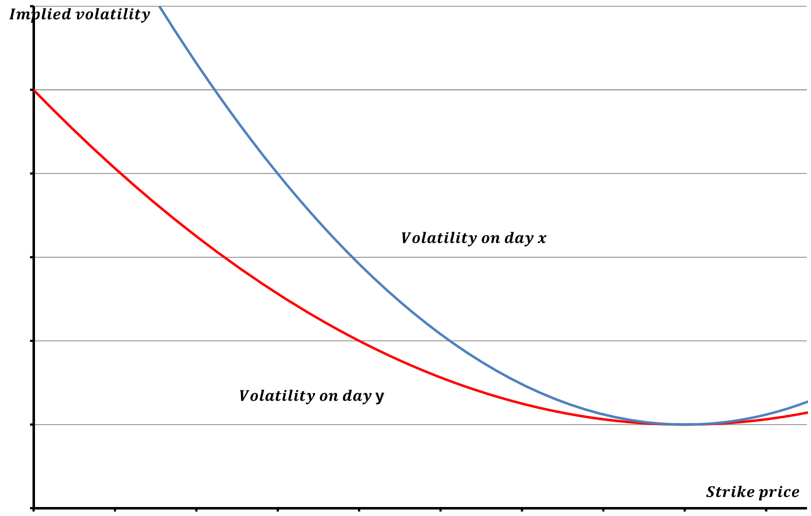
Or the volatility smile could shift horizontally.

This is illustrated by the chart below:



Or the volatility smile could change in curvature (i.e. the curve could flatten or deepen).

This is illustrated by the chart below:



In general, the graph of the volatility smile will change as a result of a combination of horizontal shifts, vertical shifts and a change in curvature.

[3]

(iv) (a) Let c be the price of the call option, S the equity price and σ_{imp} the implied volatility.

Using the chain rule the delta is equal to:

$$\Delta = \frac{dc}{ds} = \frac{\partial c}{\partial S} + \frac{\partial c}{\partial \sigma_{\text{imp}}} \frac{\partial \sigma_{\text{imp}}}{\partial S}.$$

(b) The second term in the above equation for the delta is called the hedging error. It is the quantity which modifies the theoretical value of the Black-Scholes option value due to the volatility smile.

In this term, the quantity $\partial \sigma_{\text{imp}} / \partial S$ is generally unknown but it can be approximated by $\partial \sigma_{\text{imp}} / \partial K$ where K is the strike price of the option.

This value $(\partial \sigma_{\text{imp}} / \partial K)$ can be derived from the volatility smile.

Overall this leads to a smile adjusted delta:

$$\Delta \simeq \Delta_{BS} + \upsilon_{BS} \frac{\partial \sigma_{\text{imp}}}{\partial K},$$

where Δ_{BS} is the theoretical Black-Scholes delta and υ_{BS} is the theoretical Black-Scholes vega.

The performance of a smile adjusted delta hedge relative to the theoretical Black-Scholes hedge is a significant improvement.

[3]

Part (i) should have demonstrated understanding of the core reading. Like question 1(i), the majority of students were unable to get higher than half marks as only basic definitions were given. This demonstrates the need to have more than a high level understanding of the core reading. Without access to a more detailed understanding it makes attempts at the application of core reading very difficult, as was seen in the remainder of this question.

Part (ii) was a challenging question. The solution provides one approach, but marks were also awarded for well-described alternative strategies.

Part (iii) essentially required some basic geometry, but it was generally not well answered as insufficient detail was provided. Marks were also awarded to students who explained the changes in words rather than by using graphs.

Part (iv) covered one the of the more challenging areas of the course and was one of the hardest parts on the paper. Students should expect there to be a mix of difficultness to parts of questions, as this demonstrates.

7 (i) In specifying the delta hedging strategy, would need to set down:

- Frequency of checks of actual delta vs delta of option.

- Trigger point – how big the delta mismatch needs to be to trigger a rebalancing.

- Whether, when a rebalancing is triggered, the delta is always rebalanced back to zero.

- Whether futures on the index will be the only non-zero delta hedging instrument or whether the firm will consider using options.

- Whether any cash balance or fixed interest holdings will be considered to be part of the delta hedge portfolio. [So on day one, whether the delta hedge portfolio will just be some shorted futures worth zero, or whether it will also include some bonds or cash with value equal to (say) market value of the option.]

- The volatility that should be input to the formula for the delta of the option. [Whether this should be current market implied volatility, a long term assumed market implied volatility, a long term assumed actual volatility or a day one market implied volatility.]

- Whether the delta of the option should be based on $\partial p/\partial S$, or whether it should reflect volatility skew. [i.e. if σ is a function of S, whether delta might be better defined as $\partial p/\partial S + \partial p/\partial\sigma.\partial\sigma/\partial S$.]

- Whether investment managers (say) have the authority to hold off rebalancing if they think a recent market movement is about to be reversed.

- Further points may also be valid.

[3]

(ii) The extra risks the firm is taking include:

- Vega risk. If the firm's position is marked to market on a regular basis, changes in market implied volatility will show up on the balance sheet as the option value will be sensitive to market implied volatilities, whereas the delta hedge will not.

- Gamma risk. Between rebalancing, the delta hedge will always underperform the option because of the convexity of the option value, and these costs will add up over time.

- Related to the gamma risk, the firm will be exposed to the risk of the market index being volatile and causing the firm to keep moving backwards and forwards in and out of equities, each time buying after a price increase and selling after a fall.

- Market liquidity risk. If the market falls significantly on one day, the firm may find it difficult to short the futures necessary to maintain the delta hedge. Delays like this will increase the cost of the hedging.

- Cashflow liquidity risk. If the market falls then the firm will need to keep making margin payments on the futures, which could prove to be onerous.

- Operational risk. There may well be lots of manual processes around the delta hedging and errors could be made.

[5]

(iii) A Monte Carlo simulation could be used.

The cost of delta hedging is not fixed – it will be a random variable with a probability distribution.

The Board needs to understand that distribution, so it could be presented to them in the form of mean and variance, or as certain key percentiles.

To determine the probability distribution, need to build:

- A model for how the relevant economic variables might move throughout the duration of the delta hedging plan. [This is referred to as an economic scenario generator (ESG) in the rest of this answer.]

   - As well as the equity index, relevant economic variables include interest rates and market implied volatilities.

- A model for the delta hedging.

   - This needs to model all the regular checks that are performed …

   - …and all the transactions that would take place in future....

   - ...including the responses to all the jumps and falls that will generate frictional costs,

   - … and the costs associated with managing the hedge (employees and calculation costs).

   - It needs to be able to calculate option values and deltas from the underlying economic variables.

   - It needs to be able to calculate futures payoffs from the underlying economic variables.

   - If cash or bonds are to be considered part of the delta hedge, then it needs to model any transactions in these asset classes and any payoffs from them.

The stochastic model needs to be run. Each random economic scenario is feeds into the delta hedging model and the model outputs the cost of hedging in that scenario.

This is repeated for a large number (10,000 say) of scenarios.

The results are collected together and used to calculate a probability distribution for the costs of delta hedging.

[5]

(iv) The following might also be useful for the Board:

- A full specification of the ESG underlying the determination of the possible costs.

- Explanation of the limitations arising in respect of the ESG and its use.

- Scenario tests to illustrate some of the risks.

- For example, a scenario where the market kept moving up and down, making the delta hedging expensive.

- Or a scenario where the market crashed and the hedge could not be rebalanced quickly enough as a result of illiquid markets.

- Sensitivity tests on key inputs.

- How the distribution of costs changes if the hedge rebalancing frequency or the trigger points are changed.

- How the distribution of costs changes if the definition of delta changes.

- The cost of the alternative strategy, i.e. option purchase.

- Any financial reporting implications.

- For example, if this is a financial firm and it delta hedges rather than buying the option, whether this has any impact on its regulatory capital requirements.

- An in-depth description of the processes involved.

[4]
[Total 17]

The solutions to this question were generally disappointing as not enough points were made to score reasonable marks, taking into consideration the relatively high mark allocation. The question involves a realistic application of ST6 techniques, taking into consideration real world practicalities.

Part (i) produced some bland statements around delta hedging in general rather than focussing on the specifics of this delta hedging strategy. It demonstrates one of the key skills in attempting questions in the ST series: the ability to address solutions to the specifics of a question.

The solution to (ii) suggests a range of risks, but further credit was given for other well argued risks: e.g. basis risk or rho risk.

END OF EXAMINERS' REPORT