INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

April 2015 examinations

Subject ST6 – Finance and Investment Specialist Technical B

Introduction

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and using past papers as a revision aid and also those who have previously failed the subject.

The Examiners are charged by Council with examining the published syllabus. The Examiners have access to the Core Reading, which is designed to interpret the syllabus, and will generally base questions around it but are not required to examine the content of Core Reading specifically or exclusively.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report; other valid approaches are given appropriate credit. For essay-style questions, particularly the open-ended questions in the later subjects, the report may contain more points than the Examiners will expect from a solution that scores full marks.

The report is written based on the legislative and regulatory context at the date the examination was set. Candidates should take into account the possibility that circumstances may have changed if using these reports for revision.

F Layton
Chairman of the Board of Examiners

July 2015

General comments on Subject ST6

This subject deals with mathematical techniques for the valuation and risk management of derivatives, together with some aspects of their practical application.

Different numerical answers may be obtained to those shown in these solutions, depending on whether figures were obtained from tables or from calculators. Allowance was made for these minor differences. However, candidates may have been penalised for using excessive rounding or showing insufficient working.

Derivative theory is an interesting but very exacting subject which needs to be tackled in a precise and disciplined manner. Questions that appear unfamiliar can, with some clear thinking at the start of the exam, be found to be straightforward examples of well known techniques. Thorough preparation of the course material is essential, however. This subject also requires quite a varied approach in writing solutions. Algebraic and numerical content needs to be supplemented with well reasoned but brief arguments where the question demands. Candidates should always try to provide several distinct relevant points for the discursive questions, not lengthy paragraphs developing a single idea.

Comments on the April 2015 Paper

The overall performance of students was good and towards the better end of what has been seen in the past few years. Although candidates found some questions relatively challenging, as is normally the case, well-prepared candidates scored well above the pass mark.

In terms of areas for improvement, candidates struggled more with those questions that required an element of application of Core Reading to situations that were not immediately familiar, so did not demonstrate the desired depth of knowledge and understanding. In some questions there were key details which candidates missed, which had a significant influence on the direction of their answers. Furthermore, there were significant numbers of errors in using calculators, and standard Black-Scholes formulae, which led to candidates missing out on marks.

The comments that follow the questions concentrate on areas where candidates could have improved their marks, as an attempt to help future candidates to revise accordingly and to develop their ability to apply the core reading to related situations. This skill is likely to be valuable both in exams and in using this theory in their professional careers.

1 (i) The fund will include corporate bonds and swaps

The objectives of the fund will determine the type of corporate bonds to be held:

- their credit rating
- whether to allow government bonds
- their duration
- the amount of diversification
- active or passive management; and
- benchmarks
- country / geographical region

The objectives of the fund will determine the type of swaps to be held:

- whether payer or receiver swaps
- forward starting / zero coupon

Rather than being directly invested in corporate bonds, the fund is more likely to be invested in units of a large corporate bond fund held by the fund manager.

As the fund uses swaps it is likely to hold a cash component to the fund to meet margin requirements and as collateral.

The funds are targeted at investors who are seeking:

- a broad exposure to corporate bond spreads
- without taking on any interest rate exposure

For example, a pension fund adopting a liability-driven investment strategy where the interest rate hedge is managed elsewhere in the portfolio.

(ii) (a) The change in legislation following the 2007/08 credit crisis has changed the following:

- The move to central clearing now affects swaps; and

- There is a legal requirement to report all OTC derivative transactions.

This has led to a general move by banks to use standardised collateral of cash and gilts.

As a result, it is generally difficult to transact swaps with corporate bonds as collateral.

The fund may have to post initial margin for the swaps

The fund may have to pay additional ongoing fees (e.g. to the Central Counterparty or the Clearing member)

(b) Other hedging strategies can be used instead of swaps:
- futures; and
- gilt repos.

There are other ways to obtain credit exposure, such as:
- credit default swaps
- collateralised debt obligations

(iii) The return generated by the fund is affected by the widening and tightening of credit spreads.

For example, if credit spreads widen then the fund is likely to underperform the return on short term interest rates (because of the fall in value of the corporate bonds relative to the interest rate hedge).

The return of the fund may also be affected by basis risk between government bond yields and swap yields (i.e. the swap spread).

The swaps market can become illiquid or volatile which can affect the ability of the fund to close out or execute positions.

Counterparty risk should be low due to central clearing.

But there is a still a risk that the clearing house defaults.

The fund may be exposed to currency risk.

The performance of the fund will be adversely impacted by corporate bond defaults.

Any investment in non-investment grade bonds will increase the risk of such defaults.

Market conditions can also affect the ability to trade in corporate bonds.

The asset manager may be exposed to:
- reputational risk if they need to default or change objectives.
- poor sales / withdrawals if the fund performs poorly
- operational risk (e.g. mispricing the fund)
- liquidity risk due to the need to post collateral

The investor may not be able to withdraw funds when the underlying market is illiquid.

Most students picked up some marks on this question, although few were able to link their general knowledge about the move towards central clearing of derivatives to the specifics of the given scenario, i.e. a fund that would have been using corporate bonds as collateral. Those who did were able to score highly.

2 (i) The stochastic process M_t is a martingale with respect to $\mathbb{P}$ if and only if:

$\mathbb{E}_{\mathbb{P}}\left[|M_t|\right] < \infty$, for all *t*; and

$\mathbb{E}_{\mathbb{P}}\left[M_t \mid \mathcal{F}_s\right] = M_s$, for all $s \le t$ (where $\mathcal{F}_s$ is the filtration of M_t).

(ii) (a) Assume that $\mathbb{E}_{\mathbb{P}}\left[|M_t|\right] < \infty$, for all *t*.

Let $s \le t$ and $\mathcal{F}_s$ be a filtration of M_t, then:

$$\mathbb{E}_{\mathbb{P}}\left[M_t \mid \mathcal{F}_s\right] = \mathbb{E}_{\mathbb{P}}\left[B_t^2 - t \mid \mathcal{F}_s\right],$$

$$= \mathbb{E}_{\mathbb{P}}\left[(B_t - B_s + B_s)^2 - t \mid \mathcal{F}_s\right],$$

$$= \mathbb{E}_{\mathbb{P}}\left[(B_t - B_s)^2 + B_s^2 + 2B_s\left(B_t - B_s\right) - t \mid \mathcal{F}_s\right],$$

$$= \mathbb{E}_{\mathbb{P}}[(B_t - B_s)^2 | \mathcal{F}_s] + B_s^2 + 2\mathbb{E}_{\mathbb{P}}\left[B_s\right]\mathbb{E}_{\mathbb{P}}\left[(B_t - B_s)\right] - t$$

(using independence of the Brownian motion)

$$= (t - s) + B_s^2 + 0 - t$$

$$\left(\text{as } \mathbb{E}_{\mathbb{P}}\left[B_s\right] = \mathbb{E}_{\mathbb{P}}\left[\left(B_t - B_s\right)\right] = 0 \text{ and } \mathrm{Var}_{\mathbb{P}}\left[B_t - B_s\right] = \mathbb{E}_{\mathbb{P}}\left[(B_t - B_s)^2\right] = t - s\right)$$

$$= B_s^2 - s$$

$$= M_s$$

and hence is a martingale as required.

(b) Using Ito's lemma: $dM_t = 2B_t dB_t$.

(iii) Writing this in the differential form using Ito's lemma:

$$dS_t = 2\left(B_t - \gamma t\right)dB_t - 2\gamma\left(B_t - \gamma t\right)dt.$$

$$= 2\left(B_t - \gamma t\right)\left(dB_t - \gamma dt\right)$$

Set $X_t = B_t - \gamma t$, then $dX_t = dB_t - \gamma dt$.

The above equation can then be re-written in terms of X_t:

$$dS_t = 2X_t dX_t.$$

Based on the definition of X_t, set

$$\mu_t = -\gamma.$$

Using the Cameron-Martin-Girsanov theorem, and trivially the drift is bounded, then there exists a measure $\mathbb{Q}$ such that

$W_t = B_t + \int_0^t \mu_s ds$ is a $\mathbb{Q}$-Brownian motion.

$$= B_t - \gamma \int_0^t ds$$

$$= B_t - \gamma t$$

$$= X_t.$$

The expression for dS_t can be integrated to give $S_t = W_t^2 - t$ (using the result from part (ii)(b)), which (using the result from part (ii)(a)) is a martingale as required.

Alternative solution 1:

The expression for S_t can first be rearranged to give:

$$S_t = (B_t - \gamma t)^2 - t.$$

The process can then be followed as above to demonstrate that there is a probability measure $\mathbb{Q}$ under which $X_t = (B_t - \gamma t)$ is a $\mathbb{Q}$-Brownian motion.

Hence, since $S_t = X_t{}^2 - t$ and X_t is a $\mathbb{Q}$-Brownian motion, the result from part (ii)(a) demonstrates that S_t is a martingale under probability measure $\mathbb{Q}$.

Alternative solution 2:

Rather than using the CMG approach as described above for the middle part of the derivation, an alternative approach to demonstrating that $(B_t - \gamma t)$ is Brownian motion under measure $\mathbb{Q}$ is as follows:

Under measure $\mathbb{P}$, $dB_t = dz_t$.

Moving between measures is equivalent to replacing dz_t with $dz'_t = dz_t + \mu_t dt$ for some draft μ_t, where dz'_t represents Brownian motion within measure $\mathbb{Q}$.

Setting $\mu_t = -\gamma$ gives

$$dz'_t = dz_t - \gamma dt$$

So $d\left(B_t - \gamma t\right) = dB_t - \gamma dt = dz_t - \gamma dt = dz'_t$,

which is Brownian motion within measure $\mathbb{Q}$.

(iv) Derivatives can be valued by finding a portfolio of cash and stock which replicates the derivative. The value of the derivative must be equal to the value of the replicating portfolio, otherwise arbitrage opportunities would exist.

From the previous part of the question, a measure $\mathbb{Q}$ has been found which makes the asset price S_t into a martingale.

Using the tower law, the stochastic process $Y_t = \mathbb{E}_Q\left[X \mid \mathcal{F}_t\right]$, which represents the expected value of a claim X on the asset price at time T (conditional on the history up to time t), is also a martingale. This assumes $\mathbb{E}_Q\left[|X|\right] < \infty$.

The Martingale Representation Theorem then leads to the construction of the replicating portfolio. Since Y and S are both martingales, the Martingale Representation Theorem tells us that there is a previsible process φ_t such that $dY_t = \varphi_t dS_t$ under measure $\mathbb{Q}$.

The replicating strategy is to hold φ_t units of asset at time t and $\theta_t = Y_t - \varphi_t S_t$ units of cash.
Let $V_t = \theta_t + \varphi_t S_t = Y_t$ be the value of the portfolio at time t. The portfolio is self-financing since:

$$dV_t = dY_{t,}$$

$$= \varphi_t dS_t .$$

Also, $V_T = X_T$

Hence the risk-neutral value formula for the price of the derivative is:

$$V_t = \mathbb{E}_Q\left[X \mid \mathcal{F}_t\right].$$

(v) In a non-zero interest rate (with discount rate r) world, the discounted values of the option, the cash/bond and the asset need to be modelled.

This adds an extra layer of complexity to the modelling.

The discount rate may not be constant.

Let Z_t be the discounted price of the asset with respect to the risk-free rate, then under the measure $\mathbb{Q}$:

$$Z_t = e^{-rt}(W_t^2 - t).$$

This is not a martingale and it is more difficult to convert this into a martingale for pricing, as in the above method.

Parts (i) and (ii) were relatively standard bookwork and were well answered by many candidates. However, many candidates did not appear to recognise that (iv) was also based on bookwork thus missing out on marks. Candidates were often less able to work systematically through the method in part (iii) to score highly, and in part (v) few gave sufficient reasoning to score both marks.

3 (i) Under this model, if the probability of leaving the company at time t is $p(t)$, the share options are equivalent to:

a set of European calls……
with terms τ, $\tau + 1$, $\tau + 2$, ...
which are at-the-money….
with amounts equal to $Np(\tau)$, $Np(\tau + 1)$,... where N is the total number of share options and τ is the time (in years) between the valuation date and the permitted exercise date.

But with the rule that if $t < 3$, the value of a term-t option is zero.

Each of the individual term-t options can be valued using:

- Black-Scholes
- A tree method

The CEO needs to know the current risk-free yield curve …

… and the at-the-money implied volatilities for the organisation's stock
… for all the different maturities in question.

The CEO needs to make an assumption about the organisation's future dividends, and for the time t option needs to exclude from the current share price the present value of all dividends before time t.
The deduction should include dividends that are payable just after t where the share price goes ex-dividend just before t.

(ii) Monte Carlo would be the best technique here.

Trees are generally useful when the decision on whether to exercise at a particular node depends on the value of the option at that node, e.g. useful for valuing American puts.

Monte Carlo can struggle to value options with an option to exercise early.

But in this case, the decision to exercise early is a function of the stock price, not of an option price, so this is relatively easy to model using Monte Carlo.

Monte Carlo is generally more useful for valuing derivatives with path-dependent payoffs.

In this case:

- The probability of the CEO still being in a job is path dependent: under the assumptions in the question, he is more likely to be in the job if the share price rises then falls than if it falls then rises.

- The probability of the option having been exercised early is also path dependent: if the share price is currently 175% of the exercise price and has never been above 200% then the option has not been exercised early; if it is at 175% but has been over 200% (at a previous possible exercise date) then the option will have already been exercised.

So to value the option using a tree would be cumbersome
…. since the probability of the option still being in force would need to be rolled forward within the tree.
Also, if a tree was used, would need to make sure that the 200% threshold for exercising the option early lay along a row of nodes.

While this might not be too difficult for a non-income bearing share, it would be more problematic for a dividend-paying share (as is the case here).

In contrast to trees, Monte Carlo has no such issues: within each individual run the option will be exercised at a particular point in time and only the option payoff needs to be recorded at that time.

A tree may be less complex to set up whereas a Monte Carlo model may require more calibration

Further valid points could be made.

This was one of the less well answered questions on the paper. For part (i) often candidates did not spot that this was equivalent to a series of European call options. For part (ii) many candidates did not recognise that early exercise was dependent on the stock price rather than the option price, and so Monte Carlo would be a good approach. This led them to quote bookwork results that were not relevant to the specifics of this question. Some candidates also described in detail how a tree or Monte Carlo model could be used rather than which would be the most appropriate approach, as required by the question.

4 (i) First calculate the price of zero-coupon bonds from time 3 to 3.5 using $\text{ZCB}(t) = \exp(-t.i_t)$

Term	*3.0*	*3.5*	*4.0*	*4.5*	*5.0*	*5.5*	*6.0*
Risk Free Rate	1.15%	1.32%	1.48%	1.64%	1.80%	1.94%	2.08%
ZCB	0.96609	0.95485	0.94252	0.92886	0.91393	0.89880	0.88267

Then the forward swap rate is $2 \times (\text{ZCB}_3 - \text{ZCB}_6)/(\text{ZCB}_{3.5} + ... + \text{ZCB}_6)$
= 3.02% as required

(ii) Value of a stream of payments of 0.5 at times 3.5,...,6 is

$A = 0.5 \times (\text{ZCB}_{3.5} + ... + \text{ZCB}_6) = 2.76081$

Value of the FRA is then Notional × A × (Fixed rate – Swap rate)

= 100,000,000 × 2.76081 × (3.5% – 3.02%) = \$1.321m

Alternative (incorrect) calculation method (following Hull & Core reading)

$-100\text{m} \times \text{ZCB}_3 + 100\text{m} \times (1 + 3.5\%/2)\text{^}6 \times \text{ZCB}_6 = \$1.342\text{m}$

This is an amount that the financial institution would pay to the counterparty.

(iii) Price of a receiver swaption is $LA[s_K N(-d_2) - s_0 N(-d_1)]$

Where

- $d_1 = [\ln(s_0/s_K) + \sigma^2 T/2] / \sigma\sqrt{T}$
- $d_2 = d_1 - \sigma\sqrt{T}$
- σ is Black volatility = 0.2
- s_0 is forward swap rate = 0.0302
- s_K is strike (at-the-money) = 0.0302
- L is notional = \$100m

- A is as above: value of a stream of payments of 0.5 at times 3.5, ..., 6 = 2.76081
- T = 3 years

In this case:

- $d_1 = 0.17321$
- $N(-d_1) = 0.43125$
- $d_2 = -0.17321$
- $N(-d_2) = 0.56875$

So the swaption price is

100,000,000 × 2.76081 × 3.02% × (0.56875 – 0.43125) = $1.147m

(iv) The purchase of a FRA hedges the financial institution's market risk perfectly.

The swaption, on the other hand:

- Allows the financial institution to profit from interest rate increases: if the three year swap rate at time 3 exceeds 3.5%, then the firm benefits from earning a higher rate on the $100m than it is paying out (with the FRA, the firm would be hedging away profits as well as losses).
- Under-protects the firm by having a strike rate of 3.02% rather than the desired 3.5%. If the three year swap rate at time $t = 3$ is below 3.02%, the firm will lose 0.48% on a net basis, this being the difference between the 3.02% received and the 3.5% paid. If the three year swap rate at time 3 is between 3.02% and 3.5%, the swaption expires worthless and the firm is left to pay out 3.5%, which exceeds the swap rate at that time.

As a result of the value of the under-hedging exceeding the value of the extra profits in the "interest rate up" scenarios, the FRA is more expensive than the swaption.

The firm needs to consider also the credit risk associated with the two possible hedges. Today the credit exposure is at a similar level for the two hedges. If interest rates increase above 3.5%, the FRA will become a liability to the financial institution (so not a credit risk) and the swaption would have a small time value.

If interest rates remain below 3.5%, then close to maturity the FRA would be worth more than the swaption so represents a bigger credit risk, and further from maturity the swaption will have some time value, so things will be less clear.

It is also worth noting that the FRA (but not the swaption) can become a liability to the financial institution and an asset to the counterparty. If the counterparty has insisted on credit risk mitigation processes being in place, the financial institution could find itself being called upon to post collateral with the counterparty.

The swaption will also expose the financial institution to vega risk.

Many students picked up good marks on this question, and those who worked methodically through the calculations paying attention to the detail in the question scored highly. Marks were lost through calculation errors, misuse of interest rates (despite their format being stated), and not allowing for all of the relevant details provided in the question. In (iv), only a minority of those students who identified the high level differences between the two approaches went into the required detail to score highly.

5 (i) Because the mean change in R^* needs to be equal to $-0.5aR^* = -0.5aj\,\Delta R$

$$\Delta R\, p_u - \Delta R\, p_d = -0.5aj\,\Delta R$$

or $\quad p_u - p_d = -0.5aj$

Because the variance of change in R^* needs to be equal to $\sigma^2/2$

$$p_u\,\Delta R^2 + p_d\,\Delta R^2 - 0.25a^2j^2\Delta R^2 = 0.5\sigma^2$$

"variance = mean of square less square of mean"

or $\quad p_u + p_d = 1/3 + 0.25a^2j^2$

And, obviously, $p_u + p_d + p_m = 1$

(ii) Wherever the probabilities do not make sense (i.e. are not in the range [0,1]), the branching pattern of the tree needs to be changed. Rather than having up, down and unchanged branches, there need to be an unchanged branch, a branch that takes a step of ΔR towards the middle of the tree and a branch that takes a step of $2\Delta R$ towards the middle.

In other words, for symmetry at the top of the tree there will be an extra downward branch and no upward branch; at the bottom of the tree there will be two upward branches and no downward branch.

(iii) At the top of the tree we now have p_m, p_d, p_{dd}

Considering the mean:

$$-\Delta R\, p_d - 2\Delta R\, p_{dd} = -0.5aj\,\Delta R$$

or $p_d + 2p_{dd} = 0.5aj$

Considering the variance:

$$P_d\,\Delta R^2 + 4p_{dd}\,\Delta R^2 - 0.25a^2j^2\Delta R^2 = 0.5\sigma^2$$

or $p_d + 4p_{dd} = 1/3 + 0.25a^2j^2$

And, obviously, $p_d + p_{dd} + p_m = 1$

And similarly at the bottom of the tree we now have p_m, p_u, p_{uu}:

$$\Delta \mathrm{R}\, p_u + 2\Delta R\, p_{uu} = -0.5^* aj\,\Delta R$$

or $p_u + 2p_{uu} = -0.5aj$

$$p_u\,\Delta R^2 + 4p_{uu}\,\Delta R^2 - 0.25a^2j^2\Delta R^2 = 0.5\sigma^2$$

or $p_u + 4p_{uu} = 1/3 + 0.25a^2j^2$

And, obviously, $p_u + p_{uu} + p_m = 1$

(iv) Look to transform the nodes via $R(i,j) = R^*(i,j) + \alpha(i)$, so the transformation is identical for all nodes at a particular timestep.

The $\alpha(i)$ can be determined iteratively, starting at $t = 0$ and moving forward one timestep at a time.

First, note that $\alpha(0)$ is just $R(0)$, the current six month rate.

Then given all the alphas up to time t (step $2t$), determine the value of a $t + 0.5$ year zero-coupon bond at time zero using the tree for R. This is a bond with payoffs of 1 at every node at time $t + 0.5$.

All probabilities on the R-tree (which are identical to those on the R^*-tree) are known and all values of R on the tree up to time t are also known. The Rs at time $t + 0.5$ will all be of the form $R^*(2t+1,j) + \alpha(2t + 1)$. Therefore the value of the bond will be a function of $\alpha(2t + 1)$.

Equating this to the bond's current value (discounted using today's $t + 0.5$ year interest rate) and backsolving for alpha gets us to $\alpha(2t + 1)$.

This was the most poorly answered question, with many students not even attempting it despite it being closely based on Core Reading. However, those who were well-prepared scored close to full marks. This emphasises the need to have studied all areas of the syllabus and the supporting Core Reading.

6 (i) (a) A consumption asset is an asset that is held primarily for consumption. It is not usually held for investment.

The cost of carry measures the storage cost plus the interest that is required to finance the asset less the income earned on the asset.

In terms of a futures contract, the convenience yield measures the benefits of holding the physical asset that are not obtained by the holder of a long futures contract on the asset.

(b) The convenience yield exists due to the following benefits of holding a physical asset:

- to maintain a production process (e.g. the production of batteries);
- the ability to profit from temporary shortages in the asset;
- to have fast access to an asset if needed …
- … especially if the need is difficult to predict.

The convenience yield exists because the value of the benefits listed exceeds the potential risk-free profits that could be earned selling the commodity and going long on the future.

(c) The futures contract price $F(t,T) = S_t \times e^{(T-t)(c-y)}$, where:

- t is the current time;
- T is the maturity time;
- S_t is the spot price of the underlying asset at time t;
- c is the cost of carry; and
- y is the convenience yield.

(ii) (a) The Sharpe ratio is essentially the market price of risk.

It incorporates both risk and return as opposed to just risk within the minimum variance ("MV") ratio.

The Sharpe ratio considers the return generated by the hedged portfolio in excess of the risk-free rate, and relates this to the amount of risk.

(b) Possible disadvantages of the Sharpe ratio include:

- There is no distinction between upside and downside volatility.
- As a result high outlying returns can lead to a low ratio, which is nonsensical.

- For underlying distributions of asset prices which are not normal, the variance can understate the risks present (particularly in skewed distributions).

- It doesn't produce a minimum variance outcome

- Reliance on a constant risk free rate

(c) The ratio should be maximised.

(iii) The hedge ratio is the ratio of the position taken in futures to the size of the exposure in the asset to be hedged.

The optimal hedge ratio is optimisation of the hedge ratio with respect to a given objective function.

The optimal hedge ratio could also be defined as the hedge ratio, h*, that minimises the variance of the hedged position

$$h^* = \rho \frac{\sigma_S}{\sigma_F}$$

Where

h^* = minimum variance hedge ratio
ρ = correlation between the changes in the future and spot price
$\sigma_{S,F}$ = standard deviation of the changes in the spot price and futures price, respectively

(iv) (a) The general condition is that $E(R_f) = 0$ or that ρ = +/– 1.

(b) In the case that $E(R_f) = 0$, the expected return on the futures contract is zero.

That is, the futures contract price follows a simple martingale process. The case of ρ = +/–1 implies perfect correlation which is unlikely.

(v) (a) For the minimum variance hedge ratio the optimal hedge ratio is equal to:

$$0.8 \times 0.13 / 0.15 = 0.69$$

For the Sharpe ratio the optimal hedge ratio is equal to:

$$-\frac{0.13}{0.15} \times \frac{(0.13/0.15 \times 0.027/(0.03-0.02)) - 0.8}{1 - 0.13/0.15 \times 0.027 \times 0.8/(0.03-0.02)} = 1.53$$

(b) The optimal Sharpe ratio suggests that a much greater position should be taken in the oil futures relative to the optimal MV ratio.

This also suggests that the optimal Sharpe ratio position could be riskier (in terms of exposure to derivatives) in order to generate returns.

The company may want to consider whether they need to target returns as well as focus on risk management.

The condition in (iv), $E(R_f) = 0$, is therefore not true, since the results are not equal.

(vi) Possible risk mitigation strategies:

- Negotiate a better trade agreement with the current supplier or diversify with several suppliers.
- Purchase a controlling stake in a lithium mining company or in a company which supplies lithium.
- Purchase an equity portfolio of mining companies involved in lithium production.
- Purchase a portfolio to replicate a global lithium index (this will require active management).
- Approach an investment bank for an over-the-counter forward or option.
- Negotiate better supply contracts on other input commodities to allow greater flexibility to absorb volatile lithium prices.
- Pass the costs onto the consumer.
- Partner with technology start-ups involved in areas such as new battery technology or asteroid mining.
- Through internal research and development, look for other battery technologies less reliant on lithium.
- Cross-hedge with an alternative commodity that has a higher correlation than oil.

This question was reasonably well answered by candidates, particularly the first few parts. In part (v) many candidates made calculation errors, suggesting that more care is needed. For part (vi) candidates generally could have scored more highly by generating a wider range of distinct points.

7 (i) $\sqrt{2\pi}\;N'(d_2) = e^{-d_2^2/2}$

$$= e^{-d_1^2/2} \times e^{d_1\sigma\sqrt{T}-\sigma^2 T/2}$$

since $d_2 = d_1 - \sigma\sqrt{T}$

$$= e^{-d_1^2/2} \times e^{\ln(S/K)+(r-q+\sigma^2/2)T-\sigma^2 T/2}$$

since $d_1 = [\ln(S/K) + (r - q + \sigma^2/2)T] \;/\; \sigma\sqrt{T}$

$$= e^{-d_1^2/2} \times e^{\ln(S)-qT} \times e^{-\ln(K)+rT}$$

$$= \sqrt{2\pi}\;N'(d_1) \times Se^{-qT} / Ke^{-rT}$$

Hence the result.

Other reasonable approaches to this question were possible for full marks.

(ii) Price of a call option is $c = Se^{-qT}N(d_1) - Ke^{-rT}N(d_2)$

where $d_1 = [\ln(S/K) + (r - q + \sigma^2/2)T] \;/\; \sigma\sqrt{T}$

and $d_2 = d_1 - \sigma\sqrt{T}$

$\partial c/\partial K = Se^{-qT}\,(\partial N(d_1)/\partial d_1)\,(\partial d_1/\partial K) - Ke^{-rT}\,(\partial N(d_2)/\partial d_2)\,(\partial d_2/\partial K)$
$- e^{-rT}N(d_2)$ (using the product rule)

$$= - Se^{-qT}\;N'(d_1)\,(1/K\sigma\sqrt{T}\,) + Ke^{-rT}\;N'(d_2)\,(1/K\sigma\sqrt{T}\,) - e^{-rT}N(d_2)$$

$$= (-Se^{-qT}N'(d_1) + Ke^{-rT}N'(d_2))\,(1/K\sigma\sqrt{T}\,) - e^{-rT}N(d_2)$$

$= -e^{-rT}N(d_2)$ using result from (i)

(iii) *B* would normally be positive.

This is because market implied volatility is normally a decreasing function of the exercise price.

Possible reasons that have been put forward for this include:

- Crashophobia – market fear of crashes being priced into options.
- Leverage: firms that are partly funded by debt will look more volatile as the value of their equity reduces.

- Supply and demand: most demand for options will be for out-of-the-money puts to protect capital positions.

- Traders expecting equity return distributions to have thicker tails than lognormal for market falls.

- Capital implications e.g. deeply in-the-money call options (i.e. low strikes) can be attractive as they display similar sensitivities to equities but have lower capital requirements.

(iv) Price of a call option is $c = Se^{-qT}N(d_1) - Ke^{-rT}N(d_2)$

where $d_1 = [\ln(S/K) + (\mathrm{r} - q + \sigma^2/2)T] / \sigma\sqrt{T}$

and $d_2 = d_1 - \sigma\sqrt{T}$

Here:
- σ is market implied volatility, 25% – (1.255 – 0.8)/10 = 20.45%
- S is spot, i.e. \$10m
- K is strike, i.e. \$12.55m
- r is risk-free rate, 3.3%
- q is dividend income, 1%
- T is term, 1 year

So:

$$d_1 = [\ln(10/12.55) + (0.033 - 0.01 + 0.2045^2/2)]/0.2045$$
$$= -0.89597$$

and $d_2 = -0.89597 - 0.2045 = -1.10047$

$N(d_1) = 0.1851$

$N(d_2) = 0.1356$

$$c = 10{,}000{,}000 \times e^{-0.01} \times 0.1851 - 12{,}550{,}000 \times e^{-0.033} \times 0.1356$$
$$= \$187{,}000$$

Price of a put option is

$$p = Ke^{-rT}N(-d_2) - Se^{-qT}N(-d_1)$$

where $d_1 = [\ln(S/K) + (r - q + \sigma^2/2)T] / \sigma\sqrt{T}$

and $d_2 = d_1 - \sigma\sqrt{T}$

Here:

- σ is market implied volatility, 25% – (0.8 – 0.8)/10 = 25%
- S is spot, i.e. \$10m
- K is strike, i.e. \$8m
- r is risk-free rate, 3.3%
- q is dividend income, 1%
- T is term, 1 year

So:

$$d_1 = [\ln(10/8) + (0.033 - 0.01 + 0.25^2/2)]/0.25 = 1.10957$$

and $$d_2 = 1.10957 - 0.25 = 0.85957$$

$N(-d_1) = 0.1336$
$N(-d_2) = 0.1950$

$$p = 8{,}000{,}000 \times e^{-0.033} \times 0.1950$$
$$- 10{,}000{,}000 \times e^{-0.01} \times 0.1336$$
$$= \$187{,}000$$

So value of call = value of put and therefore value of collar is zero as required.

(v) The price of the put will be unchanged …..
……since the volatility is unchanged for A = 80%.

Therefore we need the price change in call Δc to be zero.

To first order:

$$\Delta c = \partial c/\partial K \,.\, \Delta K + \partial c/\partial \sigma \,.\, \Delta\sigma$$

$$\Delta\sigma = \sigma_{new} - \sigma_{old}$$

$$= [25\% - (K_{new}/S - 80\%) \times (0.1 + \varepsilon)] - [25\% - (K_{old}/S - 80\%) \times 0.1]$$

$$= -0.1(K_{new} - K_{old})/S - \varepsilon\,(K_{new}/S - 80\%)$$

$$= -0.1\,\Delta K/S - \varepsilon\,(K_{old}/S - 80\%) \text{ to first order}$$

So for Δc to equal zero,

$$\partial c/\partial K \,.\, \Delta K + \partial c/\partial\sigma \,.\, [-0.1\,\Delta K/S - \varepsilon\,(K_{old}/S - 80\%)] = 0$$

$$\Delta K.[\,\partial c/\partial K - 0.1/S\,\partial c/\partial\sigma] - \varepsilon\,\partial c/\partial\sigma \,.\, (K_{old}/S - 80\%) = 0$$

$\Delta K = \varepsilon\, \partial c/\partial\sigma\, .\, (K_{old}/S - 80\%) / [\partial c/\partial K - 0.1/S\, \partial c/\partial\sigma]$

$= \varepsilon\, \partial c/\partial\sigma\, .\, (K_{old} - 0.8S) / [S\, \partial c/\partial K - 0.1\, \partial c/\partial\sigma]$

$= \varepsilon\, Se^{-qT} N'(d_1) \sqrt{T}\, (K_{old} - 0.8S) / [-Se^{-rT}N(d_2) - 0.1\, Se^{-qT} N'(d_1)\sqrt{T}\,]$

using result in (ii) and formula for vega

$= -\, \varepsilon\, e^{-qT} N'(d_1) \sqrt{T}\, (K_{old} - 0.8S) / [e^{-rT}N(d_2) + 0.1\, e^{-qT} N'(d_1)\sqrt{T}\,]$

Equally acceptable as answers are:

$= -\, \varepsilon\, e^{-0.01} N'(d_1) \times 4.55m / [\, e^{-0.033}N(d_2) + 0.1\, e^{-0.01} N'(d_1)]$

$= -\, \$7.6\varepsilon$ million

Candidates generally scored highly on parts (i) and (ii). In part (iii) candidates were generally aware that this was a question about skew, but many struggled to apply their knowledge to the specifics of this question sufficiently clearly. In part (iv) there were lots of errors in standard Black Scholes formulae as well as in calculations. Some students appeared to spend a lot of time recalculating their answers to correct for minor mistakes, when it may have been more efficient to move on. Candidates found part (v) difficult and only the strongest candidates scored very well in this part.

END OF EXAMINERS' REPORT