INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

April 2014 examinations

Subject ST6 – Finance and Investment
Specialist Technical B

Introduction

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and using past papers as a revision aid and also those who have previously failed the subject.

The Examiners are charged by Council with examining the published syllabus. The Examiners have access to the Core Reading, which is designed to interpret the syllabus, and will generally base questions around it but are not required to examine the content of Core Reading specifically or exclusively.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report; other valid approaches are given appropriate credit. For essay-style questions, particularly the open-ended questions in the later subjects, the report may contain more points than the Examiners will expect from a solution that scores full marks.

The report is written based on the legislative and regulatory context pertaining to the date that the examination was set. Candidates should take into account the possibility that circumstances may have changed if using these reports for revision.

D C Bowie
Chairman of the Board of Examiners

July 2014

General comments on Subject ST6

This subject deals with mathematical techniques for the valuation and risk management of derivatives, together with some aspects of their practical application.

Different numerical answers may be obtained to those shown in these solutions, depending on whether figures were obtained from tables or from calculators. Allowance was made for these minor differences. However, candidates may have been penalised for using excessive rounding or showing insufficient working.

Derivative theory is an interesting but very exacting subject which needs to be tackled in a precise and disciplined manner. Questions that appear unfamiliar can, with some clear thinking at the start of the exam, be found to be straightforward examples of well known techniques. Thorough preparation of the course material is essential, however. This subject also requires quite a varied approach in writing solutions. Algebraic and numerical content needs to be supplemented with well reasoned but brief arguments where the question demands. Candidates should always try to provide several distinct relevant points for the discursive questions, not lengthy paragraphs developing a single idea.

Comments on the April 2014 paper

While student performance was better than in the recent sessions, the pass rate remains low and disappointing. Candidates generally found this paper challenging, but well-prepared candidates scored well above the pass mark. Candidates struggled significantly with questions that required an element of application of Core Reading to situations that were not immediately familiar. For instance, candidates scored well in the first part of question 2 which required them to directly reproduce definitions from core reading. However, candidates scored much less well in the second part where they need to link their understanding of these instruments with the wider risk management knowledge contained within this course. In other words, candidates did not apply their knowledge of core reading effectively, even when they could recall the core definitions.

The comments that follow the questions concentrate on areas where candidates could have improved their marks, as an attempt to help future candidates to revise accordingly and to develop their ability to apply the core reading to related situations. This skill is likely to be valuable both in exams and in using this theory in their professional careers.

1 (i) In theory, a half-yearly coupon bearing bond could be constructed from a LIBOR-bearing deposit and a swap to pay LIBOR in return for receiving a fixed rate.

A government bond with exactly the same set of cashflows will (in normal circumstances) have less credit risk associated with it....

….along with lower operational costs and risks, and no LIBOR generation risk.

It should therefore be more expensive than the hybrid bond (and hence the swap yield curve should lie above the government bond yield curve).

(ii) During a financial crisis it might not be obvious to investors that government bonds remain *significantly* safer than banks. E.g. governments may get into financial difficulty (making them no longer "risk-free") or governments may bail out banks (making banks lower risk).

Swaps may be cheaper to deal in than government bonds for a given level of exposure – this can be attractive to institutions that need to rebalance their fixed interest portfolios regularly.

Swaps require no initial capital outlay, so will be easier to trade in from a cash management perspective.

Swap-based yield curves are being increasingly used for financial reporting. Investing in government bonds rather than in swap-based hybrid assets introduces swap-gilt spread risk to the firm, resulting in more volatile profits.

Other valid points could be made.

(iii) Arbitrageurs could borrow at LIBOR, buy gilts and enter into swaps to receive LIBOR and pay fixed, which in theory would generate a profit under the yield curve inversion.

(iv) The arbitrage opportunity is not a genuine one because:

- Government bonds are not risk-free. Payments could be missed or paid late.

- It is not always possible for banks to borrow at LIBOR.

    - LIBOR is an offer rate rather than a bid rate, so a bank asking to receive floating will receive less than LIBOR.

    - LIBOR is the result of a daily survey among banks rather than a genuine offer to deal.

- There are credit and liquidity risks associated with the swap.

    - If interest rates increase, the "arbitrageur" is exposed to the risk of the swap counterparty defaulting at a point when any deposited collateral does not fully cover the credit exposure.

    - If interest rates fall, the "arbitrageur" may find itself needing to deposit collateral with the swap counterparty; this could present it with cashflow/liquidity problems.

Also:

- During a financial crisis, a bank's attentions may be focussed elsewhere and it may not have the appetite to seek out almost-arbitrage-free profits. Its risk positions could already be close to their internal limits.

- There is the risk that (post crisis) extra regulations or capital requirements could be imposed on banks, introducing extra compliance costs around arbitraging activity.

- Transaction costs will reduce the attractiveness of any apparent arbitrage opportunity.

Candidates struggled with this question, seemingly unable to think very widely about the specific situation and so typically failed to gain many marks. In particular, candidates focussed on liability hedging rather than thinking more broadly about the assets in question, and some seemed unfamiliar with the terminology used in the question.

2 (i) (a) A survivor index is an index used as the underlying in longevity derivatives and bonds, and is proportional to the number of lives still alive at a given time out of a given population. It is typically constructed from population mortality rates.

(b) A longevity bond is a traded security with payments of $kS(t)$ at times $t = 1, 2, ..., T$ where $S(t)$ is a survivor index. k is a constant.

(c) A principal-at-risk longevity bond is a coupon-bearing bond where the principal repayable at maturity can be reduced from par if longevity experience has been worse than some pre-agreed level. Coupon payments could be fixed or floating but would be higher than those for risk-free bonds to reflect the extra risk.

(d) A mortality swap is an OTC contract where one firm pays the floating leg $kS(t)$ at times $t = 1, 2, ..., T$ where $S(t)$ is a survivor index, in return for payments $k\hat{S}(t)$ where $\hat{S}(t)$ is a fixed set of numbers agreed at time $t = 0$ representing a possible scenario for the evolution of $S(t)$. k is a constant.

(e) A survivor cap is an OTC contract that (in return for a payment at time $t = 0$) pays out the greater of $k(S(t) - \hat{S}(t))$ and zero at times $t = 1, 2, ..., T$ where $S(t)$ is a survivor index and $\hat{S}(t)$ is a fixed set of numbers agreed at time $t = 0$ representing a possible scenario for the evolution of $S(t)$. k is a constant.

(ii) Credit risk

Under any of these contracts other than the principal-at-risk bond (for which the insurer is the issuer), the insurer would have credit exposure to the counterparty. On the other hand, reinsuring would leave it with a large credit exposure to the reinsurer. The size of credit exposure to the derivative counterparty will be relatively small, though, for the mortality swap and for the survivor cap compared to the purchase of a bond. Even so, the life insurer should consider credit risk mitigation strategies, for example through the posting of collateral or the setting up of SPVs, as the underlying liability risk is often measured in $billions. This relative size of counterparty exposure is analogous to the relative risk in entering into an interest rate swap compared to purchasing a bond; with a swap it is the extent to which the derivative is in the money that determines the counterparty exposure at a point in time.

Corporate bond investment risk

Under the reinsurance deal, the firm would no longer have to hold a large portfolio of corporate bonds in an attempt to replicate annuity cashflows.

If the firm enters into a mortality swap or buys a longevity cap, then it will still need to hold corporate bonds to match the $k\hat{S}(t)$ payments. So it would still have exposure to corporate bond defaults, downgrades and spread widening. Of course, this risk is borne in exchange for the extra profits that the firm can hope to make from managing a portfolio of corporate bonds.

Basis risk

The longevity derivatives may all have payoffs linked to a survivor index based on population mortality rather than on the firm's own mortality, although longevity swaps are typically transacted on an indemnity basis. The national survivor index can evolve differently to the firm's annuity portfolio as a result of (for example) a different socio-economic makeup or as a result of random fluctuations if the annuity book is a small one. The risk that the index could evolve differently to the firm's annuity portfolio and result in a less than perfect hedge is known as basis risk.

Residual longevity risk

The derivatives considered in this question only have payoffs up to time T, so only hedge annuity payments up to time T (although it is possible to undertake an undated mortality swap, which would remove any residual longevity risk). Even then, the derivatives do not hedge all longevity experience up to time T.

For example medical advances before time T might have little impact on annuity payments up to time T but have a significant impact on annuity payments after time T and on the cost of rolling over the hedge at time T. Further, the exposure to risk at time T will depend on experience up to time T.

Candidates scored well on the book work in part (i), although a surprising number did not identify that there would be a fixed leg in the mortality swap. Many candidates were unable to apply their understanding of these hedging instruments to part (ii).

3 (i) $W_0 = 0$.

If $0 \le s < t$, then $W_t - W_s \sim N(0, t-s)$.

Future changes are independent of past and present: if $0 \le r \le s < t$, then the random variables $(W_t - W_s)$ and W_r are independent.

The paths $t \to W_t$ are continuous with probability 1.

(ii) (a) By definition $W_t \sim N(0,t)$. Using the expectation formula for a function of a variable and the expectation of W_t as an integral (from the Formula book):

$$\mathbb{E}_{\mathbb{P}}[W_t] = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} x e^{-x^2/2t}\, dx$$

$$\mathbb{E}_{\mathbb{P}}[|W_t|] = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} |x| e^{-x^2/2t}\, dx$$

(expectation for a function of a variable),

$$= \frac{2}{\sqrt{2\pi t}} \int_{0}^{\infty} x e^{-x^2/2t} dx$$

(splitting the integral and using the definition of $|x|$),

$$= 4\sqrt{\frac{t}{2\pi}} \int_{0}^{\infty} y e^{-y^2}\, dy$$ (using the change of variable $y = x/\sqrt{2t}$),

$$= 4\sqrt{\frac{t}{2\pi}} \left[-0.5 \times e^{-y^2} \right]_0^{\infty}$$ (evaluating the integral),

$$= \sqrt{\frac{2t}{\pi}}\ .$$

(b) $Var_{\mathbb{P}}\left[|W_t|\right] = \mathbb{E}_{\mathbb{P}}\left[|W_t|^2\right] - \mathbb{E}_{\mathbb{P}}\left[|W_t|\right]^2$ (by the definition of a variance),

$= \mathbb{E}_{\mathbb{P}}\left[W_t^2\right] - \mathbb{E}_{\mathbb{P}}\left[|W_t|\right]^2$ (as $|W_t|^2 = W_t^2$),

$= t - 2t/\pi$ (using $\mathbb{E}_{\mathbb{P}}\left[W_t^2\right] = Var_{\mathbb{P}}\left[W_t\right] + \mathbb{E}_{\mathbb{P}}\left[W_t\right]^2 = t + 0$),

$= t \times (1 - 2/\pi)$.

(iii) $Var_{\mathbb{P}}\left[aW_s + bW_t\right] = Var_{\mathbb{P}}\left[aW_s + b(W_s + W_t - W_s)\right]$,

$= Var_{\mathbb{P}}\left[W_s \times (a+b) + b(W_t - W_s)\right]$,

$= Var_{\mathbb{P}}\left[W_s \times (a+b)\right] + Var_{\mathbb{P}}\left[b(W_t - W_s)\right]$ (by independence)

$= (a+b)^2 s + b^2 (t-s)$.

(iv) Let $X = W_s$, let $Y = W_t - W_s$, and let $Z = W_u - W_t$. By the independence increment property of Brownian motion, X, Y and Z are independent random variables with mean 0 and $\mathbb{E}_{\mathbb{P}}\left[X^2\right] = s$, $\mathbb{E}_{\mathbb{P}}\left[Y^2\right] = t - s$ and $\mathbb{E}_{\mathbb{P}}\left[Z^2\right] = u - t$, (so all expectations in the below exist and are finite).

$\mathbb{E}_{\mathbb{P}}\left[W_s \times W_t \times W_u\right]$, $= \mathbb{E}_{\mathbb{P}}\left[X \times (X+Y) \times (X+Y+Z)\right]$

$= \mathbb{E}_{\mathbb{P}}\left[X^3 + 2X^2Y + XY^2 + X^2Z + XYZ\right]$,

$$\begin{aligned} &= \mathbb{E}_{\mathbb{P}}\left[X^3\right] + 2\mathbb{E}_{\mathbb{P}}\left[X^2\right]\mathbb{E}_{\mathbb{P}}\left[Y\right] \\ &\quad + \mathbb{E}_{\mathbb{P}}\left[X\right]\mathbb{E}_{\mathbb{P}}\left[Y^2\right] + \mathbb{E}_{\mathbb{P}}\left[X^2\right]\mathbb{E}_{\mathbb{P}}\left[Z\right] \\ &\quad + \mathbb{E}_{\mathbb{P}}\left[X\right]\mathbb{E}_{\mathbb{P}}\left[Y\right]\mathbb{E}_{\mathbb{P}}\left[Z\right] \end{aligned}$$

(by the linear properties of expectation and the independence above).

= 0 (due to the means of the variables above and the fact stated in the question).

This was one of the better answered questions in the paper, with most candidates managing to make substantial efforts at answering the question. Some candidates became bogged down in the algebra in the latter stages.

4 (i) Consider a Bermudan option on a non-dividend paying underlying asset, with expiry time *T*, and price S_t at time *t*. For simplicity assume that the possible exercise dates are single instances in time at $\{t_1, t_2, \ldots, t_n\}$ and there is a constant volatility and risk-free rate *r*. Let the payoff function for the Bermudan option be *f*(.).

Build a binomial tree of the underlying asset prices with a suitable number of steps, say N (and for simplicity assume that they are of equal length). Let $S_{t,i}$ be the asset price at the node *i* at time *t* in the binomial tree.
To model these underlying asset prices, a random process model is required which generates "up" and "down" price movements for the underlying asset at each node with given probabilities.

In order to price the option in an arbitrage free environment, the probability measure used to value the option should be risk-neutral.

A suitable number of steps should be chosen to at least coincide with the possible early exercise dates.

Let the value of the Bermudan option on the binomial tree at node *i*, time t be $V_t(S_{t,i})$. In a similar manner to pricing an equivalent European option, the terminal condition is used:

$$V_N(S_{N,i}) = f\left(S_{N,i}\right) \text{for}\, i \in \{1, 2, \ldots, N+1\}.$$

The procedure is then to work by backward induction from *N* to 0 to give a value $V_0(S_0)$. For an equivalent European option, this is modelled in the binomial tree as:

$$V_{n-1}(S_{n-1,i}) = e^{-r(T/N)}\mathbb{E}[V_n\left(S_n\right) \mid S_{n-1,i}],$$

where the expectation is with respect to the risk-neutral measure.

In order to price the Bermudan option a check needs to be made at each time period which is an exercise date to check if early exercise is preferable, i.e. the payoff is greater than the calculated value. If early exercise is preferable then the payoff $f\left(S_{n-1,i}\right)$ should be entered in the binomial tree at this node.

A crude way to create an approximate upper bound on the value of a Bermudan option with regular exercise dates is to value it as an American option with each time step being an exercise date.

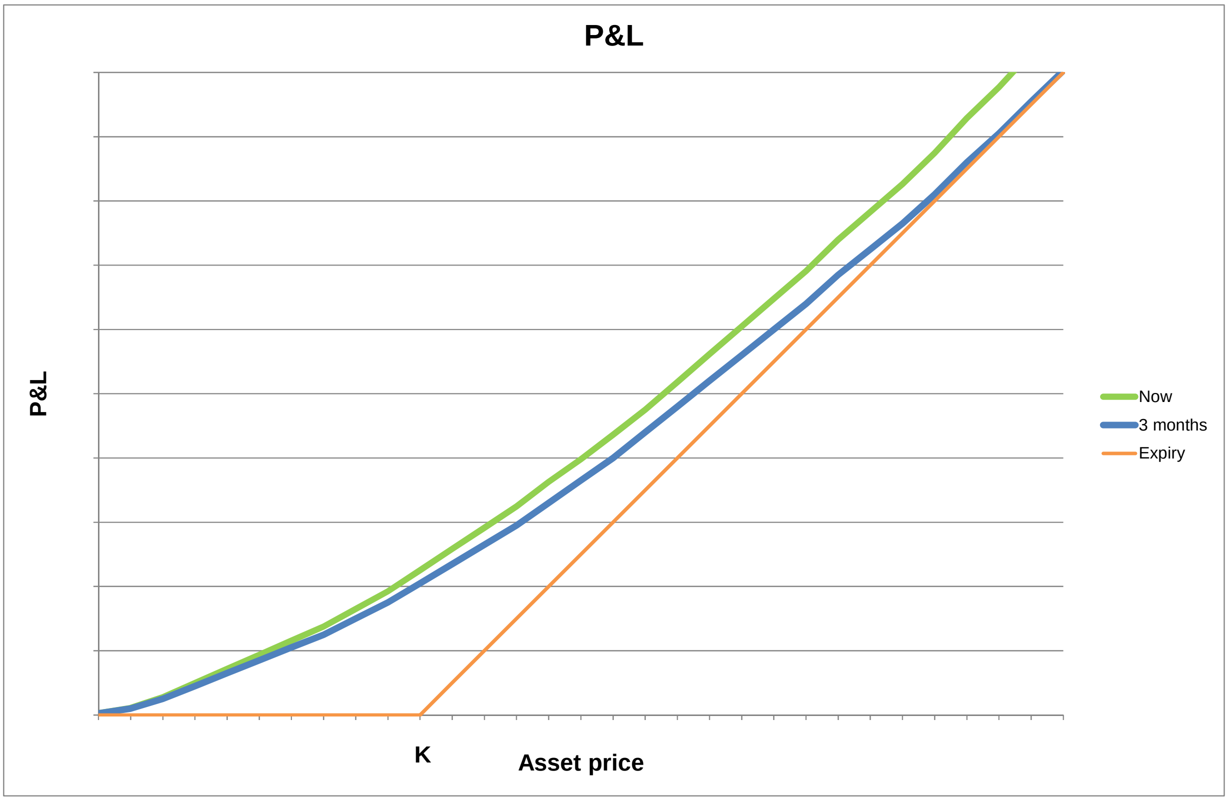
Furthermore, since the underlying stock is non-dividend paying, if it is a call option being valued then early exercise will not be optimal, so the Bermudan option is equivalent to a European option.

(ii) Any transaction costs have been ignored.

At the start of the option, before a possible exercise date, the graphs for the options are similar to those of a European option. Around the possible exercise dates the graphs are similar to those of an American option

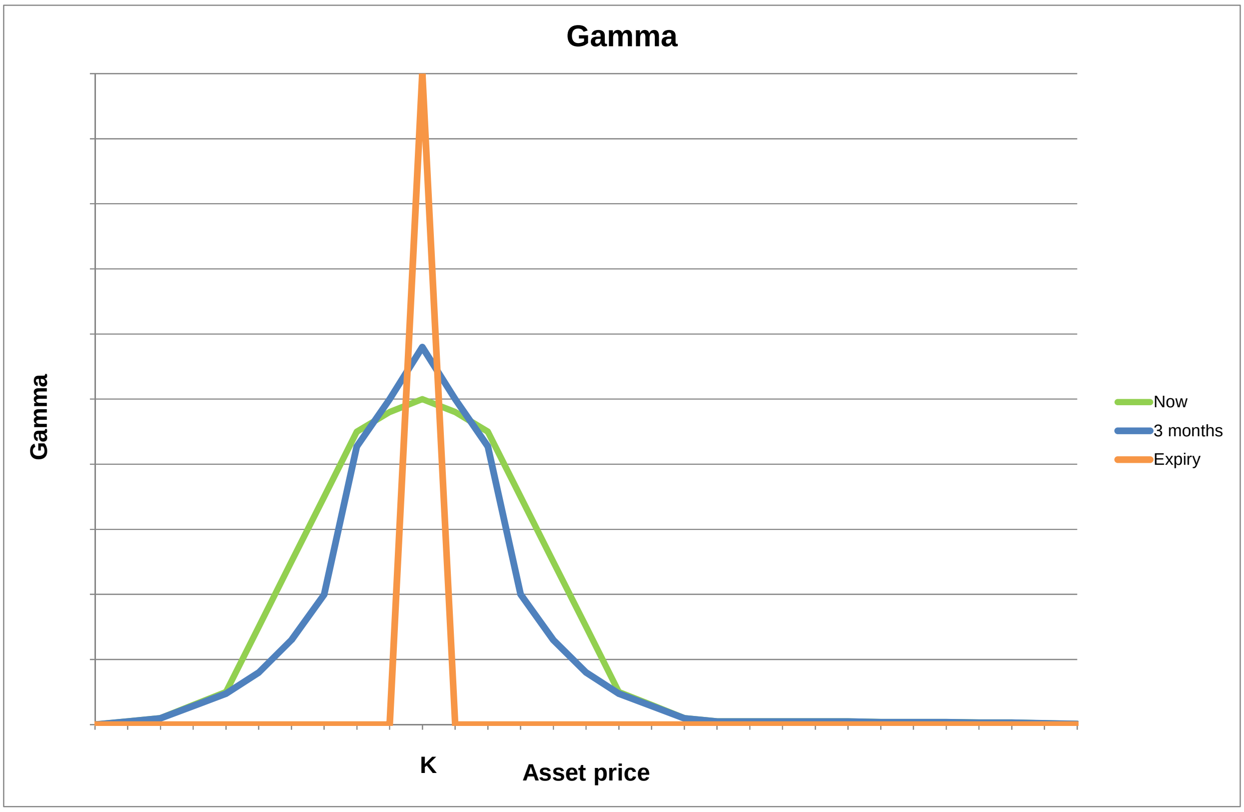
When the option is heavily in the money the "now" graph becomes asymptotically straight lines. For the "3 month" graph this approaches a straight line quicker as the option can be exercised at this time and there is less time for the option to move out of the money.

When the option is heavily out of the money there is very little profit.



The delta is the derivative of the profit and loss graph.

Similarly, the gamma is the derivative of the delta graph.



This was the best answered question by candidates, with only minor points of detail about the Bermudan option omitted (such as equivalence to a European option and choosing sufficient nodes to coincide with exercise dates). Candidates would have scored more highly had they clearly labelled graphs in part (ii).

5 (i) **Foreign exchange forward contracts** are transactions in which two parties agree to exchange a specified amount of different currencies at some future date, with the exchange rate being set at the time the contract is entered into.

(ii) Consider the following strategy at time $t = 0$:

Portfolio A: The purchase of one forward contract (i.e. the purchase of a forward to buy 1 Euro at a given dollar rate) plus Ke^{-rT} units of dollar cash; and

Portfolio B: e^{-vT} units of Euro cash.

At time T the portfolios are worth (assuming markets are frictionless):

Portfolio A: The dollar cash rolls up with interest to K dollars, and this can be used to settle the forward contract, resulting in 1 Euro.

Portfolio B: 1 Euro.

Let $F(t)$ represent the value of the forward contract at time *t*.

As both portfolios are equal at time *T*, then by the no arbitrage principle their values must be equal at any time $t < T$ (working in dollars):

$$F(t) + Ke^{-r(T-t)} = C_t^{-1}e^{-v(T-t)}.$$

At time $t = 0$, $F(0) = 0$ as it is a forward contract.

Therefore, $K = C_0^{-1}e^{(r-v)T}$.

(iii) Suppose the forward price is greater than 10*x*, e.g. 11*x* for 1 US dollar. There is an arbitrage opportunity by going long on these forwards.

At maturity the *x* can be bought via the forward at a cost of (approximately) 0.09 US dollars for 1*x*. These can then be sold in the spot market at the fixed rate of 0.1 US dollars for 1*x*, resulting in a fixed profit of (approximately) 0.01 US dollars per forward contract.

Similarly, suppose the forward price is less than 10*x*, e.g. 9*x* to the US dollar, there is an arbitrage opportunity by shorting these forwards.

At maturity the *x* can be bought in the spot market at 0.1 US dollars for 1*x* and delivered under the forward contract for an income of (approximately) 0.11 US dollars for 1*x*. This results in a fixed profit of (approximately) 0.01 US dollars per forward contract.

Hence the one month forward price of x must be 0.10 US dollars, under the principle of no arbitrage.

(iv) From the formula derived in the part (ii) of the question, this implies that the interest rates in the two countries are equal (as would be expected for a fully pegged currency).

(v) Let $C_t = C_0 \exp(\sigma W_t + \mu t)$ be the geometric Brownian motion with constants μ and σ, C_0 as defined in the question and W_t is $\mathbb{P}$-Brownian motion.

Let Z_t be the Euro-discounted value of the dollar bond:

$$\begin{aligned} Z_t &= E_t^{-1}(C_t D_t), \\ &= C_0 \exp(\sigma W_t + (\mu + r - v)t). \end{aligned}$$

(vi) Let $\gamma_t = \sigma^{-1}\left(\mu + r - v + 0.5\sigma^2\right)$.

This trivially satisfies the boundedness condition of the CMG theorem.

Then by the CMG theorem there exists a measure $\mathbb{Q}$ which is equivalent to $\mathbb{P}$, such that

$$\hat{W}_t = W_t + \int_0^t \gamma_s ds,$$

$$= W_t + \sigma^{-1}\left(\mu + r - v + 0.5\sigma^2\right)t, \text{ is } \mathbb{Q}\text{-Brownian motion.}$$

Then under $\mathbb{Q}$,

$$Z_t = C_0 \exp\left(\sigma \hat{W}_t - 0.5\sigma^2 t\right)$$

Using Ito's lemma this satisfies: $dZ_t = Z_t \sigma d\hat{W}_t$, which has zero drift so is a martingale.

(vii) Since the claim X is being discounted at Euro interest rates in the given expression, the payoff needs to be expressed in Euros (rather than dollars). One way in which to do this is to consider the opposite exchange rate forward contract, i.e. a forward contract which agrees the cost in Euros of one dollar. This must have forward price 1/K.

The payoff from this forward is therefore: $X_T = C_T - K^{-1}$.

At time *t* (using the formula in the question):

$$V_t = E_t \mathbb{E}_Q\left[E_T^{-1} X \mid \mathcal{F}_t\right],$$

$$= e^{vt}\mathbb{E}_Q\left[e^{-vT}\left(C_T - K^{-1}\right) \mid \mathcal{F}_t\right],$$

$$= \mathbb{E}_Q\left[e^{-v(T-t)}(C_T - K^{-1}) \mid \mathcal{F}_t\right],$$

$$= \mathbb{E}_Q[e^{-v(T-t)}C_T|\mathcal{F}_t] - K^{-1}e^{-v(T-t)}.$$

From part (v) we have $C_t = Z_t e^{(v-r)t}$, so therefore:

$$\mathbb{E}_Q[e^{-v(T-t)}C_T|\mathcal{F}_t] = \mathbb{E}_Q[e^{-v(T-t)}Z_T e^{(v-r)T}|\mathcal{F}_t],$$

$$= \mathbb{E}_Q[e^{-v(T-t)}(Z_T / Z_t).Z_t e^{(v-r)(T-t)}e^{(v-r)t}|\mathcal{F}_t],$$

$$= e^{-r(T-t)}C_t \mathbb{E}_Q[(Z_T / Z_t)|\mathcal{F}_t],$$

$= e^{-r(T-t)}C_t$, using the martingale result from part (vi).

Therefore $V_t = e^{-r(T-t)}C_t - K^{-1}e^{-v(T-t)}$.

At time $t = 0$, the price of the forward should be 0 which implies that $V_0 = 0$.

And thus

$K = C_0^{-1}e^{(r-v)T}$, as found in part (ii).

This was one of the more challenging questions on the paper as it required application of knowledge, but those students who had prepared well were able to apply the relevant techniques as covered in the core reading. However, many candidates appeared to be reproducing proofs from memory without understanding which led them to get their notation confused. Many did not progress to the latter stages and those that did struggled to apply their knowledge, e.g. Ito's lemma, correctly.

6 (i) (a) Binary options are options (both calls and puts) with discontinuous payoffs which pay a specified payoff or nothing.

(b) A cash-or-nothing binary option pays out a fixed amount …

… only if the option is in the money at expiry.

They are generally structured as European options.

(ii) Assumptions:

- Both options are European.
- The underlying market and derivative is frictionless.
- The underlying asset is liquid.
- The strike price (S) and expiry date (T) (current time t) are identical for both the put and call options.
- There exists a (constant) risk-free continuously compounded rate r.

Consider portfolio A consisting of a cash amount of $e^{-r(T-t)}$.

Consider portfolio B consisting of one binary call option (C^B) and one binary call put (P^B) (this portfolio yields 1 regardless of the strike price).

Both of these are therefore riskless portfolios and by the no arbitrage argument they increase at the risk-free rate and have the same value.

Equating the present value of these portfolios thus leads to the put-call parity equation:

$$C^B + P^B = e^{-r(T-t)}.$$

(iii) (a) In order to construct a cash-or-nothing binary call with strike price K and fixed payoff 1, let $CB(K)$ be the price of this binary option. Consider two plain vanilla European call options with the same time to expiry but the only difference being the strike prices. Let $C(x)$ be the price of a call with strike price x. Let the underlying asset have value S_t at time t.

Consider a bull call spread of plain vanilla European options: going long on an option with strike price K and going short on an option with strike price $(K + \varepsilon)$.

This has a payoff structure of:

0 for $S_T <= K$;
$S_T - K$ for $K < S_T < K + \varepsilon$; and
ε for $K + \varepsilon <= S_T$.

Therefore, buying $(1/\varepsilon)$ units of this portfolio will produce a maximum payoff of 1 and this is approximately the cash-or-nothing binary call.

Hence

$$CB(K) \approx (1/\varepsilon)\,(C(K) - C(K + \varepsilon))$$

(b) In the limit $CB(K) = \lim_{\varepsilon \to 0} (1/\varepsilon)\big(C(K) - C(K+\varepsilon)\big) = -\dfrac{dC(K)}{dK}$.

(c) This is unlikely to be realised in practice as the markets have limited strike intervals (not infinitesimal).

The approximation is likely to be less accurate near expiration when small moves in the underlying asset price can have very large effects on the value of the options.

This reflects the fact that the absolute value of the delta can be large close to maturity.

Additionally, the binary option delta may exhibit violent changes as the underlying price changes when the option is close to maturity (that is, the binary option is a high gamma instrument).

As $\varepsilon \to 0$ ever increasing volumes of options would need to be bought, which will not be possible in the market.

(iv) Advantages:

- Simple to use compared to many other financial products.
- Fixed margin requirements.
- A fixed level of risk and rewards.
- They usually have shorter terms compared to other investments (some are as low as seconds).
- Access to multiple asset classes in global markets.
- Customisable investment amounts.
- They can be easily used to provide protection (i.e. reduce losses) as part of a trading strategy

Disadvantages:

- Large transaction costs.
- Large spreads.
- Short-term market outlook and susceptible to random market movements.
- Non-ownership of any asset.
- It is difficult to make trading adjustments due to low (or zero) liquidity in these markets.
- Potentially large cumulative losses, for repeated trades.

(v) Let the underlying asset have value S_t at time t and T be the time at expiry of the options.

An asset-or-nothing binary option pays out an amount equal to the price of the underlying asset if the option is in the money at expiry. Let A be equal to the payoff from going long in an asset-or-nothing binary call option with strike price K and expiry time T. Then $A = S_T$ if $S_T > K$, and 0 otherwise.

Let B be equal to the payoff from going short in a cash-or-nothing binary call option with fixed payout K. Then $B = -K$ if $S_T > K$ and 0 otherwise.

Therefore $A + B = S_T - K$ if $S_T > K$ and 0 otherwise, which is the payoff from going long in a European call option with strike price K and expiry time T.

Going long in a European call option can be considered equivalent to going long in an asset-or-nothing binary call option and going short in a cash-or-nothing binary call option.

A significant number of candidates seemed to believe that this question was set in a Black-Scholes world, and so spent time covering areas that were not relevant for the mark scheme. Many candidates also appeared to broadly appreciate the essence of the question but missed out on available marks due to lack of clarity in reasoning or giving a narrow range of reasons to part (iv).

7 (i) **Basic formula**

The Black-Scholes formula for the value of a put option is

$$p = Xe^{-rT}N(-d_2) - SN(-d_1)$$

Where $d_1 = [\ln(S/X)+(r+\sigma^2/2)T]/\sigma\sqrt{T}$

And $d_2 = d_1 - \sigma\sqrt{T}$

And

- X is strike / exercise price
- S is spot price, i.e. price of the basket today
- r is one-year continuously compounded risk-free rate
- T is term of option = 1 year
- σ is implied volatility

Allow for dividends

The Black-Scholes formula above assumes that the underlying asset produces no income. In reality the equities will produce dividends, so an adjustment to the formula is necessary to allow for this.

Possible methodologies are:

- To assume dividends are paid continuously at rate q and to replace S with Se^{-qt} to get to $p = Xe^{-rT}N(-d_2) - Se^{-qT}N(-d_1)$ where $d_1 = [\ln(S/X) + (r - q + \sigma^2/2)T] / \sigma\sqrt{T}$ and $d_2 = d_1 - \sigma\sqrt{T}$.

- But better (given that there are only two equities) would be to allow for dividends explicitly by replacing S with $S - \Sigma \mathrm{div}_t \exp(-r_t t)$.

Risk-free rate

r would be derived from a yield curve that reproduces swap rates.

Implied volatility

Implied volatilities of the two equities (σ_1, σ_2) can be derived separately by looking at prices of one-year (put or call) options on the two equities and backsolving.

Historical volatility for each of the equities could be used to help set σ_1 and σ_2.

Looking at trends in historic volatility and considering future events can be used to convert the historic measure into a forward looking parameter for pricing.

The bank needs a methodology for combining σ_1 and σ_2 into a basket volatility σ.

A common approach would be to use the correlation between the equities ρ. Given individual volatilities σ_1 and σ_2 and spot prices S_1 and S_2 (excluding dividends, so either Se^{-qt} or $S - \Sigma \mathrm{div}_t \exp(-r_t t)$) then the basket volatility σ can be determined by solving for the value σ which equates the combined lognormal variances of the underlying equities (allowing for their correlation) with an assumed lognormal variance of the basket.

Or the bank might decide to use a simpler approximation such as

$$(S_1 + S_2)\sigma^2 = S_1\sigma_1^2 + S_2\sigma_2^2 + 2\rho S_1 S_2 \sigma_1 \sigma_2$$

Correlation

The bank also needs to set the correlation parameter ρ.

Correlation will vary over time and over varying market conditions (e.g. can be higher in market crashes).

If there are no traded options on the basket, as is likely, then a market implied correlation will not exist.

This leaves historic experience as the main driver of a correlation assumption.

However, because it is unhedgeable, and because price changes of the two equities in the basket will tend to increase in size and codependency in market crises where the put option is likely to be heavily in the money, the bank will probably include a safety margin.

The use of a simplified formula for basket correlation might be another reason to include an extra margin within the value chosen for ρ.

Margins

Finally, the bank will include margins in the price for:

- profit.
- expenses.
- (possibly) capital costs or hedging costs.

(ii) (a) Two Deltas = $\partial p/\partial S_1$ and $\partial p/\partial S_2$

Two Gammas = $\partial^2 p/\partial S_1^2$ and $\partial^2 p/\partial S_2^2$

Two Vegas = $\partial p/\partial \sigma_1$ and $\partial p/\partial \sigma_2$

Rho = $\partial p/\partial r$

Cross Gamma = $\partial^2 p/\partial S_1 \partial S_2$

(b) Possible hedging instruments include:

- futures/forwards on the individual equities in the basket.
- put options on the individual equities in the basket.
- or even calls on the individual equities in the basket.
- options will ideally be one-year options with similar strikes (expressed as % of spot) to the option in question.
- one year zero-coupon bonds
- or (better) swaps
- volatility index futures
- underlying assets

(c)

	Futures	*Put option (or signs reversed for shorted put)*	*Call option (or signs reversed for shorted call)*	*ZCB or receiver swap*
Delta	+	–	+	0
Gamma	0	+	+	0
Vega	0	+	+	0
Rho	+	–	+	–

(iii) Changes to the implied correlation between the two equities will be difficult to hedge. Even if another bank is prepared to create a hedging instrument, it will require compensation for writing an unhedgeable risk of its own.

The first step (because options will change the rho and delta of the hedged position) will be to hedge out the gammas and vegas using options.

The bank will need to buy four options to do this. Because a single option is unlikely to hedge out both gamma and vega at the same time, the bank will need two options on each of the shares in the basket.

Next the deltas need to be hedged (as futures will have an impact on the rho of the hedged portfolio).

The bank will calculate the deltas of the shorted OTC put and of the long positions in options (that are hedging out the gammas and vegas). It will calculate how many futures to enter into (which could be short or long positions) to cancel out these deltas.

Finally the bank will hedge rho.

Because it has a large derivative position already on its books, it probably already hedges against yield curve movements. So rather than hedge this deal separately, the rho exposure within the deal (and the hedges described) will be incorporated into its overall yield curve hedging processes.

How this is likely to work is that the bank will already have systems to:

- convert market swap rates to a zero coupon yield curve.
- value its balance sheet based on the yield curve.

With this system in place, the bank can test the sensitivity of its balance sheet to changes in individual swap rates and use this to determine any extra swaps it needs to enter into to hedge out its exposure to movements in the yield curve.

Candidates appeared largely unable to apply their general understanding of the Greeks and hedging principles to the specific scenario set out in the question. Repeating bookwork to a question which expects application of that bookwork will not score highly.

END OF EXAMINERS' REPORT