# INSTITUTE AND FACULTY OF ACTUARIES

# EXAMINERS' REPORT

April 2012 examinations

## Subject ST6 – Finance and Investment
## Specialist Technical B

**Purpose of Examiners' Reports**

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and who are using past papers as a revision aid, and also those who have previously failed the subject. The Examiners are charged by Council with examining the published syllabus. Although Examiners have access to the Core Reading, which is designed to interpret the syllabus, the Examiners are not required to examine the content of Core Reading. Notwithstanding that, the questions set, and the following comments, will generally be based on Core Reading.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report. Other valid approaches are always given appropriate credit; where there is a commonly used alternative approach, this is also noted in the report. For essay-style questions, and particularly the open-ended questions in the later subjects, this report contains all the points for which the Examiners awarded marks. This is much more than a model solution – it would be impossible to write down all the points in the report in the time allowed for the question.

T J Birse
Chairman of the Board of Examiners

July 2012

## General comments on Subject ST6

This subject deals with mathematical techniques for the valuation and risk management of derivatives, together with some aspects of their practical application.

Different numerical answers may be obtained to those shown in these solutions, depending on whether figures were obtained from tables or from calculators. Allowance was made for these minor differences. However, candidates may have been penalised for using excessive rounding or showing insufficient working.

## Comments on the April 2012 paper

The entry for this paper was smaller than in 2011, and the overall standard was slightly lower than seen in the recent past, with a reduced percentage of passes. Given that the standard of the paper was very similar to September 2011, without any especially hard questions, this outcome is a little surprising.

Derivative theory is very exacting subject which needs to be tackled in a precise and clear manner, but is also very interesting and rewarding. Questions can appear daunting, but with good preparation most will be seen to be straightforward examples of familiar techniques. Those who take ST6 for the algebraic and numerical content also need to be able to produced well reasoned brief arguments where the question demands. For the discursive questions, candidates should always try to think clearly before writing, and provide several distinct relevant points in short paragraphs, not one or two points made at length. This approach not only helps attract more marks, but also makes it quicker to write.

A number of questions picked up themes from previous papers, such as Question 1, whilst others focused on applying well established principles to topical themes, such as Question 6. Two questions stood out as discriminators of the better candidates. Question 4 looked at the use of risk and finance control techniques for derivatives, and applied these to a particular management problem. Question 7 provided an unusual approach to option pricing, essentially asking the candidate to solve for the strike price given a particular valuation, but in an unfamiliar context. Neither were well tackled overall, with several candidates skipping them completely, possibly partly in response to time pressures.

As mentioned in the September 2011 report, pacing time spent to the marks on offer is also an important examination skill. This seemed to have been better actioned in this session, with a much higher percentage of candidates achieving an FB grade than in previous sessions. Whilst this might not perhaps seem encouraging at first glance, in fact the improvement required to turn an FB into a Pass is not so great, and should be achievable with disciplined study and careful application.

The solutions below have been partly written to aid future candidates. Hence, as well as outlining a correct answer, they also often add an explanation relating to the course material from a practical perspective. As such, a study of these solutions is always beneficial to candidates preparing for ST6.

**QUESTION 1**

***Syllabus section: (h)(i)-(iii)***

***Core reading: 8, 9***

(i)

(a)

**P** and **Q** are equivalent if and only if every positive probability outcome for **P** has an equivalent positive probability outcome under **Q**. This is clearly the case because each distribution has positive probabilities over the same paths.

[*This may equally validly be expressed as* $\mathbf{P}(A)=0 \Leftrightarrow \mathbf{Q}(A)=0$ *for all paths A.*]

(b)

Starting at $t = 2$, take each branch of the tree and discount under **P** and **Q**.

Top branch: under **P**, the discounted value = 144 × 0.5 + 84 × 0.5 = 114.

Top branch: under **Q**, the discounted value = 144 × 0.6 + 84 × 0.4 = 120.

The value of the tree at $t = 1$ for that branch is 120, so **Q** gives that value not **P**.

Similarly, for the lower branch, under **P** the value is 56, under **Q** it is 60 – same result.

Finally, for the initial branch, under **P** the value is 90, under **Q** it is 100 – same result.

Hence **Q** is a martingale measure since $\mathrm{E}_{\mathbf{Q}}(S_j \mid F_i) = S_i$ for all $i < j$, but this does not apply under **P**, so **P** is not a martingale. ($F_i$ is the filtration, i.e. history up to time $i$.)

[*Note: Only one example is required of* **P** *not satisfying the martingale expectation.*]

(ii)

(a)

For the first time step, $\left.\dfrac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\right|_{t=1}^{\text{up}} = \dfrac{\frac{2}{3}}{\frac{1}{2}} = \frac{4}{3}$ and $\left.\dfrac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\right|_{t=1}^{\text{down}} = \dfrac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$

For the second time step, $\left.\dfrac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\right|_{t=2}^{\text{up,up}} = \dfrac{\frac{3}{5}}{\frac{1}{2}} \left.\dfrac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\right|_{t=1}^{\text{up}} = \frac{6}{5}\frac{4}{3} = \frac{8}{5}$, $\left.\dfrac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\right|_{t=2}^{\text{up,down}} = \dfrac{\frac{2}{5}}{\frac{1}{2}} \left.\dfrac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\right|_{t=1}^{\text{up}} = \frac{4}{5}\frac{4}{3} = \frac{16}{15}$,

$$\left.\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\right|_{t=2}^{\text{down,up}} = \frac{\frac{4}{7}}{\frac{1}{2}} \left.\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\right|_{t=1}^{\text{down}} = \frac{8}{7}\frac{2}{3} = \frac{16}{21} \text{ and } \left.\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\right|_{t=2}^{\text{down,down}} = \frac{\frac{3}{7}}{\frac{1}{2}} \left.\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\right|_{t=1}^{\text{down}} = \frac{6}{7}\frac{2}{3} = \frac{4}{7}.$$

In decimal terms, the four values are: 1.6000, 1.0667, 0.7619, 0.5714.

[*Note: The central node at t = 2 has two separate R-N derivatives, for the paths "up then down" and "down then up", which is the correct approach. The question, however, unhelpfully referred to calculating the R-N derivative at each <u>node</u> rather than along contributory <u>paths</u>. Hence the relevant marks above were given to those who combined the probabilities in the middle node to obtain a single value of* $\frac{\frac{2}{3}\frac{2}{5}+\frac{1}{3}\frac{4}{7}}{\frac{1}{2}\frac{1}{2}+\frac{1}{2}\frac{1}{2}}=\frac{32}{35}$. *This was also allowed to carry over to part (iii), where a correct answer can be achieved.*]

(b)

For the up node at $t = 1$, $\zeta_t = \mathrm{E}_{\mathbf{P}}\left[\frac{d\mathbf{Q}}{d\mathbf{P}}\,|F_t\right]=\frac{1}{2}.\frac{8}{5}+\frac{1}{2}.\frac{16}{15}=\frac{20}{15}=\frac{4}{3}$.

For the down node at $t = 1$, $\zeta_t = \mathrm{E}_{\mathbf{P}}\left[\frac{d\mathbf{Q}}{d\mathbf{P}}\,|F_t\right]=\frac{1}{2}.\frac{16}{21}+\frac{1}{2}.\frac{12}{21}=\frac{14}{21}=\frac{2}{3}$.

[*Note: these two values are also equal to* $\left.\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\right|_{t=1}^{\text{up}}$ *and* $\left.\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\right|_{t=1}^{\text{down}}$ *respectively.* $\zeta_t$ *is the history of the change of measure over time, which is a process itself. The approach of using a single R-N value for the recombining node does not work for this part.*]

(iii)

First consider the LHS. Evaluate the binomial tree under measure **Q** at $t = 0$,

$$\mathrm{E}_{\mathbf{Q}}[X]=\tfrac{2}{3}.(\tfrac{3}{5}.20+\tfrac{2}{5}.20)+\tfrac{1}{3}.(\tfrac{4}{7}.20+\tfrac{3}{7}.0)=\tfrac{2}{3}.20+\tfrac{4}{21}.20=\tfrac{6}{7}.20\ (=17.1429)$$

Now consider the RHS. Evaluate under measure **P** for the up path at $t = 1$,

$$\mathrm{E}_{\mathbf{P}}\left[\frac{d\mathbf{Q}}{d\mathbf{P}}X\right]=\tfrac{1}{2}\left.\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\right|_{t=2}^{\text{up,up}}.20+\tfrac{1}{2}\left.\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\right|_{t=2}^{\text{up,down}}.20=\tfrac{1}{2}.\tfrac{8}{5}.20+\tfrac{1}{2}.\tfrac{16}{15}.20=\tfrac{4}{3}.20\ (=26.6667)$$

and for the down path at $t = 1$,

$$\mathrm{E}_{\mathbf{P}}\left[\frac{d\mathbf{Q}}{d\mathbf{P}}X\right]=\tfrac{1}{2}\left.\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\right|_{t=2}^{\text{down,up}}.20+\tfrac{1}{2}\left.\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\right|_{t=2}^{\text{down,down}}.0=\tfrac{1}{2}.\tfrac{16}{21}.20=\tfrac{160}{21}\ (=7.6190).$$

Hence at $t = 0$, $\mathrm{E}_{\mathbf{P}}\left[\frac{d\mathbf{Q}}{d\mathbf{P}}X\right]=\frac{1}{2}.\frac{4}{3}.20+\frac{1}{2}.\frac{8}{21}.20=\frac{6}{7}.20=\frac{120}{7}(=\ 17.1429)$

*This question, looking at a numerical working of an equivalent martingale measure in discrete time, mirrored similar examples given in a past ST6 paper and in the textbook (Baxter & Rennie). Part (i) was well understood and answered correctly by almost every candidate.*

*Part (ii)(a) proved more elusive. Whilst many found the correct answers, several candidates did not evaluate the Radon-Nikodym (R-N) derivative for the whole path up to the node in question, only the most recent path from t = 1 to t = 2. This gave incorrect answers. Any mistakes in part (ii)(b) carried over to part (ii)(b).*

*Others tried to calculate a single R-N derivative for the combining central node at time t = 2, which is not correct and also led to problems in part (ii)(b). The question would have been better expressed if it had used the word "path" rather than "node", to emphasise that the R-K derivatives work on probabilities not process values. Allowance was made in the marking. Part (iii) asked the candidate to confirm that two discounted values of claim X were identical. Several candidates only evaluated the easier LHS, ignoring the harder RHS.*

**QUESTION 2**

***Syllabus section: (g) & (i)***
***Core reading: 7, 21***

(i)

The delta of a long position in one contract of the 1-year future is 10 exp[$(r-q).1$]

$= 10\ e^{0.04-0.02} = 10.202$

[*Note that the time horizon for the future is one year, not five years.*]

The delta of the (sold) 5-year Call option is $-1000\ N(d_1)\ e^{-qT}$

$$d_1 = \frac{\ln(S/K) + (r - q + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$= [\ln(1500 / 2000) + (0.04 - 0.02 + ½(0.25)^2).5]/(0.25\sqrt{5}) = -0.05623$

So delta of option is $-1000 \times N(-0.05623) \times 0.90484 = -432.1$

The bank should buy futures to hedge a short position.

Futures to buy = 432.1 / 10.202 = 42.4, say 42 contracts.

(ii)

(a)

If call price $c$ is a function of non-constant σ and other variables, then for a small change Δσ:

$$c(\sigma + \Delta\sigma) = c(\sigma) + \Delta\sigma.\frac{\partial c}{\partial \sigma} + o((\Delta\sigma)^2)$$

So, if $S$ is the stock price and $\sigma = aS + b$, then

$$\text{modified delta} = \frac{\partial c(\sigma+\Delta\sigma)}{\partial S} = \left.\frac{\partial c(\sigma)}{\partial S}\right|_{\sigma \text{ fixed}} + \frac{d\sigma}{dS}.\frac{\partial c}{\partial \sigma}$$

= unadjusted delta + $a$ × unadjusted vega, as required, since $\frac{\partial c}{\partial \sigma}$ is the unadjusted vega and

$\left.\frac{\partial c(\sigma)}{\partial S}\right|_{\sigma \text{ fixed}}$ is the unadjusted delta.

(b)

Under this proposal, $a = 0.01/100$.

Therefore, using the result from (a), the delta of the sold Call option is now

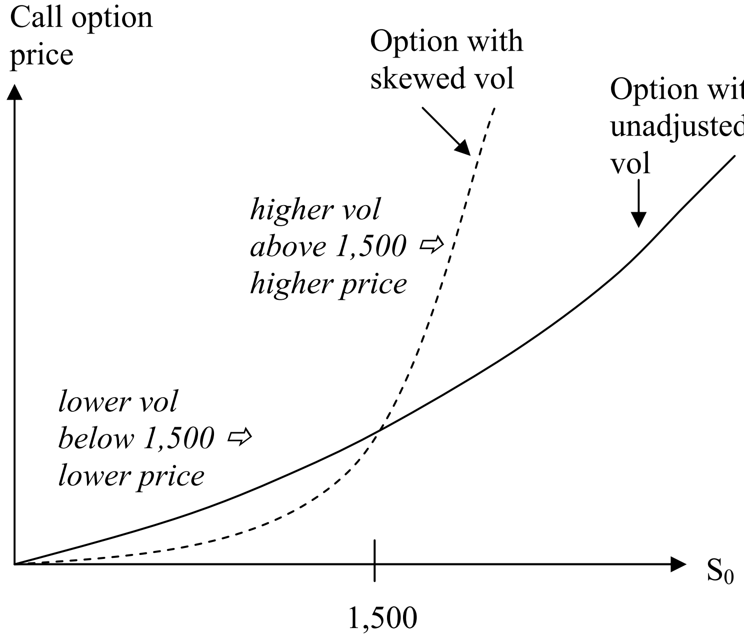
unadjusted delta + notional × $aS_0e^{-qT}\varphi(d_1)\sqrt{T}$

$$= -432.1 + (-1000).\ (0.01 / 100).\ 1500 \exp(-0.02\ .\ 5)\frac{1}{\sqrt{2\pi}}\exp(-\tfrac{1}{2}(-0.05623)^2).\sqrt{5}$$

$$= -432.1 - 150\ .\ 0.90484\ .\ 0.39831\ .\ 2.236 = -432.1 - 120.9 = -553.0$$

Revised futures to buy = 553.0 / 10.202 = 54.2, say 54 contracts.

(iii)




[*Other interpretations were accepted regarding the graph required.*]

The graph shows that skewness increases the gradient of the curve at the current index price and therefore the number of futures to be shorted.

*This question looked at delta hedging calculations under a constant and then skewed (adjusted) volatility assumption.*

*In part (i), the delta calculation was done well for the call option, but too many candidates assumed the delta of the future was 1. Some mistakenly assumed that the time horizon of the future was the same as the maturity of the option, which it was not. Several candidates seemed to think that the bank owned the call, despite what the question said.*

*Part (ii) confused some candidates, although the principle was fairly straightforward and actually the question more or less spelled out how to go about adjusting the volatility. Since the call price varies with implied volatility, a simple Taylor expansion will show the first order effect of changing the volatility assumption. Many candidates worked directly with the algebra in the Black-Scholes formula and so put more effort in than was required by the method given in the solutions.*

*For part (iii), a graph was requested showing how the option price would change for the skewed volatility, which was lower if the index was <1,500 and higher if it was >1,500. Since the call price increases in line with volatility, this should not have been too difficult, but few answered it correctly.*

**QUESTION 3**

***Syllabus section: (k)***

***Core reading: 14***

(i)

$$dS = \mu S dt + \sigma S dW_t$$

Hence using Ito:

$$d(\ln S) = \frac{1}{S}.\mu S dt + \frac{1}{S}.\sigma S dW_t + \tfrac{1}{2}\frac{-1}{S^2}.\sigma^2 S^2 dt = (\mu - \tfrac{1}{2}\sigma^2)dt + \sigma dW_t$$

Integrating between 0 and $t$:

$$\ln S_t - \ln S_0 = (\mu - \tfrac{1}{2}\sigma^2)t + \sigma(W_t - W_0)$$

But $W_t \sim N(0, t) \equiv \sqrt{t}N(0, 1)$ and $W_0 = 0$, so

$S_t = S_0 \exp[(\mu - \tfrac{1}{2}\sigma^2)t + \sigma\sqrt{t}Z]$, where $Z \sim N(0, 1)$, as required.

[*There are alternative ways of deriving this.*]

(ii)

(a)

A numeraire is a unit of account, e.g. a currency or security.

Changing the numeraire means altering the unit of account for the derivative and its underlying security …

… which is achieved through a different probability measure, usually selected to make the risk-neutral valuation simpler.

(b)

Since interest rates are deterministic, we have $dB = rBdt$ where $r$ is the risk-free rate.

Using Ito:

$$d\left(\frac{1}{S}\right) = -\frac{1}{S^2}.\mu S dt - \frac{1}{S^2}.\sigma S dW_t + \tfrac{1}{2}\frac{2}{S^3}.\sigma^2 S^2 dt = -\frac{(\mu-\sigma^2)}{S}dt - \sigma\frac{1}{S}dW_t$$

and since $B$ is not stochastic

$$d\left(\frac{B}{S}\right) = \frac{1}{S}dB + Bd\left(\frac{1}{S}\right)$$

$$= \frac{rBdt}{S} - \frac{(\mu-\sigma^2)B}{S}dt - \sigma\frac{B}{S}dW_t = \frac{(r+\sigma^2-\mu)B}{S}dt - \sigma\frac{B}{S}dW_t$$

Hence $\dfrac{B_t}{S_t}$ is a martingale when the drift is zero, i.e. $\mu = r + \sigma^2$.

[*An alternative valid approach is to refer to the Equivalent Martingale Measure (EMM) result, where the market price of risk is equal to the volatility of the underlying process, or* $\dfrac{\mu - r}{\sigma} = \sigma$*, which leads to the same result. If properly argued, this was given full marks.*]

(iii)

(a)

Given a payoff $C_T$ at time $T$, we define the price of the Call option $C_t$ denominated in units of stock price by $\dfrac{C_t}{S_t} = \mathrm{E}\left(\dfrac{C_T}{S_T} \middle| F_t\right)$ under the martingale measure in (ii)(b) and filtration to time $t$ of $F_t$. ...

... which makes $\dfrac{C_t}{S_t}$ a martingale under this measure.

Martingales are by definition arbitrage free (no portfolios can be created that will lead to arbitrage profits with non-zero probability) …

… and from the equivalent martingale theorem, we know that there is one unique martingale measure.

Hence $\frac{C_t}{S_t}$ is a martingale and is <u>the</u> price of the Call option at time $t$ denominated in units of stock price.

[*The EMM approach can also be applied to this part as it was for (ii)(b).*]

(b)

The payoff of the Call is max( $S_T - K$ , 0) …

… which is equal to $S_T - K = S_T.1 - K.1$ when $S_T \geq K$ and $0 = S_T.0 - K.0$ otherwise.

This is equivalent to saying $S_T I_{S_t \geq K} - K I_{S_t \geq K}$ as required.

(c)

$$\frac{C_0}{S_0} = \mathrm{E}\left(\frac{C_T}{S_T}\right) = \mathrm{E}\left(\frac{S_T I_{S_t \geq K} - K I_{S_t \geq K}}{S_T}\right) = \mathrm{E}\left(I_{S_t \geq K}\right) - \mathrm{E}\left(\frac{K I_{S_t \geq K}}{S_T}\right)$$

The first term of the RHS is simply the probability at time 0 that $S_T \geq K$ .

Under the martingale measure, $\mu = r + \sigma^2$ .

Now $\Pr\{S_T \geq K\} = \Pr\{S_0 \exp[(r + \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}Z] \geq K\}$ for $Z \sim \mathrm{N}(0, 1)$

$$= \Pr\{Z \geq \frac{\ln(K / S_0) - (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\}$$

$$= \Pr\{Z \leq \frac{\ln(S_0 / K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\}$$

= $\Phi(d_1)$ under the usual definition of $d_1$, as required.

*This question covered a number of topics that can be tackled with Ito's Lemma.*

*Part (i) was bookwork and well answered.*

*Part (ii) was relatively easy for those who had understood how to change numeraire. Admittedly this topic is conceptually one of the harder ones in the course, but it is central to derivative theory and important to master, at least to a basic level. Actually, there are fairly* *limited applications of this topic that are likely to come up in ST6, and use of the Equivalent Martingale Measure is a key method to apply here.*

*Part (iii) was expressed in unfamiliar terms, but was actually fairly simple. For part (iii)(c), candidates should take expectations of the LHS of the formula in part (iii)(b), giving a simple probability which could then be answered using part (i). A few candidates missed the word "Hence" and went off on the wrong track. As with most mathematical exams, algebraic questions of this type are usually set out in a logical sequence, so that earlier parts provide material than can be used in the later parts.*

## QUESTION 4

***Syllabus section: (l) & (m)***
***Core reading: 15, 16***

(i)

<u>Sensitivity reports</u>

The basic set of reports would show the following:

- The interest rate report would show basic (risk-free) interest rate and credit spread sensitivity to the value of a basis point movement in benchmark rates …

  … at varying maturities (e.g. 1, 3 and 6 months, 1, 2, 5 and 10 years) and, for credit spread, varying credit qualities.

- The equity reports would show sensitivity to benchmark indices by country and/or continent.

- The commodity reports would show sensitivity to each commodity group according to various groups.

- The FX report would show net exposure to each currency vs either the base currency or US dollar.

These reports would show all the above values against any limits imposed …
… and could also show equivalent benchmark positions (e.g. in swaps or futures terms) so that the management and/or traders can appreciate what hedge would neutralise the market risk on their books.

<u>Value at Risk (VaR) report</u>

The VaR report would show VaR usage against limit for each business unit and overall.

The firm must decide on methodology (parametric, historical simulation, Monte Carlo) ...

... and an appropriate confidence interval and observation period for the data.

Normally the VaR report would split component usage into the same factors as above.

(ii)

(a)

The basic principle of calculating P&L is that the business will have net risk positions of various kinds, plus administrative and financing costs arising over the life of a trade. The Finance department needs to record the revenue to include all the necessary factors to provide a true and fair representation of the value to the firm of each trade.

They must make the following adjustments to the P&L process:

- Prices must allow for a realistic close-out value – mid-market valuations do not reflect the eventual need to close a position, so a representative bid-offer spread must be included.

- Prices must be marked consistently – daily updates are appropriate, but timing is important so that all prices are gathered at the same end-day point.

- Prices must be marked independently of the traders – this can be performed daily if external prices are available, but more likely will be subject to regular price checks from independent sources.

- For very long-term contracts, market rates are often not liquid and quotes often unreliable, so a liquidity adjustment needs to be made.

- Models are new and (so it would seem) not fully validated – the firm needs to be sure they are accurate and theoretically correct, otherwise a model risk adjustment needs to be included in the price.

- There will also be other adjustments for:

    - financing costs (e.g. repo, unsecured borrowing, internal transfer pricing)
    - cost of capital (either economic or regulatory capital usage)
    - administrative costs (running the support functions throughout the life of the trade)
    - credit risk on counterparties, both current and future potential exposure

  It is not clear from the question whether or not these are correctly included at present.

(b)

Sensitivity reports only show the situation near to the current market value – for non-linear instruments this does not give a true picture further away where the impact of gamma might show a very different risk.

VaR is a single number based on a loss distribution and does not precisely indicate what conditions might cause a large loss.

Stress tests re-calculate P&L for prescribed larger movements than is customary for sensitivity reports, such as:

- 100 or 200 basis points movements in interest rates

- 25% rise or fall in equity prices or commodities
- 10% movement in FX rates
- large rise or fall in volatility and/or change in correlation.

Also stress tests can investigate large simultaneous contrary or similar movements in risk factors …

… plus changes in correlations from those expected.

Such testing is all the more important for complex option portfolios where hybrids are involved, since the impacts are really only seen in the tails of the distribution.

Other adverse effects occur in stress situations, such as wider bid-offer spreads, a drying up of liquidity, large cash collateral demands etc.

Stress tests can encompass general market movements or specific historical and/or hypothetical scenarios.

Stress tests enable certain hedges and contingency plans to be put in place.

(c)

The models appear to be out of line with the market, which could be because:

- they are too simple, in which case better models need to be developed

- they are incorrectly built or implemented, in which case they need to be corrected (this would have been discovered by proper validation)

- the models are not being correctly calibrated to market parameters, in which case the trader's marks need to be checked more thoroughly and often

- the market does not agree on how to price particular exotic options, in which case an accounting adjustment (model risk reserve) could be introduced to reflect the uncertainty around particular products or models …
  … together with a review of the pricing and risk management of the products to ensure the firm is not taking on unexpected or unhedgeable risks.

*This question looked at the use of risk and finance control techniques as applied to a particular management problem relating to exotic options.*

*Part (i) was standard bookwork and was well answered.*

*Part (ii)(b) was also well answered, with most candidates aware of the types of stress test that would improve the risk management framework. However, parts (ii)(a) and (c) eluded many candidates, who clearly found it hard to apply their risk management knowledge to a situation not mentioned in the reading material. Hopefully, the solution will show that the application is, in the most part, mostly common sense.*

**QUESTION 5**
***Syllabus section: (h)(i)-(iii)***
***Core reading: 8, 9***

(i)

Let $X_t = \frac{1}{\sqrt{c}} W_{ct}$.

$$X_0 = \frac{1}{\sqrt{c}} W_{c0} = \frac{1}{\sqrt{c}} W_0 = 0$$

and $X_t$ is continuous.

It is sufficient to show first two moments are as expected for Brownian motion.

For time $t \geq s$, we need to show increments are distributed as N(0, $t - s$):

$$\mathrm{E}[X_t - X_s] = \mathrm{E}\left[\frac{1}{\sqrt{c}} W_{ct} - \frac{1}{\sqrt{c}} W_{cs}\right] = \frac{1}{\sqrt{c}} \mathrm{E}[W_{ct} - W_{cs}] = 0$$

and

$$\mathrm{var}[X_t - X_s] = \mathrm{E}[(X_t - X_s)^2] = \frac{1}{c} \mathrm{E}[(W_{ct} - W_{cs})^2] = \frac{1}{c} \mathrm{var}[W_{ct} - W_{cs}] = \frac{1}{c}.(ct - cs) = t - s$$

<u>ALTERNATIVELY</u>

Or we can use simple expectation and the covariance property of Brownian motion:

$$\mathrm{E}[X_t] = \mathrm{E}\left[\frac{1}{\sqrt{c}} W_{ct}\right] = \frac{1}{\sqrt{c}} \mathrm{E}[W_{ct}] = 0$$

and

$$\mathrm{cov}[X_s, X_t] = \mathrm{cov}\left[\frac{1}{\sqrt{c}} W_{cs}, \frac{1}{\sqrt{c}} W_{ct}\right] = \frac{1}{c} \mathrm{cov}[W_{cs}, W_{ct}] = \frac{1}{c}.cs = s \text{ for } s \leq t$$

Hence $X_t$ is a standard Brownian motion.

(ii)

$W_t$ is a standard Brownian motion and $t_3 > t_2 > t_1$.

(a)

$\mathrm{E}[(W_{t_3} - W_{t_2})(W_{t_2} - W_{t_1})]$

$= \mathrm{E}[(W_{t_3} - W_{t_2})].\mathrm{E}[(W_{t_2} - W_{t_1})]$ {independence of increments}

$= 0.\ 0 = 0$ {Gaussian increments}

(b)

$\mathrm{E}[(W_{t_3} - W_{t_1})(W_{t_2} - W_{t_1}) \,\big|\, W_{t_1} = 1] = \mathrm{E}[(W_{t_3} - W_{t_1})(W_{t_2} - W_{t_1})]$

{independence of increments}

$= \mathrm{E}[\{(W_{t_3} - W_{t_2}) + (W_{t_2} - W_{t_1})\}(W_{t_2} - W_{t_1})]$

$= \mathrm{E}[(W_{t_2} - W_{t_1})^2]$ {independence of increments}

$= t_2 - t_1$ {Gaussian increments}

(c)

$\mathrm{E}[W_{t_3} W_{t_2}]$

$= \mathrm{E}[\{(W_{t_3} - W_{t_2}) + W_{t_2}\}W_{t_2}]$

$= \mathrm{E}[{W_{t_2}}^2]$ {independence of increments}

$= t_2$ {Gaussian increments}

(iii)

To prove $X_t$ is a martingale, we must show that:

(a) $X_t$ is bounded for all $t$, which it is clearly is, since it is the multiplicative combination of two Brownian motions, which are also martingales and hence bounded; and

(b) $\mathrm{E}[X_t \,|\, \mathbf{F}_s] = X_s$ for all $t \geq s$

where $\mathbf{F}_s$ is the filtration of events from both processes up to time $s$.

For part (b), firstly we note that, since $V_t$ is a Brownian motion:

$\mathrm{E}[V_t^2] = t = s + (t - s) = \mathrm{E}[V_s^2] + (t - s)$ for $t > s$

Then, for $t > s$, $\mathrm{E}[V_t^2 W_t \,|\, \mathbf{F}_s] = \mathrm{E}[V_t^2 \,|\, \mathbf{F}_s].\mathrm{E}[W_t \,|\, \mathbf{F}_s] = (V_s^2 + t - s)W_s$ and

$$\mathrm{E}\left[\int_0^t W_u du \,|\mathbf{F}_s\right] = \mathrm{E}\left[\int_0^s W_u du \,|\mathbf{F}_s\right] + \mathrm{E}\left[\int_s^t W_u du \,|\mathbf{F}_s\right] = \int_0^s W_u du + W_s \int_s^t du = \int_0^s W_u du + W_s (t-s)$$

Hence $\mathrm{E}[X_t \,|\, \mathbf{F}_s] = [(V_s^2 + t - s)W_s] - \left[\int_0^s W_u du + W_s (t-s)\right] = V_s^2 W_s - \int_0^s W_u du = X_s$

*This was a simple question based on the properties of Brownian motion and martingales.*

*Part (i) was answered well as it is a fairly standard application.*

*In part (ii)(b), candidates did not always remember that the "independence of increments" property meant that the path up to $t_1$ did not affect the solution.*

*Part (iii) was well attempted, although care was required to avoid algebraic slips when taking expectations of integrals. Quite a few tried to prove the martingale property by looking at the SDE of $X_t$ rather than the easier route of showing that $\mathrm{E}[X_t \,|\, \mathbf{F}_s] = X_s$ for all $t \geq s$.*

**QUESTION 6**

***Syllabus section: (a)-(d), (f)***
***Core reading: 1-4, 6***

(i)

(a) Contract specifications

The contract must specify:

- Underlying asset – commodity, bond, equity index, interest rate etc.
- Delivery month and year – including first (and last) settlement date(s) and first and last trading dates
- Contract size – amount of notional that is delivered per contract (e.g. \$100,000, 100 × index)
- Method of, and arrangements for, delivery – see (d) below
- Price quotation – currency and denomination (e.g. dollars and 32nds for T-bonds), price or yield (for bonds), inversion (e.g. 100 – interest rate in %)

(b) Open interest

When a contract is traded, it creates a pair of open positions – one long, for the buyer, and one short, for the seller.

The open interest is the sum total of all of these open positions (measured as the number of long contracts).

(c) Initial and variation margin

Initial margin is the minimum amount of cash that must be deposited at the exchange per contract traded. When the position is closed, initial margin is refunded, plus or minus any gains or losses that occurred since opening.

This minimum margin is determined by the futures exchange and is usually 5% to 10% of the futures contract.

Initial margin protects the exchange in the event that a participant defaults – the exchange will liquidate the positions as quickly as possible and use the initial margin to cover any losses.

Variation margin is the unrealised profit or loss on the open position, and is deducted from (for a profit) or added to (for a loss) the initial margin requirement.

[*Maintenance margin is the amount of margin the firm must maintain at the exchange before posting more cash. It is designed to avoid overly frequent margin calls. A significant unrealised loss on a futures position used as a hedge (e.g. of metal prices for a mining company) can cause cashflow problems if the firm is not able to post large amounts of cash.*]

(d) Physical vs cash settlement

Physical settlement is where the futures seller must deliver the actual underlying to a specified location on the settlement date. This is common for commodity futures and most bond futures, although much of the open interest is usually closed out before delivery takes place.

Cash settlement is where the futures buyer and seller receive cash payments at delivery, representing the market value of the contract as if they had traded it between them for cash. This is common for products with no natural physical form, such as interest rates or an equity index.

[*The exchange will specify how the cash settlement price will be calculated so the price is fair to both buyer and seller.*]

(ii)

Firms that buy and sell derivatives are exposed to replacement risk of the OTC trade if there is a credit event (such as a default) by one of its counterparties.

Signing an ISDA Master Agreement ("ISDA") helps mitigate against this risk and any adverse selection on default …

… by ensuring that all derivatives are amalgamated together for offset when there is a credit event – this amalgamation is called 'close-out netting'.

If an ISDA is signed, every OTC transaction from an agreed product list can reference one agreement with standard definitions and legally tested terms.

Confirmations are therefore simple and short.

To ensure that counterparty credit risk does not build up during the life of an OTC contract, ISDA has produced a Credit Support Annexe (CSA) whereby the party which is "out-of-the-money" posts collateral corresponding to the mark-to-market value of the OTC.

Margining continues (usually daily) according to the CSA terms, based on agreed valuations and thresholds for calling more margin.

[*CSAs are almost universal in banks and major financial institutions for derivative trades executed between them. Risk management systems assign a higher risk weight when no CSA is in place (or if the CSA has a high threshold).*]

(iii)

(a)

The concept of a clearing house, or central counterparty (CCP), is that it removes the direct bilateral credit risk between OTC counterparties.

In principle, this makes the financial "system" (the sum total of all the participants in the financial market, together with the processes they use) safer for all, because:

- <u>Variation margin</u> is paid daily so that the mark-to-market losses are covered by cash (although this already occurs with bilateral CSAs)

- If a counterparty defaults, the <u>initial margin</u> will absorbs the loss in closing out their open positions, that other counterparties might otherwise incur in a bilateral OTC world

- The CCP will have strong risk management and access to central bank funds to cover any liquidity problems that arise with a major default

- Based on the experience of exchange-traded futures and options contracts, these arrangements should also make the OTC markets more efficient as CCPs centralise the counterparty collateral calls.

(b)

Problems that arise are:

- OTC products are not as uniform as futures, leading to complexity in the contract specifications and risk management capability for CCPs

- OTC risk management was actually one part of the financial system that really worked well in the recent "credit crunch" – some losses were made, but they were relatively minor in comparison with other categories of loss

- All this is untested in practice so far, and there is little prospect of a dry run

- Standardisation of approach compared to the previously bespoke OTC market could be difficult to achieve, e.g. collateral terms, risk assessment methodologies

[*The collapse of Lehman in September 2008 led to a virtual seizure in certain markets, but that was more to do with the complexity of coping with the removal of a major financial player at a time of deep stress in the system than specifically the unwinding of OTC trades. For most banks, replacing Lehman as a counterparty was achieved within days, despite very difficult market conditions. Unwinding trades with a CCP is not guaranteed to be quicker.*]

More importantly, serious new risks introduced are:

- There are several competitive CCPs globally, plus some quite small CCPs in certain countries, so there is no real concept of a true "central" counterparty – this could become like the airline industry, where every major country wants its own CCP

- CCPs become hugely important to the system, yet they are still considering themselves as strongly profit-incentivised institutions who operate with low capital bases and great flexibility – lessons from the "credit crunch" of the large banks and monoline insurers seem not to have been learned

- The consequence of a CCP failing is almost unimaginable, far exceeding the impact of the post-Lehman problems – hence systemic risk is not removed, just transferred to the CCPs

- The margining and netting calculations CCPs involved models and assumptions, so the problem of over-reliance on models has not been eradicated

- The new approach reduces liquidity by creating non-fungible pools of collateral

Hence the systemic risk from a major counterparty default is reduced, but replaced by systemic risk of a CCP failure. Some commentators are asking if this is actually making matters worse.

*This was a bookwork question on futures and ISDA agreements, with a final part requiring some extension of the theory to a topical issue.*

*Parts (i) and (ii) were straightforward and well answered apart from Open Interest, which was often too loosely defined.*

*Part (iii) elicited some good attempts discussing the potential risk profile of a CCP (central counterparty for derivatives). The question asked about systemic risk, i.e. the risk to the whole financial system. Some candidates confused this with systematic risk, i.e. undiversifiable overall market risk (beta).*

**QUESTION 7**

***Syllabus section: (h)(iv)-(ix), (i)***
***Core reading: 10-12***

(i)

EITHER

**Put option** formula $P = Ke^{-rt}N(-d_2) - Se^{-qt}N(-d_1)$

where $d_1 = \dfrac{\ln(S/K) + (r - q + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$

Put $S$ = 5,000, $K$ = 3,000, σ = 20%, $T$ = 5, $r$ = 5%, $q$ = 2% in the above.

$d_1$ = [ln(5000/3000) + (0.05 – 0.02+ ½(0.2)$^2$)*5] / (0.2 √5) = 1.7013

$N(-d_1)$ = 0.0444

$d_2$ = 1.1407 – (0.2 √5) = 1.2540

$N(-d_2)$ = 0.1049

Hence $P$ = 3000 × $e^{-0.05*5}$ × 0.1049 – 5000 × $e^{-0.02*5}$ × 0.0444 = 44.03 as required.

OR

**Call option** formula $C = Se^{-qt}N(d_1) - Ke^{-rt}N(d_2)$

where $d_1 = \dfrac{\ln(S/K) + (r - q + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$

Put $S$ = 5,000, $K$ = 10,000, σ = 20%, $T$ = 5, $r$ = 5%, $q$ = 2% in the above.

$d_1$ = [(ln(5000/10000) + (0.05 – 0.02+ ½(0.2)$^2$)*5] / )0.2 √5) = –0.9909

$N(d_1)$ = 0.1609

$d_2$ = 0.1609 – (0.2 √5) = –1.4381

$N(d_2)$ = 0.0752

Hence $C$ = 5000 × $e^{-0.02*5}$ × 0.1609 – 10000 × $e^{-0.05*5}$ × 0.0752 = 142.13 as required.

(ii)

Value of zero coupon bond (ZCB) at $t = 5$ is 10,000 exp(–0.05 × 5) = 7,788.0

$\therefore$ Total value of portfolio = 7,788.0 + 142.13 + 44.03 = 7,974.16

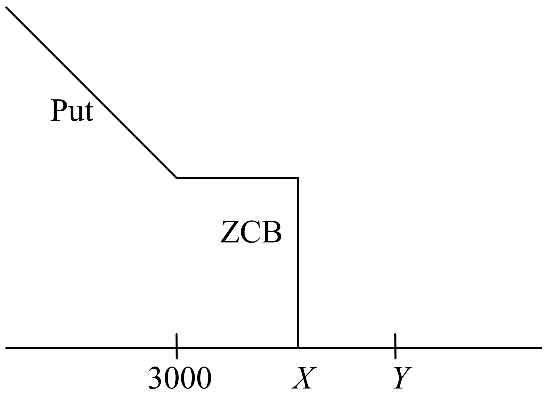
$\therefore$ Each son's one third share must be worth 7,974.16 / 3 = 2,658.05

The sons have different entitlements.

<u>Finding $X$</u>

First consider the entitlement of the youngest son, who receives everything if $S_5 < X$.

His "payoff" structure at $t = 5$ looks (approximately) like this:




which is the sum of the payoff of a Put with strike 3,000 and the ZCB if $S_5 < X$.

Taking risk-neutral expectations at time 0, we have that the value of youngest's share

= value of Put + 10,000 $e^{-0.05*5}$ Prob($S_5 < X$)

or 10,000 $e^{-0.05*5}$ $N(-d_2(X))$ = 2,658.05 – 44.03 = 2,614.02

Hence $N(-d_2(X))$ = 0.33565, and so $d_2(X)$ = 0.4244.

$$\text{Now } d_2(X) = \frac{\ln(5000/X) + (0.05 - 0.02 - \tfrac{1}{2}(0.2)^2)5}{0.2\sqrt{5}} = \frac{\ln(5000/X) + 0.05}{0.4472}$$

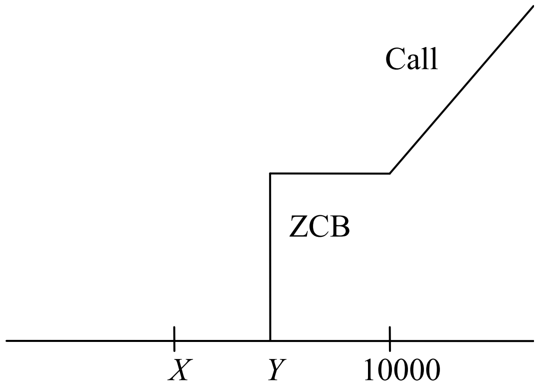
so $X = 5000\exp[-0.4472 d_2(X) + 0.05]$ = 4,348.

[*Alternatively, the ZCB entitlement could be considered as a digital Put, which gives the same formulation and result.*]

Finding $Y$

Now consider the entitlement of the eldest son, who receives everything if $S_5 > Y$.

His "payoff" structure at $t = 5$ looks (approximately) like this:




which is the sum of the payoff of a Call with strike 10,000 and the ZCB if $S_5 > Y$.

Taking risk-neutral expectations at time 0, we have that the value of eldest's share

= value of Call + 10,000 $e^{-0.05*5}$ Prob($S_5 > Y$)

or 10,000 $e^{-0.05*5}$ $N(d_2(Y))$ = 2,658.05 – 142.13 = 2,515.92.

Hence $N(d_2(Y)) = 0.32305$, and so $d_2(Y) = -0.4592$.

$$\text{Now } d_2(Y) = \frac{\ln(5000/Y) + (0.05 - 0.02 - \frac{1}{2}(0.2)^2)5}{0.2\sqrt{5}} = \frac{\ln(5000/Y) + 0.05}{0.4472}$$

so $Y = 5000\exp[-0.4472 d_2(Y) + 0.05]$ = 6,455.

[*Alternatively, the ZCB entitlement could be considered as a digital Call, which gives the same formulation and result.*]

(iii)

(a)

If the yield on the FTSE were higher, this reduces the risk-neutral expected FTSE growth rate, increasing the risk-neutral probability (and value) of low FTSE outcomes and reducing the risk-neutral probability (and value) of high FTSE outcomes.

If $X$ and $Y$ stay unchanged, this increases the value of the youngest son's inheritance and reduces the value of the oldest son's.

To maintain equality between the three, $X$ and $Y$ would both need to be reduced.

(b)

If implied volatility is increased, this increases the value of both options.

It also increases the value of the digital options that pay 10,000 if the FTSE finishes outside the range [$X$, $Y$] (this only being true because the risk-neutral expected FTSE after five years lies within that range).

If $X$ and $Y$ stay unchanged, this increases the value of the youngest and oldest son's inheritance, so to maintain equality between them, $X$ would need to reduce and $Y$ would need to increase.

(c)

When volatility skew is introduced, the market places a higher value on payoffs in low FTSE scenarios and a lower value on payoffs in high FTSE scenarios.

If $X$ and $Y$ stay unchanged, this increases the value of the youngest son's inheritance and reduces the value of the oldest son's.

To maintain equality between the three, $X$ and $Y$ would both need to be reduced.

*This question, evaluating a portfolio of options handed on to a deceased investor's children, was clearly found unfamiliar by many candidates, with poor attempts particularly to part (ii).*

*Part (i) was a simple Black-Scholes evaluation, and hence well answered.*

*Part (ii) required careful thought about how the three sub-portfolios involving uncertain outcomes interacted to make up the entire portfolio. The key was to think of the payoffs to the three sons, then realise that taking expectations of each generated a call option on one side, a put option on the other, and a various forward contracts in the middle. Once this was clear, the evaluation was relatively straightforward, albeit quite onerous for the marks allocated. Many candidates omitted this part in its entirety.*

*Part (iii) was based around inputs to the Black-Scholes model. It was better answered and actually offered some fairly easy marks, independently of whether part (ii) was successfully attempted or not.*

**QUESTION 8**
***Syllabus section: (e) & (j)***
***Core reading: 5, 13***

(i)

UK gilt STRIPS

These are zero coupon bonds created by stripping (detaching cashflows) from a gilt (fixed interest UK Government bond).

The term STRIPS stands for Separately Traded and Registered Interest and Principal Securities, which was designed to fit the stripping process.

Each coupon becomes an individual "interest only" (IO) strip ...

… with coupons paid on the same actual date from different bonds being allowed to amalgamate into the same strip issue.

Each maturity amount becomes a "principal only" (PO) strip, not fungible with the coupons of the same date.

To enhance liquidity in the strip market, the UK Government has issued bonds with coupons on the same sets of dates in the year. Principal and interest are not fungible to avoid altering the balance between the smaller coupons and much larger final payment – otherwise, reconstituting might not be possible and market values could diverge widely from the theoretical price.

Vanilla Interest Rate Swaps (IRS)

An IRS is a swap of fixed rate for a floating rate over a given maturity.

It is transacted between two parties, A & B, usually governed by an ISDA agreement.

Every coupon period, A pays B a fixed rate of interest on a specified notional
i.e. fixed payment = $(1 / n)$ * fixed swap rate * notional, where $n$ = no of payments per year. [$(1 / n)$ *is the tenor*]

In return, B pay*s* A the prescribed LIBOR rate on the same notional

i.e. floating payment = $(1 / n) * L$ * notional, where $L$ is LIBOR (or other equivalent reference rate) set in arrears.

The payments are netted against each other (so there is only one cash payment every period, which could be in either direction).

Usually an IRS is instigated with no upfront payment or exchange of notional, i.e. the (fixed) swap rate being set so that the fixed and floating legs have the same value.

There is no exchange of notional at expiry.

(ii)

For gilt STRIPS, the sum of the prices of a set of strips that replicates a gilt needs to be equal to the price of the gilt, i.e. for a gilt with exactly $T$ years to maturity and annual coupon $g$:

$$\text{Gilt price} = \frac{g}{2}\left[Z_I(\tfrac{1}{2}) + Z_I(1) + ... + Z_I(T)\right] + Z_P(T)$$

where the $Z_I(t)$ and $Z_P(t)$ are the prices of gilt STRIPS maturing at times $t$, interest only and principal only respectively.

The swap fixed rate will be the coupon on a bond that would be valued at par by the bank's risk-free yield curve, i.e. swap rate $R$ is the solution to:

$$R\left[e^{-r(1)} + ... + e^{-Tr(T)}\right] + e^{-Tr(T)} = 1$$

where $T$ is term of swap and $r(t)$ is bank's risk-free yield curve.

[*Note: The question could be interpreted as asking generally about arbitrage pricing, and this was accepted provided the formula were meaningful. Also, the above gives an example of a swap paying annually; other examples could be given.*]

(iii)

For banks, gilt STRIPS are not great hedging instruments because:

- They are not very liquid, so of little use for dynamic hedging

- In pricing and valuing derivatives, banks will use a risk-free yield curve that is based on LIBOR and swap rates: there would be basis risk involved in using gilt STRIPS to hedge against movements in this yield curve.

Gilt STRIPS could, however, be useful to a firm with a set of illiquid cashflows and that (for whatever reason) is content to earn gilt returns rather than bond returns. In this case the ZCB-nature of the strips makes it easier to replicate cashflows, e.g. a life insurer or pension scheme might use strips to hedge annuities in payment.

Swaps are a commonly used hedge instrument for banks, their positive features being

- Their liquidity and low transaction costs

- Their interaction with the banks' risk-free yield curve, in terms of both LIBOR payoffs and the use of swap rates to derive the yield curve.

- Swaps are useful to convert a fixed or floating liability to the opposite type.

(iv)

The bank will create a yield curve from a set of money-market instruments and swaps (using a "boot-strapped" yield curve model).

The yield curve will be used to mark-to-market the bank's derivative positions.

The bank will also perform a sensitivity analysis, looking at the impact of a small change in each input rate (with all other rates staying fixed) …

… and therefore calculate the mark-to-market sensitivity of the portfolio of derivatives to changes in each benchmark rate.

These sensitivities would be compared to the corresponding sensitivity for each individual benchmark swap and money-market instruments, thereby determining a set of equivalent hedging transactions.

The bank may well select benchmark rates only when calculating its yield curve, thereby restricting the hedges to the most liquid instruments in the market.

*This question was largely bookwork, comparing gilt zero-coupon bonds (strips) with interest rate swaps. Given that the course material on strips is fairly sparse, candidates tended to cover these in less detail than the swaps, or else focus too much on a description of the underlying UK gilt market rather than the zero coupon aspects. Those that tried to tailor their answers to the question, with appropriate balance, achieved many of the marks available for parts (i), (ii) and (iii).*

*Part (iv) was often poorly answered or omitted. Candidates should note the solution given, though, as this outlines in a few words the very important standard techniques for managing market risk on interest rate swaps.*

## END OF EXAMINERS' REPORT