INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

September 2011 examinations

Subject ST6 — Finance and Investment
Specialist Technical B

Purpose of Examiners' Reports

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and who are using past papers as a revision aid, and also those who have previously failed the subject. The Examiners are charged by Council with examining the published syllabus. Although Examiners have access to the Core Reading, which is designed to interpret the syllabus, the Examiners are not required to examine the content of Core Reading. Notwithstanding that, the questions set, and the following comments, will generally be based on Core Reading.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report. Other valid approaches are always given appropriate credit; where there is a commonly used alternative approach, this is also noted in the report. For essay-style questions, and particularly the open-ended questions in the later subjects, this report contains all the points for which the Examiners awarded marks. This is much more than a model solution – it would be impossible to write down all the points in the report in the time allowed for the question.

T J Birse
Chairman of the Board of Examiners

December 2011

General comments on Subject ST6

This subject deals with mathematical techniques for the valuation and risk management of derivatives, together with some aspects of their practical application.

Different numerical answers may be obtained to those shown in these solutions, depending on whether figures were obtained from tables or from calculators. Allowance was made for these minor differences. However, candidates may have been penalised for using excessive rounding or showing insufficient working.

Comments on the September 2011 paper

The paper was tackled better than many recent ST6 papers, and the examiners were therefore able to pass a higher percentage of candidates. There were few really hard questions in the paper, but nevertheless there were still a number of difficult sections and as usual the material needed to be covered in a precise and clear manner.

This subject undoubtedly attracts candidates who enjoy algebra and numerical calculation. However, it is also important to be able to summarise points in a reasoned argument where the question demands this. It has often been commented in these Examiner Reports that for such questions candidates should try to provide several distinct points in short paragraphs, not one point made at length. They should aim to demonstrate their understanding of as many different relevant issues as the marking scheme would seem to require.

Question 6 appeared to be the toughest, asking for an application of risk mitigation techniques in an unfamiliar setting. Given that several candidates were short of time, it was natural that they tended to avoid this question, although in facts marks were really not that hard to obtain. Those candidates on the borderline of passing could perhaps consider whether they might have set aside a few minutes to attempt at least part of this question and thereby garner additional marks.

Pacing time spent to the marks on offer is an important examination skill. There are 100 marks available in total, which are intended to be roughly of equal difficulty throughout the paper. This implies spending just under two minutes for each mark, so a ten mark question should take 18 minutes on average. A few candidates wrote very lengthy answers to the first two questions, which inevitably meant they ran out of time in the later questions.

QUESTION 1

Syllabus section: (a)-(d), (f)
Core reading: 1-4, 6

(i)

Arbitrage is a technique for taking advantage of two or more securities (or derivatives on those securities) being mispriced relative to each other.

If an arbitrage exists, there is a trading strategy that makes a riskless (or much reduced risk) profit by buying the cheap instrument and selling the expensive one, possibly in different markets, timezones or physical manifestations.

Arbitrageurs seek to make profits from setting up arbitrage trades.

Their role is not entirely self-serving, since by their actions arbitrageurs keep cash and derivatives markets in line, and remove or reduce arbitrage.

Often significant amounts of risk capital are required to make arbitrage profits, which tend to be small relative to the size of the deals undertaken (since they are low risk), so arbitrage opportunities tend to be the province of global firms.

Transaction costs, financing costs, liquidity and the ability to execute deals simultaneously are key issues for arbitrageurs.

(ii)

(a)

Basis is the difference between the spot price [of oil] and the [oil] futures price.
Basis risk is the risk arising from exposure to variations in the basis.

(b)

The arbitrageur will (in theory) avoid basis risk.

An arbitrage opportunity by definition should not involve basis risk. In practice, some positions that involve small basis risk will be (inaccurately) labelled as arbitrage opportunities.

The hedger may view basis risk as undesirable but (occasionally) unavoidable. They will be using derivatives to hedge out first order market risks; the residual basis risks are likely to be of second order.

On the other hand, they may have had the opportunity to hedge using more bespoke OTC derivatives but decided that the extra cost of hedging out the basis risk was not justified.

The speculator does not primarily carry basis risk since any basis position is implied and at most of second order importance: the intention is to make profit by upward or downward price movements in the commodity, which dominate any basis effect.

(iii)

(a)

Value of physical oil for future delivery = 50,000 × \$85.50 = \$4,275,000
Financing cost of holding physical oil = \$4,275,000 × 2% × 47 / 365 = \$11,010
Delivery value of futures contracts = 50,000 × \$86.20 = \$4,310,000
Hence expected profit = \$4,310,000 – \$4,275,000 – \$11,010 = \$23,990

[*The above calculation uses the normal method of simple interest with annual compounding. Equally valid is using direct annual compounding, which would make the financing cost:* $\$4{,}275{,}000 \times [(1.02)^{47/365} - 1] = \$10{,}915$ *and the profit equal to* \$24,085.]

[*It is not correct to use say 252 business days for financing instead of 365, or to use continuous compounding based on* exp(2% × 47/365).]

(b)

The trade is a straightforward arbitrage, so *apart from transaction costs* should make the profit intended (which is quite small).

It does not require shorting physical oil, which would be difficult.

However, it relies on either being able to deliver the physical commodity (oil) into the futures contract, which will incur delivery costs ...
... or using the cash settlement facility for Brent Crude, which means the physical oil has to be sold, which may not be possible at the arbitrage price as it is a different market place.

[*For this reason, most futures arbitrage that takes place in commodities is confined to those firms who are dealing in the physical commodity on a daily basis.*]

Transaction costs will be incurred setting up the trade, reducing the profit margin.

If financing or storage costs are variable, this will affect the profit on the arbitrage.

If the futures price rises substantially, more variation margin will need to be posted, which could increase the funding cost of holding the position.

This was a straightforward question about arbitrage, asking for definitions of market participants in parts (i) and (ii) and a worked example in part (iii). The question was generally well answered overall, but considering that there were 11 marks on offer, including 7 for bookwork, several marks were dropped unnecessarily by candidates answering imprecisely. Parts (ii)(b) and (iii)(b) distinguished the candidates who knew the subject matter better.

In part (iii)(a), the model solution highlights that financing costs are assessed on a simple interest basis for durations under a year. In part (iii)(b), the examiners were looking for candidates to explore practical problems in achieving a theoretical arbitrage outcome. This type of question arises frequently, and even if the answer is not readily known it should be possible to formulate a few appropriate ideas. The key is to visualise what actually would have to be done to close the trade at the theoretical profit, and look for any obstacles.

Several candidates appeared to be confused between "basis", "change in basis" and "basis risk". Some attention to the definitions of these topics from the Core Reading might bear fruit.

QUESTION 2
Syllabus section: (h)(i)-(iii)
Core reading: 8, 9

(i)

Process $W(t)$ in $t \geq 0$ is a **P**-Brownian motion **if and only if**:

- $W(t)$ is continuous
- $W(t) = 0$
- $W(t) - W(s) \sim N(0, t - s)$, $s < t$, under probability measure **P** and is independent of its prior path up to time s.

(ii)

Looking at the third condition in (i), suppose without loss of generality that $0 \leq s < t$.

Then $E[W(s).W(t)] = E[W(s)^2 + W(s)(W(t) - W(s))] = E[W(s)^2] + 0$
since $E[W(s)] = 0$ and W has independent increments.

Hence $\text{cov}[W(s), W(t)] = \text{var}[W(s)] = s$.

If, conversely, $0 \leq s < t$, then similarly $\text{cov}[W(s), W(t)] = t$, hence the result.

[*Other approaches are valid, e.g. using decomposition of variance.*]

(iii)

Linear combinations or transformations of Normal distributions are also Normal, so we only need to demonstrate mean = 0, variance = t and covariance = s.

(a)

$\text{E}[-B_t] = -\text{E}[B_t] = 0$
$\text{var}[-B_t] = \text{var}[B_t] = t$

For $0 \le s < t$, $\text{cov}[-B_s, -B_t] = \text{cov}[B_s, B_t] = \min(s, t) = s$

Hence this is a standard Brownian motion. [*Also, value is zero at t = 0.*]

(b)

$\text{E}[t\, B_{1/t}] = t\, \text{E}[B_{1/t}] = 0$

$\text{var}[t\, B_{1/t}] = t^2\, \text{var}[B_{1/t}] = t^2 . \dfrac{1}{t} = t$

For $0 \le s < t$, $\text{cov}[s\, B_{1/s}, t\, B_{1/t}] = s\, t\, \text{cov}[B_{1/s}, B_{1/t}] = s\, t \min(\dfrac{1}{s}, \dfrac{1}{t}) = s\, t\, \dfrac{1}{t} = s$

[*Strictly we should show the value approaches zero as* $t \to 0$*, but this is not required.*]

Hence this is a standard Brownian motion.

This question developed a familiar result for Brownian motion and applied it to two fairly simple processes. Good marks were achieved. Examiners were expecting that candidates would use the covariance result from part (ii) in part (iii), as it made the solutions easier, but an approach from first principles was equally valid.

In part (iii), quite a number of candidates put forward the suggestion that, for t > s, B(s) – B(t) ~ N(0, s – t), a distribution with negative variance!

QUESTION 3

Syllabus section: (k)

Core reading: 14

(i)

Cox-Ingersoll-Ross avoids having negative rates because ...
... the mean reversion (autoregressive) nature of the drift causes r_t to be pulled upwards if rates fall ...
... and, furthermore, the volatility is dampened down when r_t is small.

Provided θ is large enough in relation to σ^2, r_t is always positive.

(ii)

Consider the two variables $X_t = e^{\kappa t}$ and $Y_t = r_t$.

Then $dX_t = \kappa e^{\kappa t} dt$ and $dY_t = dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}\, dW_t$.

Differentiating, $d(XY) = X\,dY + Y\,dX$:

$$\begin{aligned} d(X_tY_t) &= e^{\kappa t}dr + r\kappa e^{\kappa t}dt \\ &= e^{\kappa t}\kappa(\theta - r_t)dt + e^{\kappa t}\sigma\sqrt{r_t}\,dW_t + r\kappa e^{\kappa t}dt \\ &= e^{\kappa t}\kappa\theta\, dt + e^{\kappa t}\sigma\sqrt{r_t}\,dW_t \end{aligned}$$

Hence, integrating:

$$\left[e^{\kappa u}r_u\right]_0^t = \left[\theta e^{\kappa u}\right]_0^t + \sigma\int_0^t e^{\kappa u}\sqrt{r_u}\,dW_u$$

$\Rightarrow\ e^{\kappa t}r_t - r_0 = \theta e^{\kappa t} - \theta + \sigma\int_0^t e^{\kappa u}\sqrt{r_u}\,dW_u$ and hence the required result.

(iii)

$\mathrm{E}[r_t] = \theta + (r_0 - \theta)e^{-\kappa t}$ since $\mathrm{E}[W_t] = 0$

$$\begin{aligned} \mathrm{var}[r_t] &= \mathrm{E}[r_t - \mathrm{E}(r_t)]^2 \\ &= \sigma^2 e^{-2\kappa t}\mathrm{E}\left[\left(\int_0^t e^{\kappa u}\sqrt{r_u}\,dW_u\right)^2\right] \\ &= \sigma^2 e^{-2\kappa t}\mathrm{E}\left[\int_0^t e^{2\kappa u} r_u\,du\right] \\ &= \sigma^2 e^{-2\kappa t}\int_0^t e^{2\kappa u}\mathrm{E}(r_u)\,du \end{aligned}$$

using $(dW_t)^2 = dt$ from properties of Brownian motion.

Substituting $\mathrm{E}(r_t)$ from above gives:

$$\begin{aligned} \mathrm{var}(r_t) &= \sigma^2 e^{-2\kappa t}\int_0^t e^{2\kappa u}\mathrm{E}(r_u)\,du \\ &= \sigma^2 e^{-2\kappa t}\int_0^t e^{2\kappa u}\left[\theta + (r_0 - \theta)e^{-\kappa u}\right]du \\ &= \sigma^2 e^{-2\kappa t}\left[\theta\left[\frac{1}{2\kappa}e^{2\kappa u}\right]_0^t + (r_0 - \theta)\left[\frac{1}{\kappa}e^{\kappa u}\right]_0^t\right] \\ &= \frac{\sigma^2\theta}{2\kappa}(1 - e^{-2\kappa t} - 2e^{-\kappa t} + 2e^{-2\kappa t}) + \frac{\sigma^2}{\kappa}r_0 e^{-2\kappa t}(e^{\kappa t} - 1) \\ &= \frac{\sigma^2\theta}{2\kappa}(1 - e^{-\kappa t})^2 + \frac{\sigma^2}{\kappa}r_0 e^{-\kappa t}(1 - e^{-\kappa t}) \end{aligned}$$

[*Full marks were given for obtaining any of the final three representations of the integral.*]

In this question on the Cox-Ingersoll-Ross model, most candidates were confident on the reasons why negative rates are avoided, i.e. mean reversion and volatility dampening. However, several said that rates could not go negative because $\sqrt{r}$ would then not be a real number. This did not explain why rates would not go negative, only what problem might occur with the model if they did.

Part (ii) was attempted well, being a simple non-stochastic differentiation by parts. In part (iii), candidates derived the expectation easily enough but lost their way a little evaluating the integral to obtain the variance. In such questions, it is always good practice to start with a clearly written definition before following through the algebra carefully.

QUESTION 4

Syllabus section: (e) + (j)

Core reading: 5, 13

(i)

Let the principal amount of the floor be L and the strike rate R. [*There could be a different strike for each floorlet, but this is unusual.*]

For each floorlet in a floor, say $k = 1, 2, ... n$, let time to expiry be t_k, let the underlying reference interest rate be r_k, i.e. the rate that applies between times t_k and t_{k+1}, and let F_k be the equivalent forward rate at time t_k.

Let the volatility of each r_k be σ_k, and let $\delta_k = t_{k+1} - t_k$.

[*Note that the volatility is time-dependent, i.e. σ_k and not just σ.*]

The Black model states that:

Price of kth caplet at time 0 = $L\delta_k P(0,t_{k+1})\left[RN(-d_2) - F_k N(-d_1)\right]$

where $d_1 = \dfrac{\ln\left(F_k/R\right) + \frac{1}{2}\sigma_k^2 t_k}{\sigma_k\sqrt{t_k}}$, $d_2 = d_1 - \sigma_k\sqrt{t_k}$,

and $P(0, t_{k+1})$ is the price of a zero-coupon bond paying 1 at time t_{k+1}.

The value of the floor is the sum of the value of each of the floorlets.

(ii)

We have L = A\$10m, $R = 0.05$, $F_k = 0.05$, $\sigma_k = 0.18$, $t_k = 1$ and $\delta_k = t_{k+1} - t_k = 0.25$.

Then $d_1 = \dfrac{\ln\left(F_k / R\right) + \frac{1}{2}\sigma_k^2 t_k}{\sigma_k \sqrt{t_k}}$ = (0 + 0.0162) / 0.18 = 0.09

and $d_2 = d_1 - \sigma_k \sqrt{t_k}$ = 0.09 – 0.18 = –0.09

P(0, t_{k+1}) = $(1 + 0.05 / 4)^{-5}$ = 0.93978 using quarterly compounding

Hence floorlet price

= A$ 10,000,000 × 0.25 × 0.93978 [0.05 × N(0.09) – 0.05 × N(–0.09)]
= A$ 2,349,450 [0.05 × 0.53586 – 0.05 × 0.46414] = A$ 8,425

(iii)

(a)

The cash flows of the reverse floater are:

Quarters 1 to 20:	$2.5 - \dfrac{f_i}{4}$	i = 1 to 20	(quarterly coupons)	(A)
+ Quarter 20:	100		(redemption)	(B)

where f_i are the forward rates at time i in quarter years.

But the coupons on a 10% per annum quarterly paying fixed-floating swap are the same as in (A) ....
... and the 5-year zero coupon bond pays 100 at the end of quarter 20 as in (B).

Hence the decomposition required.

(b)

Discount quarterly = v = 1 / (1 + 0.05 /4) = 0.98765

Value of 5-year zero coupon bond = 100 v^{20} = 77.994

Value of 5-year 10% swap = value of fixed payments – value of floating payments

= [2.5 ($v + v^2 + v^3 \ldots + v^{20}$)] + [100 v^{20} – 100]
= 2.5 (1 – v^{20}) / (0.05 / 4) + [77.994 – 100]
= 44.012 – 22.006 = 22.006

Hence value of reverse floater = notional × (value of swap + value of 5-year zero)

= A$10,000,000*(22.006 + 77.994) / 100 = A$10,000,000.

[*It might seem surprising that the value is exactly par, but this derives from the fact that the 10% fixed rate on the reverse floater is twice the 5% forward rates, and so 10% – 5% = 5%.*]

(iv)

The floor is a straightforward way of protecting a deposit, very liquid ...
... but is an option, so there is a premium cost that depends on implied volatility.

The reverse floater is also simple but more of a speculative instrument, with less liquidity.

Although the example reverse floater is currently valued at par, and its initial coupon is the same as on a normal floater, it does not behave like a normal floater. [*In fact, its duration is about twice that of a normal floater.*]

There are different initial outlays, giving different funding and capital requirements.

[*Other valid points could be made.*]

This question dealt with an interest rate floor and contrasted it with a "reverse floater" bond (the latter fully defined in the question). Both instruments provide protection when rates fall, but in different ways, as explored in part (iv).

Parts (i) and (ii) were straightforward bookwork and numerical calculation based on Black's model, and presented few problems for the well prepared candidate.

Part (iii) required candidates to understand the concept of the "reverse floater" and apply the valuation technique for interest rate swaps. For those candidates uncertain how to proceed here, writing down a quick algebraic representation of the "reverse floater", or working out a simple numerical example, might have helped.

QUESTION 5

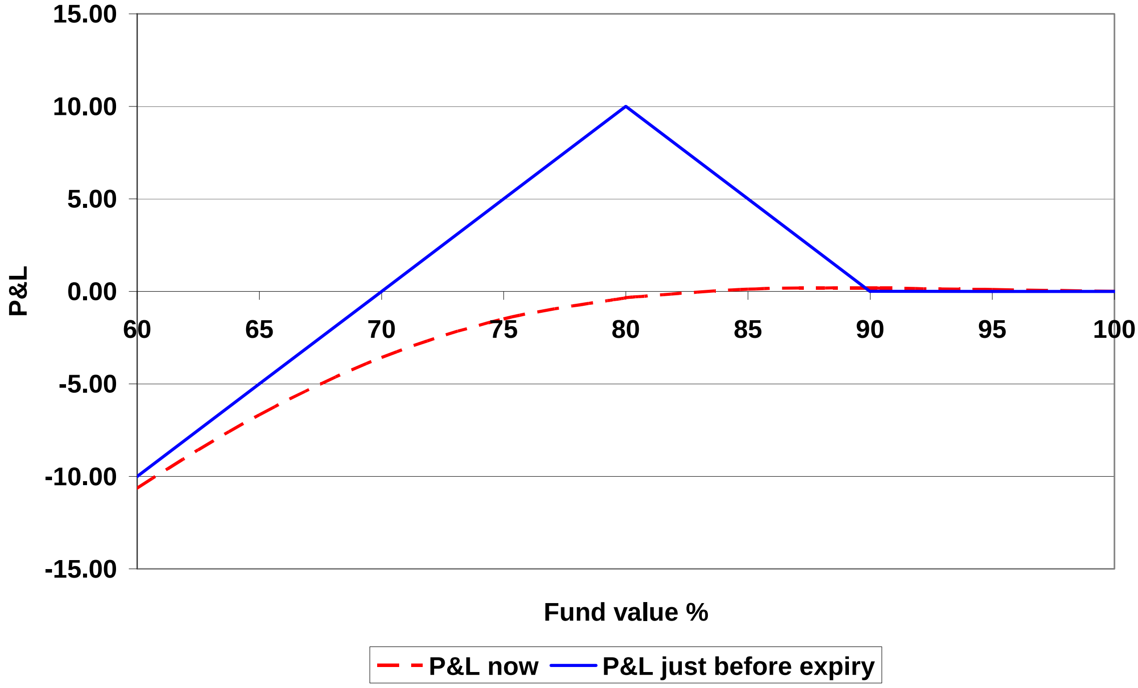
Syllabus section: (g) + (i)

Core reading: 7, 12

(i)

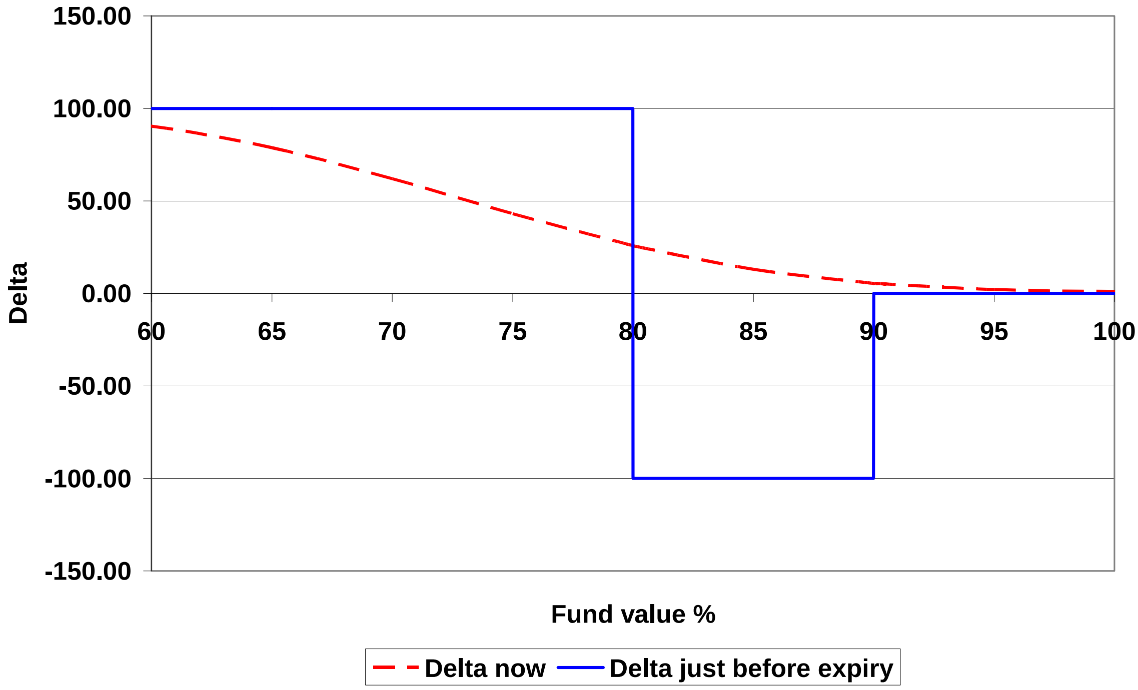
[*Note: the diagrams below were drawn by computer – the examiners are not expecting such precise sketches. Y-axis labelling is not needed.*]

(a)

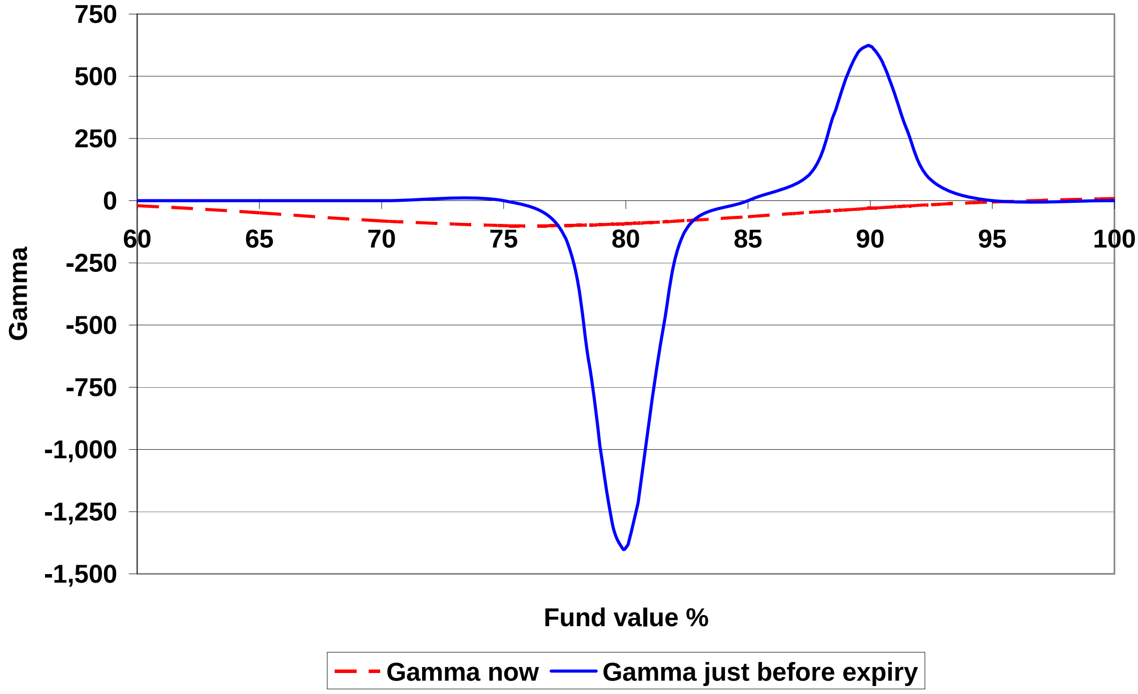


[*As there is no net premium paid, the blue solid line is zero above 90%. Also, although not very clear in the above graph, the red dotted line moves above zero between 85% and 95%.*]

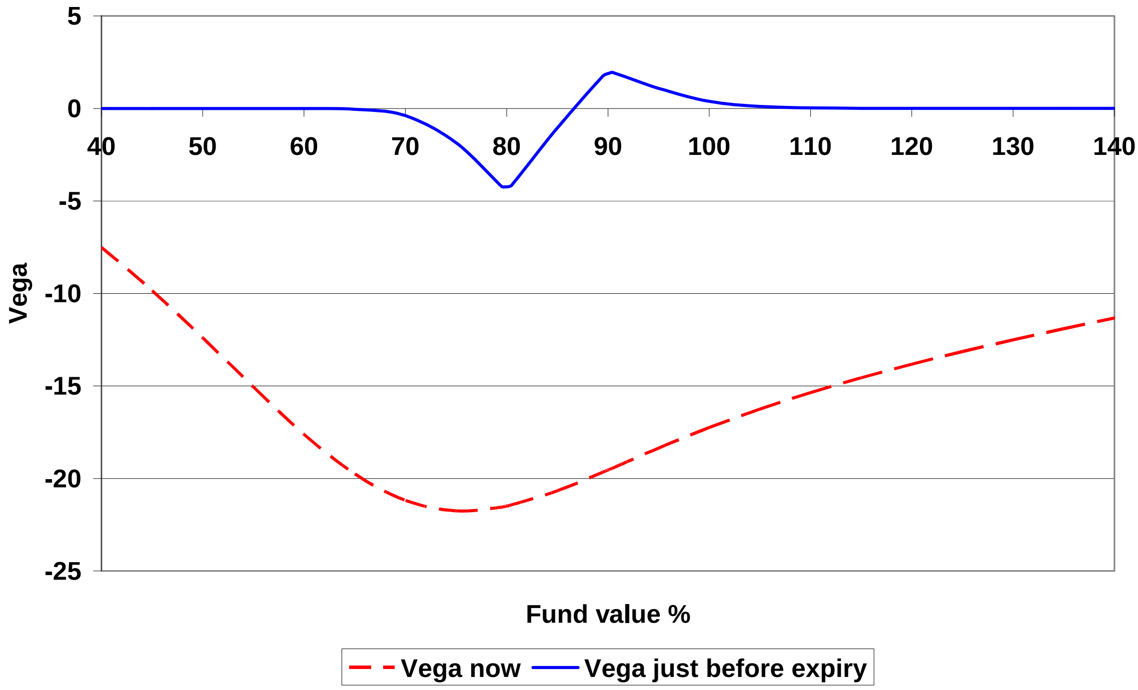
(b)


150.00
100.00
50.00
0.00
-50.00
-100.00
-150.00
Delta
60
65
70
75
80
85
90
95
100
Fund value %
Delta now
Delta just before expiry

(c)


750
500
250
0
-250
-500
-750
-1,000
-1,250
-1,500
Gamma
60
65
70
75
80
85
90
95
100
Fund value %
Gamma now
Gamma just before expiry

(d)



(ii)

Fund value has dropped from 100% to 85% (we should work in % of par).

$T - t = 3/12 = 0.25$ and $r = q = 0$ (zero rates and dividends).

For in-the-money options (strike 90), $\sigma_I = 0.25$, so $\sigma_I \sqrt{T-t} = 0.125$

For out-of-the-money options (strike 80), $\sigma_O = 0.35$, so $\sigma_O \sqrt{T-t} = 0.175$

For strike 90,

$$d_1 = \frac{\ln(S/90) + \frac{1}{2}\sigma_I^2 (T-t)}{\sigma_I \sqrt{T-t}} = [\ln(85/90) + 0.5 \text{ x } (0.125)^2] / 0.125$$

$$= [-0.05716 + 0.007813] / 0.125 = -0.39480$$

and $d_2 = d_1 - \sigma_I \sqrt{T-t} = -0.51980$

Then $N(-d_1) = 0.65350$ and $N(-d_2) = 0.69840$
so Put (strike 90) = $90 \times 0.69840 - 85 \times 0.65350 = 7.308$

For strike 80,

$$d_1 = \frac{\ln(S/80) + \frac{1}{2}\sigma_O^2 (T-t)}{\sigma_O \sqrt{T-t}} = [\ln(85/80) + 0.5 \text{ x } (0.175)^2] / 0.175$$

$$= [0.06063 + 0.01531] / 0.175 = 0.43394$$

and $d_2 = d_1 - \sigma_I \sqrt{T-t} = 0.25894$

Then $N(-d_1) = 0.33217$ and $N(-d_2) = 0.39784$
so Put (strike 80) = 80 × 0.39784 – 85 × 0.33217 = 3.593

Hence profit = 7.308 – 2 × 3.593 = 0.122, since initial cost was zero.

(iii)

(a)

Positive aspects

- No net cost so it is a cheap strategy to protect the portfolio down to 70% at expiry
- If the market falls, the strategy can be rolled to a lower level, in principle at no net cost
- Provided the market does not break the 70% level, the strategy provides significant protection at expiry …
- … which would help protect the underlying maturity guarantees.

Negative aspects

- Initially the transaction will be delta positive, gamma negative and vega negative so that the "hedge" will increase losses in the event of short term falls in the market
- Unlimited downside risk
- Not necessarily a perfect (or even good) hedge for the fund – only an equity part
- Basis risk exists between the index and the equities in the fund
- Capital requirements will increase
- Each time a "roll" down is required, the market exposure of the portfolio increases … as will capital requirements
- If the market continues to fall, the insurer will not have enough capital to continue the strategy and will have to close the hedge taking a significant loss

- The next "roll" may not be available at zero net cost

- Short term market volatility will cause a loss on the hedge position since initially the transaction will be vega negative and gamma negative

- Significant risk of margin calls which might lead to asset liquidation in a falling market

- No protection for the first 10% of a market fall at expiry

(b)

The proposal is unwise and should not be pursued.

One way of looking at it is that the company would be purchasing a limited risk / limited benefit put spread plus writing a naked put option. This cannot be a good hedge for the underlying assets especially in volatile conditions.

In fact, this strategy is best used to gain exposure to the market in volatile conditions, or where low volatile range trading conditions are expected.

The strategy is gambling on the survival of the company. If the company has sufficient resource to execute the roll in need then a loss on the hedge transaction can be avoided.

However, if the company runs out of resources to maintain the strategy a significant loss will occur and that would be in addition to the loss on the underlying portfolio.

This question required the drawing of option diagrams in part (i). Candidates are very familiar with what needs to be done here, but it is important to consider carefully the differences between diagrams at expiry and before expiry. As can be seen from the solution given, curves relating to a time long before expiry do not hug the expiry lines anything like as closely as some candidates surmised.

Part (ii) provided an easy five marks for a Black-Scholes calculation with zero dividends and risk-free rates. Since the option hedge was put on at no cost, the profit was simply the net value of the options after three months.

For part (iii), six marks were available which meant a reasonably long answer was expected. The basic idea was that the hedge did not provide full downside protection, so chasing the market down by doubling up was likely to be a bad strategy.

QUESTION 6
Syllabus section: (l) + (m)
Core reading: 15, 16

[*There are six possible answers listed below – other strategies and points may be valid.*]

Assessment Key: ☑ = positive, ☒ = negative, ☒☒ = very negative (for the company)

(1) <u>Replace some (say 20%) of the corporate bonds with government bonds of comparable duration and credit quality (and in the same currency)</u>

- ☑ would reduce credit capital requirements by say £240m
- ☑ easy to execute
- ☑ easy to continue the interest rate sensitivity management strategy
- ☒☒ would reduce revenue by around £40m per annum (assuming no defaults)
- ☒☒ creates a loss which will itself consume capital on an ongoing basis

(2) <u>Replace some (say 20%) of the corporate bonds with Southern European government bonds</u>

- ☑ will increase revenue by £20m per annum for no extra risk charge
- ☑ will probably have reasonable liquidity in a stable market ...
- ☒ ... but it's a bet on government funding and politics in the Southern European nations
- ☒ liquidity would probably be poor if a crisis developed, making this position hard to get out of quickly
- ☒ there is additional currency risk, which could be hedged (at a cost)
- ☒ duration matching may not be possible, based on the range of bonds available

(3) <u>Replace some (say 20%) of the corporate bond portfolio with collateralised funding transactions</u>

- ☑ replaces £240m of credit spread capital requirements with a counterparty capital requirement that is likely to have a much smaller use of capital
- ☑ can use swaps to manage any interest rate mismatch arising
- ☒ liquidity will probably be limited
- ☒ will require some effort for the company to understand the structures involved and execute necessary documentation
- ☒ unlikely to be long enough maturities
- ☒ similar issues as (2) above regarding support infrastructure and collateral

(4) Replace long dated corporate bonds with shorter dated corporate bonds

- ☑ maximises spread capture at 10 years with current yield curve shape
- ☑ will reduce spread capital requirements significantly ...
- ☒ ... but introduces a spread reinvestment risk ...
- ☒ ... and some interest rate mismatch risk that will need to be managed via a swap overlay

(5) Purchase credit protection on the corporate bonds, say through credit default swaps (CDS)

- ☑ strategy is relatively simple and well understood
- ☒ the question indicates that the cost of such protection will be significant; it probably only makes sense if the company is concerned about economic credit risk on a particular pool of assets
- ☒ infrastructure required to process derivatives, as well as to ensure hedging is effective and allowed under rules
- ☒ CDS contracts generally have low liquidity and may not be available for all bonds
- ☒ collateral margining may be required by the counterparty to cover mark-to-market movements

(6) Purchase out-of-the-money credit protection on the corporate bonds, for example through a senior tranche (such as a 9-15% loss attachment)

- ☑ cost is reduced, since really only catastrophe-level protection is purchased
- ☒ it is not at all clear the hedge has any real value to the company
- ☒ basis risk against the underlying portfolio will need to be allowed for
- ☒ unlikely to be liquid enough market in each name required
- ☒ so, if using macro hedges (e.g. global credit indices), could have a second order (quanto) component, or even a first order currency risk component

[*Note: in the above, it was not necessary to include estimates of the revenue or capital impacts. The question asked for four strategies only – a quarter of the total marks were allocated for each one.*]

This question was deemed the hardest by most candidates, and many avoided it altogether. Essentially, it asked for the application of risk mitigation techniques to an unfamiliar problem, which the well-informed might have recognised as the new Solvency II regulatory regime. It was not, however, necessary to have any prior knowledge about Solvency II to answer the question.

Although the description of the problem was long, it was not that hard to postulate at least one or two of the four strategies required. The basic premise was the new regime penalises corporate bonds over government bonds, and longer-dated bonds over shorter-dated. Hence strategies to reduce capital charge would be to switch into government bonds, shorten *duration and/or buy credit protection on the corporate bonds to reduce the net exposure. Several candidates described these strategies.*

Candidates then needed to assess their advantages and disadvantages of each strategy. Points that were relevant revolved around the impact on revenue, the resulting match against liabilities, and other considerations such as liquidity or administrative capability.

Quite a few candidates thought that a low spread was desirable as it indicated less credit risk. This did not, however, reflect the need to focus on the capital requirement, which was not based on the spread but on the asset class.

QUESTION 7

Syllabus section: (h)(iv)-(ix)
Core reading: 10-12

(i)

(a)

Let c_t be the fair value price at time t of the European Call, strike K, maturity T, and p_t the price of the equivalent European Put. As stated later in the question, we use C_t and P_t for the equivalent American Call and Put fair value prices.

From Black Scholes:

$$c_t = S_t N(d_1) - Ke^{-r(T-t)}N(d_2) \text{ and } p_t = Ke^{-r(T-t)}N(-d_2) - S_t N(-d_1)$$

where as usual N is the cumulative standard Normal distribution and

$$d_1 = \frac{[\ln(S_t/K) + (r + \tfrac{1}{2}\sigma^2)(T-t))]}{\sigma\sqrt{T-t}} \text{ and } d_2 = d_1 - \sigma\sqrt{T-t}\ .$$

[*Candidates are not required to write these out, as they are given in the Actuarial Formulae and Tables book.*]

But, since $N(-x) = 1 - N(x)$,

$$p_t = Ke^{-r(T-t)}N(-d_2) - S_t N(-d_1) = Ke^{-r(T-t)}[1 - N(d_2)] - S_t[1 - N(d_1)]$$

so

$$c_t - p_t = S_t N(d_1) - Ke^{-r(T-t)}N(d_2) - Ke^{-r(T-t)}[1 - N(d_2)] + S_t[1 - N(d_1)]$$

i.e. $c_t - p_t = S_t - Ke^{-r(T-t)}$, which is the Put-Call parity formula.

[*Other approaches are acceptable, e.g. considering two portfolios.*]

(b)

There are three steps to obtaining the formula for rho.

(I) Since $N'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$,

$$N'(d_1) = N'(d_2 + \sigma\sqrt{T-t}) = \frac{1}{\sqrt{2\pi}} \exp\left[-\tfrac{1}{2}\left(d_2^2 + d_2 2\sigma\sqrt{T-t} + \sigma^2(T-t)\right)\right]$$

$$= N'(d_2)\exp\left[-\left(d_2\sigma\sqrt{T-t} + \tfrac{1}{2}\sigma^2(T-t)\right)\right] = N'(d_2)\frac{Ke^{-r(T-t)}}{S_t}$$

with the last step using the definition of $N(d_2)$ and taking exponentials; hence $S_t N'(d_1) = Ke^{-r(T-t)} N'(d_2)$.

(II) Since d_1 and $d_2 = \frac{r(T-t)}{\sigma\sqrt{T-t}}$ + terms not involving r,

we have $\frac{\partial d_1}{\partial r} = \frac{\partial d_2}{\partial r}\left[= \frac{T-t}{\sigma\sqrt{T-t}} = \frac{\sqrt{T-t}}{\sigma}\right]$.

(III) Taking partial derivatives of c_t wrt r:

$$\frac{\partial c}{\partial r} = S_t N'(d_1)\frac{\partial d_1}{\partial r} - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial r} + K(T-t)e^{-r(T-t)}N(d_2)$$

$$= K(T-t)e^{-r(T-t)}N(d_2)$$

using (I) and (II)

so rho = $K(T-t)e^{-r(T-t)}N(d_2)$.

[*The usual formula for rho, e.g. the one that Hull quotes, is obtained by setting t = 0. It is acceptable to set t = 0 throughout.*]

(ii)

(a)

Since a European Put is always worth more than zero for $t < T$, by Put-Call parity:

$$c_t > S_t - Ke^{-r(T-t)}.$$

But we must have $C_t \geq c_t$, because an American Call has all the attributes of a European Call plus the early exercise possibility.

So $C_t \geq c_t > S_t - Ke^{-r(T-t)} \geq S_t - K$ since $r \geq 0$.

But on early exercise, $C_t = S_t - K$. Hence it can never be optimal to exercise early, which implies $C_t = c_t$ for all $0 \le t \le T$.

(b)

From Put-Call parity, $c_t - p_t = S_t - Ke^{-r(T-t)}$, and we know that $C_t = c_t$ (from (a) above) and $P_t \ge p_t$ (an American Put has all the attributes of a European Put plus the early exercise possibility), so:

$C_t - P_t \le S_t - Ke^{-r(T-t)}$: this is the RHS of the inequality to be proved.

Now, consider at time t a portfolio A consisting of one European Call option and cash of K.

At time T, portfolio A will be worth:

$$\max(S_T - K,\, 0) + Ke^{r(T-t)} = \max(S_T, K) + K(e^{r(T-t)} - 1)\,.$$

Consider a portfolio B consisting of one American Put option plus one share. If it is not exercised early, its value at time T is $\max(S_T, K)$.

Hence at time T, portfolio B is worth less than portfolio A.

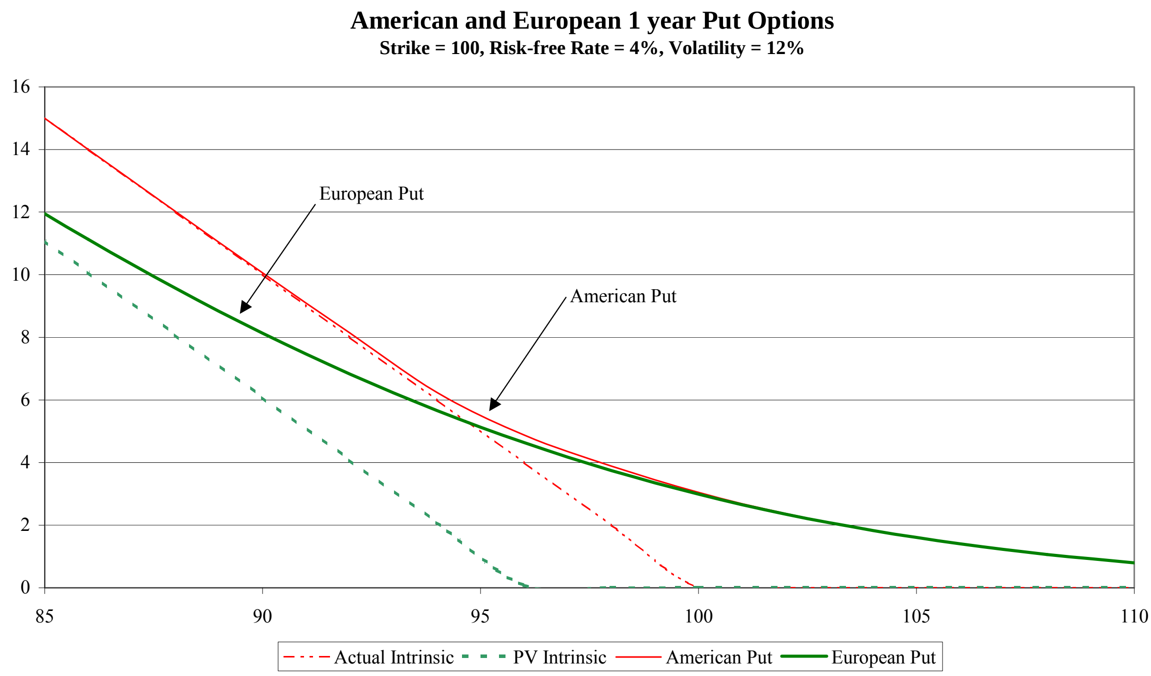
But if the American Put is exercised early, portfolio B is only worth K and portfolio A is always worth at least K.

So portfolio B is never worth more than portfolio A, i.e. $c_t + K \ge P_t + S_t$
or, since $C_t = c_t$, $C_t - P_t \ge S_t - K$, which is the LHS of the inequality.

(iii)

[*The required graph is shown below for a particular example. The exact details are less important than the relative positions of the lines. Separate graphs for the European Put and American Put are also acceptable provided they are labelled consistently.*]

[*"PV Intrinsic" has been added to illustrate how intrinsic value moves over time: it is not required for the solution.* A*s time to expiry decreases, "PV Intrinsic" approaches "Actual Intrinsic". For r = 0, which also the case for margined options on futures, they co-incide.*]



Part (i) of this question asked for a derivation of rho, the sensitivity of an option to interest rates. This was familiar and well answered, although a few candidates jumped stages of the algebra by simply quoting interim results (and lost a few marks).

Both sub-sections of part (ii) on option inequalities are bookwork, but needed careful logic based on the hints given. For this type of question it is often best to set out the steps on a separate sheet of paper before writing in the answer booklet. Some candidates presented confused (and confusing) arguments.

The subject of part (iii), the way European and American Puts approach their expiry lines, is important to grasp conceptually. With non-zero interest rates, the European expiry line actually moves towards the intrinsic value line over time.

QUESTION 8

Syllabus section: (h)(i)-(iii)
Core reading: 8, 9

(i)

As S_t follows Brownian motion, $S_t = S_0 \exp(\mu t + \sigma W_t)$

Let $L_t = \ln(Z_t)$. Then

$$\begin{aligned} L_t &= \ln(B_t^{-1} S_t) = \ln(B_t^{-1}) + \ln(S_t) = \ln(e^{-rt}) + \ln(S_0 \exp(\mu t + \sigma W_t)) \\ &= -rt + \ln S_0 + \mu t + \sigma W_t \end{aligned}.$$

Hence:

$$dL_t = (\mu - r)dt + \sigma\, dW_t\,.$$

Now as $Z_t = f(L_t) = \exp(L_t)$, we apply Ito's formula to get:

$$dZ_t = \left((\mu - r) f'(L_t) + \tfrac{1}{2}\sigma^2 f''(L_t)\right)dt + \left(\sigma f'(L_t)\right)dW_t$$

Substituting $f'(L_t) = f''(L_t) = \exp(L_t) = Z_t$ leads to

$$dZ_t = Z_t(\mu - r + \tfrac{1}{2}\sigma^2)dt + \sigma Z_t dW_t$$

[*Other simple derivations are possible.*]

(ii)

Consider a dynamic portfolio (φ_t, ψ_t) consisting of φ_t units of S_t and ψ_t units of B_t.

A portfolio is <u>self-financing</u> if and only if changes in its value depend only on changes in the prices of the assets constituting the portfolio.

Mathematically, if V_t denotes the value of the portfolio (φ_t, ψ_t), then the portfolio is self financing if and only if

$$dV_t = \varphi_t dS_t + \psi_t dB_t$$

where φ_t and ψ_t are previsible.

[*Alternatively, for stocks paying dividends at rate q,* $dV_t = \phi_t\left[dS_t + qS_t dt\right] + \psi_t dB_t$.]

A <u>replicating</u> strategy for X is a strategy which involves investing in previsible quantities (φ_t and ψ_t) of stock and risk-free bonds, such that:

- the portfolio of (φ_t, ψ_t) of stocks and bonds will be self-financing
- the portfolio (φ_t, ψ_t) will have terminal value equal to the magnitude of the claim, i.e. $V_T = \varphi_T S_T + \psi_T B_T = X$, which means that the portfolio cash flows at claim exercise date match the cash flows under the claim.

(iii)

Steps are as follows.

(1) Apply the Cameron-Martin-Girsanov theorem to Z_t. This states that there exists a new probability measure, say Q, equivalent to the current measure, such that Z_t is a Q-Martingale, i.e.:

$dZ_t = \sigma Z_t d\widetilde{W}_t$ where $\widetilde{W}_t$ is a standard Brownian motion under Q.

(2) Form the discounted expected claim process E_t under measure Q:

$$E_t = \mathrm{E}[B_T^{-1} X \mid \mathbf{F}_t]$$

where $\mathbf{F}_t$ is the history of the process up to time t.

This process is a Q-Martingale, which can be demonstrated using the properties of Martingale and the Tower Law of conditional probabilities.

(3) Invoke the Martingale Representation Theorem (MRT) which states that there is a previsible function φ_t such that:

$$dE_t = \varphi_t dZ_t$$

(4) Consider a replication strategy of holding φ_t units of stock, where φ_t is chosen based on the MRT, and $\psi_t = E_t - \varphi_t Z_t$ of risk free bonds.

Firstly, we show that this portfolio replicates the value of the claim.

The value of the portfolio at any time t can be written:

$V_t = \varphi_t S_t + \psi_t B_t = B_t E_t$ (substituting from the definition of ψ_t)

so $V_T = B_T E_T = \mathrm{E}[X \mid \mathbf{F}_t] = X$.

Secondly, differentiating V_t gives:

$dV_t = d(B_t E_t) = B_t dE_t + E_t dB_t$ (using the product rule, as B_t is non-stochastic)

$\Rightarrow$ $dV_t = \varphi_t B_t dZ_t + (\varphi_t Z_t + \psi_t) dB_t$ (substituting dE_t and E_t from the above)

$\Rightarrow$ $dV_t = \varphi_t (B_t dZ_t + Z_t dB_t) + \psi_t dB_t = \varphi_t dS_t + \psi_t dB_t$

(substituting $dS_t = d(B_t Z_t) = B_t dZ_t + Z_t dB_t$, as B_t is non-stochastic),

hence the portfolio is self-financing.

(5) Since the portfolio replicates the claim without any additional funds (generated or required), the arbitrage-free condition requires that the value of the claim equals the value of the replicating strategy.

The final parts of the paper can sometimes be quite long and involved, but this question was a straightforward application of stochastic processes to the pricing of contingent claims. The emphasis here on bookwork made up for a lower amount elsewhere in the paper.

The average achieved for Q8 was 70% of the marks allotted, which suggests that most candidates were well versed in the theory. The balance of marks available may have been a little generous for part (i), but the better candidates ensured that they filled out their answers with the requisite number of distinct and relevant points, so as to secure a full allocation.

END OF EXAMINERS' REPORT