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SYMMETRY IN CENTRAL POLYNOMIAL INTERPOLATION

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We use the term 'central' to indicate that the formulae considered are understood to be applied so that the centre of each segment interpolated is the same as the centre of the data from which it is calculated. This is of course the normal use of formulae for subdivision of intervals, whether expressed in terms of central differences or written in some other form. When, in the course of a proof, it is necessary to mention a formula that is not applied centrally, this will be indicated by the text.

The formulae of different types that have been devised for subdivision of intervals have invariably been symmetrical in the sense that they yield the same numerical results if the order of the series is reversed, and this symmetry is normally a necessary result of the basis of the formula. Using the letter a for the central point of any given segment of the data, we take as one illustration the ordinary formula, correct to fifth differences, for filling the space between $u_{a-\cdot 5}$ and $u_{a+\cdot 5}$ from given values of

$$u_{a-2\cdot 5},\quad u_{a-1\cdot 5},\quad u_{a-\cdot 5},\quad u_{a+\cdot 5},\quad u_{a+1\cdot 5},\quad u_{a+2\cdot 5}.$$

The process ascertains a polynomial curve of the fifth order which will pass through six given points and, since there is only one such curve, the interpolated numbers must be the same whether we work from left to right along the series or from right to left. In the same way, for Sprague's formula there are six conditions to satisfy from six given values, so that there is only one curve which will satisfy the conditions, and we must get the same result from working in either direction. Again, in formulae that have been deduced for minimizing the summed squares of an order of differences the conditions to be met are the same if the order of the series is reversed. In each case there is a unique solution and the same symmetrical property exists. The direction which we treat as negative is arbitrary, and in all these cases reversing it cannot alter the result.

Admittedly, it is mathematically possible to construct an unsymmetrical polynomial central formula, but no one appears to have found reason for doing so. It can be done only if the order of the polynomial stipulated is higher than can be fully determined by the data, so that there is a multiplicity of solutions. In that case a reversal of order can bring us from one solution to another. Apart from this theoretical proviso the symmetrical case covers the field.

We propose to show that certain properties, usually attributed to particular formulae, are in fact general, and can be deduced directly from the principle of symmetry.

PROPOSITION I

(*a*) *The sum of the terms interpolated centrally, from values of u given at $2n$ equidistant points, by a symmetric formula correct to $(2n-2)$ differences, is the same for all formulae.*

(*b*) *The term interpolated at the centre is the same for all formulae.*

Assume that we are given $2n$ values of u, at the points $(a-n+\cdot 5)$, $(a-n+1\cdot 5)$, etc., up to $(a+n-\cdot 5)$; let v_{a+x} denote the term interpolated at the point $(a+x)$ by any given formula of the type described; and let w_{a+x} denote the term interpolated by another such formula.

Two interpolations, each correct to $(2n-2)$ differences, can vary from one another only by some compound of differences of order $(2n-1)$; and in this case there is only one difference of that order in the range of the data, namely, $\delta^{2n-1}u_a$ (where δ indicates a central difference).

Therefore we can write

$$v_{a+x}-w_{a+x}=f(x)\,\delta^{2n-1}u_a, \tag{1}$$

where $f(x)$ is a polynomial.

The coefficients and degree of $f(x)$ will depend on the particular formulae that have been given, but for any pair of formulae $f(x)$ is a determinate function.

Now, if we reverse the direction of working, by hypothesis v_{a+x} and w_{a+x} will not alter; but the sign of δ^{2n-1} will change, and its coefficient will become $f(-x)$. It follows that we can also write

$$v_{a+x}-w_{a+x}=-f(-x)\,\delta^{2n-1}u_a. \tag{2}$$

Hence $f(x)$ must equal $-f(-x)$ (for all values of x); and it follows that $f(x)$ cannot contain a constant. Therefore $f(0)$ must vanish; and, putting $x=0$ in equation (1), we find $v_a=w_a$. This proves (*b*).

Now if any number of equidistant terms are interpolated centrally by the given formulae, say between $u_{a-\cdot 5}$ and $u_{a+\cdot 5}$, the values of $f(x)$ to right and left of $f(0)$ cancel out in pairs, and the sum of $f(x)$ over the segment is zero. From (1) it follows that the sum of v_{a+x} is the same as that of w_{a+x}. If a continuous curve is interpolated, the same statement applies to the definite integral. This proves (*a*).

Rider

The sum of the terms interpolated must equal the mean sum of corresponding terms obtained, by two ordinary interpolations correct to $(2n-2)$ *differences, from* $(2n-1)$ *of the given values centring at* $(a-\frac{1}{2})$ *and* $(a+\frac{1}{2})$ *respectively.*

This is really a particular case of the Proposition. The two ordinary formulae are off-centre to the same extent in opposite directions; and their mean constitutes a single symmetric formula, correct to $(2n-2)$ differences, centring in a. The main Proposition therefore establishes this Rider.

(This Proposition and Rider do not apply to formulae based on an odd number of points, because an even order of differences does not change sign when the direction is reversed.)

Particular applications

To take a simple illustration, Proposition I shows that, if say u_{-1}, u_0, u_1 and u_2 are given, and the gap between u_0 and u_1 is filled (1) by ordinary third-difference interpolation, (2) by King's formula, (3) by second-difference interpolation from u_{-1}, u_0 and u_1, and (4) by second-difference interpolation from u_0, u_1 and u_2, then the sums of the terms interpolated by (1) and by (2) (and by any other symmetric formula correct to second differences) will be the same, and will equal the mean sum of the terms interpolated by (3) and (4). The Rider shows that the same equality must apply to the interpolated value of the individual term $u_{\cdot 5}$.

For further illustration, among six-term formulae we have ordinary fifth-difference interpolation, Sprague's formula (which has second-order osculation), Shovelton's formula (first-order osculation), any minimizing formula correct to fourth differences—in fact, any formula of special type that may be devised, provided it is symmetric and correct to fourth differences. The sum of the terms interpolated by each of them is the same, and equals the mean sum of the terms derived by two ordinary fourth-difference interpolations such as described.

The centre term will be the same in every case, so that in halving an interval it would be futile to use a formula of special type.

PROPOSITION II

An n-term interpolation, by a formula with osculation of the rth order, correct to $(n-2)$ differences, centring at a, must at the point $(a-\cdot 5)$ have r derivatives in common with an ordinary interpolation correct to $(n-2)$ differences, centring at $(a-\cdot 5)$; and must have a similar property at point $(a+\cdot 5)$.

We write $I(a-\cdot 5,\ x+\cdot 5)$ to denote the value for u_{a+x}, as interpolated by the ordinary formula, correct to $(n-2)$ differences, centring at $(a-\cdot 5)$; and $I(a+\cdot 5,\ x-\cdot 5)$ for u_{a+x}, as interpolated by the similar formula centring at $(a+\cdot 5)$.

The value of u_{a+x}, interpolated by any given osculatory formula such as described, can (from previous reasoning) be written as

$$I(a-\cdot 5,\ x+\cdot 5)+g(x)\,\delta^{n-1}u_a, \tag{3}$$

where $g(x)$ is a polynomial as before.

In the reverse direction, this becomes

$$I(a+\cdot 5,\ x-\cdot 5)\pm g(-x)\,\delta^{n-1}u_a. \tag{4}$$

(The sign here will be plus if n is odd, minus if n is even.)

The principle of an osculatory formula is, of course, to interpolate the segment of values between the points $(a-\cdot 5)$ and $(a+\cdot 5)$ from data centring at a. The adjoining segment to the left, from $(a-1\cdot 5)$ to $(a-\cdot 5)$, is interpolated from data centring at $(a-1)$. At the junction-point of the two segments, that is, at point $(a-\cdot 5)$, the two segments must meet and have r derivatives in common. Similar conditions must of course hold on the right-hand side, but it is sufficient to examine one side since the other follows from symmetry.

An expression for u_{a+x}, as it would be calculated for the left-hand segment, can be obtained from (4) by substituting $(a-1)$ for a, and $(x+1)$ for x. This gives

$$I(a-\cdot 5,\ x+\cdot 5)\pm g(-1-x)\,\delta^{n-1}u_{a-1}. \tag{5}$$

Subtracting (5) from (3), we get

$$g(x)\,\delta^{n-1}u_a \mp g(-1-x)\,\delta^{n-1}u_{a-1}. \tag{6}$$

Expression (6) and r derivatives thereof must vanish at the point $(a-\cdot 5)$; this can hold, for all series, only if *both* terms of (6) vanish (because the two terms involve different u's). Therefore $g(x)$ and its first r derivatives must vanish at the junction-point $(a-\cdot 5)$.

From (3) it follows at once that at the left-hand junction-point the term interpolated by our given osculatory formula and its first r derivatives must be the same as for $I(a-\cdot 5,\ x+\cdot 5)$. This establishes the Proposition.

SPRAGUE'S FORMULA

Writers have distinguished between the 'classical' and more recent osculatory formulae on the ground that the fundamental principle of the former was that derivatives at the junction-points should have certain predetermined values, and that this restriction was later abandoned. This is a false distinction, since for every formula there is a definite value for such derivatives. The line of approach to a formula by the original research worker is of historic interest, but should not affect the classification of the formula at the present time.

The first osculatory formula devised (that of Sprague in *J.I.A.*, **22**, 270) was deduced without regard to correctness to any order of differences. The object was in effect to interpolate, between u_0 and u_1, a polynomial of the fifth order, calculated from given values of u_{-2}, u_{-1}, u_0, u_1, u_2 and u_3, in such manner that at the points 0 and 1 the first and second derivatives should equal those of adjoining segments interpolated in the same way. As the simplest way of securing this Sprague made two derivatives at each end of a segment of interpolated values the same as those of an ordinary fourth-difference formula centring at that point. The resulting formula, written in terms of advancing differences, is

$$\begin{aligned} u_x = u_{-2} + (x+2)\Delta u_{-2} + (x+2)(x+1)\Delta^2 u_{-2}/2 + (x+2)(x+1)x\Delta^3 u_{-2}/6 \\ + (x+2)(x+1)x(x-1)\Delta^4 u_{-2}/24 + x^3(5x-7)(x-1)\Delta^5 u_{-2}/24. \end{aligned}$$

Written as above, it is obvious that the formula is in fact correct to fourth differences; but the formula was written in the original paper in a form that did not make this evident.

Sprague's formula could equally well have been deduced simply as an osculatory formula correct to fourth differences. $g(x)$ in equation (3) is in this case a polynomial of the fifth order, and so involves six coefficients which must be fixed to meet six conditions, namely, that at each end the curve should meet the adjoining segment and have two derivatives in common with it. Therefore there is only one possible formula, correct to fourth differences, of the specified range, degree and order of osculation. Proposition II shows that the derivatives of such a formula must have the values stipulated by Sprague. Further, it is of course known that there is only one formula answering to the conditions imposed by Sprague. The same formula must therefore emerge from the above approach and from Sprague's.

One authority has remarked that Sprague's selection of the predetermined values seems rather arbitrary, but the above shows that they are the only values consistent with correctness to fourth differences; so that, on retrospective examination, there is good reason for the values chosen.

The most appropriate classification of Sprague's formula would be simply as the six-term formula, of fifth degree, correct to fourth differences, with osculation of the second order.

The usual derivation of the formula taught to students seems to be to start with 'fixed' or 'predetermined' values for derivatives at the ends of the segment and so deduce the formula, which is then found to be correct to fourth differences as it were by accident. It is suggested that a sounder course would be to describe the basis of the formula on the lines of the last preceding paragraph. If it is desired to use the end-values in the proof, it should first be shown that they are a necessary consequence of correctness to fourth differences.

SHOVELTON'S FORMULA

It has been stated as a distinctive feature of Shovelton's formula that the average value of the terms interpolated equals the average of the mean of two ordinary fourth-difference interpolations. Proposition I, however, shows that this is not peculiar to this one formula, but is also a property of Sprague's formula and others.

Shovelton's formula (*J.I.A.*, 47, 284) was designed to secure one continuous derivative (in lieu of Sprague's two), and hence the polynomial interpolated could be made a degree lower than in Sprague's case. The formula can be written

$$u_x = u_{-2} + (x+2)\Delta u_{-2} + (x+2)(x+1)\Delta^2 u_{-2}/2 + (x+2)(x+1)x\Delta^3 u_{-2}/6$$
$$+ (x+2)(x+1)x(x-1)\Delta^4 u_{-2}/24 + x^2(x-1)(x-5)\Delta^5 u_{-2}/48.$$

There is a peculiarity in Shovelton's formula in that it is specified as of the fourth degree (and could not be of lower degree since it is correct to fourth differences). This, without other proviso, would have left five coefficients to determine with only four conditions to satisfy, namely, that the curve should pass through two given points and have one predetermined derivative at each such point. For this reason no doubt, in order to make the formula determinate Shovelton added the condition of average value. It should be made clear, however, that this does not make the formula different from others; on the contrary, it puts it on the same basis. Proposition I shows that Shovelton's extra condition is in fact equivalent merely to stipulating that the formula must be symmetrical; and it will be found that, if the requirement of symmetry is substituted for that of average value, Shovelton's formula will still emerge as the sole solution.

With regard to predetermined values of the first derivative similar remarks to those made on Sprague's formula apply to this case.

Shovelton's formula should therefore be classed as the only symmetrical six-term formula, of the fourth degree, correct to fourth differences, with osculation of the first order.

KING'S (OR KARUP'S) FORMULA

Similar remarks to those on Sprague's formula apply to this, which is a four-term formula, of the third degree, correct to second differences, with osculation of the first order.

To illustrate the use of symmetry in abbreviating a proof, the derivation of this formula will now be considered, the data being taken as u_{-1}, u_0, u_1, u_2.

The ordinary advancing difference formula correct to third differences is

$$u_x = u_{-1} + (x+1)\Delta u_{-1} + (x+1)x\Delta^2 u_{-1}/2 + (x+1)x(x-1)\Delta^3 u_{-1}/6,$$

and the only modification we can make to this is to add an expression of the form $h(x)\Delta^3 u_{-1}$, where $h(x)$ is a polynomial of the third degree.

From Proposition I(*b*), $h(x)$ must vanish when $x=0$, 1 or $\cdot 5$, and so must be of the form $Ax(x-1)(2x-1)$. Now the d.c. of the required curve when $x=0$ must (from Proposition II) be that of the ordinary second-difference formula represented by the first three terms in the above expansion. Hence when $x=0$ the d.c. of $\{Ax(x-1)(2x-1)+(x+1)x(x-1)/6\}$ must be zero. This d.c. is simply the coefficient of x in the expression, so we can at once write $A-\frac{1}{6}=0$. The adjustment therefore is

$$\tfrac{1}{6}x(x-1)(2x-1)\Delta^3u_{-1},$$

agreeing with the result reached at the end of para. 25, *Mathematics for Actuarial Students*, **2**, 150.

MORE RECENT OSCULATORY FORMULAE

The orders of difference to which the formulae of Sprague, Shovelton and Karup are correct, coupled with the degree of the polynomial stipulated, leave only one solution in each case. The relaxation of condition which enables further formulae to be produced does not consist in abandoning a principle of predetermined values for the d.c.'s. The actual relaxation has been in accepting correctness to a lower order of differences, which leaves a multiplicity of solutions from which simple forms can be selected. For example, the earlier formula of Jenkins, quoted in *Mathematics for Actuarial Students*, **2**, 153, is correct to third differences as compared with fourth differences for Sprague.

By increasing the order of the polynomial permitted, further coefficients can be put at disposal so that an infinite number of solutions are possible, and in the same way the order of contact can be made as high as desired.

A further relaxation, of course, occurs in some formulae of Jenkins, where the curve does not pass through the given points and an element of graduation is thereby introduced.