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TEMPORARY SELECTION

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The origin of these notes was the idea that in a search for a 'law of mortality' allowance should be made for the fact that persons living at any age may be divided into two essentially different groups, viz. those who are in their usual state of health (or are perhaps temporarily unwell) and those who have contracted a fatal illness. Doubtless in both groups there is considerable variability, but this will not usually be so important as the difference between the two groups. *A priori* it seemed possible that the presence of 'damaged lives' subject to a rate of mortality appropriate to their illness might make the shape of an ultimate mortality experience different from that of recently selected lives and that the latter might follow a simpler 'law'. Even if this proved not to be the case an investigation might throw some light on the relative rates of mortality to be expected in the select and ultimate experience.

These ideas are fundamentally similar to those of R. D. Anderson set out in his paper on Select Mortality Tables (*J.I.A.* **68**, 223). Anderson assumed that select lives were subject to two decremental forces, one a force of mortality and the other a force of transfer to the category of 'damaged lives' which were subject to a high constant force of mortality. In the present notes the idea of a force of transfer from select lives to damaged is retained, but the problem is approached from the standpoint that the number of damaged lives at any age must depend partly on the shape of the frequency curve of deaths at that age when these deaths are arranged according to duration of illness. If suitable data were available it should be possible to infer the shape of this frequency curve from the manner in which the select rates of mortality progressed to the ultimate rates.

The first step therefore seemed to be to study the effects of selection and hence obtain a formula for the relative numbers of select and damaged lives living at any age, and this led inevitably to Sprague's theory that $l_x - l_{[x]}$ is the number of damaged lives contained in the l_x persons aged x in the ultimate column of a select mortality table formed from life assurance data.

Sprague's theory has been criticized on various grounds which mainly involve the consideration that in practice the data available are not such that the theory can be true. Spurious selection may arise from different causes, e.g. the amalgamation of mortality experience over a period of varying mortality or from class or medical selection varying in amount with time or age. In addition, there is the possibility that withdrawals from the experience may have been selective or that mortality at any age depends partly on the year of birth. Also Sprague's theory was only intended to apply to the selection of healthy lives. In the case of selection of lives because they are impaired other considerations will apply.

However, although there may be difficulty in applying Sprague's theory in practice, with suitable data it can be axiomatic. If, for example, we have a large group of persons and select from these all those who are not suffering from a fatal illness, then the mortality amongst the selected group will for a time be lighter than, and ultimately will be identical with, the mortality of the unselected group.

At this stage may be introduced definitions of the expressions and symbols used:

'Select lives' is applied to persons who are not suffering from a fatal illness and 'Damaged lives' to those who have contracted a fatal illness. γ_x is the rate (or force) at which select lives contract a fatal illness (including death by accident).

u_t is the proportion of damaged lives who survive time t after contracting a fatal illness. It is evident that we may find u_t to be a function of x as well as of t.

Now by way of illustration let us suppose that out of every 100 persons who contract a fatal illness in any policy year 50 die in that year, 25 die in the next year, 15 die in the next and the remaining 10 in the fourth year. Let us suppose further in order to simplify the arithmetic that $\gamma_x l_{[x]}$ is constant and equals 100. Then it is evident that in the years immediately following selection there will be successively 50 deaths, 75 deaths, 90 deaths and in the fourth and subsequent years 100 deaths. Thus it is possible to infer the shape of the frequency curve of deaths arranged according to duration of illness from the manner in which select mortality progresses to ultimate mortality, but since available data only show this progression of mortality at annual intervals we can only make a rough estimate of the shape.

It is shown later, according to the somewhat scanty evidence provided by the A1924–29 data of select mortality, that approximately $u_t = \kappa e^{-at}$ which formula is arrived at on the hypothesis that out of those who contract a fatal illness at any moment the proportion $1-\kappa$ have such a low expectation of life that it can be neglected, whilst the remaining proportion κ are subject to a constant force of mortality $=a$. It is not thought that this 'law' can be any more than an approximation because it seems more likely that the mortality increases with age, and also the formula makes no obvious allowance for the possibility of a person contracting a long-term fatal illness and subsequently dying from a short-term illness.

Let us suppose, therefore, that select lives aged x are subject to a force of mortality of $\overline{1-\kappa}\gamma_x$ representing short-term illness and to a force of transfer to the category of damaged lives of $\kappa\gamma_x$, and further that damaged lives are subject to a force of mortality of $a+\overline{1-\kappa}\gamma_x$. Then if we have a mixture of $l_{[x]}$ select lives and $\lambda_x = l_x - l_{[x]}$ damaged lives

$$\frac{d\lambda_x}{dx} = \kappa\gamma_x l_{[x]} - (a+\overline{1-\kappa}\gamma_x)\,\lambda_x,$$

and also
$$\frac{d\lambda_x}{dx} = \gamma_x l_{[x]} - \mu_x l_x,$$

therefore
$$l_x(a-\mu_x+\overline{1-\kappa}\gamma_x) = a l_{[x]}. \tag{1}$$

Since these equations relate to any mixture of select and damaged lives they may be applied to select mortality. Thus at age $[x]+t$, where there is a mixture of $l_{[x+t]}$ select lives and $l_{[x]+t} - l_{[x+t]}$ damaged lives, we have

$$l_{[x]+t}(a-\mu_{[x]+t}+\overline{1-\kappa}\gamma_{x+t}) = a l_{[x+t]}.$$

Some adjustment will be necessary to make $\mu_{[x]+t}$ join smoothly with the ultimate table.

Now if we differentiate equation (1) with respect to x and then eliminate l_x and $l_{[x]}$, we have for ultimate mortality

$$\frac{d\mu_x}{dx} - \overline{1-\kappa}\,\frac{d\gamma_x}{dx} = (\gamma_x - \mu_x)(a - \mu_x + \overline{1-\kappa}\,\gamma_x). \tag{2}$$

This equation may be altered by the substitution $\mu_x - \overline{1-\kappa}\,\gamma_x = a - \frac{a}{w_x}$ to the form $\frac{dw_x}{dx} + w_x(a - \kappa\gamma_x) = a$ and thence to $\frac{d}{dx}\{w_x e^{ax - \int\kappa\gamma_x dx}\} = a\,e^{ax - \int\kappa\gamma_x dx}$, which is integrable if the form of γ_x is known. But if in the absence of better information as to the law for γ_x we assume that the ratio μ_x/γ_x is constant, it will be found that we arrive at Perks's modification of the Gompertz law. Thus if $\mu_x = b\gamma_x$ equation (2) becomes $\frac{d\gamma_x}{dx} = \frac{(1-b)a}{b-1+\kappa}\gamma_x\left(1 - \frac{b-1+\kappa}{a}\gamma_x\right)$, and putting $\frac{(1-b)a}{b-1+\kappa} = \log_e c$ we find on integration that

$$\gamma_x = \frac{Bc^x}{1 + \dfrac{\kappa}{a + \log_e c}Bc^x} \quad \text{and} \quad \mu_x = \frac{a + \overline{1-\kappa}\log_e c}{a + \log_e c}\,\gamma_x.$$

It should be noted that the assumption that μ_x/γ_x is constant would not be suitable in all circumstances, e.g. in the case of an experience of impaired lives.

According to the foregoing result, therefore, if we fit the Perks curve μ_x or $q_x = \frac{A + Bc^x}{1 + Dc^x}$ to the ultimate experience of a reasonably homogeneous mortality table we should find that the ratio D/B was approximately equal to $\frac{\kappa}{a + \overline{1-\kappa}\log_e c}$. Later in these notes this Perks curve for q_x is fitted to the A 1924–29 ultimate experience of whole-life with-profit policies, and the values found for the ratio D/B range between 1·75 and 2·39 with an average for the six years of 2·03.

From the manner in which the A 1924–29 select mortality rates progress to the ultimate rates the values of the 'constants' are $a = \cdot 3285$, $\kappa = \cdot 5943$; so that the expected value of the ratio $D/B = 1{\cdot}59$ (taking $\log_e c = \cdot 1095$). However, since q_x and not μ_x was fitted to the whole-life experience it would be more consistent to use the annual rate of mortality corresponding to a constant instantaneous rate $a + \overline{1-\kappa}\log_e c$. With this adjustment the expected value of D/B becomes 1·91, which is in good agreement with the mean calculated value of 2·03. It must be pointed out that this close agreement is probably partly fortuitous because a considerable variation in A, B and D can result merely from the omission of one or more age groups from the graduation.

The result supports the supposition that κ is approximately constant at the older ages, but independent evidence of this is desirable. This might perhaps be obtained from a study of causes of death at different ages.

Equation (2) shows that even if κ and γ_x are both constant, μ_x is not necessarily constant, while if κ is increasing, as it must be at the younger ages (say 15 to 35), μ_x may actually decrease for a time as x increases. This may be seen by assuming that over a section of the mortality curve κ increases in arithmetical progression. Substituting hx for κ in equation (1) and proceeding as before we have

$$\frac{d\mu_x}{dx}-(1-hx)\frac{d\gamma_x}{dx}=(\gamma_x-\mu_x)\{a-\mu_x+(1-hx)\gamma_x\}-h\gamma_x.$$

THE A1924–29 DATA

In deciding how the foregoing results might be tested it was desired to choose, from the A1924–29 data, experience which was sufficiently extensive and at the same time seemed likely to be reasonably homogeneous. The whole-life assurance experience for durations 5 and over is sufficiently extensive, but the select experience is small. For endowment assurances the select experience is more extensive, but there is little experience at the older ages.

The most satisfactory solution seemed to be to use the experience for medically examined lives of the 52 original offices 1924–26 and 50 original offices 1927–29. The whole-life with-profits assurance data have been used for the purpose of calculating the constants according to Perks's law, and the corresponding endowment assurance with-profits data have been used to investigate the experience of damaged lives.

Table 1 shows the endowment assurance experience for the six years.

It seemed likely that fluctuations in the mortality from year to year were due to the combined results of random variation and what may be termed environmental fluctuations. In order, therefore, to minimize the effect of the latter the expected deaths for each duration and quinquennial age group were calculated separately for each year according to the experience of that year for durations 5 and over.

The graduated percentages in the table were obtained as follows. It was assumed that in the proportion $1-\kappa$ of those who contract a fatal illness at any moment, the duration of illness is so short that it may be neglected, and that in the remaining proportion κ the numbers living decrease in geometrical progression so that $u_t=\kappa e^{-at}$. On the assumption that fatal illnesses are contracted uniformly throughout the year, out of those who contract a fatal illness in the first policy year $\int_0^1 (1-\kappa e^{-at})\,dt$ per unit will die in that year, $\int_0^1 \kappa(e^{-at}-e^{-a(t+1)})\,dt$ will die in the next year, and so on. If γ_x is a relatively small quantity we shall by summing these expressions obtain approximately the yearly proportions of select annual mortality to ultimate:

$$\text{Duration 0:}\quad 1-\frac{\kappa}{a}+\frac{\kappa}{a}e^{-a};\qquad \text{Duration 1:}\quad 1-\frac{\kappa}{a}e^{-a}+\frac{\kappa}{a}e^{-2a};$$

$$\text{Duration 2:}\quad 1-\frac{\kappa}{a}e^{-2a}+\frac{\kappa}{a}e^{-3a};$$

and so on.

Evidently these expressions are only approximate, but the data did not seem to warrant a more accurate treatment.

The values found by fitting the expressions to the experience were

$$\kappa=\cdot 5943,\quad a=\cdot 3285.$$

A test was made to see whether the experience of lives selected before 1924 was different from those selected in 1924 or after, but this subdivision did not

Table 1. Medically examined lives. 52 original offices 1924–26. 50 original offices 1927–29. With-profit endowment assurance experience

Duration of assurance	Ages 21–45			Ages 46–65			All ages 21–65			Graduated percentage
	Actual deaths	Expected deaths	%	Actual deaths	Expected deaths	%	Actual deaths	Expected deaths	%	
0	280	554	51	171	386	44	451	939	48	49·4
1	341	571	60	296	427	69	637	997	64	63·5
2	475	588	81	334	461	72	809	1049	77	73·7
3	479	654	73	434	512	85	913	1166	78	81·1
4	607	739	82	527	581	91	1134	1320	86	86·4

seem to have any significant effect on the percentages of actual to expected deaths.

The ultimate experience of with-profit whole-life assurances was next graduated by the Perks formula

$$q_x = \frac{A + Bc^x}{1 + Dc^x}.$$

The experience for each of the six years 1924–29 was graduated separately, and in order that no bias should arise in the calculation of the constants an identical method was used for each year. The exposed-to-risk and deaths were grouped in the quinquennial age-groups adopted in the official publication of data and the arithmetical mean age of the exposed-to-risk was calculated for each group. Previous investigations had shown that a reasonable value of c was 1·1157, and this value was used throughout. The exposed-to-risk and deaths for each age-group were multiplied by the appropriate value of c^x, and the constants A, B and D were found by the method of moments.

Table 2 gives a summary of the constants and Table 3 of the graduations.

Table 2. Medically examined lives

Graduation of whole-life with-profit ultimate experience for the years 1924–29 by $q_x = (A + Bc^x)/(1 + Dc^x)$ with $c = 1{\cdot}1157$.

Year	A	B	D/B
1924	·00244	·0000259	1·75
1925	·00207	·0000266	2·28
1926	·00254	·0000247	2·39
1927	·00180	·0000267	2·03
1928	·00219	·0000250	1·97
1929	·00246	·0000274	1·79

CONCLUSION

From the foregoing it is thought it may be concluded that the difference to be expected between the select and ultimate mortality of a homogeneous experience may be expressed in terms of κ, a and $\log_e c$ and variations in κ and a will form a factor in the variations in mortality observed in the past.

As to a 'law of mortality' it seems that such a law must be very complicated. Apart from variations in κ, a, A and B if the law took the form of some exponential formula involving Bc^x it must be assumed that c varies with individuals in order to describe the effect of the known fact of inheritance of longevity. Quite a small change in c could mean several years' difference in the expectation of life.

Allowance would also have to be made for what may be termed a dispersion effect measuring the rate of change of variability amongst select lives. Even if it were possible to observe a number of identical lives in a constant environment it is reasonably certain that the lives would not remain identical (apart from fatal illness) with the lapse of time.

These two effects, a variability of c in individuals and a tendency to dispersion, might tend to act in opposite directions, with the result that c might appear to remain constant.

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Table 3. Medically examined lives

Table showing actual deaths and deviations (= expected deaths − actual deaths). Whole-life with-profit ultimate experience for the years 1924–29, graduated by $q_x = (A + Bc^x)/(1 + Dc^x)$ with $c = 1{\cdot}1157$.

Approximate mean age	1924		1925		1926		1927		1928		1929		1924–29	
	Deaths	Deviation	Deaths	Deviation	Deaths	Deviation	Deaths	Deviation	Deaths	Deviation	Deaths	Deviation	Deaths	Deviation
24	2	—	4	−2	3	—	1	1	5	−3	3	—	18	−4
28	4	2	7	−1	5	2	7	−1	8	−1	6	2	37	3
33	15	3	14	2	11	8	12	2	7	8	21	−5	80	18
38	54	−8	31	11	33	11	34	1	28	8	35	3	215	26
43	93	6	105	−15	101	−11	68	8	84	−9	66	13	517	−8
48	189	7	171	13	182	−2	187	−25	150	5	183	−21	1,062	−23
53	374	−6	357	−5	322	11	312	2	292	6	322	−5	1,979	3
58	670	2	687	−39	639	−36	589	−3	543	5	596	−10	3,724	−81
63	1,195	−30	1,111	10	1,071	−42	1,020	−1	991	−43	974	40	6,362	−66
68	1,840	−26	1,698	75	1,603	32	1,575	66	1,532	−14	1,590	34	9,838	167
73	2,198	7	2,195	−20	2,046	−9	2,153	−29	1,993	14	2,254	−69	12,839	−106
78	1,960	59	2,021	−43	1,863	13	2,047	−30	1,925	32	2,173	−3	11,989	28
83	1,305	16	1,258	6	1,170	52	1,339	−4	1,322	−34	1,405	18	7,799	54
88	593	−30	498	14	538	−30	542	17	522	37	627	13	3,320	21
92	132	—	123	−10	104	7	136	−4	142	−3	176	−12	813	−22
97	18	−3	8	4	19	−5	18	—	26	−7	16	1	105	−10
101	—	1	2	—	2	−1	1	—	2	−1	—	1	7	—