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Testing, communicating and justifying your Internal Model
Life Aggregation and Simulation Techniques Working Party

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Agenda

1. Introduction
2. Setting the scene
3. Copulas
   - Allowing for tail dependence
   - Testing the copula
   - Focus on how statistical techniques can inform judgements
4. Proxy models

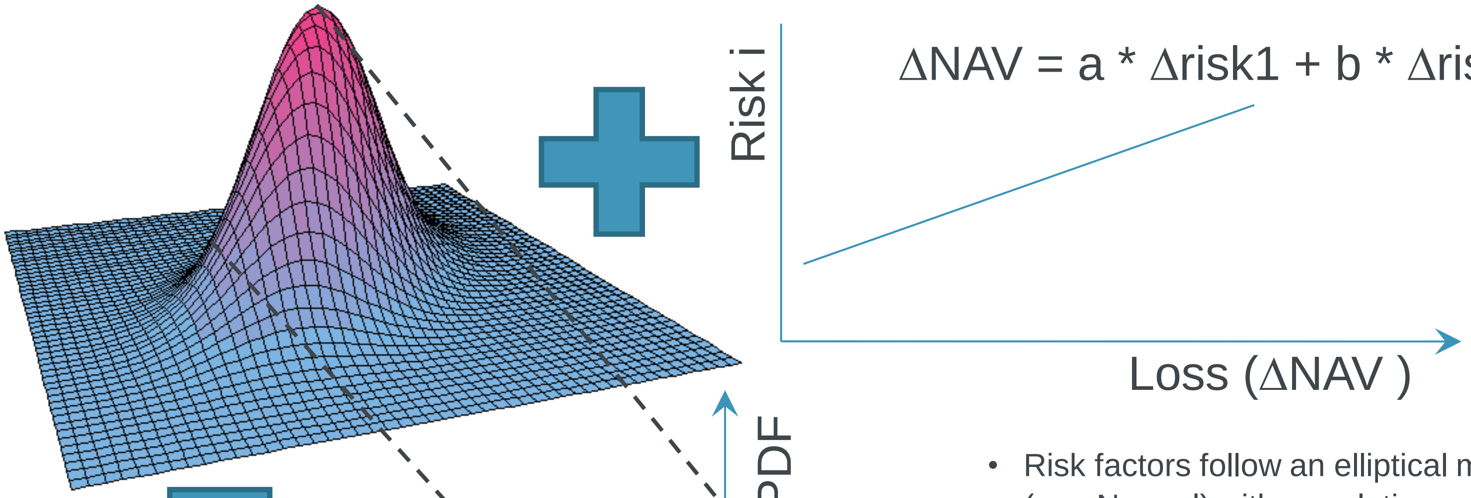
Introduction

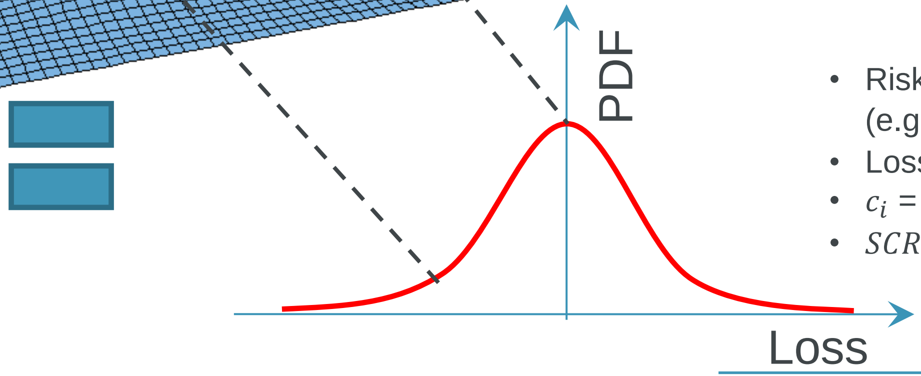
- Background
  - Increasing sophistication in modelling of effects of diversification in economic capital models
  - Move away from closed form techniques based on correlation matrices to simulation based techniques using copulas and proxy models
  - Increased scrutiny of choices made by stakeholders (model validators, Boards, supervisory authorities…)
  - Regulatory minimum standards if internal model to be used to calculate Solvency II SCR

- Objectives of Working Party
  - Investigate how actuaries can assess and choose between the techniques available
  - How choices made can be tested, communicated and justified to stakeholders
  - Technical details and merits of techniques not main focus – plenty of other papers and textbooks on this!

Setting the scene

Theory behind correlation matrix approach to capital aggregation



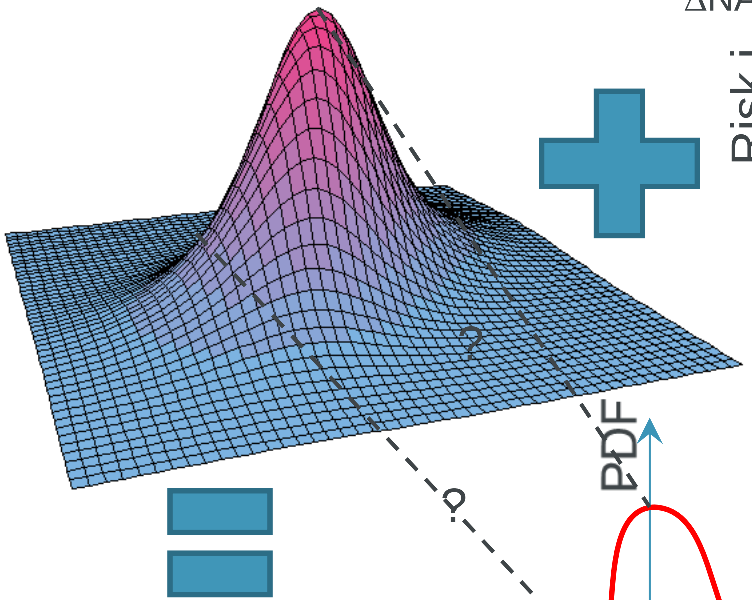


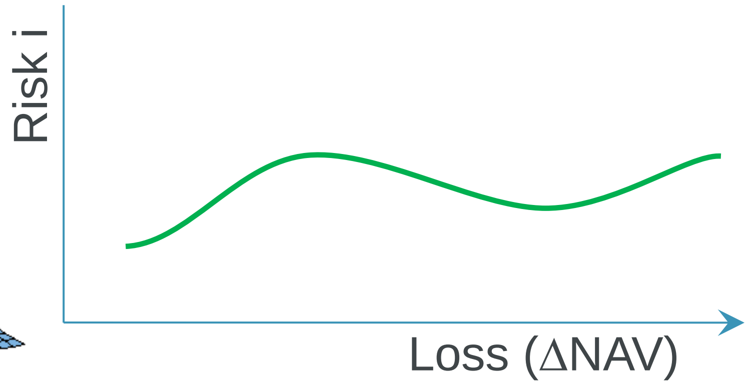
- Risk factors follow an elliptical multivariate distribution (e.g. Normal) with correlation matrix $\left(R_{i,j}\right)$
- Losses are linearly related to risks and no interactions
- c_i = standalone capital requirement for risk i
- $SCR = \sqrt{\sum R_{i,j} c_i \, c_j}$

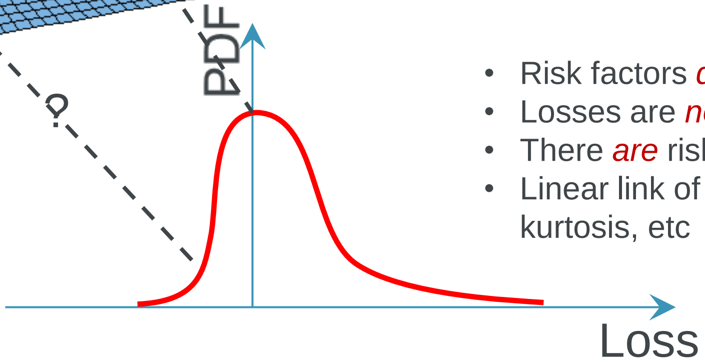
Setting the scene

Reality . . .

$\Delta NAV = a * \Delta risk1 + b * \Delta risk2 + c * \Delta risk1 * \Delta risk2 + d * (\Delta risk1)^{2} + \ldots$





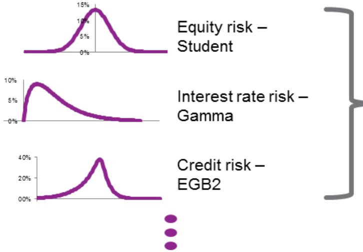


- Risk factors *do not* follow an elliptical multivariate distribution
- Losses are *not* linearly related to risks
- There *are* risk interactions
- Linear link of risks to losses is broken; there may be skewness, kurtosis, etc

Setting the scene

Copula + proxy model approach to capital aggregation

Marginal risk distributions



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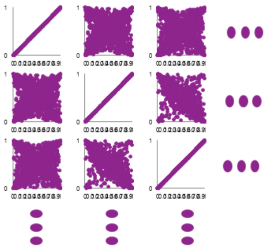
Dependency



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Risk Universe for simulations

sim number	equity returns	interest rates	credit spreads	expense level	...	lapse rates
1	-10%	3% up	50 bps	10% up	...	+3%
2	+3%	1% up	2 bps	2% down	...	-2%
3	-45%	1% down	400bps	7% up	...	+10%
...	...	...	...	...	...	...
999,999	+1%	4% up	20bps	15% up	...	+3%
1,000,000	-4%	0.5% down	80bps	0% up	...	-10%

\+ **proxy model**

ΔNAV(Equity, IntRates, Credit, ..., Lapses) = 251 Equity - 45 Equity^2 - 25 IntRates + . . . - 32.7 IntRates Credit +...-... +234.15 Lapses

=

simulation number	capital requirement
1	1,252,993
2	- 2,949,843
3	47,837,447
...	...
999,999	- 9,857,849
1,000,000	6,884,929

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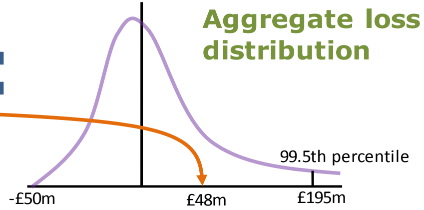
Aggregate loss distribution



- Most common approach adopted by UK life insurers intending to use an internal model to calculate their Solvency II SCR
- Solvency II Directive requires internal models to use a “full” Probability Distribution Forecast

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Copulas

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Copulas – reminder

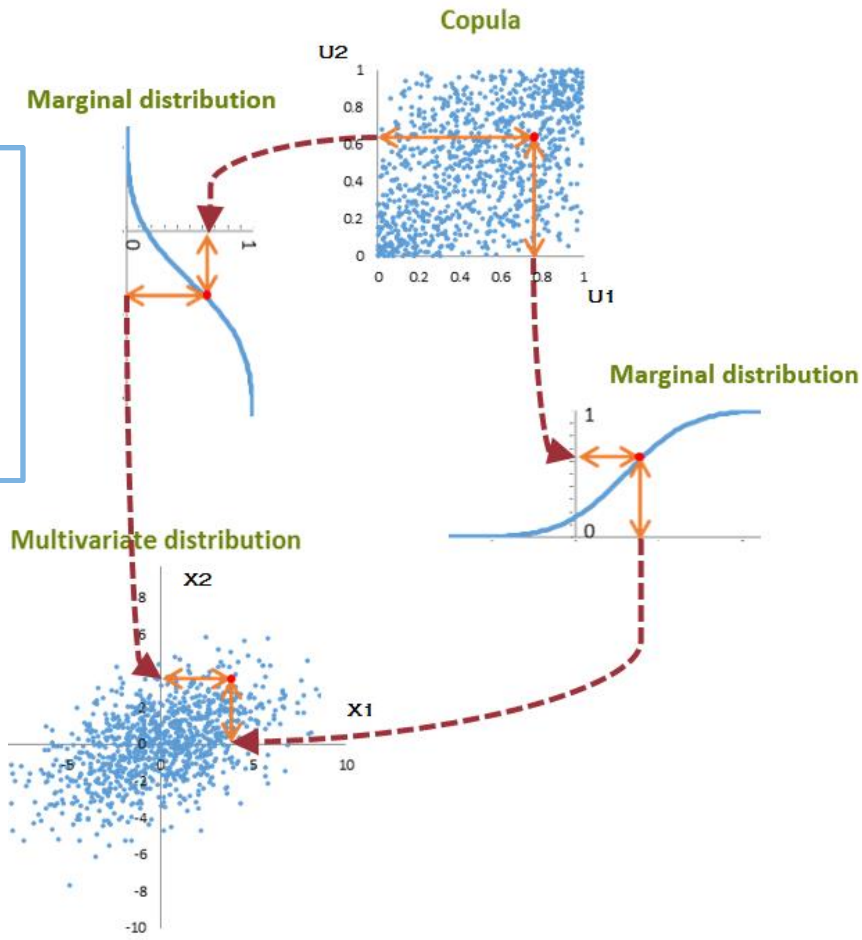
What is a copula?

- A d-dimensional copula is a multivariate distribution function on $[0,1]^d$ with uniform marginals
- Copulas providing a rule for matching ranks.
- This can be used to glue together 1-dimensional distributions to create a multivariate "meta-distribution" with the required dependency structure. Every multivariate distribution can be decomposed in this way. (Sklar's theorem)

Class	Example
Implicit	Gaussian (Normal)
	Student's T
	Individuated T (IT)
Explicit	Clayton
	Gumbel
Empirical	

Parametric

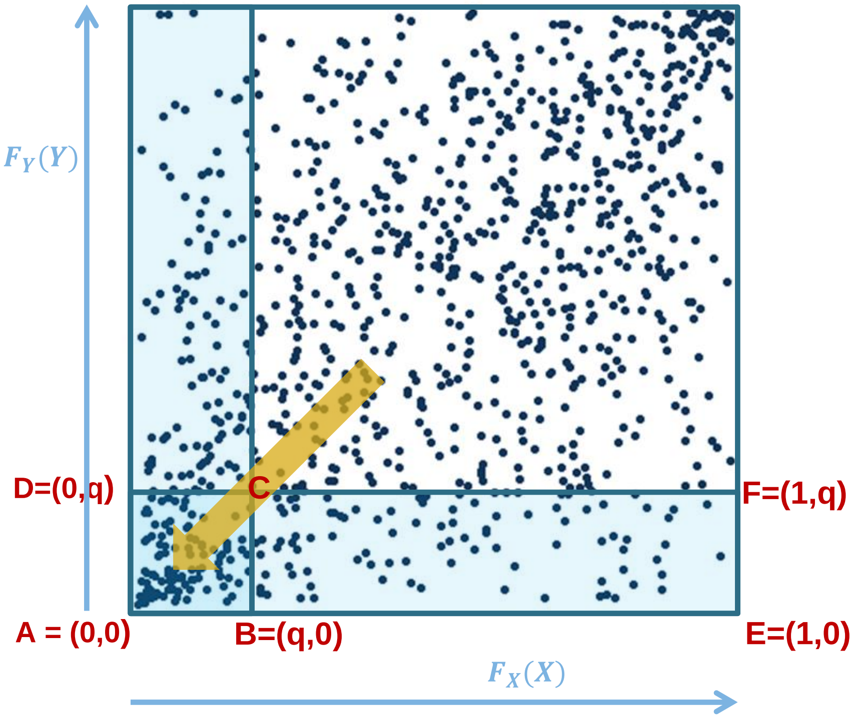
Most common choice for S2 internal models

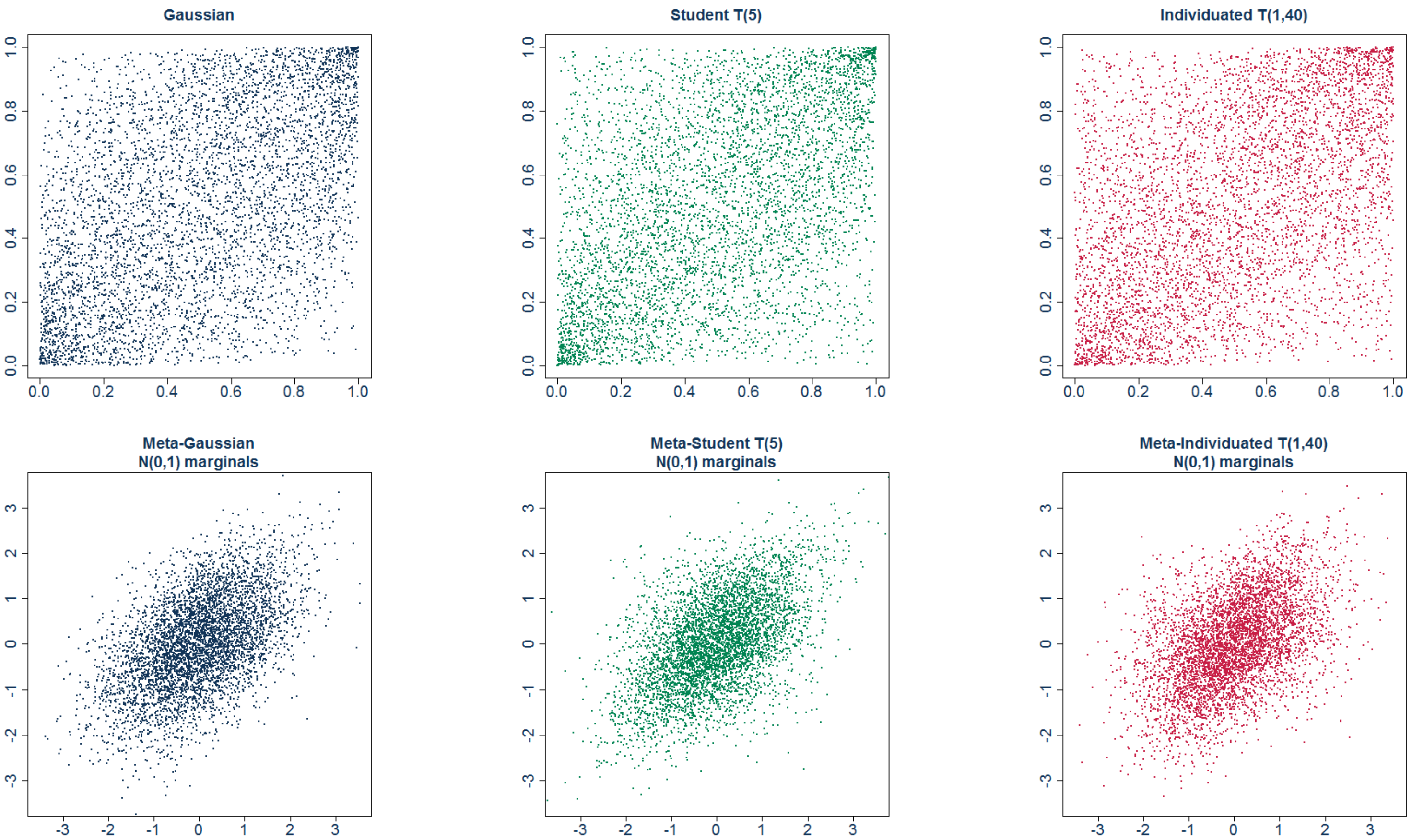


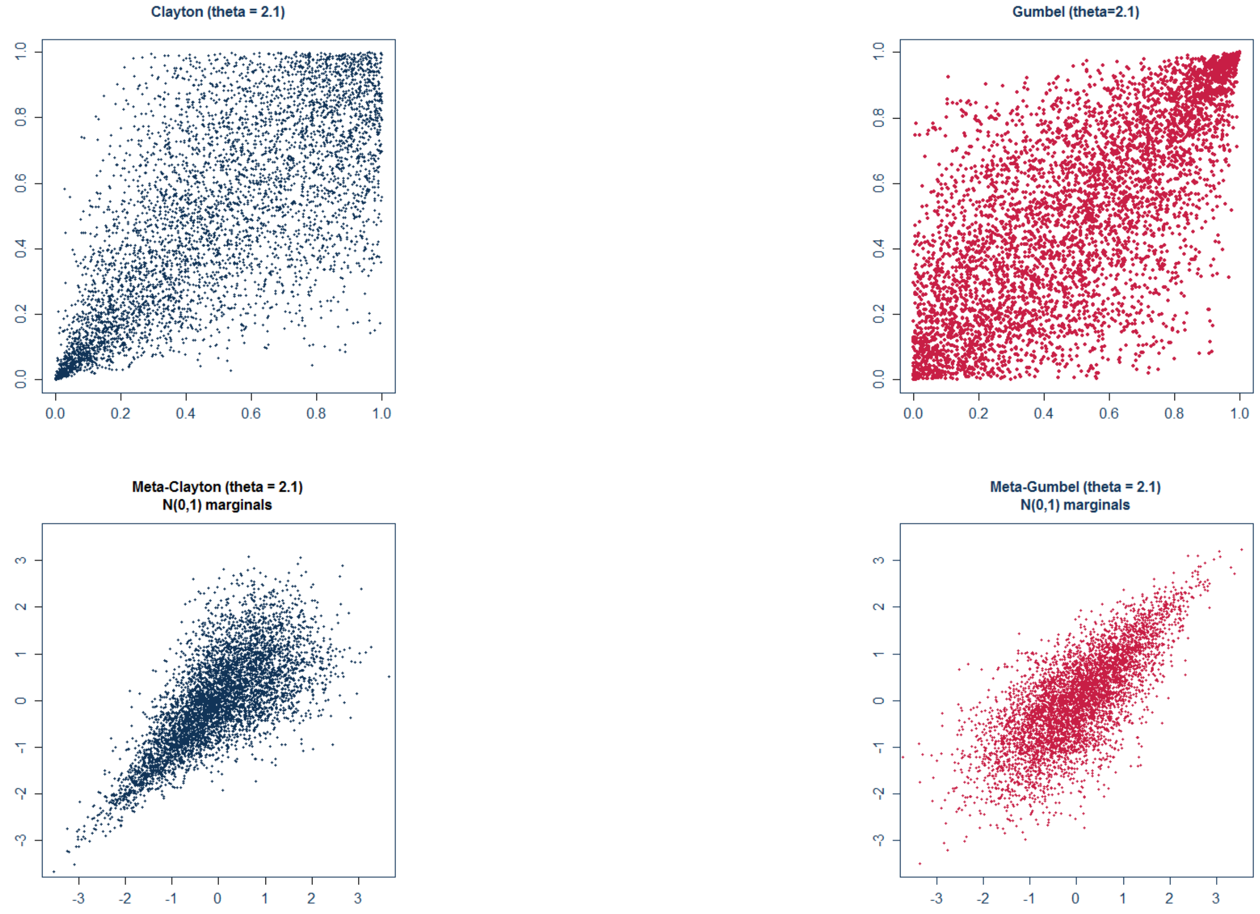
Tail dependence

Reminder

- Measure of probability of simultaneous occurrence of extreme movements in two risk factors
- Related to conditional probabilities
- Coefficient of *finite* lower tail dependence
  $\hat{\lambda}(q)$= Pr(event in square ABCD)/Pr(event in rectangle AEFD)
- Coefficient of lower tail dependence λ_L = limiting value of $\hat{\lambda}(q)$ as q tends to zero (i.e. as little square ABCD and rectangle AEFD shrink)
- Analogous definition of coefficient of upper tail dependence
- In practice, the coefficient of *finite* lower tail dependence $\hat{\lambda}$ is a more useful measure than the asymptotic value λ
- For Gaussian, λ is zero unless correlation = +1 (λ = 1)
- For Student's T, λ is non-zero (see e.g. McNeil for formula)




Gaussian
Student T(5)
Individuated T(1,40)
0.0
0.2
0.4
0.6
0.8
1.0
Meta-Gaussian
N(0,1) marginals
Meta-Student T(5)
N(0,1) marginals
Meta-Individuated T(1,40)
N(0,1) marginals
-3
-2
-1
0
1
2
3


Clayton (theta = 2.1)
Gumbel (theta=2.1)
Meta-Clayton (theta = 2.1)
N(0,1) marginals
Meta-Gumbel (theta = 2.1)
N(0,1) marginals
0.0 0.2 0.4 0.6 0.8 1.0
-3 -2 -1 0 1 2 3

Tail dependence

Why it is important

Ratio of joint exceedance probabilities of T-copula to Gaussian

(99th percentile)

Correlation/DOF	2	5	10	30	Gaussian
0%	18.46	7.45	3.72	1.74	0.00010
25%	6.30	3.33	2.12	1.35	0.00044
50%	3.05	2.00	1.52	1.17	0.00129
75%	1.78	1.41	1.22	1.08	0.00317
95%	1.20	1.11	1.06	1.02	0.00670

Table shows:

- joint exceedance probs (i.e. prob X and Y both exceed their q^{th} percentiles) for Gaussian
- How much you need to multiply these by to get corresponding value for a T-copula
- With correlation parameter of 50%, a T copula with 5 degrees of freedom has double the probability of a Gaussian
- Moving from a Gaussian to a T-copula turns a "1 in 770 year event" into a "1 in 385 year event"

Tail dependence influences the probability of joint occurrence of extreme events
If using Gaussian, may need to adjust correlations to allow for tail dependence

Choosing and parameterising a copula

Main challenges

Parameterisation

- Challenges similar to those for correlation matrix
  - Lack of data
  - Judgement
- Tail dependence
- Solvency II
  - Statistical quality standards
  - Expert judgment
  - Supervisory scrutiny

Choice

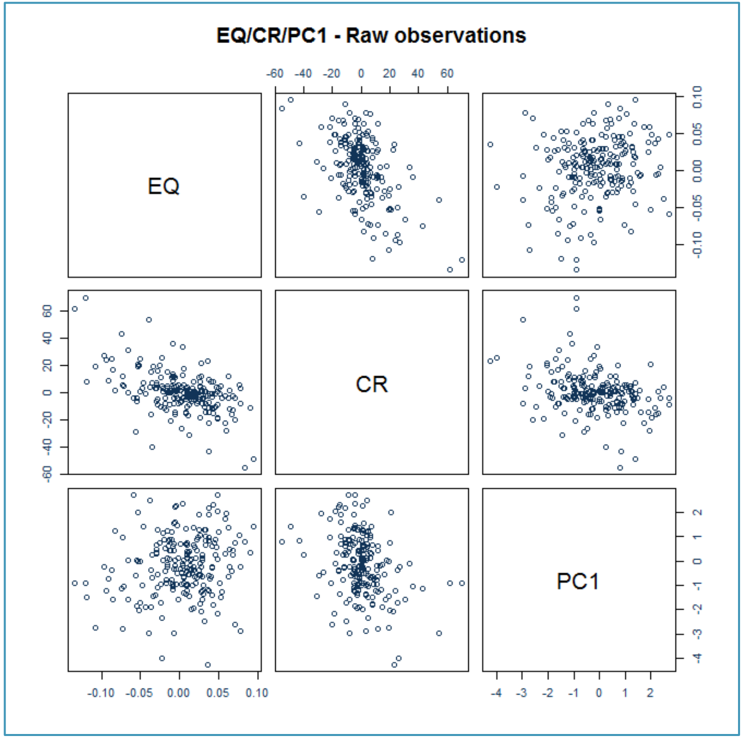
- In practice choice affected by factors such as:
  - Richness of data
  - Number of dimensions required
  - Transparency/complexity
  - Use test
  - Evolution of actuarial practice
  - Proprietary aggregation packages
- Increasing flexibility in future?

To what extent can statistical methods help inform judgements?

Copula parameterisation
Example

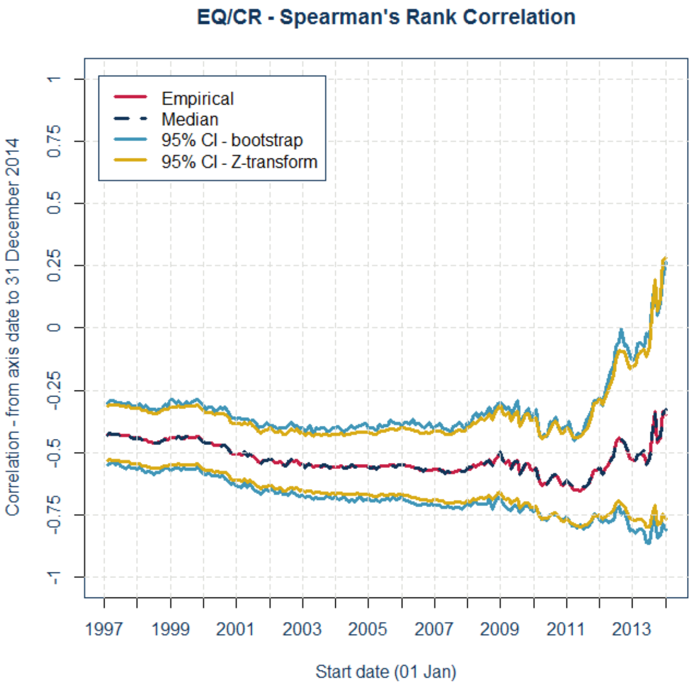
Data

- 31.12.1996 to 31.12.2014
- Monthly increases in:
  - FTSE-All Share index
  - Corporate Bond Spreads (widely used index)
  - PC1 (Bank of England gilts spot curve)
- 216 data points



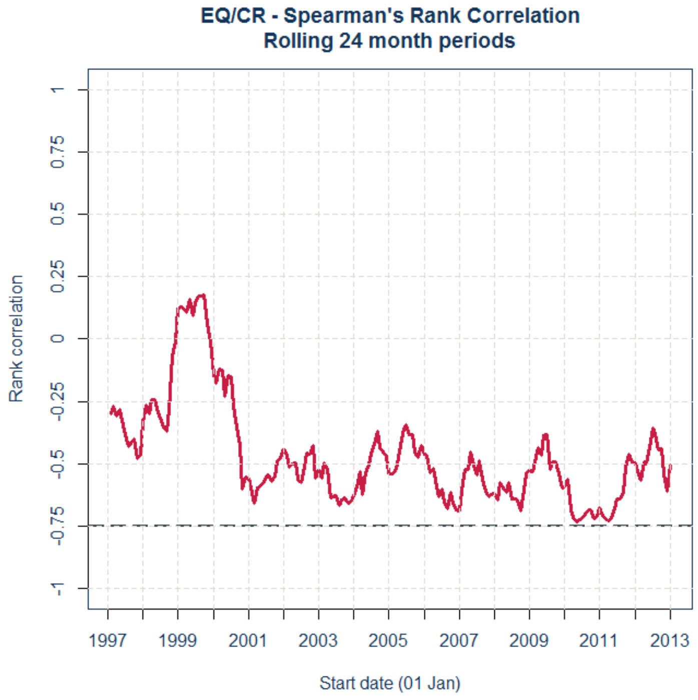
Copula parameterisation

Different time periods



- Spearman's rank correlation for periods starting 01 January YYYY to 31 December 2014
- Confidence intervals generated using
  - Fisher Z-transformation
  - Bootstrapping with re-sampling
- Charts can highlight any trends
- Illustrate uncertainty
- Can inform choice of correlation
- Values over longer periods more useful in informing "best estimate" view rather than allowance for tail dependence

Copula parameterisation
Different time periods



- Looking at correlations over shorter periods may give more information about behaviour in adverse conditions and help inform any allowance for tail dependence
- But less data means greater sampling error and greater uncertainty around conclusions
- Chart shows 24 month rolling correlation for EQ/CR reached around -75% in financial crisis

Copula parameterisation

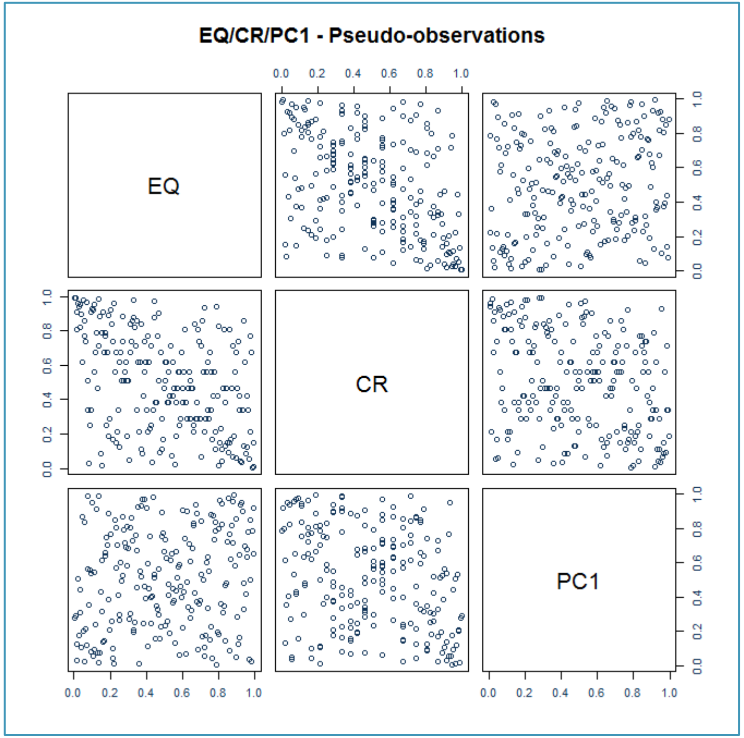
Statistical fitting techniques

Method of Moments

- Formulae based on rank correlations
  - Inverse Spearman
  - Inverse Tau
- Use to parameterise correlation matrix.
- Need to use other techniques to estimate other parameters (e.g. MLE for Degrees of Freedom of a T-copula)

Maximum Pseudo-Likelihood

- Transform data to pseudo-observations in range (0,1) by converting to ranks and dividing by (N+1)
- Fit parametric copula to pseudo-observations using MLE



Copula parameterisation

Bivariate fits

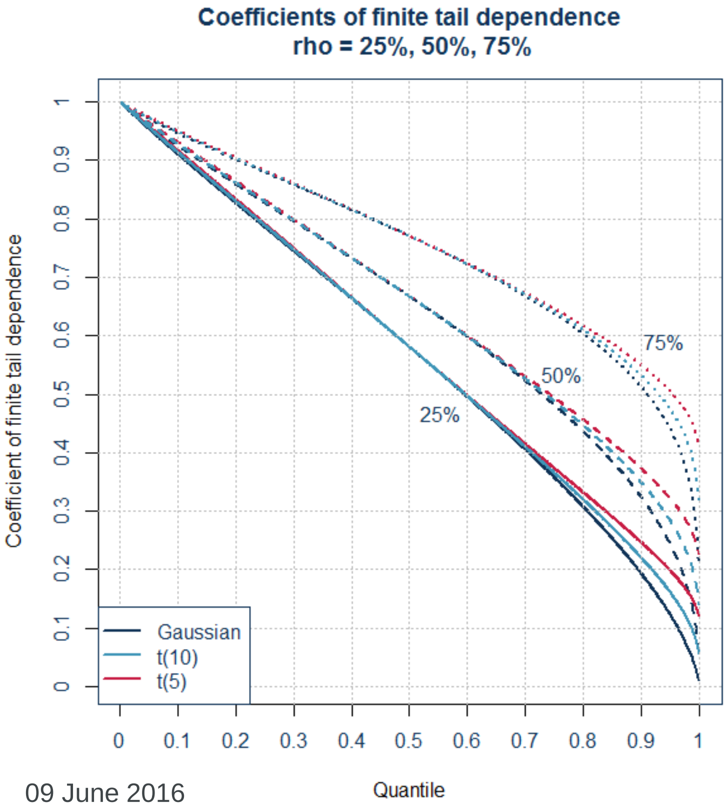
inverse Tau	EQ/CR		CR/PC1		PC1/EQ	
Copula	Rho	DOF	Rho	DOF	Rho	DOF
Gaussian	-46.6%		-29.6%		16.0%	
T	-46.6%	2.60	-29.6%	11.30	16.0%	4.43

MPL	EQ/CR			CR/PC1			PC1/EQ		
Copula	Rho	DOF1	DOF2	Rho	DOF1	DOF2	Rho	DOF1	DOF2
Gaussian	-48.8%			-31.8%			16.8%		
T	-46.5%	2.60		-31.2%	9.40		16.6%	6.08	
IT	-40.3%	2.50	1.68	-31.2%	5.99	12.82	21.0%	41.91	0.59

Note small DOF parameter for EQ/CR – indicator of tail dependence?

Coefficients of finite tail dependence

Definition

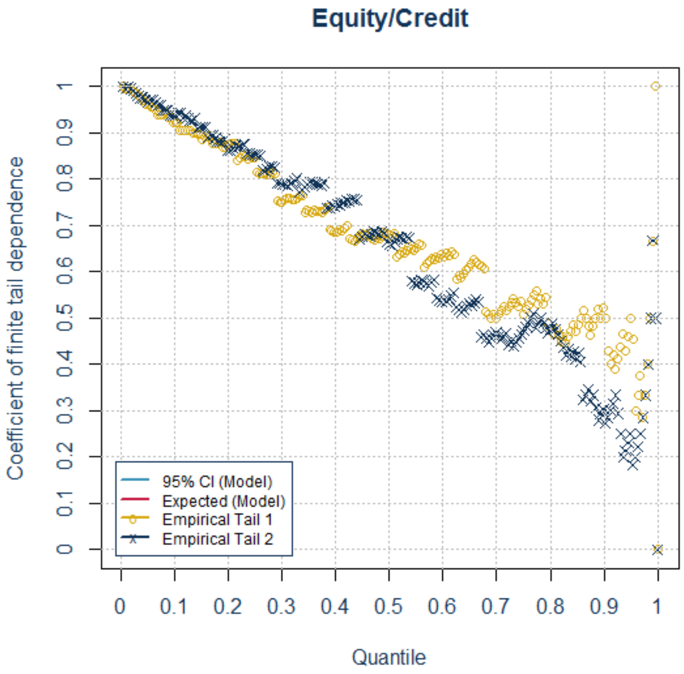


- Function showing how conditional probability depends on quantile
- Easier to work with finite values of q as can compare with conditional probabilities from data
- Richer information (function rather than single value)
- Provides graphical tool to assist in assumption selection
- $\hat{\lambda}_U(q) = \Pr((F_X(X) > q | F_Y(Y) > q)$
- $\hat{\lambda}_L(q) = \Pr((F_X(X) < (1-q) | F_Y(Y) < (1-q))$

- Value for Gaussian tends to 0 as q tends to 1
- Value for t tends to non-zero value as q tends to 1
- Correlation is principal driver of conditional probabilities across distribution
- Effect of DOF parameter becomes apparent in tail

Allowing for tail dependence

Targeting conditional probabilities

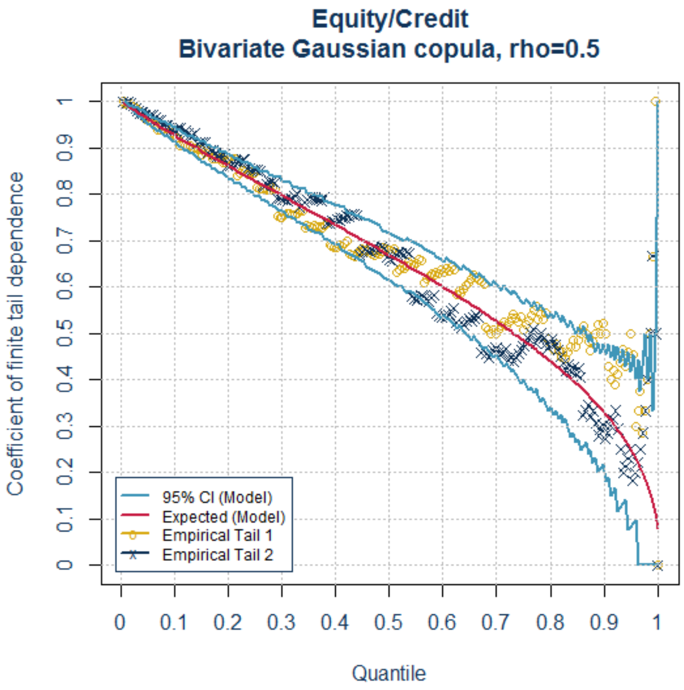


Step 1 – Plot data

- Chart empirical coefficients of finite tail dependence
- i.e. conditional probabilities derived from sample data
- (May need to change signs of one data set if correlation is negative)
- This is data and remains fixed throughout the analysis
- Gold circles correspond to events where credit spreads widen and equity to values fall
- Blue crosses represent opposite tail
- Some evidence of asymmetry in tails

Allowing for tail dependence

Targeting conditional probabilities

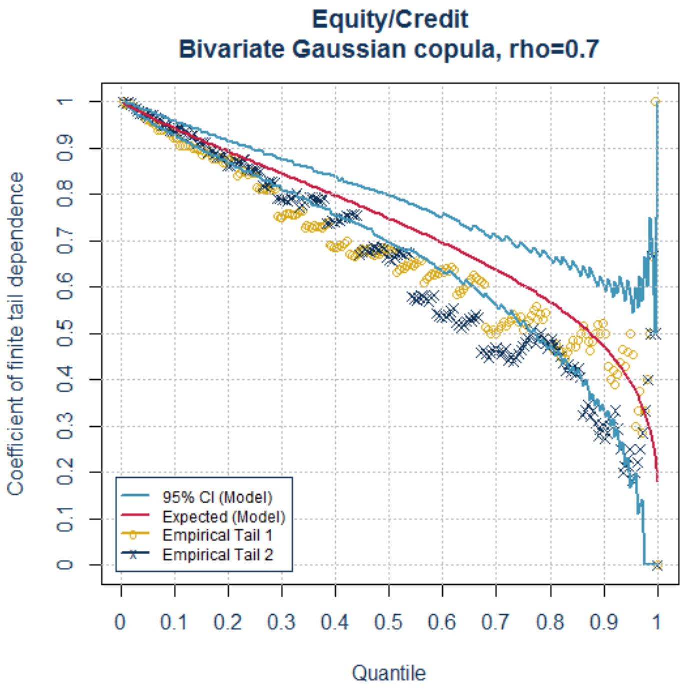


Step 2 – superimpose model with confidence intervals

- Choose a copula model and parameterisation
- In this case a Gaussian model with correlation parameter of 50%
- Plot the coefficient of finite tail dependence for the model (red line)
- Draw confidence intervals (blue lines) around model values using bootstrapping techniques. These illustrate the range you might expect observed values to fall into if the process followed the assumed model
- Red line drops towards zero as expected for model with zero coefficient of tail dependence
- Only one red line as upper and lower coefficients of finite tail dependence are equal for a radially symmetric copula

Allowing for tail dependence

Targeting conditional probabilities



Step 3 - Test alternative models/parameterisations

- If using Gaussian model, can use to inform allowance in correlation for tail dependence
- Can use knowledge of percentile of interest (e.g. based on biting scenario derived from averaging simulations in window around SCR) to inform choice.
- 70% may be adequate if biting scenario contains EQ or CR at around 95th percentile
- But even poorer fit in body of distribution
- 70% is more than 20% greater than MPL estimate of 46.5%. Reflection of low degrees of parameter from MPL estimate of T-copula
- Test sensitivity of SCR and results at other percentiles
- Is the assumption consistent with your view of the future?

Allowing for tail dependence
Arachnitude*

$$\text{arachnitude} = \frac{45}{4N(N^2-1)(N^2-4)}\left[\sum_{k=1}^{N}(2R_k - N - 1)^2(2S_k - N - 1)^2 - \frac{N(N^2-1)^2}{9}\right]$$

Where R_k and S_k are the ranks of the sample data, $1 \leq k \leq N$

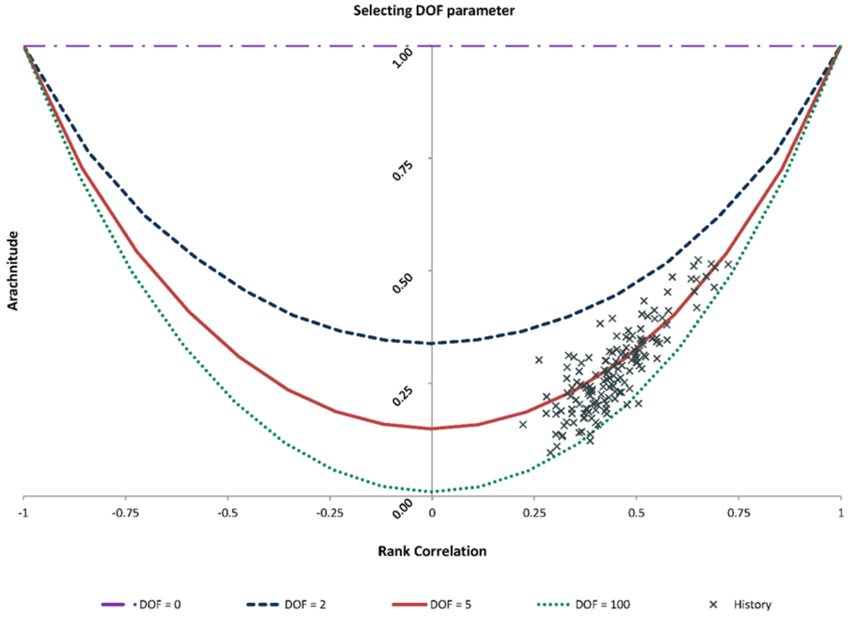
Arachnitude – what is it? how is it used?

- It is a statistic used in conjunction primarily (but not only) with the t-copula,
- Can be used to determine what is the “appropriate” number of degrees of freedom (DOF).
- Takes values between -1 and 1;
  - tends to 1 when extreme values of X coincide with extreme values of Y, even when not in the main diagonal;
  - tends to 0 when X and Y are independent, but an arachnitude of zero does not mean independence.
  - tends to approximately -7/8 when extreme values of X are associated with moderate ranks of Y.
  - The -1 value is obtained only for a pair-sample of size 4 where extremes are connected with non-extremes.

* Reproduced with kind permission from the 2010 paper by Shaw/Smith/Spivak

Allowing for tail dependence

Arachnitude – practical example*



The problem

For d risks we have d*(d-1)/2 pairs of risks, each pair giving rise to a DoF parameter; we want to identify a single and suitable DoF parameter for a t-copula.

A possible (and not unique) solution algorithm

1. Plot the actual combinations of (RCorr, Arachnitude) pairs from the data;
2. Fix DoF, and plot (rank correlation, arachnitude) pairs for all corrs for a t-copula (from -1 to 1, say step by 5%); can do this e.g. by simulation in R
3. Repeat step 2 for a wide-enough range of DoFs; in this example we have done this for DoF = 0, 2, 5 and 100 which gives the four lines shown.
4. Use judgement to identify which DoF parameter best fits the data.
5. Back-solve to identify an optimal correlation parameter for the t-copula.

*Reproduced with kind permission from the 2010 paper by Shaw/Smith/Spivak

Selecting a copula
Statistical tests

- Gaussian copula
  - Mardia’s test – tests whether ranks of observations are consistent with a multivariate Normal distribution
  - Malvergne-Sornette – if copula is Gaussian, then a certain test statistic derived from the pseudo-observations has a specific distribution (χ^2). Can test using various version of Kolmogorov-Smirnov and Anderson-Darling statistics.
- Student’s T copula
  - Kole-Koedijk-Verbeek – if copula is Student’s T, then a then a certain test statistic derived from pseudo-observations has a specific distribution (F). Can test using various version of Kolmogorov-Smirnov and Anderson-Darling statistics.
- Blanket tests of Genest & Remillard
  - Not specific to a particular family of parametric copulas
  - Based on measure of the distance (Cramer von Mises statistic) between the empirical copula derived from sample and a parametric copula fitted using MPL techniques
- Akaike Information Criterion
  - Penalised pseudo-likelihood where penalty depends on number of parameters
- Likelihood Ratio Test for nested models
  - Tests whether addition of further parameters is statistically significant

Goodness of fit tests

Results

	EQ/CR		EQ/CR/PC1	
	Gaussian – reject?	Student-T – reject?	Gaussian – reject?	Student-T – reject?
Mardia	N	N/A	N	N/A
KS-1	N	N	N	N
KS-2	N	N	N	N
AD-1	N	N	N	Y
AD-2	N	N	N	N
CvM	Y	Y	Y	Y
AIC prefers		Student T		Student T
Likelihood ratio test prefers		Student T		Student T

Mixed results and for only subset of risks $\Longrightarrow$ need to rely on judgement and prior beliefs
Greater transparency and use test considerations have led to majority of UK life insurers using Gaussian copula for Solvency II internal models with allowance for tail dependence in correlations

Top-down validation

Is the model fit for purpose?

- Sensitivities
  - What are the financially most material assumptions?
  - Does the analysis and rationale for those assumptions stack up?
- Process
  - Has appropriate use been made of expert judgement?
  - Was the governance process appropriate?
- PSD adjustments
  - What is the distribution of the movements? (e.g. draw a histogram)
  - Are there any large movements? What risks are involved? Is the impact acceptable?
  - Financial impact (e.g. estimate using a correlation matrix approach)
- Check the simulated scenarios in a window around the SCR
  - Do they appear plausible given your knowledge of risk exposures? (Combination of risks and relative severity.)
  - Does the “smoothed biting scenario” appear reasonable?

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Proxy Models

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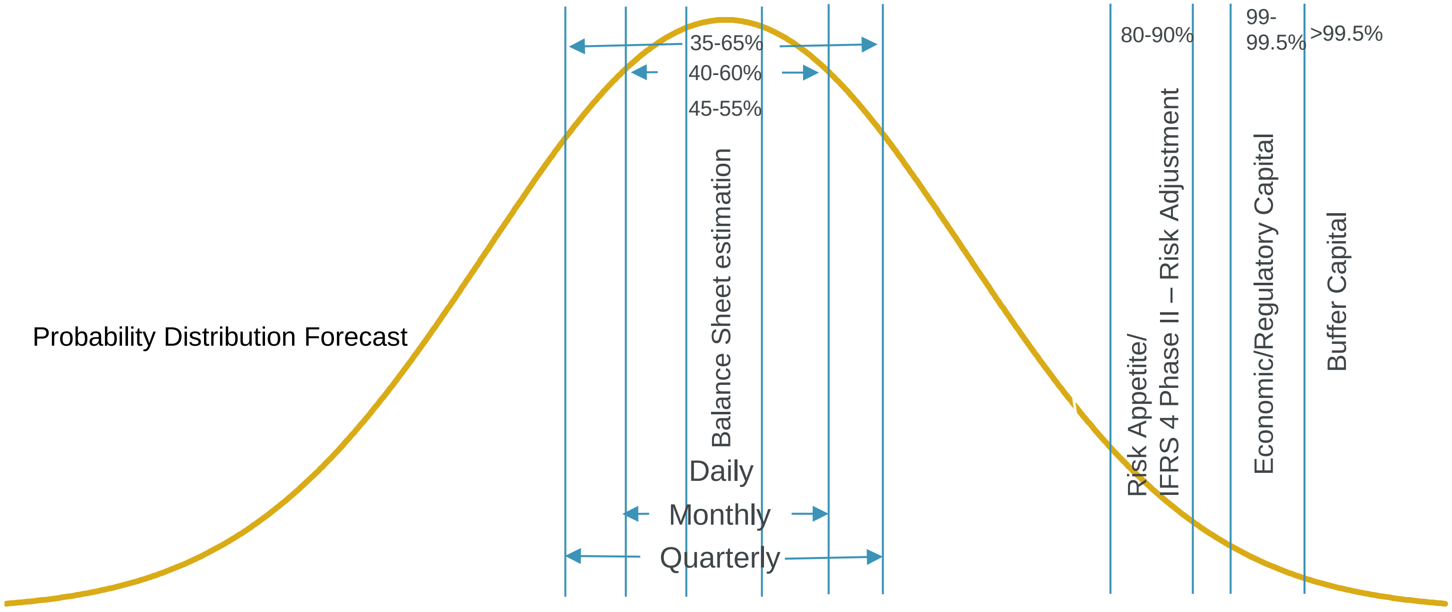
Uses of the proxy model

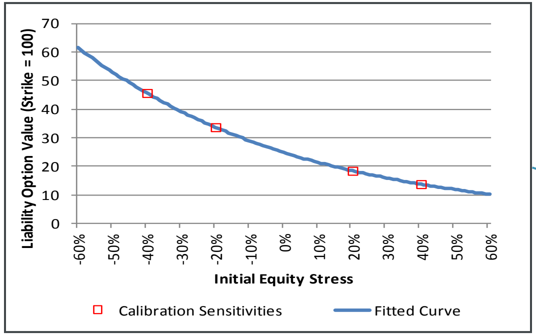


Proxy Models: types and terminology

Proxy Model: Formulaic representation of Financial Metric

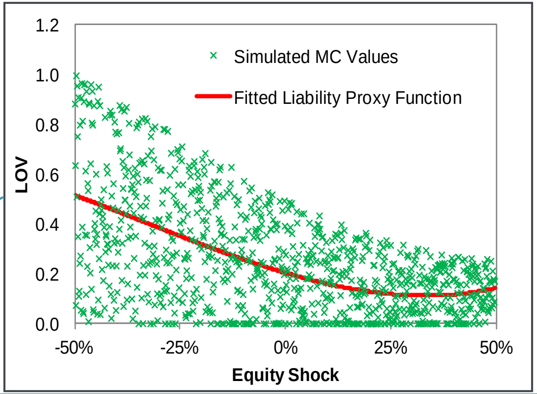
Proxy Function: Mathematical function fit to modelled data

Replicating Portfolio: Mathematical asset valuation formula for set of replicating assets

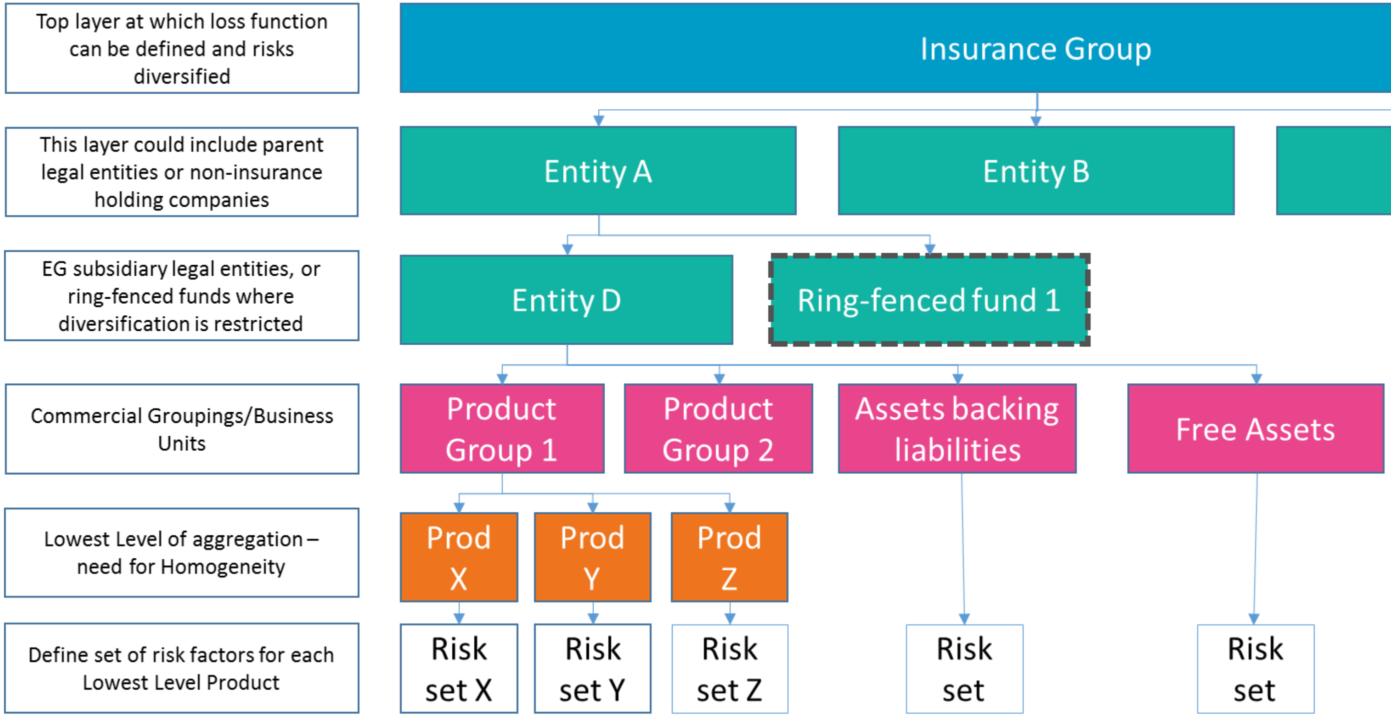


Curve Fitting: Fit curves through a relatively small number of data points

Least Squares Monte Carlo: Fit curves to a cloud of data points



Designing a proxy model



Other Design Points

- Tax – gross/net/both
- Reinsurance
- Complex risks e.g. Operational

Fitting Approach

Objectives

- Accuracy
- Parsimony
- No Overfitting
- Practically applicable

Fitting Methodology

- Top Down (Theory Driven)
- Bottom-Up (data driven)
- What Level of hierarchy to fit

Fitting points

- Univariate
- Identified joint terms
- Multivariate
- Randomised or hand-selected

Fitting Tools

- Regression method (OLS/WLS)
- Stepwise algorithm or manual judgement
- In-sample fitting statistics (R, max error tolerances)
- Visual inspection!

Influencing factors

Fitting Methodology:
Time available
Complexity of business
Uses of model
Expert judgement vs. statistical optimisation

Fitting points:
Expert judgement
Computing capacity
Ease of validation of heavy models (multivariate)

Fitting Tools:
Software
Time available

Intended use(s) of the model

Communication and Validation Challenges

Proxy modelling represents a step change in how insurers value assets and liabilities, hence management need confidence in the techniques chosen.

- General acceptance that proxy models are needed to fulfil regulatory capital calculations for internal models
- But use is widening with more onus on the centre of the PDF – for estimating balance sheet and risk management. Level of accuracy required likely to be higher.
- As such, validation needs to be:
  - Use focussed: What are we using the proxy model for? And what area(s) of the distribution should we be testing?
  - Informative: what are the potential sizes of model error? In what parts of the PDF, or for what particular risks? How can we improve the model?
  - Clearly presented: easy to understand, and with clear conclusions

Validation against stated objectives

Objective	Validation Tools
Accurate	In-sample testing Out-of-Sample Testing Profit and Loss Attribution/Backtesting
Parsimonious	Bayes/Akaike Information Criteria
No overfitting	Visual inspection Out of sample testing
Practical	Driven by capabilities of firms fitting and aggregation software

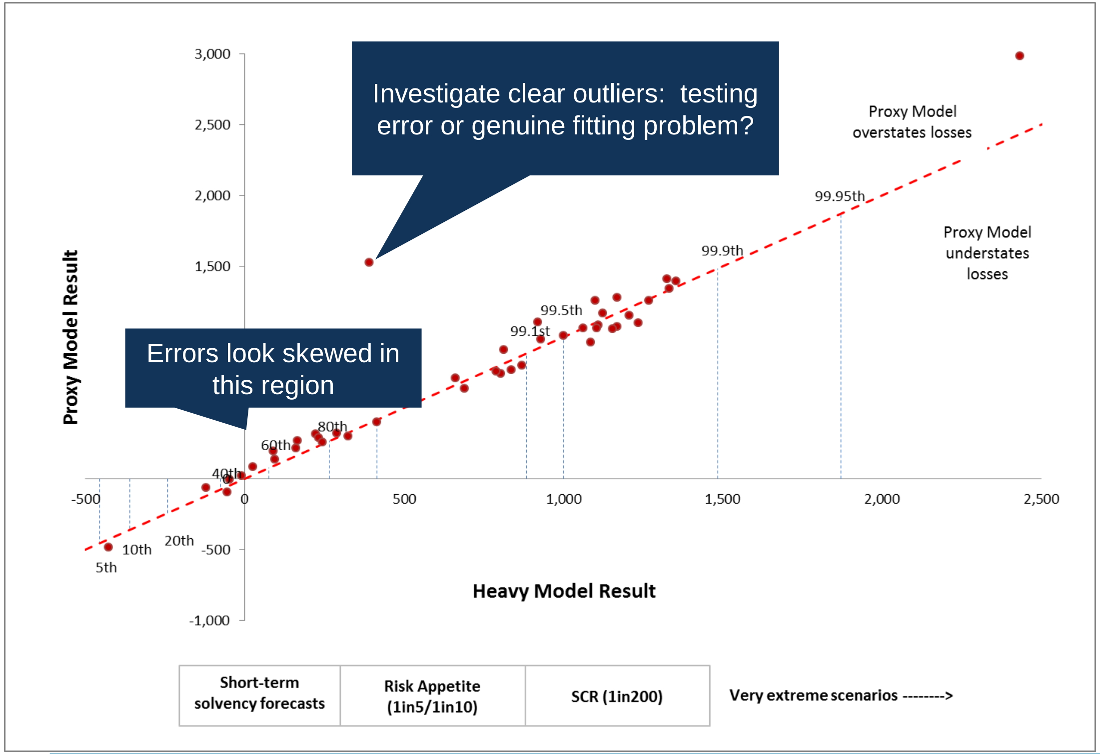
- Validating different uses of the model: Testing intensity to focus on relevant areas of interest
- Solvency II Statistical Quality Standards:
    - "Ability to rank risk" – ensure testing validates that individual risks are well fit, and that the relative importance of risks in multivariate scenarios is appropriate
    - "resulting capital requirements are appropriate" – Out-of-Sample testing: would the pattern of any errors suggest that re-ranking the simulations would materially move the SCR?

Out of Sample Testing considerations

Consideration	Decisions
Number of testing points	Should increase with scale and complexity of the business, and the number of modelled risk factors. Will be limited by computing power. Industry survey*: 40 to 80 was the interquartile range (median 50).
Univariate/bivariate/multivariate	Multivariate tests ‘real’ scenarios, and therefore interactions. Univariate tests may be more appropriate in assessing ability to rank risk, and for risk appetite. Uni/bivariate useful for investigating poor fit in multivariate scenarios.
Location of testing points on the PDF	Driven by use of model, however most firms will have a single model so will need an appropriate range and intensity. EG 50% of points around the 99.5th percentile for SCR, 20% close to the median (for roll forward use), and the remainder evenly scattered. Mixture of selected and randomised scenarios
Level of testing	Entity and Group to validate disclosed results Lower down e.g. business units or products for internal risk management Lower levels more likely to bring out fitting issues – these may offset higher up (have we been lucky with the scenarios tested?)

* Source: Deloitte survey 2014

Out of Sample Testing



Diagnostic tools:

Finding issues:

- Visual inspection !
- Points outside tolerance e.g. 5%
- Clustering above or below the line

Diagnostic statistics:

- Mean (unbiased?)
- MSE/SD (quantum of error)
- Skewness of errors

Can look at full sample or subset around a given percentile.

How to make sense of out of sample results

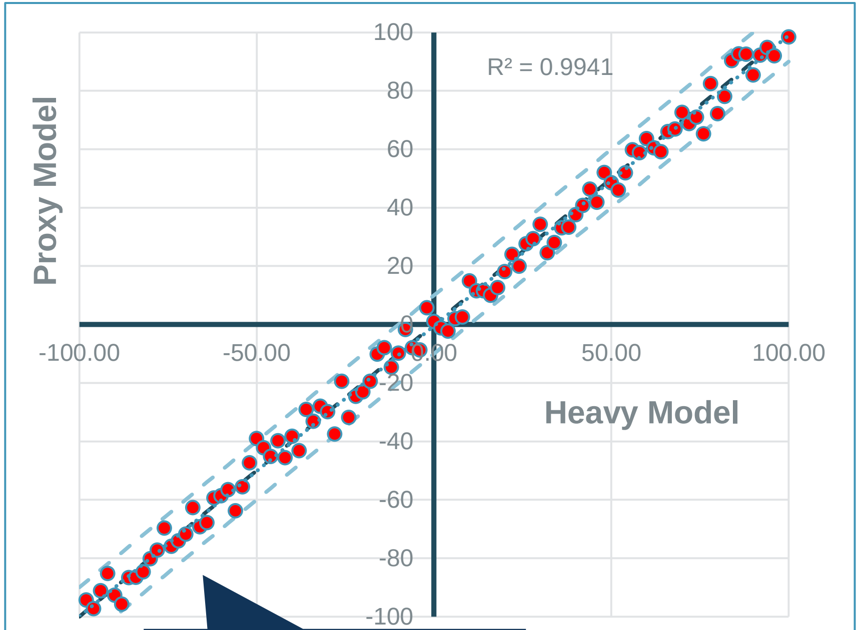
Two questions

1. Is the model unsuitable for use?
2. Can the model be improved?

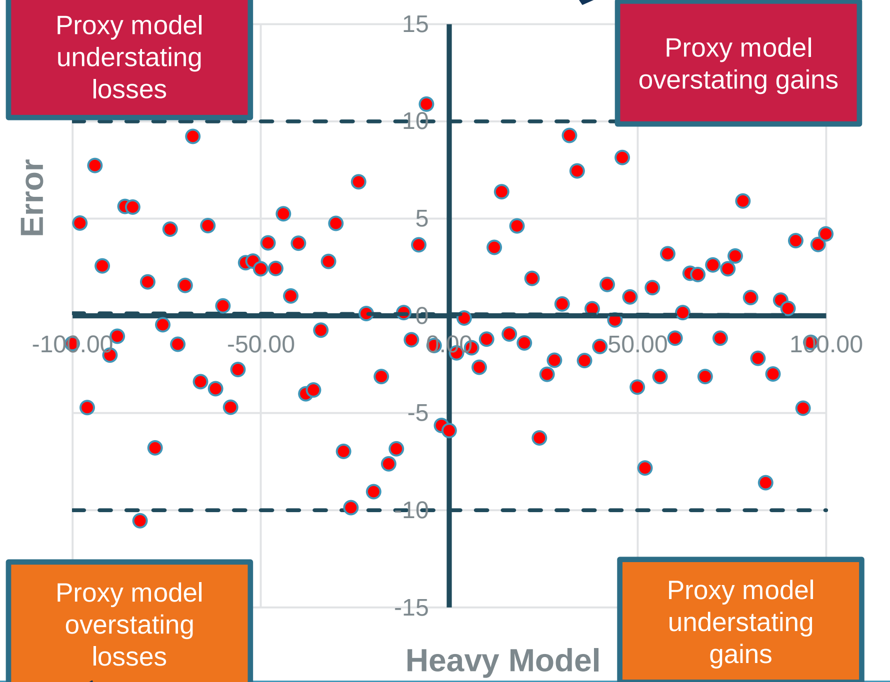
- Statistics can be informative, but shouldn't rely on these:
  - The fitting errors are not random (plenty of human/computer influence)
  - Low sample sizes
  - Can show up issues but not diagnose them
- Be clear on why you have set tolerances:
  - Setting tolerances on multiple statistics – does a single one mean failure? All?
  - Tolerances should be used to identify areas for investigation
- Errors are expected, but bias and trends in the errors are not. Test the properties of the errors:
  - Skewness – statistically significant?
  - Test whether Positive Error ~ Binomial (p = 0.5)
  - Test for autocorrelation – errors and profit/loss are correlated

Out of Sample testing - Presentation

In this presentation, the patterns in the observed residuals become clearer



Make any tolerance limits clear



Proxy model understating losses

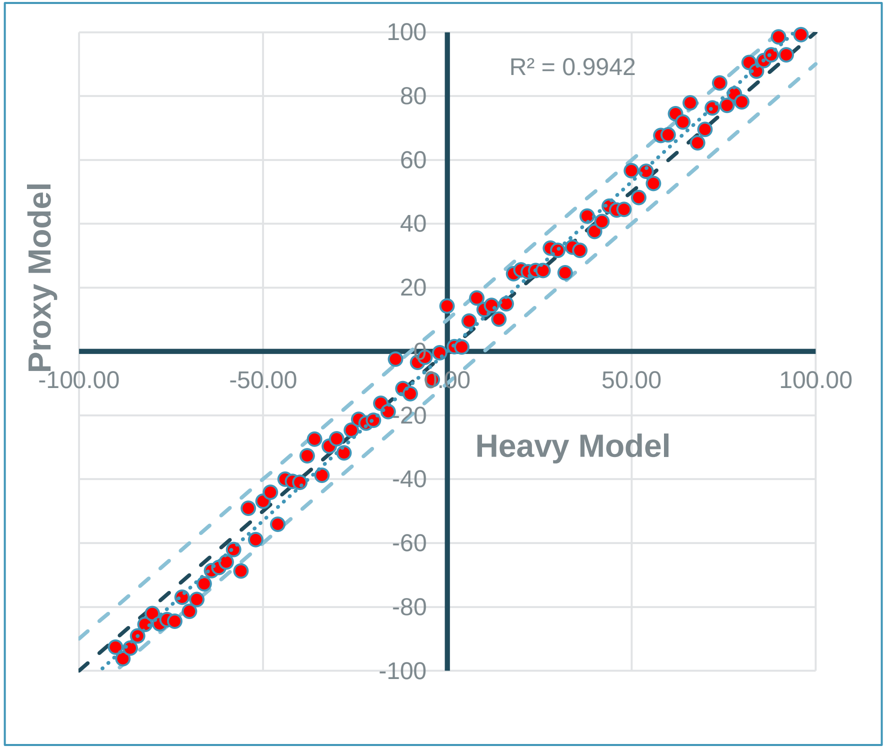
Proxy model overstating gains

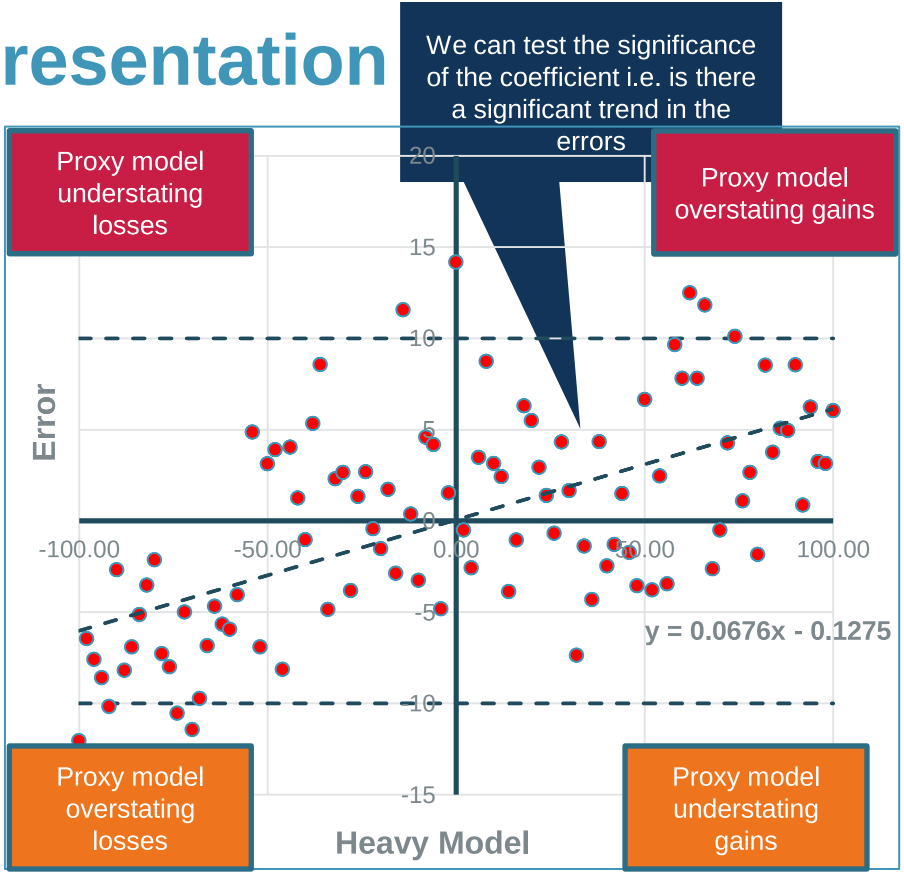
Proxy model overstating losses

Proxy model understating gains

Make clear what dots in each segment represent

Out of Sample testing - Presentation





Using out of sample data

- In order for clear decisions to be made, firms need to agree on pass/fail criteria.
- Multiple statistics make this difficult. There is no one statistic that is better than all others. Some methods currently in use that allow firms to adjust results if material:
  - Maximum or average error in particular percentile range (could be excessively prudent)
  - Include the error terms in a multivariate empirical proxy function, using interpolation to estimate and add the error in each simulation. This would create a revised PDF, and hence new results. Requires a high enough sample size.
  - Fit a trendline to the errors against the proxy model loss (similar to previous slide), and use this to scale the proxy functions to give an adjusted PDF.
- In the above cases, the firm could set limits linked to wider materiality policy
- Results would be adjusted if materiality threshold breached
- Biggest constraint would be time, and ability to do this ‘in-cycle’

Modelling errors in the aggregation

1. Empirically

Scenario	Risk 1	Risk 2	….Risk R	Error
1	-40%	+2.5%	….X	+£10m
2	+10%	+1.3%	….Y	+20m
...N	+25%	-0.6%	….Z	-15m

The out-of-sample set gives us an N data points in an (R+1) dimensional space.

The estimated error in the non-sampled points is an interpolation over this multidimensional space.

Available interpolation methods include Delaunay Triangulation, and Shepard's Inverse-distance weighting

2. Scaled proxy functions

Regress the heavy model results on the proxy model results

h = Mp+C

Where h is the heavy model result and p is the proxy model result.

Existing proxy functions can then be scaled (factor of 1/M) and shifted (- C/M), so that the adjusted proxy result includes an estimation of the error.

This can be performed at any level of the proxy model, but ideally at lower levels to give more accurate results from the bottom up.

Each method effectively re-includes the out-of-sample points in the aggregation model. These provide estimates of the error to be used in each simulation, and importantly can be produced more quickly than a re-calibration of the model.

Members of Working Party

- Nikos Katrakis (Chair)
- Taras Androshschuck
- Ruth Dodgshun
- Shaun Gibbs
- Jonathan Lau
- Rob Harris
- Steven Oram
- Lyle Semchyshyn
- Phil Raddall
- David Stevenson
- Joshua Waters

Includes joint work with Stephen Makin on copulas

Questions

Comments

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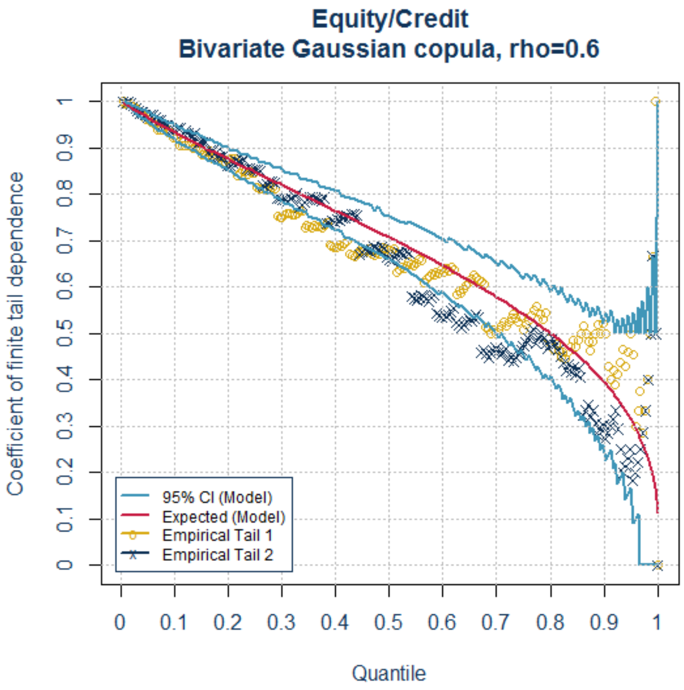
Appendix

Allowing for tail dependence

Targeting conditional probabilities

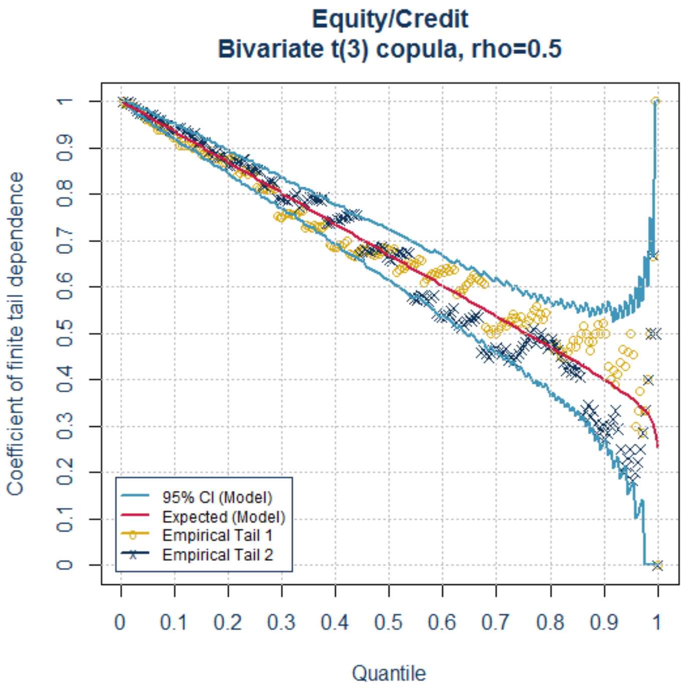


Step 3 - Test alternative models/parameterisations

- Gaussian with rho = 60%
- Shifts tail dependence function for model upwards
- Improved fit in tail of interest… but further away in body compared to rho = 50%

Allowing for tail dependence

Targeting conditional probabilities

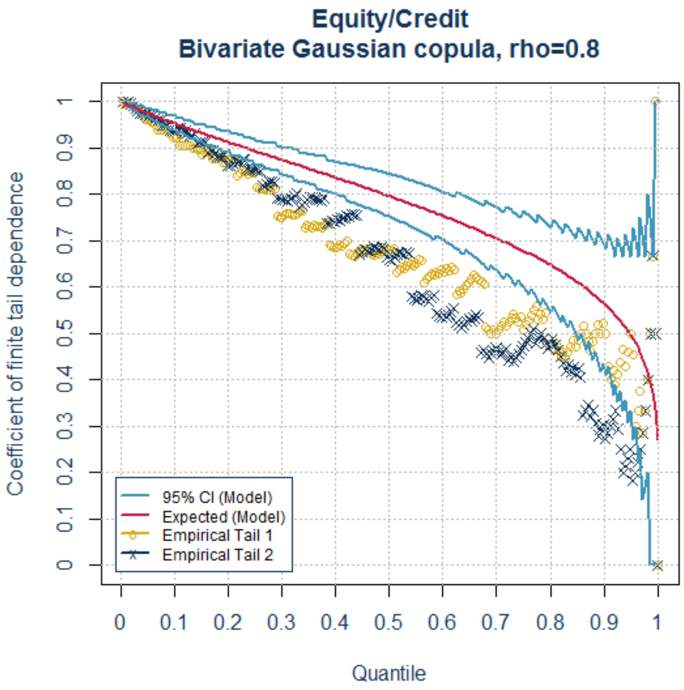


Step 3 - Test alternative models/parameterisations

- T copula with rho = 50% and 3 degrees of freedom
- Improved fit in tail
- Without losing fit in body
- But additional parameter to estimate
- Is the same degrees of freedom parameter appropriate for all risk pairs? May need to compromise somewhere on less material pairs.

Allowing for tail dependence

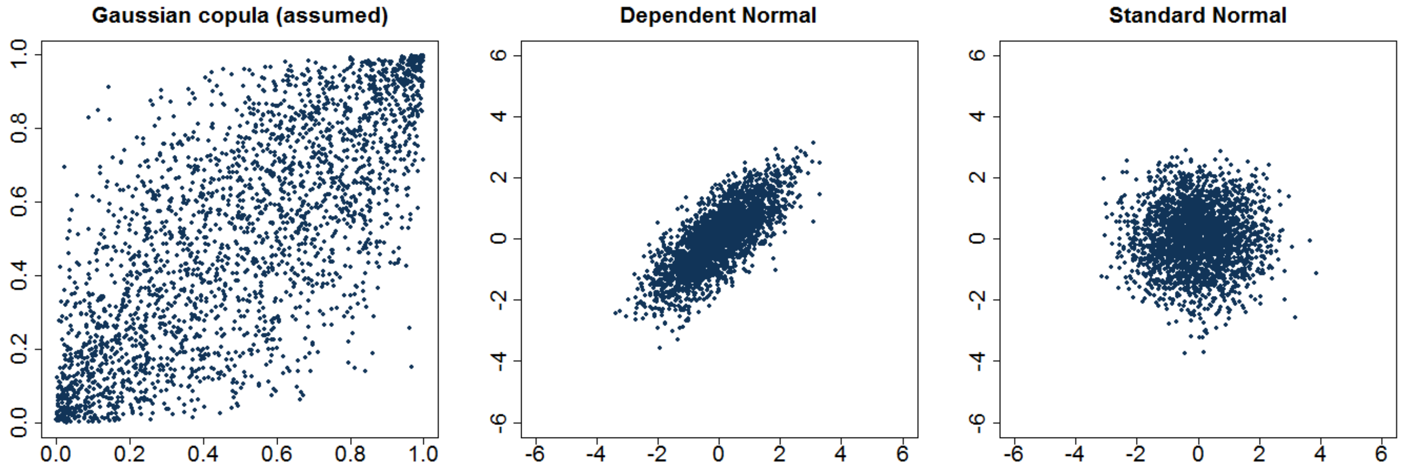
Targeting conditional probabilities



Step 3 - Test alternative models/parameterisations

- Gaussian with rho = 80%
- Overshooting?

Goodness of fit – Gaussian
Tests of Mardia and Malvergne/Sornette



Pseudo-observations (POBS)

$$\boldsymbol{U}_i = \binom{u_i}{v_i}$$

N observations from d-dimensional copula

Normalised POBS

$$\boldsymbol{X}_i = \begin{pmatrix}\Phi^{-1}(u_i)\\ \Phi^{-1}(v_i)\end{pmatrix}$$

Mardia's test for multivariate normality ($\boldsymbol{b_d}$ and $\boldsymbol{k_d}$)

Standard Normal

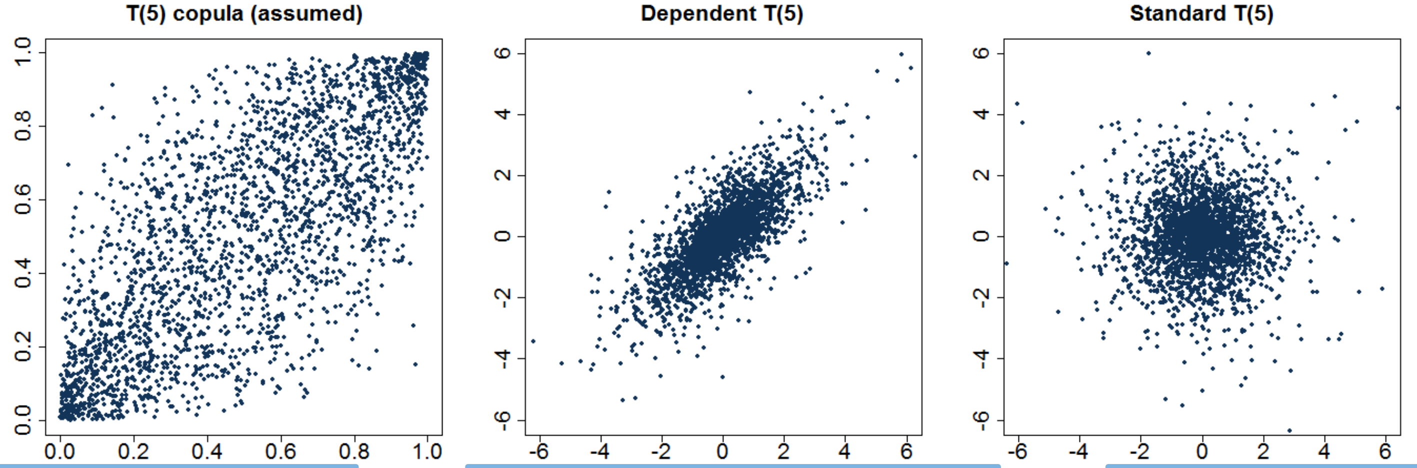
$$\frac{1}{\sqrt{1-\rho^2}}\begin{pmatrix}\Phi^{-1}(u_i)-\rho\Phi^{-1}(v_i)\\ \Phi^{-1}(v_i)-\rho\Phi^{-1}(u_i)\end{pmatrix}$$

$$(\boldsymbol{X}_i-\overline{\boldsymbol{X}})^T S^{-1}(\boldsymbol{X}_i-\overline{\boldsymbol{X}})$$

$$\boxed{\sim\chi_d^2 \text{ as } \mathrm{N}\rightarrow\infty}$$

Goodness of fit – Student's T

Test of Kole/Koedijk/Verbeek



Pseudo-observations (POBS)

$$\boldsymbol{U}_i = \binom{u_i}{v_i}$$

Studentised POBS

$$\boldsymbol{X}_i = \begin{pmatrix} t_\nu^{-1}(u_i) \\ t_\nu^{-1}(v_i) \end{pmatrix}$$

Standard T

$$\frac{1}{d}(\boldsymbol{X}_i - \bar{\boldsymbol{X}})^T S^{-1}(\boldsymbol{X}_i - \bar{\boldsymbol{X}}) \boxed{\sim F_{d,\nu}}$$

as $n \to \infty$

Goodness of fit tests

Testing χ^2 and F

- Under H_0: Gaussian $\quad \mathrm{TS} = (\boldsymbol{X}_i - \overline{\boldsymbol{X}})^T S^{-1}(\boldsymbol{X}_i - \overline{\boldsymbol{X}}) \sim \chi_d^2$ as $N \to \infty$
- Under H_0: $t(\nu)$ $\quad \mathrm{TS} = \frac{1}{d}(\boldsymbol{X}_i - \overline{\boldsymbol{X}})^T S^{-1}(\boldsymbol{X}_i - \overline{\boldsymbol{X}}) \sim F_{d,\nu}$ as $N \to \infty$
- We can test these using standard techniques

Approach	Anderson-Darling (AD)	Kolmogorov-Smirnov (KS)
1	$\max_x \frac{\lvert F_{TS}(x) - F_{TD}(x)\rvert}{\sqrt{F_{TD}(x)(1 - F_{TD}(x))}}$	$\max_x \lvert F_{TS}(x) - F_{TD}(x)\rvert$
2	$\int \frac{\lvert F_{TS}(x) - F_{TD}(x)\rvert}{\sqrt{F_{TD}(x)(1 - F_{TD}(x))}} dF_{TD}(x)$	$\int \lvert F_{TS}(x) - F_{TD}(x)\rvert dF_{TD}(x)$

- Parameters estimated from data $\Rightarrow$ bootstrapping needed

Goodness of fit tests

Cramer von Mises (Genest, Remillard)

- Start with pseudo-observations – these define an empirical copula
- Fit assumed parametric copula using MPL – this is our Null Hypothesis
- Test Statistic = square of usual Euclidean distance between empirical copula and fitted copula
- Generate distribution of TS under Null Hypothesis by generating large number of simulations with same number of observations as in original sample
- For each set of simulations, determine the corresponding empirical copula and re-fit the parametric copula using MPL
- Re-calculate the TS
- The simulated values generate an empirical distribution of the TS
- Calculate p-value