JIA 73 (1947) 0427-0429



# THIRD-ORDER MOMENTS OF A BIVARIATE FREQUENCY DISTRIBUTION

By A. W. JOSEPH, M.A., B.Sc., F.I.A.
*Assistant Actuary of the Wesleyan and General Assurance Society*

THIS note will show how to obtain all the moments up to the third order of a bivariate frequency distribution $f$ or correlation table. If $x$ and $y$ are the variates, horizontal totals give the data according to $x$ and vertical according to $y$. The totals obtained by summing the data diagonally upwards from left to right set out the data according to $x+y$ and those obtained by summing the data diagonally upwards from right to left set out the data according to $x-y$. The four sets of totals are summed successively in reverse order in the usual way so as to obtain respectively:

$$\begin{array}{lllll}
\Sigma f, & \Sigma xf, & \Sigma x_{(2)}f, & \Sigma x_{(3)}f, & \ldots, \\
\Sigma f, & \Sigma yf, & \Sigma y_{(2)}f, & \Sigma y_{(3)}f, & \ldots, \\
\Sigma f, & \Sigma (x+y)f, & \Sigma (x+y)_{(2)}f, & \Sigma (x+y)_{(3)}f, & \ldots, \\
\Sigma f, & \Sigma (x-y)f, & \Sigma (x-y)_{(2)}f, & \Sigma (x-y)_{(3)}f, & \ldots.
\end{array}$$

The four totals $\Sigma f$ give a primary check on the work. The relationships,

$$\Sigma (x+y)f = \Sigma xf + \Sigma yf,$$

and

$$\Sigma (x-y)f = \Sigma xf - \Sigma yf,$$

give two more checks. A further check comes from dual computations of $\Sigma xyf$, for

$$\Sigma (x+y)_{(2)}f = \Sigma x_{(2)}f + \Sigma xyf + \Sigma y_{(2)}f,$$

so that

$$\Sigma xyf = \Sigma (x+y)_{(2)}f - \Sigma x_{(2)}f - \Sigma y_{(2)}f,$$

also

$$\begin{aligned}
\Sigma (x-y)_{(2)}f &= \Sigma x_{(2)}f + \Sigma x(-y)f + \Sigma (-y)_{(2)}f \\
&= \Sigma x_{(2)}f - \Sigma xyf + \Sigma (y+1)_{(2)}f, \\
&= \Sigma x_{(2)}f - \Sigma xyf + \Sigma y_{(2)}f + \Sigma yf.
\end{aligned}$$

The third-order moments $\Sigma x_{(2)}yf$ and $\Sigma xy_{(2)}f$ come from

$$\Sigma (x+y)_{(3)}f = \Sigma x_{(3)}f + \Sigma x_{(2)}yf + \Sigma xy_{(2)}f + \Sigma y_{(3)}f,$$

i.e. $\Sigma x_{(2)}yf + \Sigma xy_{(2)}f = \Sigma (x+y)_{(3)}f - \Sigma x_{(3)}f - \Sigma y_{(3)}f,$

and

$$\begin{aligned}
\Sigma (x-y)_{(3)}f &= \Sigma x_{(3)}f - \Sigma x_{(2)}yf + \Sigma x(y+1)_{(2)}f - \Sigma (y+2)_{(3)}f \\
&= \Sigma x_{(3)}f - \Sigma x_{(2)}yf + \Sigma xy_{(2)}f + \Sigma xyf - \Sigma y_{(3)}f - 2\Sigma y_{(2)}f - \Sigma yf,
\end{aligned}$$

i.e. $-\Sigma x_{(2)}yf + \Sigma xy_{(2)}f = \Sigma (x-y)_{(3)}f - \Sigma x_{(3)}f - \Sigma xyf + \Sigma y_{(3)}f + 2\Sigma y_{(2)}f + \Sigma yf.$

The fourth-order moment $\Sigma x_{(2)}y_{(2)}f$, which is sometimes required, may be obtained from

$$\Sigma x_{(3)}yf + \Sigma x_{(2)}y_{(2)}f + \Sigma xy_{(3)}f = \Sigma\,(x+y)_{(4)}f - \Sigma x_{(4)}f - \Sigma y_{(4)}f,$$

and
$$-\Sigma x_{(3)}yf + \Sigma x_{(2)}y_{(2)}f - \Sigma xy_{(3)}f = \Sigma\,(x-y)_{(4)}f - \Sigma x_{(4)}f - \Sigma x_{(2)}yf + 2\Sigma xy_{(2)}f$$
$$+\Sigma xyf - \Sigma y_{(4)}f - 3\Sigma y_{(3)}f - 3\Sigma y_{(2)}f - \Sigma yf.$$

The following example illustrates the preceding algebra. It should be noted that the origin of the distribution according to $x-y$ is in the centre of the distribution, not at the beginning as with the distributions according to $x$, $y$ and $x+y$. It is necessary, therefore, in this case to make the progressive totals in two halves which, as a matter of fact, reduces the numerical work.

The distribution $f$ is shown inside the rectangle marked by heavy lines.
The sums of the rows are shown on the right.
The sums of the columns are shown at the bottom.
The distribution according to $x+y$ is shown along the top.
The distribution according to $x-y$ is shown on the left.
There are four totals of 151 which form a primary check.

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 8 | | | | | | | | | | | | |
| 10 | | | | | | | | | | | | |
| 16 | | | | | | | | | | | | |
| 14 | | 10 | 11 | 4 | 25 | 13 | 40 | 14 | 14 | 12 | 8 | Total 151 |
| 38 | $x$ $y$ | 0 | 1 | 2 | 3 | 4 | 5 | | | | | |
| 23 | 0 | 10 | 7 | 3 | 5 | 6 | 8 | 39 | | | | |
| 22 | 1 | 4 | 0 | 7 | 2 | 8 | 4 | 25 | | | | |
| 5 | 2 | 1 | 6 | 2 | 9 | 3 | 3 | 24 | | | | |
| 14 | 3 | 7 | 2 | 8 | 5 | 7 | 6 | 35 | | | | |
| 1 | 4 | 1 | 7 | 2 | 4 | 6 | 8 | 28 | | | | |
| Total 151 | | 23 | 22 | 22 | 25 | 30 | 29 | Total 151 | | | | |

| $x$ | $f$ | | | |
|---|---|---|---|---|
| 4 | 28 | 28 | 28 | 28 |
| 3 | 35 | 63 | 91 | 119 |
| 2 | 24 | 87 | 178 | — |
| 1 | 25 | 112 | — | — |
| 0 | 39 | — | — | — |
| | 151 | 290 | 297 | 147 |
| | $\Sigma f$ | $\Sigma xf$ | $\Sigma x_{(2)}f$ | $\Sigma x_{(3)}f$ |

| $y$ | $f$ | | | |
|---|---|---|---|---|
| 5 | 29 | 29 | 29 | 29 |
| 4 | 30 | 59 | 88 | 117 |
| 3 | 25 | 84 | 172 | 289 |
| 2 | 22 | 106 | 278 | — |
| 1 | 22 | 128 | — | — |
| 0 | 23 | — | — | — |
| | 151 | 406 | 567 | 435 |
| | $\Sigma f$ | $\Sigma yf$ | $\Sigma y_{(2)}f$ | $\Sigma y_{(3)}f$ |

| $+y$ | $f$ | | | |
|---|---|---|---|---|
| 9 | 8 | 8 | 8 | 8 |
| 8 | 12 | 20 | 28 | 36 |
| 7 | 14 | 34 | 62 | 98 |
| 6 | 14 | 48 | 110 | 208 |
| 5 | 40 | 88 | 198 | 406 |
| 4 | 13 | 101 | 299 | 705 |
| 3 | 25 | 126 | 425 | 1130 |
| 2 | 4 | 130 | 555 | — |
| 1 | 11 | 141 | — | — |
| 0 | 10 | — | — | — |
| | 151 | 696 | 1685 | 2591 |
| | $\Sigma f$ | $\Sigma(x+y)f$ | $\Sigma(x+y)_{(2)}f$ | $\Sigma(x+y)_{(3)}f$ |

| $x-y$ | $f$ | | | |
|---|---|---|---|---|
| $-5$ | 8 | 8 | 8 | 8 |
| $-4$ | 10 | 18 | 26 | 34 |
| $-3$ | 16 | 34 | 60 | 94 |
| $-2$ | 14 | 48 | 108 | 202 |
| $-1$ | 38 | 86 | 194 | 396 |
| | 86 | 194 | 396 | 734 |
| 4 | 1 | 1 | 1 | 1 |
| 3 | 14 | 15 | 16 | 17 |
| 2 | 5 | 20 | 36 | — |
| 1 | 22 | 42 | — | — |
| 0 | 23 | — | — | — |
| | 65 | 78 | 53 | 18 |
| | $+86$ | $-194$ | $+396$ | $-734$ |
| | 151 | $-116$ | 449 | $-716$ |
| | $\Sigma f$ | $\Sigma(x-y)f$ | $\Sigma(x-y)_{(2)}f$ | $\Sigma(x-y)_{(3)}f$ |

$$\Sigma xyf = 1685 - 297 - 567 = 821.$$

*Checks*

| | |
|---|---|
| $\Sigma xf$ | 290 |
| $\Sigma yf$ | 406 |
| $\Sigma\,(x+y)f$ | 696 |
| $\Sigma\,(x-y)f$ | $-116$ |

| | |
|---|---|
| $\Sigma x_{(2)}f$ | 297 |
| $\Sigma y_{(2)}f$ | 567 |
| $\Sigma yf$ | 406 |
| | 1270 |
| $\Sigma xyf$ | 821 |
| $\Sigma\,(x-y)_{(2)}f$ | 449 |

$$\Sigma x_{(2)}yf + \Sigma xy_{(2)}f = 2591 - 147 - 435 = 2009,$$

$$-\Sigma x_{(2)}yf + \Sigma xy_{(2)}f = -716 - 147 - 821 + 435 + 1134 + 406 = 291.$$

Therefore $\Sigma x_{(2)}yf = 859,$

and $\Sigma xy_{(2)}f = 1150.$

The moments up to the third order are:

| $\Sigma f$ | $\Sigma xf$ | $\Sigma yf$ | $\Sigma x_{(2)}f$ | $\Sigma xyf$ | $\Sigma y_{(2)}f$ | $\Sigma x_{(3)}f$ | $\Sigma x_{(2)}yf$ | $\Sigma xy_{(2)}f$ | $\Sigma y_{(3)}f$ |
|---|---|---|---|---|---|---|---|---|---|
| 151 | 290 | 406 | 297 | 821 | 567 | 147 | 859 | 1150 | 435 |

The same device may be used to find the third moments of the sums assured of whole-life assurances for the purpose of applying Perks's or Henry's method to a valuation. In *J.I.A.* Vol. LXXII, pp. 498–499, I gave four methods, (A), (B), (C) and (D), for tabulating information on cards, and I implied that if third-order moments are required it is necessary to tabulate $x$S on the cards. I now propose a fifth method (E), which enables third-order moments to be obtained without tabulating $x$S, by using four classification books. Three are as described in (D), viz. year of entry, year of birth and age at entry. The fourth shows the information tabulated according to *the year in which the duration of the policy equals the age at entry*. Very little thought will show that this classification enables the $x-t$ moments to be obtained, and the computations proceed as described earlier in this note.