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# TIME-CHANGES IN THE MORTALITY RATE: AN EXPERIMENTAL FORMULA

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ALTHOUGH this paper deals with the application of mathematical formulae to mortality data, it is not concerned with 'graduation' if that word is held to imply the fitting of a particular curve to a particular experience with the object of satisfying statistical tests. Nor is the paper concerned with the development of any philosophical theory of mortality. The experiments which it describes were undertaken in the hope of finding a standard type of curve which would give a good over-all representation of adult mortality in general. If such a curve could be found, an examination of the variations in its parameters might contribute something to an understanding, not of the nature of mortality itself, but of the differences—more particularly the secular differences—between one mortality experience and another.

2. In view of their success in dealing with the behaviour of the rate of mortality at the higher ages, the natural starting-point for such a project is a consideration of the formulae developed by Perks in 1931. These formulae involve the use of at least four parameters—a feature which, for the present purpose, is important for two reasons. First, the investigation of a large number of experiences by such means would involve a great deal of work. Secondly, while the flexibility of a curve defined by numerous parameters may be essential for the representation of the particular features of an isolated experience with sufficient fidelity to satisfy graduation tests, it may be a disadvantage if the objective is merely to study major changes of a secular nature. It seems reasonable to suppose that the smaller the number of parameters employed for this purpose, the more consistent and regular will be their movements.

3. Other points may be noted in connexion with the Perks formulae. First, in fitting them, the exponential parameter $c$ is not derived directly from the data; the remaining parameters are so derived on the basis of trial values of $c$. In Beard's recent paper (1951), three such values are used for each of a number of experiences. Separate sets of values of the other parameters are worked out for each trial value of $c$, and the operator (or the reader) is presented with a choice of alternative graduations of the same experience by the same basic formula.

4. Secondly, the formulae owe their success at the advanced ages to the fact that they provide a point of inflexion in the curve of $q$ or $\mu$ very late in life (see Isaac's remarks in opening the discussion on the 1931 paper). This feature of the formulae is, of course, justified by the good results which they produce. On the other hand, as Trachtenberg pointed out in the same discussion, a graph of the derivative or the first difference of log $\mu$ provides, in many experiences, clear visual evidence of a maximum point (and therefore of a point of inflexion in the curve of the log itself) in the neighbourhood of

age 70. In this region data are more abundant, and their characteristics less likely to be obscured by chance fluctuations, than in the 80's and 90's—a point which has some relevance in an investigation concerned primarily with broad obvious features.

5. Finally, one parameter in the Perks formulae could be dispensed with if it were accepted as axiomatic that $q \to 1$, $\mu \to \infty$ as $x \to \infty$. Also, if for the sake of simplicity we are content to assume that for all values of $x$ deaths are evenly distributed over the year of age $x$ to $x+1$, an upper limit of 2 can be assigned to the central death-rate $m_x$.

6. With these considerations in mind, it was thought that a suitable formula for the purpose described in paragraph 1 might be

$$y = \frac{A}{1 + Dc^x}, \tag{1}$$

where $y = \text{colog}\, \frac{1}{2}m_x$. This formula (the reciprocal of which is an ordinary Makeham expression) can be written in a way which shows more clearly, perhaps, the form of the curve which it represents. If $A$, $D$ and $c$ are exchanged for three other parameters $x_0$, $y_0$, $\alpha$ such that

$$A = 2y_0, \quad c = e^{2\alpha}, \quad D = e^{-2\alpha x_0},$$

the expression becomes

$$y = y_0\{1 - \tanh \alpha(x - x_0)\}, \tag{2}$$

where $y = y_0$ when $x = x_0$ and $y \to 2y_0$ and 0 as $x \to \mp\infty$ respectively. Alternatively, we can write

$$\frac{1}{y} = r_x = M + Ne^{2\alpha x}, \tag{3}$$

where

$$M = \frac{1}{2y_0}, \quad N = \frac{e^{-2\alpha x_0}}{2y_0}.$$

7. The geometry of the curve is shown by the diagram (Fig. 1). The

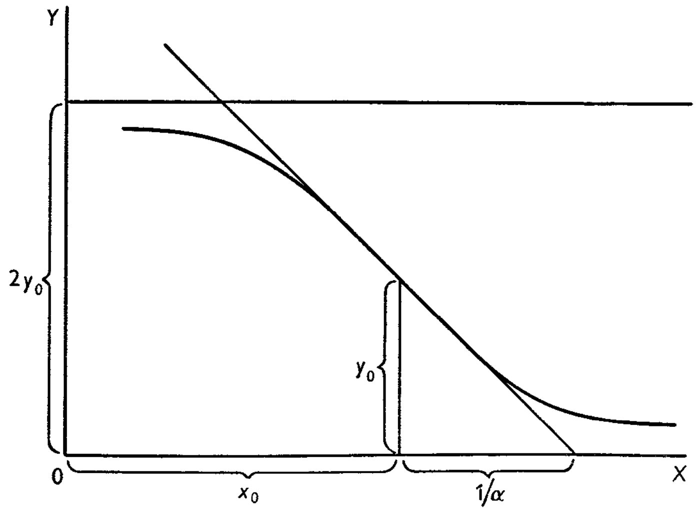


Fig. 1

parameters $x_0, y_0$ in (2) are the co-ordinates of the point of inflexion. The third parameter $\alpha$ defines the path taken by any point $(x, y)$ as it travels from $(-\infty, 2y_0)$ through $(x_0, y_0)$ to $(+\infty, 0)$; and since the derivative

$$y' = -\alpha y_0 \operatorname{sech}^2 \alpha(x - x_0),$$

the slope of the curve at the point of inflexion is $-\alpha y_0$, so that $| 1/\alpha |$ is the length of the subtangent at this point. This suggests that it might be more useful to think of the third parameter as the reciprocal of $\alpha$ rather than $\alpha$ itself. If the formula is written as

$$\frac{y - y_0}{y_0} = -\tanh \frac{x - x_0}{\alpha^{-1}},$$

it bears some resemblance to a regression equation in which the variables are measured from some origin such as the mean and standardized by dividing by a measure of dispersion.

8. A convenient method of fitting the formula is as follows. If any series of equidistant values of $r_x = M + Ne^{2\alpha x}$ is divided into three groups corresponding to values of $x$,

$$x,\ x+a,\ x+2a,\ \ldots,\ x+(n-1)\,a;$$
$$x+b,\ x+b+a,\ x+b+2a,\ \ldots,\ x+b+(n-1)\,a;$$
$$x+2b,\ x+2b+a,\ x+2b+2a,\ \ldots,\ x+2b+(n-1)\,a,$$

and if $S_1$, $S_2$, $S_3$ are the sums of the three groups of values of $r_x$, then

$$\left.\begin{aligned} S_1 &= nM + N' \\ S_2 &= nM + N'e^{2b\alpha} \\ S_3 &= nM + N'e^{4b\alpha} \end{aligned}\right\}, \quad \text{where} \quad N' = N\frac{e^{2\alpha x}(e^{2na\alpha} - 1)}{e^{2a\alpha} - 1},$$

whence

$$e^{2b\alpha} = \frac{S_3 - S_2}{S_2 - S_1}, \tag{4}$$

$$\frac{1}{S_2 - nM} = \frac{1}{S_2 - S_1} - \frac{1}{S_3 - S_2}. \tag{5}$$

9. If both (4) and (5) are used to give unique values of $\alpha$ and $M$, the corresponding unique value of $N'$ (and therefore of $N$) follows at once from the three group equations; or a 'best' value for $N$ can be found by reverting to the individual values of $r_x$. Alternatively, if $\alpha$ is found from (4), the individual values of $r_x$ can be used to give 'best' values of $M$ and $N$. Or again, if $M$ (and therefore $y_0$) is found from (5), then by writing

$$\tanh^{-1}\left(1 - \frac{y}{y_0}\right) = \alpha(x - x_0)$$

and obtaining the left-hand side by entering inversely a table of hyperbolic tangents, 'best' values of $\alpha$ and $x_0$ can be found by fitting a straight line. None of these methods is very laborious. The simplest method, of course, is not to 'fit' the formula to the individual values of $r_x$ but to determine all three parameters from the three group equations.

10. The first trial of the formula was made on the quinquennial pivotal values of the central death-rate $m$ used in the construction of English Life Tables 8, 9 and 10. In order to keep well away from any complications in the

behaviour of $m$ at the early adult ages, the fitting was confined to ages over 35. In Table 1 the 'actual' and 'expected' values of $m$ are shown side by side, the column $100(A-E)/E$ being added to give some indication of the degree of correspondence.

11. At this stage of the work the idea (see paragraph 1) that the formula was not to be required to pass graduation tests was very much in mind. There

*Table* 1. Comparison of pivotal values ($A$) of $m_x$ with expected values ($E$) derived from the formula

$$y = y_0\{1 - \tanh \alpha(x - x_0)\}, \quad \text{where} \quad y = \text{colog}\left(\tfrac{1}{2}m_x\right)$$

England and Wales population data

| Age $x$ | Men | | | Women | | |
|---|---|---|---|---|---|---|
| | $A$ | $E$ | $100\dfrac{A-E}{E}$ | $A$ | $E$ | $100\dfrac{A-E}{E}$ |
| | | | 1910–12 | | | |
| 36 | ·00662 | ·00681 | −3 | ·00550 | ·00556 | −1 |
| 41 | ·00861 | ·00865 | — | ·00696 | ·00693 | — |
| 46 | ·0116 | ·0115 | +1 | ·00908 | ·00897 | +1 |
| 51 | ·0160 | ·0157 | +2 | ·0122 | ·0122 | — |
| 56 | ·0230 | ·0224 | +3 | ·0175 | ·0172 | +2 |
| 61 | ·0333 | ·0327 | +2 | ·0252 | ·0254 | −1 |
| 66 | ·0484 | ·0490 | −1 | ·0370 | ·0381 | −3 |
| 71 | ·0727 | ·0740 | −2 | ·0592 | ·0585 | +1 |
| 76 | ·111 | ·111 | — | ·0919 | ·0904 | +2 |
| 81 | ·166 | ·166 | — | ·144 | ·138 | +4 |
| 86 | ·237 | ·239 | −1 | ·203 | ·206 | −1 |
| 91 | ·335 | ·336 | — | ·289 | ·299 | −3 |
| | | | 1920–22 | | | |
| 39 | ·00658 | ·00637 | +3 | ·00513 | ·00509 | +1 |
| 44 | ·00843 | ·00839 | — | ·00635 | ·00650 | −2 |
| 49 | ·0110 | ·0115 | −4 | ·00855 | ·00871 | −2 |
| 54 | ·0165 | ·0165 | — | ·0124 | ·0122 | +2 |
| 59 | ·0237 | ·0244 | −3 | ·0175 | ·0179 | −2 |
| 64 | ·0373 | ·0372 | — | ·0278 | ·0275 | +1 |
| 69 | ·0564 | ·0574 | −2 | ·0430 | ·0432 | — |
| 74 | ·0901 | ·0887 | +2 | ·0718 | ·0689 | +4 |
| 79 | ·138 | ·136 | +1 | ·114 | ·109 | +5 |
| 84 | ·208 | ·202 | +3 | ·178 | ·171 | +4 |
| 89 | ·289 | ·292 | −1 | ·253 | ·256 | −1 |
| 94 | ·401 | ·405 | −1 | ·357 | ·371 | −4 |
| | | | 1930–32 | | | |
| 37 | ·00475 | ·00498 | −5 | ·00393 | ·00394 | — |
| 42 | ·00641 | ·00647 | −1 | ·00487 | ·00497 | −2 |
| 47 | ·00929 | ·00883 | +5 | ·00670 | ·00659 | +2 |
| 52 | ·0130 | ·0127 | +2 | ·00945 | ·00925 | +2 |
| 57 | ·0191 | ·0190 | +1 | ·0139 | ·0137 | +1 |
| 62 | ·0292 | ·0295 | −1 | ·0213 | ·0212 | — |
| 67 | ·0468 | ·0469 | — | ·0338 | ·0342 | −1 |
| 72 | ·0752 | ·0752 | — | ·0559 | ·0562 | −1 |
| 77 | ·120 | ·119 | +1 | ·0945 | ·0925 | +2 |
| 82 | ·185 | ·185 | — | ·151 | ·150 | +1 |
| 87 | ·274 | ·277 | −1 | ·233 | ·234 | — |
| 92 | ·387 | ·396 | −2 | ·338 | ·349 | −3 |

were, however, other reasons why a comparison of actual and expected deaths was not attempted. Population data are particularly subject to age errors; it cannot be assumed that these are entirely disposed of by quinary grouping and it would be desirable to take specific account of the point in applying statistical tests. The somewhat similar problem arising from the presence of duplicates in experiences of assured lives and annuitants has been dealt with—albeit on certain arbitrary assumptions—by Seal (1940) and others; but an analogous technique has yet to be devised for dealing with age errors. Further, while the standard practice of obtaining population mortality rates by dividing the deaths of the three years around the census date by three times the census population is a perfectly good and reasonable means of measuring national mortality for historical record or practical working purposes, it inevitably complicates the application of statistical tests. This has already been pointed out by Daw (1944).

12. The results shown in Table 1 were encouraging; but before passing judgment on the suitability of the formula not only (i) as a means of portraying age-variations in mortality, but also (ii) as a mirror of secular change, it was necessary to obtain a time series of parameter values by bringing additional experiences into the picture. Data in respect of pre-1910 and post-1932 mortality were therefore obtained by the methods described in Appendix 1. The results of applying the formula to this material are shown in Tables 2 and 3.

13. An obvious comment on Table 2 is that when a formula containing three unknowns is fitted to not more than five or six observations the results are unlikely to be very wide of the mark unless the formula is fundamentally unsuitable. The value of the pre-1910 material lies in the contribution it makes to the series of parameter values. As regards Table 3, the percentage deviations (at any rate those relating to men) are greater than those of Table 1, but perhaps not unreasonably so in view of the provisional nature of much of the material. On the whole it was felt that the formula could be regarded as satisfying requirement (i) of paragraph 12. It is, of course, quite possible that this requirement would be as well or better served by other formulae; some remarks on this subject will be found in Appendix 2.

14. The left-hand portion of Table 4 gives the parameter values for all the population experiences. It will be seen that

(i) $x_0$ and $y_0$ show steady trends consistent with the continued improvement in mortality over the period covered by the table;

(ii) the progression of $\alpha$ is more uncertain—there was a reversal of trend during the period 1900–30; apart from this, the tendency is to decline;

(iii) during the same period there was a decrease in the rate of change of $x_0$ and $y_0$, followed in the case of men by an acceleration in the movement of all three parameters;

(iv) over most of the period $x_0$ (women) exceeds $x_0$ (men) by something like three years of age, with the result that $y_0 = \text{colog}\,(\frac{1}{2}m_{x_0})$ is much the same for each sex.

15. These features are interesting, but a system of parameters, two of which move in opposite directions while the third shows a wavy tendency, cannot be said to provide a very simple or clear-cut picture of secular trend. It was hinted in paragraph 2 that the movement of parameters may tend to be obscured if there are too many of them. The next step, therefore, was to

inspect the table for any evidence of correlation between the parameters which would enable one or more of them to be dispensed with.

16. The last two columns of Table 4, which show the results of this inspection, suggest that all three parameters are linearly interrelated. The

*Table* 2. Comparison of pivotal values ($A$) of $m_x$ with expected values ($E$) derived from the formula

$$y=y_0\{1-\tanh\alpha(x-x_0)\},\quad \text{where}\quad y=\text{colog}\,(\tfrac{1}{2}m_x)$$

England and Wales population data

| Age $x$ | Men $A$ | Men $E$ | Men $100\frac{A-E}{E}$ | Women $A$ | Women $E$ | Women $100\frac{A-E}{E}$ |
|---|---|---|---|---|---|---|
| | | | 1838–54 | | | |
| $39\frac{1}{2}$ | ·0128 | ·0126 | +2 | ·0127 | ·0122 | +4 |
| $49\frac{1}{2}$ | ·0185 | ·0182 | +2 | ·0157 | ·0173 | −9 |
| $59\frac{1}{2}$ | ·0319 | ·0319 | — | ·0281 | ·0293 | −4 |
| $69\frac{1}{2}$ | ·0682 | ·0676 | +1 | ·0610 | ·0603 | +1 |
| $79\frac{1}{2}$ | ·156 | ·159 | −2 | ·141 | ·140 | +1 |
| $89\frac{1}{2}$ | ·358 | ·357 | — | ·320 | ·317 | +1 |
| | | | 1870–72 | | | |
| $39\frac{1}{2}$ | ·0142 | ·0140 | +1 | ·0121 | ·0116 | +4 |
| $49\frac{1}{2}$ | ·0196 | ·0198 | −1 | ·0155 | ·0163 | −5 |
| $59\frac{1}{2}$ | ·0336 | ·0335 | — | ·0281 | ·0279 | +1 |
| $69\frac{1}{2}$ | ·0690 | ·0687 | — | ·0604 | ·0589 | +3 |
| $79\frac{1}{2}$ | ·155 | ·159 | −3 | ·140 | ·142 | −1 |
| | | | 1880–82 | | | |
| $39\frac{1}{2}$ | ·0128 | ·0128 | — | ·0109 | ·0105 | +4 |
| $49\frac{1}{2}$ | ·0192 | ·0191 | +1 | ·0148 | ·0155 | −5 |
| $59\frac{1}{2}$ | ·0339 | ·0337 | +1 | ·0277 | ·0275 | +1 |
| $69\frac{1}{2}$ | ·0692 | ·0693 | — | ·0594 | ·0579 | +3 |
| $79\frac{1}{2}$ | ·153 | ·153 | — | ·134 | ·135 | −1 |
| | | | 1890–92 | | | |
| $39\frac{1}{2}$ | ·0130 | ·0131 | −1 | ·0109 | ·0107 | +2 |
| $49\frac{1}{2}$ | ·0214 | ·0209 | +2 | ·0161 | ·0168 | −4 |
| $59\frac{1}{2}$ | ·0392 | ·0388 | +1 | ·0318 | ·0313 | +2 |
| $69\frac{1}{2}$ | ·0791 | ·0798 | −1 | ·0683 | ·0673 | +1 |
| $79\frac{1}{2}$ | ·169 | ·169 | — | ·151 | ·152 | −1 |
| | | | 1900–02 | | | |
| $39\frac{1}{2}$ | ·0110 | ·0112 | −2 | ·00887 | ·00887 | — |
| $49\frac{1}{2}$ | ·0187 | ·0184 | +2 | ·0142 | ·0145 | −2 |
| $59\frac{1}{2}$ | ·0350 | ·0349 | — | ·0275 | ·0277 | −1 |
| $69\frac{1}{2}$ | ·0719 | ·0724 | −1 | ·0602 | ·0598 | +1 |
| $79\frac{1}{2}$ | ·153 | ·153 | — | ·134 | ·134 | — |

next operation would therefore appear to be to express two of them in terms of the third, to fit the formula as thus revised to each of the experiences, and to examine the time trend of the new values of the remaining parameter. It is obvious, however, that the one-parameter formula could only be fitted by trial and error methods based on prepared tables, and that some criterion other than the percentage ratios used in Tables 1, 2 and 3 would have to be employed to determine for each experience which value of the parameter gave the best over-all fit.

17. A more direct course would be to assume that the interrelationship of the parameters reflected a linear relation between each of them and time $t$, and

*Table* 3. Comparison of pivotal values ($A$) of $m_x$ with expected values ($E$) derived from the formula

$$y=y_0\{1-\tanh\alpha(x-x_0)\},\quad \text{where}\quad y=\operatorname{colog}(\tfrac{1}{2}m_x)$$

England and Wales population data

| Age $x$ | Men | | | Women | | |
|---|---|---|---|---|---|---|
| | $A$ | $E$ | $100\frac{A-E}{E}$ | $A$ | $E$ | $100\frac{A-E}{E}$ |
| | | | 1935–37 | | | |
| 37 | ·00426 | ·00451 | −6 | ·00352 | ·00346 | +2 |
| 42 | ·00577 | ·00604 | −4 | ·00444 | ·00456 | −3 |
| 47 | ·00861 | ·00847 | +2 | ·00620 | ·00627 | −1 |
| 52 | ·0132 | ·0124 | +6 | ·00900 | ·00904 | — |
| 57 | ·0193 | ·0189 | +2 | ·0129 | ·0136 | −5 |
| 62 | ·0299 | ·0295 | +1 | ·0209 | ·0211 | −1 |
| 67 | ·0451 | ·0468 | −4 | ·0329 | ·0337 | −2 |
| 72 | ·0739 | ·0746 | −1 | ·0547 | ·0540 | +1 |
| 77 | ·121 | ·117 | +3 | ·0907 | ·0861 | +5 |
| | | | 1942–44 | | | |
| 37 | ·00343 | ·00362 | −5 | ·00270 | ·00268 | +1 |
| 42 | ·00482 | ·00505 | −5 | ·00353 | ·00353 | — |
| 47 | ·00749 | ·00729 | +3 | ·00509 | ·00490 | +4 |
| 52 | ·0118 | ·0110 | +6 | ·00734 | ·00713 | +3 |
| 57 | ·0181 | ·0170 | +7 | ·0108 | ·0109 | −1 |
| 62 | ·0274 | ·0269 | +2 | ·0167 | ·0173 | −3 |
| 67 | ·0423 | ·0429 | −1 | ·0278 | ·0281 | −1 |
| 72 | ·0635 | ·0682 | −7 | ·0460 | ·0462 | — |
| 77 | ·105 | ·107 | −2 | ·0771 | ·0760 | +1 |
| | | | 1946–48 | | | |
| 37 | ·00269 | ·00286 | −6 | ·00227 | ·00229 | −1 |
| 42 | ·00398 | ·00428 | −7 | ·00301 | ·00308 | −2 |
| 47 | ·00680 | ·00658 | +3 | ·00457 | ·00436 | +5 |
| 52 | ·0108 | ·0103 | +5 | ·00672 | ·00646 | +4 |
| 57 | ·0172 | ·0164 | +5 | ·00987 | ·0100 | −1 |
| 62 | ·0277 | ·0264 | +5 | ·0158 | ·0166 | −5 |
| 67 | ·0407 | ·0421 | −3 | ·0254 | ·0269 | −6 |
| 72 | ·0639 | ·0661 | −3 | ·0450 | ·0450 | — |
| 77 | ·101 | ·102 | −1 | ·0762 | ·0746 | +2 |

thus to arrive at a formula containing no parameters, six constants and two independent variables $x$ and $t$. In theory, the constants should be determined by using all the values of colog $(\frac{1}{2}m_x)$ in all the experiences; in practice, they would be found by fitting straight lines $a+bt$ to the series of parameter values shown in Table 4.

18. Table 4 seemed at any rate to establish the three-parameter logistic as a promising instrument for investigating the secular trend of mortality. To be of any practical value, such an investigation ought to include the most recent and most reliable data available. The population data had served well for experimental purposes; but the national mortality rates for the last twenty years have been based on denominators which are estimates and not recorded facts, and it now seemed essential to switch to the Continuous Mortality Investigation of assured lives and its predecessors, despite the long time-interval between the $O^M$ and C.M.I. experiences. It was unfortunate that this interval coincided with the period (see paragraph 14) in which the population parameters appeared to deviate from their general trends; but this did not seem to be a sufficient reason for leaving the nineteenth-century assured lives data out of the picture.

*Table* 4. Parameters for England and Wales population data

| | | $x_0$ | $y_0$ | $\alpha$ | $x_0+\frac{1}{\alpha}$ | $y_0+\cdot 05x_0$ |
|---|---|---|---|---|---|---|
| Men: | 1838–54 | 76·5 | 1·211 | ·0310 | 108·8 | 5·036 |
| | 1870–72 | 77·5 | 1·173 | ·0315 | 109·2 | 5·048 |
| | 1880–82 | 75·9 | 1·240 | ·0279 | 111·7 | 5·035 |
| | 1890–92 | 72·7 | 1·293 | ·0254 | 112·1 | 4·928 |
| | 1900–02 | 71·6 | 1·371 | ·0239 | 113·4 | 4·951 |
| | 1910–12 | 69·9 | 1·470 | ·0244 | 110·9 | 4·965 |
| | 1920–22 | 69·6 | 1·518 | ·0250 | 109·6 | 4·998 |
| | 1930–32 | 69·4 | 1·532 | ·0268 | 106·7 | 5·002 |
| | 1935–37 | 67·5 | 1·610 | ·0251 | 107·3 | 4·985 |
| | 1942–44 | 65·5 | 1·730 | ·0235 | 108·1 | 5·005 |
| | 1946–48 | 59·5 | 1·982 | ·0207 | 107·8 | 4·957 |
| Women: | 1838–54 | 78·0 | 1·209 | ·0308 | 110·5 | 5·109 |
| | 1870–72 | 77·8 | 1·216 | ·0319 | 109·1 | 5·106 |
| | 1880–82 | 77·0 | 1·263 | ·0293 | 111·1 | 5·113 |
| | 1890–92 | 74·1 | 1·308 | ·0272 | 110·9 | 5·009 |
| | 1900–02 | 73·5 | 1·385 | ·0254 | 112·9 | 5·060 |
| | 1910–12 | 72·5 | 1·475 | ·0256 | 111·6 | 5·100 |
| | 1920–22 | 73·0 | 1·504 | ·0270 | 110·0 | 5·154 |
| | 1930–32 | 72·4 | 1·534 | ·0284 | 107·6 | 5·154 |
| | 1935–37 | 70·3 | 1·639 | ·0252 | 110·0 | 5·154 |
| | 1942–44 | 70·8 | 1·688 | ·0258 | 109·6 | 5·228 |
| | 1946–48 | 69·9 | 1·742 | ·0257 | 108·8 | 5·232 |

19. It was also decided to bring in the young adult age-groups previously omitted; the fact that the formula could not provide for undulations in the mortality curve in the age-range 20–30 was thought to be of minor importance in an investigation of general trends over a long period. To save labour, the parameters for all the assured lives experiences were determined, not by 'fitting', but from the three group equations in paragraph 8.

20. The unadjusted data of six experiences were used:

$H^M$ Aggregate, pp. 94, 95 of King's *Life Contingencies* (second edition); $O^M$ whole-life, with-profits, excluding first ten years of assurance, p. 154 of *Principles and Methods*;

Continuous Mortality Investigation, life, with profits, medical, durations 3 and over: 1924–28, 1929–33, 1934–38, *J.I.A.* LXXI, 267; 1947–48, *J.I.A.* LXXVII, 112; the exposures being obtained by dividing the group deaths by the group rates.

Quinquennial pivotal values of $\theta_x$ and $E_x - \frac{1}{2}\theta_x$ were used throughout. 'Pivotal' here denotes merely the operation of deducting $\frac{1}{25}$th of the second difference, without dividing by 5.

21. Actual and expected deaths were compared by means of the formula $(\theta - E'm)/\sqrt{(E'm)}$,* where $E' = E - \frac{1}{2}\theta$. This function falls short of

$$(\theta - Eq)/\sqrt{(Epq)}$$

to an extent which increases steadily with age, so that the use of the sum of its squares as a substitute for $\chi^2$ would exaggerate the goodness of fit. On the other hand, it seems fair to point out that in the present work goodness of fit has to some extent been deliberately sacrificed by the labour-saving device mentioned in paragraph 19. Also, it would seem that in applying a formula to the logarithm of a mortality rate a standard of fit equivalent to that obtainable by the direct use of $E$ and $\theta$ could only be secured by introducing a system of weights representing the varying statistical importance of $E$ and $\theta$ at different ages. Minimizing the sums of the squares of the differences between the expected and actual values of the logarithms of $\theta$ and $Eq$ is very different from minimizing the sums of the squares of the differences between $\theta$ and $Eq$.

22. There seems to be no reason why a 'quasi-$\chi^2$' derived from

$$(\theta - E'm)/\sqrt{(E'm)}$$

for quinary groups should *not* incorporate Beard's (1951) factor 1·08 and—in the case of the four C.M.I. experiences—a $k$-factor of say 1·5 for duplicates.

23. Table 5 shows

(i) the values of $\alpha$, $x_0$, $y_0$ derived from the six experiences;

(ii) $\sum \frac{(\theta - E'm)^2}{E'm}$ without adjustment for the points referred to in paragraphs 21 and 22, the number of degrees of freedom (= number of groups − 3) being stated in brackets;

(iii) for experiences other than $H^M$, the 'expected' values of the parameters derived from the 'best' lines

$$\left.\begin{aligned} \frac{1}{\alpha} &= 37{\cdot}7 + {\cdot}585t \\ x_0 &= 71{\cdot}1 - {\cdot}865t \\ y_0 &= 1{\cdot}494 + {\cdot}0442t \end{aligned}\right\}, \quad \text{where} \quad t = \tfrac{1}{5}\,(\text{central year of experience—1900}).$$

24. The parameters are not comparable with those in Table 4 for contemporary population data because they are derived from a longer range of ages. Parameters derived from annuitant data from age 50 or thereabouts would again be different; but unless there are *a priori* reasons for assuming that the secular factors influencing the mortality of one class of lives are basically different from those influencing the mortality of another class, it

* Cf. Cramér's formula in $\mu$ mentioned by Seal (1940).

seems preferable to base the investigation of time-trend in both classes on a formula derived from one age-range only—the more extensive the better.

25. The points emerging from Table 5 are:

(i) the similarity—as indicated by their respective parameters—of $H^M$ and $O^M$. This enables the $O^M$ experience to be adopted as the starting-point for trend purposes;

(ii) the reasonably good fit of the three-parameter formula in all cases;

(iii) the reasonably good fit, also, of the straight lines defining the trends of the parameters.

26. The basic formula has now become

$$\text{colog}\,(\tfrac{1}{2}m_{x,t}) = (1\cdot494 + \cdot0442t)\left\{1 - \tanh\frac{x - 71\cdot1 + \cdot865t}{37\cdot7 + \cdot585t}\right\},$$

where $t = \frac{1}{5}$ (year of experience—1900).

*Table* 5. Experiences of assured lives

| | Calculated | | | $\Sigma\frac{(\theta - E'm)^2}{E'm}$ | $t$ | Expected | | |
|---|---|---|---|---|---|---|---|---|
| | $\alpha$ | $x_0$ | $y_0$ | | | $\alpha$ | $x_0$ | $y_0$ |
| $H^M$ | ·0287 | 75·0 | 1·299 | 13·7 (11) | | | | |
| $O^M$ | ·0285 | 75·0 | 1·308 | 25·7 (12) | $-4\cdot4$ | ·0285 | 74·9 | 1·300 |
| C.M.I.: 1924–28 | ·0241 | 65·7 | 1·739 | 15·9 (11) | $+5\cdot2$ | ·0246 | 66·6 | 1·724 |
| 1929–33 | ·0244 | 66·3 | 1·721 | 47·8 (11) | $+6\cdot2$ | ·0242 | 65·7 | 1·768 |
| 1934–38 | ·0242 | 65·0 | 1·799 | 43·3 (11) | $+7\cdot2$ | ·0239 | 64·9 | 1·812 |
| 1947–48 | ·0230 | 62·9 | 1·951 | 45·8 (11) | $+9\cdot5$ | ·0231 | 62·9 | 1·914 |

The application of this formula to the five experiences from which it has been derived by the stages described in earlier paragraphs may be expected to yield, for each experience, a total of expected deaths which will not be very different from the total actual deaths. Part of the difference will be due to incidental fluctuations about the long-term trend due to the irregular incidence of influenza epidemics and severe winters. If the difference is spread arbitrarily over the age-range by multiplying the expected deaths in each age-group by the ratio $\frac{\text{total actual deaths}}{\text{total expected deaths}}$ (this adjustment is equivalent to the introduction of one parameter derived from the individual experience), we should expect the age-group figures to stand up fairly well to a quasi-$\chi^2$ test.

27. The detailed results are shown in Table 6. They may be summarized as follows:

| | Adjusting factor | $\Sigma\frac{(\theta - E'm)^2}{E'm}$ |
|---|---|---|
| $O^M$ | ·963 | 21·5 |
| C.M.I.: 1924–28 | 1·035 | 29·8 |
| 1929–33 | 1·060 | 45·8 |
| 1934–38 | 1·023 | 42·1 |
| 1947–48 | ·923 | 40·4 |

To indicate the extent of short-term fluctuations in national mortality, the Registrar-General for England and Wales publishes each year a 'mortality ratio' which expresses the 'all ages' mortality of each year (or period) as a ratio to that of the preceding year (or period) after adjustment for age differences in the populations exposed to risk. For the period to which the five assured lives experiences relate the averages of the male mortality ratios (which contain a minor element of trend) are ·975, ·996, 1·004, ·982 and ·960 respectively. The variance of this series is considerably less than that of the set of adjusting factors given above, but the rank order is the same in each case —1929–33 largest, 1924–28 next largest, and so on. It happens, therefore, that the adjusting factors in the above table reflect—although in an exaggerated fashion—the ups and downs of the contemporary national mortality experience.

28. The age-group comparisons in Table 6 speak for themselves, but it is interesting to see that in four cases out of the five the value of $\sum \frac{(\theta - E'm)^2}{E'm}$ is less than that yielded by the original three-parameter formula in $x$ only (see Table 5).

29. The formula in $(x, t)$ could be applied to the various other groups of assured lives (durations 3 and over) covered by the Continuous Mortality Investigation. The mortality of most of these groups probably differs little from that of the group (whole life, with profits, medical) from which the formula has been derived; and one would expect that a fairly good representation of the actual experience could be obtained by the use of a single adjusting parameter. Division of the parameter for one experience by the parameter for another (contemporary) experience would yield a measure of the over-all difference in mortality; the factors, implicit in each parameter, representing epidemic and meteorological fluctuations would cancel out.

30. It is more interesting to consider the application of the $(x, t)$ formula to an experience which differs from the standard in a way which cannot be assumed to be independent of age. For example, if duration 0 mortality is compared age-group by age-group with mortality at durations 3 and over, one would expect the ratio of the former to the latter to be greater at the younger ages than at the older ages. In such a case a good representation of the select experience could be obtained, if at all, only by introducing into the $(x, t)$ formula some function of $x$ which fitted fairly well the series of ratios of actual deaths to unadjusted expected deaths.

31. Table 7 illustrates this process. The function $y = ae^{b/x}$ suggested itself as more descriptive of the progress of the ratios than a Makeham curve. The factor $a$ must be deemed to contain an adjustment for the incidental fluctuations of the mortality of the period. Without calculating $(\theta - E'm)/\sqrt{(E'm)}$ it can be seen that the agreement between columns (2) and (6) is remarkably close.

32. A similar experiment with population data (Table 8) was less successful. The general run of the ratios of actual deaths to unadjusted expected deaths suggested the same type of curve as was used in Table 7, but there are considerable irregularities at the younger ages, and the curve inevitably fails to reproduce the remarkable steadiness of the ratios from age 62 onwards. If only for this reason, the process seems more than a little artificial.

*Table* 6. Application of $(x, t)$ formula to experiences of assured lives

($E'm$ = expected deaths as adjusted by factor given in paragraph 27)

OM

| Central age of group | $\theta$ | $E'm$ | $\frac{\theta - E'm}{\sqrt{(E'm)}}$ | Central age of group | $\theta$ | $E'm$ | $\frac{\theta - E'm}{\sqrt{(E'm)}}$ |
|---|---|---|---|---|---|---|---|
| 27 | 87 | 90 | − ·3 | 67 | 21,528 | 21,560 | − ·2 |
| 32 | 804 | 792 | + ·4 | 72 | 20,886 | 20,733 | +1·1 |
| 37 | 3,076 | 3,087 | − ·2 | 77 | 16,223 | 16,393 | −1·3 |
| 42 | 6,463 | 6,319 | +1·8 | 82 | 9,999 | 10,248 | −2·5 |
| 47 | 9,761 | 9,720 | + ·4 | 87 | 3,821 | 3,905 | −1·3 |
| 52 | 13,065 | 13,081 | − ·1 | 92 | 935 | 898 | +1·2 |
| 57 | 16,535 | 16,431 | + ·8 | 97 | 78 | 61 | +2·1 |
| 62 | 19,680 | 19,573 | + ·8 | Total | 142,941 | 142,891 | |

| Central age of group | $\theta$ | $E'm$ | $\frac{\theta - E'm}{\sqrt{(E'm)}}$ | $\theta$ | $E'm$ | $\frac{\theta - E'm}{\sqrt{(E'm)}}$ |
|---|---|---|---|---|---|---|
| | C.M.I. 1924–28 | | | C.M.I. 1929–33 | | |
| 27½ | 44 | 38 | +1·0 | 66 | 50 | +2·3 |
| 32½ | 80 | 93 | −1·3 | 88 | 93 | − ·5 |
| 37½ | 200 | 218 | −1·2 | 191 | 186 | + ·4 |
| 42½ | 506 | 460 | +2·2 | 403 | 388 | + ·8 |
| 47½ | 934 | 938 | − ·1 | 815 | 784 | +1·1 |
| 52½ | 1,760 | 1,770 | − ·2 | 1,619 | 1,608 | + ·3 |
| 57½ | 3,334 | 3,221 | +2·0 | 3,047 | 3,057 | − ·2 |
| 62½ | 5,680 | 5,495 | +2·5 | 5,188 | 5,004 | +2·6 |
| 67½ | 8,671 | 8,729 | − ·6 | 8,100 | 8,267 | −1·9 |
| 72½ | 11,164 | 11,014 | +1·4 | 11,462 | 11,255 | +2·0 |
| 77½ | 10,363 | 10,596 | −2·3 | 11,320 | 11,545 | −2·1 |
| 82½ | 6,666 | 6,714 | − ·6 | 7,898 | 7,968 | − ·8 |
| 87½ | 2,758 | 2,838 | −1·5 | 3,365 | 3,478 | −1·9 |
| 92½ | 617 | 632 | − ·6 | 933 | 821 | +3·9 |
| Total | 52,777 | 52,756 | | 54,495 | 54,504 | |
| | C.M.I. 1934–38 | | | C.M.I. 1947–48 | | |
| 27½ | 71 | 54 | +2·4 | 9 | 6 } | − ·4 |
| 32½ | 111 | 103 | + ·8 | 18 | 23 } | |
| 37½ | 138 | 164 | −2·0 | 49 | 54 | − ·7 |
| 42½ | 277 | 297 | −1·2 | 71 | 89 | −2·0 |
| 47½ | 539 | 592 | −2·2 | 126 | 139 | −1·1 |
| 52½ | 1,193 | 1,218 | − ·7 | 222 | 239 | −1·1 |
| 57½ | 2,613 | 2,523 | +1·8 | 548 | 512 | +1·6 |
| 62½ | 4,617 | 4,560 | + ·8 | 1,097 | 1,062 | +1·1 |
| 67½ | 7,093 | 7,095 | — | 2,152 | 2,062 | +2·0 |
| 72½ | 9,258 | 9,570 | −3·2 | 2,841 | 3,077 | −4·3 |
| 77½ | 10,679 | 10,373 | +3·0 | 3,281 | 3,336 | − ·9 |
| 82½ | 7,605 | 7,637 | − ·4 | 2,651 | 2,542 | +2·2 |
| 87½ | 3,630 | 3,630 | — | 1,468 | 1,404 | +1·7 |
| 92½ | 967 | 975 | − ·3 | 435 | 424 | + ·5 |
| Total | 48,791 | 48,791 | | 14,968 | 14,969 | |

*Table* 7. A 1924–29, all classes, duration 0

| Central age of group | Actual deaths | Expected deaths by $(x, t)$ formula for assured lives | (2)/(3) | $\cdot 387e^{(17\cdot 5)/x}$ | Adjusted expected deaths (3)×(5) |
|---|---|---|---|---|---|
| (1) | (2) | (3) | (4) | (5) | (6) |
| 22½ | 204 | 240 | ·850 | ·842 | 202 |
| 27½ | 199 | 276 | ·721 | ·731 | 202 |
| 32½ | 189 | 287 | ·659 | ·663 | 190 |
| 37½ | 202 | 331 | ·601 | ·615 | 204 |
| 42½ | 226 | 388 | ·588 | ·584 | 227 |
| 47½ | 226 | 426 | ·531 | ·559 | 238 |
| 52½ | 298 | 531 | ·561 | ·540 | 287 |
| 57½ | 243 | 450 | ·540 | ·524 | 236 |
| 62½ | 132 | 252 | ·524 | ·512 | 129 |
| 67½ | 43 | 95 | ·453 | ·502 | 48 |
| 72½ | 6 | 15 | ·400 | ·493 | 7 |
| 77½ | 1 | 4 | ·250 | ·485 | 2 |
| Total | 1969 | | | | 1972 |

*Table* 8. Male mortality, England and Wales.
1931 Census population and 1931 deaths

| Central age of group | Actual deaths | Expected deaths by $(x, t)$ formula for assured lives | (2)/(3) | $\cdot 952e^{(19\cdot 5)/x}$ | Adjusted expected deaths (3)×(5) |
|---|---|---|---|---|---|
| (1) | (2) | (3) | (4) | (5) | (6) |
| 22 | 5,709 | 2,377 | 2·404 | 2·311 | 5,494 |
| 27 | 5,405 | 2,806 | 1·928 | 1·960 | 5,501 |
| 32 | 5,223 | 3,155 | 1·656 | 1·749 | 5,518 |
| 37 | 6,207 | 3,776 | 1·644 | 1·611 | 6,082 |
| 42 | 8,191 | 5,094 | 1·607 | 1·516 | 7,710 |
| 47 | 11,399 | 7,212 | 1·581 | 1·441 | 10,397 |
| 52 | 14,997 | 10,352 | 1·449 | 1·384 | 14,330 |
| 57 | 19,319 | 14,489 | 1·334 | 1·339 | 19,407 |
| 62 | 22,940 | 18,409 | 1·247 | 1·303 | 23,992 |
| 67 | 27,778 | 22,388 | 1·242 | 1·274 | 28,517 |
| 72 | 28,666 | 23,660 | 1·211 | 1·248 | 29,529 |
| 77 | 24,921 | 20,232 | 1·233 | 1·227 | 24,827 |
| 82 | 15,615 | 12,736 | 1·225 | 1·208 | 15,386 |
| 87 | 6,850 | 5,598 | 1·225 | 1·191 | 6,667 |
| 92 | 1,706 | 1,371 | 1·244 | 1·178 | 1,615 |
| 97 | 177 | 189 | ·937 | 1·164 | 220 |
| Total | 205,103 | | | | 205,192 |

33. While the paper was being written the publication of the unadjusted data for office annuitants for the period 1921–48 provided further material for experiment. The ultimate exposures and deaths for men were pivoted round central ages in the usual way and expected deaths were computed by the $(x, t)$

formula derived from the assured lives experiences and rated down to produce equality with the total actual deaths at all ages. The results shown in Table 9 may be summarized as follows. There are 8 age-groups, and therefore 7 degrees of freedom, in each case:

| Period | Adjusting factor | $\Sigma \frac{(\theta - E'm)^2}{E'm}$ |
|---|---|---|
| 1921–25 | ·974 | 9·7 |
| 1926–30 | ·889 | 17·4 |
| 1931–35 | ·983 | 14·4 |
| 1936–40 | ·983 | 26·3 |
| 1941–45 | ·969 | 33·1 |
| 1946–48 | ·898 | 8·9 |

The ratio of actual to expected deaths did not appear to vary systematically with age. The over-all average was ·95, so that if the adjusting factors shown above are divided by this figure, the result should, in theory, be a measure of incidental fluctuations due to influenza and weather. Unfortunately, no comparison can be made in this respect with the experiences of assured lives, because the periods in which the data are grouped differ; nor, in view of the limited span of ages to which the above figures relate, is comparison with the Registrar-General's all-ages mortality ratio relevant. The run of the adjusting factors is certainly rather odd; the two periods 1926–30 and 1946–48, for which the factors are lowest, contain the 'bad' years 1929 and 1947.

34. The age-fit is not remarkably good; it would have been more disappointing than it is if the 'quasi-$\chi^2$' function had continued its steady increase instead of reverting to a low figure in the last period. The extent of duplication in the C.M.I. annuitant experience is unknown; but it would need to be fairly considerable to make the quasi-$\chi^2$'s present a really good picture. At the same time, the figures do not seem quite bad enough to justify the despondent terms used by the Committee in their report when describing the male data. To provide a more practical yardstick than the quasi-$\chi^2$ test, expectations of life (i.e. annuity-values at 0% interest) were roughly computed for 1941–45 (the worst case) at various age-points by means of abridged life tables based on the actual death-rates and the death-rates implicit in the equalized actual deaths. At ages under 80 the differences between the two series of expectations were of the order of 1% only. This tends to support the view that statistical tests of the graduation of mortality rates—elegant, theoretically correct and intellectually stimulating as they are—can sometimes be excessively stringent from the point of view of the practical purposes which the rates have to serve.

35. The $(x, t)$ formula was also applied to the ultimate (after 5 years) office annuitant experiences of 1863–93 and 1900–20. Details are given in Table 10, the summary particulars being as follows:

| Period | Adjusting factor | $\Sigma \frac{(\theta - E'm)^2}{E'm}$ |
|---|---|---|
| 1863–93 | ·949 | 13·9 (9 d.f.) |
| 1900–20 | ·950 | 65·1 (10 d.f.) |

The adjusting factors are reasonable enough if it can be assumed that (*a*) the mortality of annuitants has always been about 5% less than that of assured lives and (*b*) incidental annual fluctuations would tend to cancel out over the fairly long periods to which the experiences relate. While the fit for 1863–93

*Table* 9. Life-Office annuitants (men) 1921–48. Comparison of actual deaths with expected deaths by $(x, t)$ formula for assured lives

($E'm$ = expected deaths as adjusted by factor given in paragraph 33)

| Central age of group | $\theta$ | $E'm$ | $\dfrac{\theta-E'm}{\sqrt{(E'm)}}$ | $\theta$ | $E'm$ | $\dfrac{\theta-E'm}{\sqrt{(E'm)}}$ |
|---|---|---|---|---|---|---|
| | 1921–25 | | | 1926–30 | | |
| 57½ | 19 | 33 | −2·3 | 34 | 20 | +3·5 |
| 62½ | 109 | 111 | − ·2 | 76 | 72 | + ·5 |
| 67½ | 284 | 310 | −1·4 | 229 | 227 | + ·1 |
| 72½ | 500 | 505 | − ·2 | 558 | 534 | +1·0 |
| 77½ | 728 | 699 | +1·1 | 712 | 700 | + ·5 |
| 82½ | 618 | 599 | + ·8 | 614 | 662 | −1·8 |
| 87½ | 346 | 339 | + ·4 | 358 | 370 | − ·6 |
| 92½ | 121 | 128 | − ·6 | 112 | 108 | + ·4 |
| Total | 2725 | 2724 | | 2693 | 2693 | |
| | 1931–35 | | | 1936–40 | | |
| 57½ | 20 | 23 | − ·6 | 44 | 34 | +1·7 |
| 62½ | 72 | 73 | − ·1 | 142 | 128 | +1·3 |
| 67½ | 266 | 241 | +1·6 | 310 | 325 | − ·9 |
| 72½ | 611 | 547 | +2·8 | 641 | 734 | −3·4 |
| 77½ | 851 | 876 | − ·8 | 1013 | 958 | +1·8 |
| 82½ | 803 | 829 | − ·9 | 1052 | 1036 | + ·5 |
| 87½ | 478 | 513 | −1·5 | 582 | 601 | − ·8 |
| 92½ | 164 | 166 | − ·2 | 233 | 201 | +2·3 |
| Total | 3265 | 3268 | | 4017 | 4017 | |
| | 1941–45 | | | 1946–48 | | |
| 57½ | 41 | 44 | − ·4 | 17 | 16 | + ·2 |
| 62½ | 179 | 145 | +2·8 | 50 | 57 | − ·9 |
| 67½ | 361 | 423 | −3·0 | 166 | 187 | −1·5 |
| 72½ | 871 | 856 | + ·5 | 418 | 444 | −1·2 |
| 77½ | 1191 | 1168 | + ·7 | 698 | 661 | +1·4 |
| 82½ | 935 | 1013 | −2·4 | 673 | 682 | − ·3 |
| 87½ | 742 | 663 | +3·0 | 395 | 387 | + ·4 |
| 92½ | 155 | 166 | − ·8 | 147 | 130 | +1·5 |
| Total | 4475 | 4478 | | 2564 | 2564 | |

is fairly good, that for 1900–20 is extremely bad; and it will be recalled that it was during this period that the population-data parameters (particularly $\alpha$) were found to deviate from their general trend. If the values of $x_0$ and $y_0$ implicit in the $(x, t)$ formula are retained and a value of $\alpha$ is obtained from the 1900–20 experience itself, $\sum \dfrac{(\theta-E'm)^2}{E'm}$ can be reduced to 28·9.

36. It was made clear at the beginning of the paper that no attempt was being made to discover a 'law of mortality'. The $(x, t)$ formula as stated at the beginning of paragraph 26 would, indeed, defy any attempt to provide it with a philosophical background. There is also the consideration that the formula bears no sort of relation to the mortality of childhood and adolescence. It may be, of course, that over the range of ages to which it has been applied the formula merely provides approximate values of an entirely different function which would be more susceptible of *a priori* interpretation.

*Table* 10. Life Office annuitants (men) 1863–93 and 1900–20. Comparison of actual deaths with expected deaths by $(x, t)$ formula for assured lives

($E'm$ = expected deaths as adjusted by factor given in paragraph 35)

| Central age of group | 1863–93 | | | Central age of group | 1900–20 | | |
|---|---|---|---|---|---|---|---|
| | $\theta$ | $E'm$ | $\frac{\theta-E'm}{\sqrt{(E'm)}}$ | | $\theta$ | $E'm$ | $\frac{\theta-E'm}{\sqrt{(E'm)}}$ |
| 42 } 47 } | 9 | 12 | −1·0 | $47\frac{1}{2}$ | 45 | 23 | +4·4 |
| 52 | 25 | 17 | +2·0 | $52\frac{1}{2}$ | 72 | 66 | + ·8 |
| 57 | 67 | 48 | +2·7 | $57\frac{1}{2}$ | 218 | 189 | +2·1 |
| 62 | 115 | 108 | + ·7 | $62\frac{1}{2}$ | 505 | 438 | +3·2 |
| 67 | 231 | 240 | − ·6 | $67\frac{1}{2}$ | 944 | 827 | +4·0 |
| 72 | 451 | 456 | − ·2 | $72\frac{1}{2}$ | 1321 | 1383 | −1·7 |
| 77 | 612 | 610 | + ·1 | $77\frac{1}{2}$ | 1735 | 1773 | − ·9 |
| 82 | 532 | 548 | − ·7 | $82\frac{1}{2}$ | 1593 | 1652 | −1·4 |
| 87 | 318 | 324 | − ·3 | $87\frac{1}{2}$ | 989 | 1019 | − ·9 |
| 92 | 88 | 85 | + ·3 | $92\frac{1}{2}$ | 272 | 323 | −2·8 |
| | | | | $97\frac{1}{2}$ | 46 | 49 | − ·4 |
| Total | 2448 | 2448 | | Total | 7740 | 7742 | |

37. All that has been achieved is a demonstration that on a particular mathematical hypothesis the variations in a particular type of mortality with both age and time have followed a fairly regular and consistent pattern. If the experiments which have been described were put on a more substantial footing (which should include women as well as men), it could become a matter of routine to write mortality functions in terms of two major variables $x$ and $t$ instead of $x$ only as at present; and with an additional source of variation covered in this way it might be easier to study systematically the subsidiary sources of variation—selection, class differences and so forth.

38. The two-variable formula could obviously be written in terms of the year of birth and the year of experience instead of the age and the year of experience; and a study could be made, by means of partial derivatives, of the relative importance of birth-time and time passed through in determining the course of the mortality rate. A link is thus provided with the 'generation theory'.

39. No formula can foretell the future. It is emphasized that the work described in this paper was undertaken in the hope of gaining a better understanding of secular changes in mortality in the past. Pollard (1949) and other writers mentioned in his paper have also investigated this subject—in some cases with the declared object of obtaining a reasonable basis for projection

purposes. Projections of mortality rates are frequently in demand; actuaries are without question best qualified to attempt them; and a formula which has been found to represent fairly well the changes in mortality over a considerable period inevitably invites examination as a potential means of projection.

40. One way of doing this would be to analyse the behaviour of the two-variable function and its various derivatives; but the mathematics is rather cumbrous, and a simpler and more practical method is to take cross-sections, at various points of future time, of the surface which the formula represents. Such cross-sections are provided in Table 11. The further improvements in mortality which the table suggests for the next century are not obviously absurd, and the time-series of rates at the younger ages show obvious signs of approaching asymptotic values; Makeham curves fit well at these ages. At the older ages there is some slight suggestion of an accelerated rate of decline towards the end of the period covered; and it may be that if the range of the table were extended, a similar feature would appear at the younger ages also. No reasonable being, however, would demand a forecast more than a hundred years ahead, and, if calculations involving future rates of mortality have to be made, it may be thought that the formula provides a basis which is not altogether unreasonable. The figures for 1980 are broadly comparable with the projections (by an entirely different method) of male population group death-rates given for 1978 on page 79 of a memorandum, *The Course of Mortality in Great Britain*, appearing in the volume of Royal Commission on Population papers mentioned in Appendix 1 to this paper. The figures for 2040 are considerably higher than those given in the memorandum for 2048.

*Table* 11. Projection of $10^3 m_x$* by the $(x, t)$ formula

| $x$ | Year of experience | | | | | |
|---|---|---|---|---|---|---|
| | 1940 | 1960 | 1980 | 2000 | 2020 | 2040 |
| 20 | 1·03 | ·65 | ·45 | ·33 | ·25 | ·20 |
| 30 | 1·66 | 1·17 | ·89 | ·72 | ·62 | ·55 |
| 40 | 3·16 | 2·52 | 2·12 | 1·89 | 1·76 | 1·67 |
| 50 | 7·18 | 6·30 | 5·77 | 5·50 | 5·38 | 5·31 |
| 60 | 18·6 | 17·6 | 16·8 | 16·5 | 16·6 | 16·4 |
| 70 | 50·6 | 49·2 | 47·8 | 47·2 | 47·1 | 46·8 |
| 80 | 130 | 126 | 121 | 119 | 117 | 114 |
| 90 | 286 | 277 | 264 | 255 | 248 | 239 |
| 100 | 530 | 508 | 482 | 461 | 444 | 426 |

* The rates refer to medically examined lives assured under whole-life with-profit policies (durations 3 and over).

41. Actuarial arithmetic (particularly if it is concerned with curve-fitting, graduation testing and similar subjects) is often carried out to several decimal places. In a recent paper Barnett (1950) suggested that this was a matter 'worthy of consideration'. In the present paper a severe economy in arithmetic has been practised, partly because of the wide range of experiences covered and partly to bring most of the computing work within the range of Cotsworth's multiplication tables. It will have been observed that parameter values are stated throughout to three places or three significant figures; similar practices have been observed in the calculations leading to these

values. On such questions as the 'right' number of places to be used in recording the quotient of two figures of particular orders of magnitude some people hold strong views, while others decline to commit themselves. In the present instance the object was to investigate as much data as possible without excessive labour, without much regard to the question how the significance of a particular result would be affected by the inclusion or exclusion of an extra digit, and without any desire to contribute anything to the establishment of a doctrine on these matters.

42. It seems reasonable, however, to suggest that a good deal of unnecessary refinement may be introduced into arithmetical processes—at any rate where graduation is concerned—in the name of 'smoothness'. The *principle* of smoothness, surely, is automatically observed whenever a mathematical function is used to express a statistical series. Perfect smoothness in a series of irrational numbers could only be achieved by the use of an infinite number of decimal places. The *practical* objective, however, is to obtain regularity and consistency in tables of premium rates or other derived monetary functions; and it would seem simpler and more effective to secure this objective by 'smoothing' the tables themselves rather than the ingredients from which they have been concocted.

43. Some office colleagues have helped the preparation of this paper by their readiness to discuss ideas and make suggestions; others have lent a hand with the computing. I am deeply grateful to them all, and to the Government Actuary for his permission to refer to unpublished material in the files of the Department. My sincere thanks are due also to Messrs Perks and Redington, whose criticisms of an early draft were both forthright and encouraging, and to numerous others—including, possibly, some who are not mentioned in the list of references—whose published work has helped me to conceive and carry out the investigation described in the paper.

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## APPENDIX 1

### *Bases of Tables 2 and 3*

The pre-1910 material used consisted of

(*a*) the E.L. 3 data given on pages xviii and xix of Farr's *English Life Table* (Longman, 1864);

(*b*) the census populations (England and Wales) of 1871, 1881, 1891 and 1901 and the deaths of 1870–72, ..., 1900–2, given on pp. 14 and 36 of the *Eighty-Second Statistical Abstract for the United Kingdom* (Cmd. 5903, 1939).

These data are, for the most part, given in 10-year age-groups, and it was therefore necessary to use 10-year pivotal values. In extracting these values the group '95 and over' in (*a*) and the group '85 and over' in (*b*) were treated as closed 10-year groups.

The post-1932 data consisted of

(*c*) the mid-year estimates of population in age-groups, and the statistics of deaths at each age, published annually in the *Annual Reviews* of the Registrar-General for England and Wales;

(*d*) the hypothetical life-table for Great Britain (1942–44) prepared by the Government Actuary's Department for the Royal Commission on Population (*Papers of the Royal Commission on Population: Reports and Selected Papers of the Statistics Committee*, II, 54).

As regards (*c*) it would have been possible to calculate the parameters from quinary pivotal values of both population and deaths, but for the fact that the population estimates provide no detail beyond age 85. As it seemed desirable to bring the experience at ages 85 and over into the picture, this age-group was treated as a closed 10-year group and 10-year pivotal values were obtained at ages $39\frac{1}{2}$, $49\frac{1}{2}$, ..., $79\frac{1}{2}$ from which the parameters were derived. These parameters were then used to calculate 'expected' quinary values of $m$ at ages 37, 42, ..., 77 for comparison with 'actual' pivotal values at these ages derived directly from the data. The material actually used consisted of the deaths and the mid-year population estimates for the two triennia 1935–37 and 1946–48.

As regards (*d*), the 1942–44 life table was completed at the old ages by the customary National Life Table device of grafting on a Gompertz curve. For the purpose of the paper, it was thought desirable to go back to the unadjusted data and to use the '85 and over' group, in conjunction with 10-year groups of population and deaths from age 45 onwards, to obtain pivotal values at $59\frac{1}{2}$, $69\frac{1}{2}$ and $79\frac{1}{2}$. The methods used in constructing the 1942–44 table made it impossible to derive values of $m$ at ages $39\frac{1}{2}$ and $49\frac{1}{2}$ from pivotal values of populations and deaths; they were therefore obtained from the tabular $q$'s by the formula $m = 2q/(2-q)$. The five actual values of $m$ were then adjusted to an England and Wales basis by multiplying by $q$ (E.L. 10)/$q$ (G.B. 1930–32), the denominators of this fraction being obtained from a table constructed in the Department after the 1931 census for internal working purposes. When the parameters had been derived from these five values they were used, as in (*c*), to calculate 'expected' quinary values of $m$ at ages 37, 42, ..., 77, the corresponding 'actual' values being obtained from the tabular $q$'s at those ages.

## APPENDIX 2

### *Characteristics of the logistic and other curves*

(1) The formula used in the paper may be written

$$y = y_0\,(1 - \tanh u),$$

where $u$ represents the 'standardized' age $\alpha(x - x_0)$. Tanh $u$ is an infinite series in odd powers of $u$; its derivative is therefore an infinite series in even powers of $u$, so that the curve of $\frac{1}{m}\frac{dm}{du}$ is symmetrical and bell-shaped. It would be difficult to explain this symmetry by any *a priori* considerations of the nature of mortality; moreover, a symmetrical bell-shaped curve naturally suggests a frequency distribution, and $\frac{1}{m}\frac{dm}{du}$ cannot be thought of in this fashion. But it can at least be said that there seems to be no obvious reason in nature why $\frac{1}{m}\frac{dm}{du}$ should not be symmetrical.

(2) A further feature of some interest is the steady decline in the age ($x_0$) at which $\frac{1}{m}\frac{dm}{du}$, i.e. the rate of age-increase in the rate of mortality relative to itself, is greatest. This invites a study of secular changes in the proportions and age-patterns of deaths from the different causes.

(3) The conception of the logarithm of a mortality rate as a series in powers of $x$ recalls some remarks of Trachtenberg already referred to at the beginning of the paper. (Reference should also be made to the contributions of Lidstone and Perks himself to the same discussion.) Trachtenberg, however, envisaged the series as a polynomial of low degree. The idea of an infinite series emerges immediately from the Perks formula (whether it is applied to a mortality rate or to its logarithm) if $c^x$ is written as $e^{\delta x}$, as has been done ($\delta = 2\alpha$) in this paper. This is, of course, equally true of the Gompertz and Makeham formulae. By introducing a denominator involving $c^x$, Perks arrived at the logistic as a suitable instrument for mortality analysis; and it is on this idea that the whole of the present paper has been based.

(4) The logistic, however, is only one of a number of curves possessing the features described above. The general equation of the logistic is

$$y = a + b \tanh \alpha(x - x_0), \quad \text{or} \quad y = a + b \tanh u.$$

The curve of the hyperbolic tangent presents a similar appearance to the curves of the inverse of the natural tangent ($\tan^{-1} u$), the inverse of the hyperbolic sine ($\sinh^{-1} u$), the probability integral ($\operatorname{erf} u$) and—if attention is confined to a single period of the wave—to the curve of sines or simple harmonic motion ($\sin u$). The general similarity can also be seen by expanding these functions in powers of $u$; or, if tabular values of the functions are extracted for a series of positive and negative values of $u$, then, by choosing suitable values for $a$ and $b$ in each case, values of $y = a + bf(u)$ can be obtained which resemble each other fairly closely over a considerable part of that range of values of $u$ which corresponds to the particular $\alpha$'s, $x$'s and $x_0$'s used in the paper.

(5) The choice of one or other of these forms as an alternative means of expressing mortality in terms of age would depend on the particular mortality function chosen. Thus, if it is regarded as essential doctrine that $\mu \to \infty$, the form which suggests itself is $\sinh^{-1} u$, since this also $\to \infty$.* Similarly, if $q$ or $m$ is used, $\sin u$ is a *prima facie* possibility although—even for the broadest and most utilitarian of purposes—the association of mortality with a particular segment of the curve of wave-motion seems very artificial. When the choice of function has been made, consideration of the limiting values should enable either $a$ or $b$ to be dispensed with or allotted a predetermined value, thus bringing the formula to a three-parameter basis.

(6) Some of these formulae have been applied to mortality data with good—in some cases remarkably good—results; but no attempt has been made to use any one of them for a series of mortality experiences in order to investigate parameter changes. The use of the logistic has the substantial advantage that, as is shown in the paper, the parameters can readily be estimated direct from the values of the logarithm of the mortality function. This advantage is shared by the sine curve; in the other cases it seems necessary to work on the first derivative of the logarithm—a method which is liable to break down unless the data run very smoothly. As regards the use of the derivative it may be remarked that some curves of the kind under discussion can be regarded as generated by the integration of $(1+u^2)^{-n}$. If this integral is denoted by $I(n)$, then $I(1)=\tan^{-1} u$, and $I(\tfrac{1}{2})=\sinh^{-1} u$. Again,

$$\lim_{n\to\infty}\left(1+\frac{u^2}{n}\right)^{-n}=e^{-u^2} \quad \text{(cf. earlier reference to erf } u\text{).}$$

(7) It may also be noted that if the logarithm of a mortality function ($q$, $m$ or $\mu$) is expressed in the form $a+bf(x, x_0, \alpha)$ the function itself can be written as $Bc^{f(x, x_0, \alpha)}$, of which the Gompertz expression $Bc^x$ is a particular case. For the three-parameter form $a\{1-f(x, x_0, \alpha)\}$ the $B$ disappears and if $\phi(x, x_0, \alpha)=1-f(x, x_0, \alpha)$,

$$\log_a q \text{ (or } m \text{ or } \mu)=\phi(x, x_0, \alpha),$$

$$q \text{ (or } m \text{ or } \mu)=a^{\phi(x, x_0, \alpha)}.$$

* Since $\sinh^{-1} u=\log_e[u+\sqrt{(1+u^2)}]$ an association with $\log\mu$ would enable logs to be dispensed with on both sides and lead to a quadratic or other polynomial relation between $\mu$ and $u$.

## ABSTRACT OF THE DISCUSSION

**Mr L. G. K. Starke,** in introducing the paper, said that few, if any, actuaries would care to put their money on the chance that the trend of mortality would be reversed and begin to go up instead of down, or even on the chance that mortality rates would remain indefinitely at their current level. It seemed to him that calculations involving the mortality rate should, if they were to be realistic, take into account, in however crude and conjectural a way, the probability that there would be some further decline. He suggested that an approach of that kind was more in keeping with the actuary's claim to be a practical man than a preoccupation with the minor idiosyncrasies of the mortality curve at any particular moment. They had got to the stage when, having constructed a mortality table from recent experience, they had to consider as a separate process (and with no very well-established technique at their disposal) how to make it a more suitable instrument for the particular set of financial prognostications they wished to make. Although ample scope had always to be left for individual judgment in individual circumstances, might it not be advantageous to be able to consult, as a matter of routine, a table which made some attempt to take account of the secular element?

Consideration of the secular aspect of mortality inevitably raised the question of the construction of life tables. Someone (he thought it was probably Dr Farr) had said that a life table depicted the march of a generation through time. At the time it was made that remark was probably a good epigram; but to-day he would be inclined to describe a life table constructed in the traditional manner as showing a generation which insisted on marking time while the rest of the world went by, or a collection of people who were forever condemned to march in the wrong direction on a moving staircase. Despite the violent state of motion which that implied those people were described, in the actuary's curious idiom, as a stationary community.

He often felt that he would like to meet one of the million people whose life history was recorded, say, in English Life Table No. 7. At the time of speaking such a person would be approaching middle age, but he still lived in a world of hansom cabs and horse buses; he had escaped both world wars and would escape any future war; he had never heard, and would never hear, of wireless, the aeroplane, penicillin, the atom bomb, national insurance, or anything else that had happened in the past forty-five years.

**Mr R. D. Clarke,** in opening the discussion, recalled that there were two classic methods of approach in the search for a formula in $x$ and $t$ which would represent rates of mortality that varied both with age and time. The first was implicit in the work of Derrick (*J.I.A.* LVIII, 117), which had been developed mathematically by Kermack, McKendrick and McKinlay in their paper published in the *Journal of Hygiene* in 1934. It consisted in regarding the rate of mortality as a product of two independent functions, one of which varied with age and the other with time.

The second approach, which had been used by the author in his paper, was that which had been adopted by Cramér and Wold in their historic work on Swedish death-rates published in *Skandinavisk Aktuarietidskrift* in 1935. Those authors had proceeded by two stages. First, they had fitted a series of Makeham formulae to $\mu_x$ at successive periods of time and so obtained what amounted to three time-series corresponding to the $A$, $B$ and $c$ of the Makeham formula.

To each of those they had then fitted a logistic curve varying with time, and had thus been able to derive a comprehensive formula involving both $x$ and $t$ as variables. Into that method the author had introduced two major developments. He had replaced the rate of mortality (or central death-rate) by its logarithm and he had used a logistic curve as his primary mathematical model varying in $x$.

It had long puzzled him why actuaries had not paid more attention to $\log q_x$. The first mortality formula to find general acceptance had been that of Gompertz, who had represented the force of mortality by $Bc^x$. If, instead of $\mu_x$, $q_x$ were represented by the Gompertz expression, the formula was easily written in the form

$$\log q_x = \beta + \gamma x,$$

where $\gamma = \log c$. Makeham's modification of Gompertz's formula could not, of course, be transformed in that way, and in consequence later research had tended to ignore $\log q_x$. Nevertheless, he had often wondered how actuarial theory might have developed if $\log q_x$ had held the centre of the field. The advantage of the logarithmic function was that it brought the mortality curve into a new focus. The rapid upward sweep at the later ages was telescoped, while the variations at younger ages, formerly insignificant, took on a new importance. When it was considered how fruitful the logarithmic transformation had been in other fields of statistical inquiry—for example, in R. A. Fisher's $z$-distribution for the variance ratio and in the investigation of dosage mortality by means of probit analysis—it certainly seemed that $\log q_x$ might well merit a little more of their attention.

When the Gompertz formula was used, $\log c$ measured the slope of the straight line representing $\log q_x$. Although that relationship was no longer precise for the later developments of Gompertz—namely, the Makeham and Perks formulae—it was still approximately valid. With the lightening of mortality, the slope of $\log q_x$ had necessarily become steeper (since $q_x$ had to reach unity at the end of the life table, and since experience had shown that the limit of life remained nearly constant) and so over the years $\log c$ had tended to become progressively larger. That result was demonstrated by Cramér and Wold in the paper he had mentioned, and it had, indeed, been very generally recognized by actuaries. At the same time, it was interesting to note from Table 2 that the population mortality in England and Wales had in the nineteenth century been by no means undergoing a continuous reduction. A glance at the male rates of mortality for the period 1890–92 showed that they had been higher than in any of the previous periods in the table. In fact, the value of $\log c$ appeared to have fallen during the greater part of the nineteenth century and to have increased again in the twentieth.

In the diagram at the foot of p. 172 the author's adoption of colog $(\frac{1}{2}m_x)$ as his primary function had had the effect of turning the curve upside down, but essentially it was still the curve of $\log q_x$. The slope of the curve could be measured by the slope of the tangent at the point of inflexion. Now the tangent was the reincarnation of their old friend the Gompertz formula, and one of the merits of the diagram was that it illustrated clearly how the logistic curve could be regarded as a refinement of the Gompertz straight line. It could be easily demonstrated that, in the recently published mortality of assured lives for 1947–48, a good fit could be obtained between ages 40 and 80 from a Gompertz curve, with $\log c$ equalling ·0424. Outside those ages, however, Gompertz was quite useless. The curve of $\log q_x$ flattened out at either end—and the logistic was admirably suited to represent just that effect. Nevertheless, it seemed to him both sensible and appropriate to consider the slope of the tangent as a basic parameter of the logistic curve corresponding to log $c$ in the Gompertz formula. In terms of the author's symbols, that quantity was $-\alpha y_0$, as stated near the top of p. 173. If they were considering the actual curve of log $q_x$, the negative sign would disappear, and so he proposed to speak simply of $\alpha y_0$. He had calculated the values of that function from the data in Table 4, and it was interesting to note that for male lives they began at ·0375 for the period 1838–54, steadily declined to ·0328 in 1890–92 and later rose to ·0410 in 1946–48. Those values bore an obvious similarity to the magnitude which they were accustomed to find for log $c$ in their traditional graduations, and in his view they helped to establish the priority of $\alpha y_0$ as the basic characteristic of the curve.

After the slope of the tangent, the next important feature seemed to be the vertical range. That was equal to $A$ in formula (1) or to $2y_0$ in terms of formula (2). Broadly speaking, the range would increase as mortality declined. That did not mean that the slope and the range were completely inter-dependent, though there was undoubtedly a limited dependence between them. Thus, if the minimum rate of mortality fell from ·01 to ·001 its common logarithm fell from $-2$ to $-3$, so that there was a 50% increase in the range, and, at the same time, the slope of the curve must grow steeper in order to traverse the increased range within the span of life. But, as the diagram demonstrated, the range determined the position of the point of inflexion, the ordinate at that point being equal to $y_0$. If the range were increased while the tangent remained fixed, the point of inflexion would occur at an earlier age. Thus variations in the range had the effect of shifting the age group at which the steepest part of the curve was to be found.

Another interesting characteristic of the curve was the point at which the tangent cut the $x$ axis. That point appeared in Table 4 in the column headed $x_0+1/\alpha$. As mortality lightened, there was some tendency for that function to decrease and, as could be seen from Table 4, it had diminished for male lives from 113·4 in 1900–2 to 107·8 in 1946–48. But the trend had not been continuous, and what was chiefly remarkable was its relative constancy round about 110. It was perhaps of interest that he himself, in fitting Pearson Type III curves to the distribution of deaths, had found that to secure a good fit it was necessary to put the upper limit of the curve for male lives somewhere in the same neighbourhood; moreover, paradoxical though it might seem, that upper limit had also tended to diminish with reducing mortality. He did not profess to attribute any particular significance to that parallel; but it was not inconceivable that a profound mathematician could trace an underlying identity of method between the treatment of the curve of deaths with the gamma function and the fitting of the logistic to log $q_x$.

The logistic curve appropriate to a given series of log $q_x$ could be fully defined by the slope, the range and the tangent intersection; or in the symbolism of the paper $\alpha y_0$, $2y_0$ and $x_0+1/\alpha$. Those might be termed the geometric parameters of the curve corresponding to the algebraic parameters which emerged naturally from formula (2).

The calculated series for $x_0$, $y_0$ and $\alpha$ shown in Table 4 were interesting. Not only were $x_0$ and $\alpha$ closely correlated—which followed inevitably from the near-constancy of $x_0+1/\alpha$—but $x_0$ and $y_0$ were negatively correlated. A glance at the parameter values for male lives in 1946–48 showed that a large increase in $y_0$ was accompanied by a sharp fall in $x_0$. That, of course, would be expected, once it was known that the slope ($\alpha y_0$) changed very little from the previous period. In terms of the diagram the ordinate at the point of inflexion increased as it moved to the left. Why there should have been that remarkable increase in $y_0$ for male lives in 1946–48, when the corresponding increase for females was in no way exceptional, was a mystery which he had been unable to solve.

When the author came to the next stage in the investigation, namely, the fitting of mathematical formulae, having time as the independent variable, to the calculated parameter values, he turned from population data to assured lives and annuitants. It was worth noting, however, that the population data in Table 4 would not have led to simple linear relationships such as those in paragraph 23. The first differences of $x_0$, $y_0$ and $1/\alpha$ were all erratic, and more complex formulae would have been required. Even with the assured lives, the author decided to discard the $H^M$ because, although much earlier than the $O^M$, it provided very similar parameter values. The author was thus left with the $O^M$ centred on the year 1888 and with four recent C.M.I. experiences extending from 1924 to 1948. To those five sets of parameters the author fitted the linear equations in paragraph 23. The fit was perhaps rather better than suggested by Table 5, since the values of $\alpha y_0$, which were not given there, were in fact fairly closely reproduced. He thought the point of sufficient interest to justify quoting the actual figures:

| Experience | Values of $\alpha y_0$ | |
|---|---|---|
| | Calculated | Expected |
| $O^M$ | ·0373 | ·0371 |
| 1924–28 | ·0419 | ·0421 |
| 1928–33 | ·0420 | ·0428 |
| 1934–38 | ·0435 | ·0438 |
| 1947–48 | ·0449 | ·0442 |

Again it was seen that improving mortality was reflected in a steady increase in the index of slope—the index which was more familiar to them as log $c$. At the same time, inspection of $y_0$ showed that the range was also growing steadily, a feature to be expected on general grounds of reasoning. Nevertheless, the remarkable movement in both $x_0$ and $y_0$ which was observed in the male population for 1946–48 was not repeated in the 1947–48 assured lives.

He next came to a feature of the paper which had caused him some difficulty. The author had developed a mortality formula in $x$ and $t$. However, to improve the closeness of fit, he had introduced the concept of an adjusting factor which was in fact the ratio of the actual to the expected deaths. As was stated in paragraph 26, that was equivalent to introducing an extra parameter at each point of time at which data were assembled. The need for those adjusting factors was attributed to climatic variations and epidemics. However, they seemed to him to detract from the essential value of the method, since they destroyed the simple relation of a two-variable formula in $x$ and $t$. The first three C.M.I. periods were each of 5 years' duration, so that climatic and epidemic variations had a fair chance of cancelling out; furthermore, the $O^M$ experience covered 30 years. He could only regard the adjusting factors as a weakness which diminished the success of the author's approach.

It was hardly to be expected that a formula derived from the experience of one class of lives would be appropriate to another class without modification.

When the author applied the assured lives formula to examining annuitants' mortality, it was certainly reasonable that some additional factor should be needed to allow for the class differential. The adjusting factors given at the foot of p. 184 combined the effect of the class differential with the basic factor for climatic and epidemic variations. To get a satisfactory measure of the class differential in itself, the annuitants' adjusting factors should be divided by those for assured lives for the same periods. In view of the complicated situation created by the adjusting factors, he doubted whether the author was altogether justified in questioning the report of the Mortality Committee in regard to the intractability of the data for male annuitants. By equating actual to expected deaths, the author had indeed achieved a good fit to his formula. But the real difficulty with the male annuitants' data was their failure to exhibit improvement over the period considered. That feature seemed to him to be absorbed by the author into his adjusting factors, which had the effect of forcing the secular trend into a preconceived mould.

He heartily welcomed the statement at the top of p. 186 that the $(x, t)$ formula was not to be regarded as a law of mortality. The substitution of a mathematical model for raw statistical data was never more than a process of abstraction—an endeavour to preserve broad outlines while eliminating local and idiosyncratic features. For example, the logistic curve contained a single point of inflexion, whereas in practice there might be more than one. In the 1947–48 assured lives' mortality, there were three major points of inflexion in the actual curve of $\log q_x$—at ages 44, 68 and 75. At 44 and 75 the function $d/dx \log q_x$ had a maximum value and at 68 it had a minimum value. The author's logistic curve gave a single point of inflexion at age 63, at which point $d/dx \log q_x$ was a maximum. But there was no real objection to simplification of that kind provided that they were clear as to what exactly was being done.

There were many other aspects of the paper which he would have liked to consider if time had permitted. In particular, he felt that there was a great deal to be said about quasi-$\chi^2$, but that he must leave to other speakers in the discussion.

**Mr H. A. R. Barnett** did not agree entirely with the author in his liking for the use of the logarithmic function. By the usual relationship between $m$ and $q$,

$$\operatorname{colog}(\tfrac{1}{2}m_x) = \log(1+p_x) - \log(1-p_x) = 2p_x + \tfrac{2}{3}p_x^3 + \tfrac{2}{5}p_x^5 + \ldots.$$

He saw no advantage in the author's somewhat abstruse function given on p. 172. He would probably have arrived at similar results by operating on the more familiar $p_x$ and $q_x$ and the trends would have been simpler to follow.

That brought him to a comparison between the author's formula and the Perks family of formulae. If $p_x$ followed the Starke form $A/(1+Dc^x)$, clearly $q_x$ followed a Perks form in which $D=B$ and *vice versa*, and he had no doubt that that was what the author had in mind when he stated that if it were accepted as axiomatic that $q_x$ approached 1 as $x$ approached infinity, then one constant in the Perks curve could be discarded. But there were certain properties of the Perks curve which did not apply to the Starke curve if fitted to the function $\operatorname{colog} \tfrac{1}{2}m_x$. In the discussion on Perks's paper, it had been pointed out by Lidstone that the Perks formulae could be expanded into multiple Makeham form, the reason being that at all ages met in mortality tables (and even beyond) $Dc^x$ proved to be not greater than 1, so that $(1+Dc^x)^{-1}$ could be expanded by the binomial theorem. He thought that that might lead to a possible philosophical interpretation of the Perks formula which was not similarly available for the Starke formula, because, with the constants of Table 4, $Dc^x$ had a value more than 1 at about half the ages, and it was not valid, therefore, to expand the Starke formula as applied to the colog function and to say that that was also of a multiple Makeham form. He had not had time to try whether the same objection applied to the Starke formula fitted to $q_x$.

In the meantime, he had made certain experiments with fitting a type of 'Perks expansion' in the form of a double Makeham. Considering the multiple Makeham which was the binomial expansion of the Perks formula, it had occurred to him that the reason Perks curves appeared to fit mortality data so well might be that they were an approximation to something else, and it might well be (Lidstone had also hinted at that) that most of the infinite number of expressions in the expanded Perks formula could be discarded. He had made experiments with a double Makeham, going as far as $A+Bc^x-Dk^x$. That had five constants; to avoid introducing too many constants, he had tried putting $k=c^2$, and he had further reduced the number of constants to 3 by attempting to fit the curve $A+Bc^x-(\frac{1}{2}B)^2c^{2x}$. He had tried that on some of the recent annuity data, and the rough attempts which he had made as a start had been most encouraging.

He suggested that if it were required to dispense with the constant $D$ in the double Makeham, there were certain advantages if the multiplier of $c^{2x}$ were of a $B^2$ form, as it meant that the relationship would be unaffected by a change of origin. If the $D$ in $-Dc^{2x}$ were a function of $A$ and $B$ instead of a function of $B^2$, then the relationships between the constants could be altered more or less at will by altering the origin, but if it were of a form such as $\frac{1}{2}B^2$ or $\frac{1}{4}B^2$, then a change of origin brought a similar change in the multipliers of both $c^x$ and $c^{2x}$. He put that forward as a possible formula for the rate of mortality with only three parameters, and one which might indicate secular trends clearly if fitted to the data the author had used.

Although Perks (*J.I.A.* LXIII, 31) had pointed out the dangers of attempting a philosophical interpretation of a mathematical formula, he, the speaker, suggested the following interpretation of a curve of the type described above. $A$ corresponded to those causes of death which did not appear to be correlated with age, the rate of which could as well be represented by a straight line as anything. Part of the function $Bc^x$ represented the causes of death for which the rate increased throughout life, and the remainder of $Bc^x$, less $Dc^{2x}$ (or $\frac{1}{4}B^2c^{2x}$), would represent those causes of death which after rising to a maximum somewhere around the middle of life gradually lost their effect—and that would probably include most industrial diseases. It might also include cancer, which had a heavy effect at the middle ages. Although, as he gathered from the paper, the author disagreed with him, he thought that if secular trends were really to be interpreted it was impossible completely to get away from a philosophical interpretation, and that was why, in the first place, it was better to use a formula of such a form that it was possible to give it such interpretation, and why, in the second place, it was desirable to operate on a function whose meaning could be visualized clearly.

**Mr F. J. Lloyd** said that the author, with considerable mathematical ingenuity, had painted in a new medium a picture of the improvement in mortality of the last 100 years. He had emphasized that that new medium was designed to help the understanding of the secular changes in mortality. As he had read the paper, his interest had developed, but not, he had to confess, in the direction of the author's new medium. It was the subject-matter of the picture which had fascinated him. As he had studied the first three tables in the paper, he had been struck afresh by the remarkable improvement in mortality over the last 100 years. The rates of mortality in 1949 had been, broadly speaking, at age 40 about one-third, at age 50 about one-half and at age 60 about two-thirds of those ruling 100 years earlier. He had found the examination of those tables no less instructive than Table 4 setting out the changes of the author's parameters.

He asked himself what had caused that remarkable improvement in mortality, and it seemed to him that the main causes could be put into three broad categories. First, there was the improvement in medical science which had been fully discussed in 1947 on the occasion of Sir John Conybeare's paper (*J.I.A.* LXXIV, 57). Secondly, there was the improvement in working conditions. The Registrar-General's Report on Occupational Mortality in 1931 had shown marked differences between the rates for different occupations, even when the considerable difficulties of obtaining the true exposed to risk were taken into account. However, if they went farther back, to the beginning of the 100 years, it was perhaps worth recording that the Mines Act, 1842, had prevented the employment underground of women and children under the age of 10, and, as a result of the Ten Hours Bill, the employment of women and children for more than 10 hours a day in factories was prohibited. An Act of 1951, which had the awful title of the Pneumoconiosis and Byssinosis (Benefit) Act, was an indication of the serious occupational diseases which existed currently among members of the community. Thirdly, there was the improvement in general living conditions, in which he included the elimination of poverty. That improvement was illustrated by the three social surveys of Mr Seebohm Rowntree about the City of York. He had made those surveys in 1899, 1935 and 1948, and had shown, for example, that in the period between the two last surveys the proportion of the population in York living below what he termed the minimum living standard had decreased from about 30% to less than 3%.

In each of those three broad categories there had been notable steps forward, and there were still many stages which it was hoped to reach in the future. Each step forward, however great it might be in itself, could only, of course, have a gradual impact on the secular trend of mortality. If, however, they were considering a smaller group of lives than the whole nation, then it was almost certain that the standards of selection of entry into that group of lives would have varied at different periods of time. That feature, for example, was well illustrated in the Life Office annuitants' mortality experience.

In his paper, the author had applied his mathematical methods to what might be termed a heterogeneous group of data. In trying to understand what caused the secular trends and changes in mortality, another method of approach was first to separate the data into groups which were as homogeneous as possible, bearing in mind the factors which they knew affected mortality. Those groups might then be analysed with due regard to the causes of death in broad diagnostic groups.

Perhaps he might mention that recent experiments in the allied field of the analysis of sickness absence experience had strengthened his belief that occupation was an important factor in causing mortality differences. Those experiments had also demonstrated that the crude rates produced when a large body of data was carefully separated into groups which were reasonably homogeneous showed a surprising smoothness. In fact, his experience had usually been that if the rates did not progress smoothly, a heterogeneous element had been introduced in separating the data into groups.

He was sure that the vast body of data on assured lives and annuitants would, if analysed with some regard to occupation and cause of death, add materially to the knowledge of the underlying causes of mortality and the improvements which had taken and were taking place in the secular trend.

**Mr R. E. Beard,** in closing the discussion, said that, as one who had done some experimenting with rates of mortality, he had a high regard for the work which Mr Starke had done. Unfortunately, he had not himself had a chance of doing any arithmetic on the paper and therefore his observations would not be as complete as he would have liked. But it was important that the approach to the paper should be to consider whether some of the aspects of it were misleading relative to the Perks system of curves. He had no particular predilection for the Perks system and he had tried hard to find an alternative approach to the subject, though without success.

In paragraph 3 the author commented that in the recent paper which he, the speaker, had submitted, he had used three values of $c$ in the experiments. Of course, in fitting a Perks curve from exposed to risk and deaths the value of $c$ could not be found explicitly. In practice three of the constants were found for a number of values of $c$, and then from the mesh of values the value of $c$ could be determined by an additional criterion. His own paper and, he thought, Perks's paper beforehand, had not discussed the criteria except in so far as he had himself devised four tests of a graduation which implied that there was an optimum value for $c$ defined by some function of the deviations between the actual and the expected deaths. He did not think that paragraph 3 really brought out that concept.

On paragraph 4 he was again in some difficulty about the precise meaning to be attached to the comments on the points of inflexion. A Perks curve fitted to $q_x$ showed a point of inflexion in $q_x$ very late in life, and also a point of inflexion in the derivative of $\log q_x$ at about age 70. It seemed to him that, if a Perks curve were fitted to $q_x$, the use of both those points of inflexion was implied. The author seemed to feel that he was using the data round about that maximum point, but having made that comment, he then ignored it because he fitted to $m_x$ and the only weighting he used was the arbitrary weight of 1 and 0, depending on whether the values were used or not.

In paragraph 5, the author stated that one parameter of the Perks formula could be dispensed with if it were adopted as axiomatic that $q_x$ approached 1 as $x$ approached infinity. Perks, and later Anderson, had put forward some suggestions as to why $\mu_x$ need not go to infinity and $q_x$ accordingly need not go to 1 as an upper limit. Although other comments had been made upon this, it went back as far as Farr, who had said that the last illness was about two years in duration. There was a lot of indirect evidence to suggest that it was quite acceptable to have an upper limit of $\mu_x$ which was not infinity. None of the subsequent work would be at all impeded by that assumption, except that one would have to supply an adjustment in one of the constants. If $\mu_x$ went to an upper limit, $m_x$ would also go to a fixed upper limit, and some of the mathematical philosophy behind it would be simplified and it would not be necessary to make so many assumptions.

The method of fitting described in paragraph 8 could be used with a Perks curve. If it were assumed that $q_x$ had an upper limit of 1 then the reciprocal of $p_x$ was a Makeham expression. He had himself avoided that method because he considered that it was important to pay regard to the weight of the data when dealing with a long series. He noticed that in some of the fittings there were only five values used; it would be interesting to know which five the author used and how the equations were modified.

He wished to raise a point concerning Table 3. If the deviations between actual and expected values of $m_x$ were studied as they progressed in time, it would be found that a pronounced bias was developing. For example, in the 1946–48 figures, the deviations in the males were negative to begin with, then were positive and subsequently became negative again. Such a feature could not be ignored in the use of the formula for extrapolation. There were signs that the formula was beginning to get away from the real underlying shape of the data, so giving a warning for due care to be exercised in its further use.

In paragraph 14 (iv) it was stated that '$x_0$ (women) exceeds $x_0$ (men) by something like three years of age'. It was important to notice that that difference tended to increase steadily, a feature that might be important in looking at those figures over a long period of time. There might well be a trend which had been ignored.

In paragraph 20, a rather heterogeneous set of assured lives mortality experiences had been brought together. Although, no doubt, some of the reasons were obvious, it would have been valuable to indicate why those particular data were selected.

On paragraphs 21 and 22 he would first comment that Hardy had had a lot to say about fitting logarithmic functions. Next, on the question of $\chi^2$, it was essential, when applying a statistical test, that a hypothesis should be set up so that the test had a real meaning; he did not know what was being tested there. It was called a 'quasi-$\chi^2$' test, but that did not alter the fact that the statistical validity of what was being done was open to question. Futhermore, he could not agree with the calculation of the degrees of freedom. Three parameters were calculated from a derived function and the process regarded as imposing linear constraints. The answer might be approximately right, but certainly his statistical teaching led him to beware of such assumptions, for which he could see no justification. Nevertheless, it seemed that a statistical test could be applied to the assumption that the observed values had arisen from the set of graduated values, in which case he would not think that any reduction in the number of degrees of freedom would be necessary.

Reference had already been made to the arbitrariness of the adjusting factors described in paragraph 26; nevertheless, study of those factors showed a regular variation in time, first a tendency to increase, then to decrease. Such a regular variation should have been a warning that the trend had not been eliminated in the mathematical formulae being used. The factors were supposed to be taking care of random fluctuations, but they exhibited no such sign. They suggested that the formula was not adequate over the range and indicated, once again, that care had to be taken in extrapolation.

He was afraid that his remarks had been largely critical, but it was important that they should be made, because projection was a difficult subject. He agreed entirely with Mr Barnett that for going ahead into the future it was desirable to build on more than just arithmetic. Until there was a sound philosophical basis for any particular formula, extrapolation was essentially a tentative process. In that particular case, it was true, a formula had been devised by some general, broad considerations, much in the way that Pearson had developed his system of frequency curves. However, for reasons advanced earlier, there were grounds for criticizing extrapolations based on it.

Recently, in discussing the mortality of annuitants, he had indicated that some experiments he had made with the fitting of the curve of deaths with a gamma function had led him to a formula which had given a surface for mortality over about 100 years of time. However, he was not satisfied with his formula other than as a piece of arithmetic, because it had no philosophical foundation. Nevertheless, he thought that there were some reasons for believing that the curve of deaths might be a useful function to use for extrapolation. He found that over a period of time the mode of the curve of deaths progressed uniformly with time and that the limiting age, which was necessary to the formula, also progressed at about the same rate. Those features led to a simple method of describing mortality, but the method was still largely experimental because of the difficulty of estimating the parameters from observations.

In his opinion a better philosophical foundation for mortality than the one put forward by Gompertz as modified by Makeham had not been found. He thought that if it were possible to get a truly homogeneous group of lives, it might be found that their mortality was Makeham in form. If a longevity factor were postulated, distributed in the population in a certain way, and if it were assumed that the Makeham $c$ was constant for the population, it could be shown that the resulting group mortality was Perks in form; a Perks mortality formula could thus be derived from simple Makeham assumptions, and it seemed to be an approach which might be worth developing further.

A further point was that a large number of mathematical expressions could be derived by considering the differential equation between the derivative and some simple function of $\mu_x$. It was possible to get a set of formulae with slightly different characteristics from the Makeham and Perks formulae, which would fit over a limited range. Therefore, while the author's formula did fit over a limited range it was important to keep well in mind the fact that it was only one formula of a whole group suitable for describing mortality. Furthermore, there were strong signs at the boundaries that the author's formula was veering away from the facts. For that reason, although the author had commented that his 1980 forecasts happened to be near those put forward by the Statistics Committee of the Royal Commission on Population, he would himself feel very nervous of using the formula outside the limited range, at least until he had seen more arithmetic.

The idea of trying to express mortality by a formula with time as the variable was attractive, but there was also a practical end in view. His own recent experiments with the gamma function had been strictly concerned with the practical end, because he had been thinking in terms of the calculation of actuarial functions. He fully agreed with the author that it would have been useful to have developed the calculations in generation form.

The simple concept underlying the formulae was the search for an S-shaped form. Appendix 2 really covered that aspect and was valuable, although he thought that it did not bring out into relief the simple concepts involved.

**The President (Mr F. A. A. Menzler, C.B.E.)** expressed his great personal pleasure at proposing a vote of thanks to the author for his paper. Mr Starke was an old friend of his, and he had worked with him in days gone by in the Government Actuary's Department.

When he had got half-way through the paper, he had had a sudden vision of a two-variable graduation formula to end all graduation formulae. He had seen the more mathematical portions of what was now designated as Part III, B(1) of the examinations simplified out of existence. In his dream, or mirage, he had been greatly encouraged by the last sentence of paragraph 34 of the paper: 'This tends to support the view that statistical tests of the graduation of mortality rates—elegant, theoretically correct and intellectually stimulating as they are—can sometimes be excessively stringent from the point of view of the practical purposes which the rates have to serve.' To the general tenor of that some of the less actuarially cultured of them, at any rate, might be prepared to subscribe!

It would represent a great stride forward if the author's general formulae—though they did not satisfy the more exacting tests of goodness of fit—provided them with values of familiar functions within 1% of those resulting from a graduation which would satisfy one or more of Mr Beard, Mr Barnett and Mr Perks. But he realized that when they got working on the next experience from, should they say, the Joint Mortality Investigation, the professional graduators would still feel impelled to apply with even increased enthusiasm their more refined methods of smoothing the data in the hope that they could demonstrate that the error was well over 1%. They could therefore look forward with the utmost certainty to further papers on graduation.

The author suggested that his formulae provided a not altogether unreasonable basis for projecting rates of mortality. Certainly he would agree that he would just as soon use the author's formulae as some other methods of which he was aware; at any rate, they had rather more statistical foundation than some of the methods which had been adopted in the past. As a very humble student of those matters, he had found the paper extremely interesting. The author had put forward his researches with disarming modesty; he hoped, nevertheless, that students of graduation and mortality would feel impelled to follow up the hints given in paragraph 37 for possible further systematic researches in regard to what the author termed the subsidiary sources of variation, such as selection and class differences.

**Mr Starke,** in reply, referred to the opener's mention of the high mortality of 1890–92. Although he had had to use the deaths of those years, in conjunction with the 1891 Census, to obtain parameters for his secular series, he doubted whether they were truly representative of the mortality of the period. At the risk of being thought obsessed with weather conditions, he would mention that he believed that one, if not two, of the winters of the triennium in question was exceptionally severe. He was interested in the opener's geometrical rendering of the parameters; it had not occurred to him to use the combinations to which the opener had referred instead of $x_0$, $y_0$ and $\alpha$.

He agreed with Mr Lloyd that, given the necessary data and the necessary patience, the ideal way to study mortality trends was to make a detailed analysis of the kind that he (Mr Lloyd) had suggested.

Mr Beard had given the paper a searching scrutiny from his own particular angle, and if he viewed it from the same standpoint he would be bound to agree with many of Mr Beard's criticisms. But his object had been to give the subject the broad approximate treatment which was possible only if too much regard was not paid to the niceties of statistical testing. He had felt, however, that the paper would not find favour if he ignored that aspect altogether. In the result he seemed to have fallen between two stools; but he would like to say how very much he appreciated the remarks of the President, which defined so admirably the spirit in which the paper was written.

As Mr Beard had disclaimed any wish to defend the Perks formulae, he might be permitted to say that the purpose of his paper was not to attack them. He had, however, tried to make a rather simpler use of the logistic for his own purposes.

**Mr Starke** subsequently wrote as follows:

Mr Clarke's main criticism of the paper concerned the introduction of adjusting factors (i) to allow for non-systematic fluctuations attributable to epidemics and weather conditions, and (ii) to adapt the $(x, t)$ formula to mortality of a type other than that from which the formula was derived. In both cases I made use of the idea of equating actual and expected deaths. I agree that there is no particular merit in this equation; it had, in fact, occurred to me after the paper was written that a possible, and perhaps a better, alternative would have been to obtain adjusting factors of type (i) by a 'minimum-$\chi^2$' approach. As regards type (ii) it seemed to me that the idea of a standard $(x, t)$ formula would be of very limited interest or value if it were necessary to establish a separate formula of this kind for every different class of mortality; hence came the experiments, described in the paper, which I admit were neither very extensive nor very conclusive.

Much is learnt by writing a paper and more by hearing it discussed; and I am now inclined to think that a much better line of development would be (*a*) to bring several different types of mortality into the study of parameters and (*b*) instead of seeking to express all the parameters in terms of $t$, to leave at least one of them free to be determined from the individual experience. It is fair comment on the paper to say that by aiming at over-simplification in the first instance, I put myself under the necessity of introducing some rather cumbrous and not very successful complications later on.

Mr Barnett's remarks were concerned to some extent with his attempts to view my formula in the context of the Perks formulae and the discussion on the paper in which those formulae were presented. I cannot think that this is a very profitable pursuit; a representation of log $q_x$ by a logistic seems unlikely to have much in common with a representation of $q_x$ by the same type of curve. If Mr Barnett can find a simple formula expressive of secular trend and also susceptible of philosophical interpretation it will be of very considerable interest. My formula is avowedly empirical; but I do agree with Mr Barnett and Mr Beard that to be able to discern some meaning behind the symbols is desirable at all times, and particularly desirable if the formula is to be used for extrapolation over an interval of any length. In fact, I said so in the paper which I submitted in 1949.

I doubt whether detailed comments on the numerous methodological points raised by Mr Beard would serve any very useful purpose. The opening paragraphs of the paper make it reasonably clear that my experiments were not concerned with graduation in the ordinary sense, but were an attempt to find a simple formula which would give a good over-all representation of the mortality curve. They were definitely not an attempt to compete with the professional graduator (to use the President's term) on his own ground; and a dissection of them under the microscope must inevitably reveal much that is distasteful to the specialist in orthodox statistical analysis.

The course which I tried to steer is defined in the last sentence of paragraph 34 of the paper. If all members of the profession were compelled to record either agreement or disagreement with the view there expressed, the results would, I think, be interesting. I adhere strongly to the words in that sentence which are in parentheses; at the same time, I cannot escape the feeling that, so long as the other factors which determine the economic price of an annuity or an assurance are so much a matter for speculation, an over-preoccupation with the finer points of the curve which represents the mortality operating at the time when the contract is made must savour of the academic.