# TWO-VARIABLE DEVELOPMENTS OF THE $n$-AGES METHOD

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THIS paper is devoted mainly to the extension of the '$n$-ages method'* of approximation for product-sums to summations in respect of two variables; some uses in valuation work of the formulae developed are indicated and illustrated arithmetically. Hitherto, apart from a special formula devised for the particular purpose of illustrating the possibilities of valuing whole-life assurances by limited payments grouped by years of entry (*J.I.A.* Vol. LXIV, p. 325), the $n$-ages method has been confined to one variable.

H. G. Jones, in his note on the $n$-ages method (*J.I.A.* Vol. LXIV, p. 318), developed his very valuable one-variable formulae up to the third and fourth moments by generalizing the method in the following form:

$$\Sigma u_x f(x) = \Sigma u_x\,[a_1 f(x_1) + a_2 f(x_2) + \dots + a_n f(x_n)],$$

where $\Sigma a = 1$ and the values of $a_1$ and $x_1$, etc., are expressed in terms of the moments of $u_x$, regarded as a frequency distribution.

In the same way the $n$-ages method for two variables can be expressed in general terms as follows:

$$\Sigma u_{xy} f(x, y) = \Sigma u_{xy}\,[a_1 f(x_1, y_1) + a_2 f(x_2, y_2) + \dots + a_n f(x_n, y_n)],$$

where $\Sigma a = 1$ and the values of $a_1$, $x_1$ and $y_1$, etc., are to be expressed in terms of the moments of the distribution $u_{xy}$ with regard to $x$ and $y$ separately and also with regard to $x$ and $y$ taken together, i.e. the successive product moments, of which the first $(\Sigma xy\, u_{xy}/\Sigma u_{xy})$ provides the coefficient of correlation between $x$ and $y$.

It is an important feature of the method that $u_{xy}$ need not be continuous or regular (e.g. in the valuation problem it may represent the total sums assured for age at entry $x$ and duration $y$), while $f(x, y)$ must be a smooth function (e.g. the valuation factors) which can be assumed with sufficient accuracy to be of a given order in $x$ and $y$. In the formulae given below, $\bar{x}$ and $\bar{y}$ are written for the means of the distribution $u_{xy}$ according to $x$ and $y$ respectively; $\sigma_x$ and $\sigma_y$ for the respective standard deviations; $\mu_3^x$ and $\mu_3^y$, and $\mu_4^x$ and $\mu_4^y$ for the respective third and fourth moments; $r$ for the coefficient of correlation between $x$ and $y$; $\mu_{x^t y^s}$ for the higher product moments and $r_{x^t y^s}$ for $\mu_{x^t y^s}/\sigma_x^t \sigma_y^s$; and $x_1$, $x_2$, etc., $y_1$, $y_2$, etc., are measured from $\bar{x}$ and $\bar{y}$ respectively. Taking first the simplest assumption for $f(x, y)$, i.e. $\mathrm{A} + \mathrm{B}x + \mathrm{C}y$, it is obvious that there is a one-term formula in which

$$a_1 = 1, \quad x_1 = 0, \quad y_1 = 0. \tag{1}$$

* In view of the various developments of the method, the name '$n$-ages' has ceased to be appropriate. '$n$-variates' would be a better name, but, as Elderton indicated (*J.I.A.* Vol. LXIV, p. 309) and as will further appear from comments in this paper, the method is closely akin to quadrature. Accordingly the suggestion is made that the name be changed to 'weighted quadrature'. [But see p. 398 where '$n$-point method' is suggested.—Eds. *J.I.A.*]

$f(x, y) = A + Bx + Cy + Dxy$

The minimum number of terms on this assumption is two and the simplest two-term, equally weighted, formula is easily seen to be as follows:

$$\textit{When } r \textit{ is positive: } \left.\begin{aligned} a_1 &= \cdot 5, & x_1 &= \sqrt{(r)}\,\sigma_x, & y_1 &= \sqrt{(r)}\,\sigma_y, \\ a_2 &= \cdot 5, & x_2 &= -\sqrt{(r)}\,\sigma_x, & y_2 &= -\sqrt{(r)}\,\sigma_y. \end{aligned}\right\} \quad (2)$$

$$\textit{When } r \textit{ is negative: } \left.\begin{aligned} a_1 &= \cdot 5, & x_1 &= \sqrt{(-r)}\,\sigma_x, & y_1 &= -\sqrt{(-r)}\,\sigma_y, \\ a_2 &= \cdot 5, & x_2 &= -\sqrt{(-r)}\,\sigma_x, & y_2 &= \sqrt{(-r)}\,\sigma_y. \end{aligned}\right\} \quad (3)$$

Generally, for a two-term equally weighted formula there are four unknowns and only the three conditions

$$x_1 + x_2 = 0, \quad y_1 + y_2 = 0, \quad \text{and} \quad x_1 y_1 + x_2 y_2 = 2r\sigma_x\sigma_y.$$

There is thus an unlimited number of solutions. Formulae (2) and (3) are symmetrical both with regard to and between $x$ and $y$, and result as a solution when those conditions are added.

$f(x, y) = A + Bx + Cx^2 + Dy + Exy$

On this assumption there are four conditions for four unknowns in a two-term equally weighted formula, the additional condition being $x_1^2 + x_2^2 = 2\sigma_x^2$.

The following formula results:

$$\left.\begin{aligned} a_1 &= \cdot 5, & x_1 &= \sigma_x, & y_1 &= r\sigma_y, \\ a_2 &= \cdot 5, & x_2 &= -\sigma_x, & y_2 &= -r\sigma_y. \end{aligned}\right\} \quad (4)$$

This formula is symmetrical with regard to $x$ and $y$, but by the nature of the basic assumption is not symmetrical between $x$ and $y$.

$f(x, y) = A + Bx + Cx^2 + Dy + Ey^2 + Fxy$

Before proceeding to develop various formulae on this assumption, it will be helpful for exposition, and it may be helpful to others, as it has been to the writer, in visualizing the various formulae, to give graphic expression to the ideas involved. In the figure, the origin represents the point $(\bar{x}, \bar{y})$, and measurements along the $x$-axis are in the scale of $\sigma_x = 1$ and those along the $y$-axis are in the scale of $\sigma_y = 1$. The frequency surface represented by $u_{xy}$ can be visualized as running along the tops of ordinates erected vertically from the plane of the paper at each point $(x, y)$ in the figure, the height of each ordinate being proportional to the corresponding value of $u_{xy}$.

The circle is drawn with the origin as centre and a radius of $\sqrt{2}$, and the lines AC and BD are drawn at 45° to the axes.

The co-ordinates of the points A, B, C and D are $(\sigma_x, \sigma_y)$, $(-\sigma_x, \sigma_y)$, $(-\sigma_x, -\sigma_y)$ and $(\sigma_x, -\sigma_y)$ respectively, and those of the points E, F, G and H are $(\sqrt{(2)}\,\sigma_x, 0)$, $(0, \sqrt{(2)}\,\sigma_y)$, $(-\sqrt{(2)}\,\sigma_x, 0)$ and $(0, -\sqrt{(2)}\,\sigma_y)$ respectively.

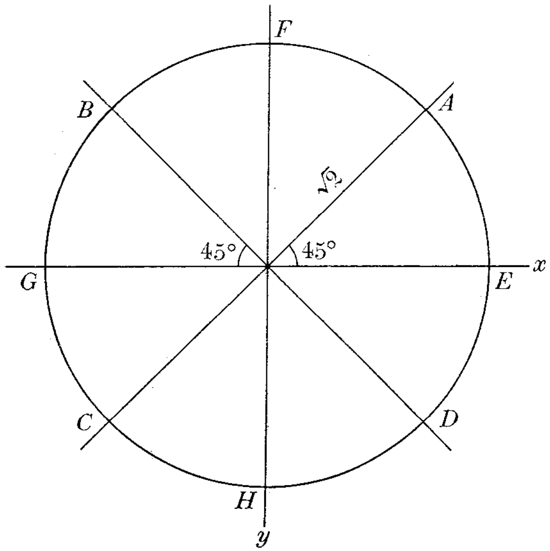
Proceeding now to the formulae on the assumption that $f(x, y)$ is of the second order in both $x$ and $y$ as in the heading of this section of the paper, we may consider the special cases of perfect correlation and no correlation.

It is clear that, if $r = 1$, the points A and C provide a two-term, equally weighted, symmetrical formula, i.e.

$$\left.\begin{aligned} a_1 &= \cdot 5, & x_1 &= \sigma_x, & y_1 &= \sigma_y, \\ a_2 &= \cdot 5, & x_2 &= -\sigma_x, & y_2 &= -\sigma_y. \end{aligned}\right\} \quad (5)$$

Similarly, if $r = -1$, the points D and B provide a corresponding solution, i.e.

$$\left.\begin{array}{lll} a_1 = \cdot 5, & x_1 = \sigma_x, & y_1 = -\sigma_y, \\ a_2 = \cdot 5, & x_2 = -\sigma_x, & y_2 = \sigma_y. \end{array}\right\} \tag{6}$$




If $r = 0$, a two-term formula does not seem to be possible, but it is obvious that there are two simple equally weighted, symmetrical four-term formulae represented by the four points A, C, D and B, and E, G, H and F respectively, i.e.

$$\left.\begin{array}{lll} a_1 = \cdot 25, & x_1 = \sigma_x, & y_1 = \sigma_y, \\ a_2 = \cdot 25, & x_2 = -\sigma_x, & y_2 = -\sigma_y, \\ a_3 = \cdot 25, & x_3 = \sigma_x, & y_3 = -\sigma_y, \\ a_4 = \cdot 25, & x_4 = -\sigma_x, & y_4 = \sigma_y, \end{array}\right\} \tag{7}$$

and

$$\left.\begin{array}{lll} a_1 = \cdot 25, & x_1 = \sqrt{(2)}\,\sigma_x, & y_1 = 0, \\ a_2 = \cdot 25, & x_2 = -\sqrt{(2)}\,\sigma_x, & y_2 = 0, \\ a_3 = \cdot 25, & x_3 = 0, & y_3 = -\sqrt{(2)}\,\sigma_y, \\ a_4 = \cdot 25, & x_4 = 0, & y_4 = \sqrt{(2)}\,\sigma_y. \end{array}\right\} \tag{8}$$

Formulae (5), (6), (7) and (8), being symmetrical in both $x$ and $y$, not only satisfy the $x$, $x^2$, $y$ and $y^2$ conditions but also, when $u_{xy}$ is symmetrical, satisfy the corresponding $x^3$ and $y^3$ conditions and all the conditions represented by the higher odd powers of $x$ and $y$. Formula (7) represents the combination in equal proportions of formulae (5) and (6), that is, the $xy$ condition is satisfied when $r=0$ by the combination in equal proportions of the formulae appropriate to the cases where $r=1$ and $r=-1$. This suggests that a simple four-term solution exists, when $r$ is not $\pm 1$ or nil, by taking a combination of formulae (5) and (6) in suitable proportions. The appropriate proportions are easily seen to be $\cdot 5\,(1+r)$ and $\cdot 5\,(1-r)$, thus embracing formulae (5), (6) and (7) and satisfying the $xy$ condition.

The resulting four-term formula* is as follows:

$$\left.\begin{aligned} a_1&=\cdot 25\,(1+r), & x_1&=\sigma_x, & y_1&=\sigma_y,\\ a_2&=\cdot 25\,(1+r), & x_2&=-\sigma_x, & y_2&=-\sigma_y,\\ a_3&=\cdot 25\,(1-r), & x_3&=\sigma_x, & y_3&=-\sigma_y,\\ a_4&=\cdot 25\,(1-r), & x_4&=-\sigma_x, & y_4&=\sigma_y.\end{aligned}\right\} \tag{9}$$

In a four-term formula, there are four values each of $a$, $x$ and $y$, and, as there are only six elements in $f(x, y)$ and hence only six conditions, it is obvious that there is an unlimited number of other solutions. If, however, we satisfy the constant element and fix the four values of $a$ by postulating an equally weighted formula, and secure symmetry with regard to $x$ and $y$ by making $x_2=-x_1$, $x_4=-x_3$, $y_2=-y_1$ and $y_4=-y_3$, thus satisfying the $x$ and $y$ conditions (and those represented by all other odd powers of $x$ and $y$ when $u_{xy}$ is symmetrical), the number of unknowns is reduced to four and the number of conditions to three, i.e.

$$x_1^2+x_3^2=2, \quad y_1^2+y_3^2=2, \quad x_1y_1+x_3y_3=2r,$$

where $x_1$ and $x_3$ are measured in units of $\sigma_x$, and $y_1$ and $y_3$ in units of $\sigma_y$.

To obtain a solution which is symmetrical *between* $x$ and $y$ (i.e. for which the $x$-solution will be identical in form with the $y$-solution), one of the following pairs of conditions may be added:

$$\text{(i) } x_1=y_1 \text{ and } x_3=y_3, \text{ or (ii) } x_1=y_1 \text{ and } x_3=-y_3,$$
$$\text{or (iii) } x_1=y_3 \text{ and } x_3=y_1, \text{ or (iv) } x_1=y_3 \text{ and } x_3=-y_1.$$

The first and fourth of these conditions are seen at once to be incompatible with the three equations. Taking the second additional condition the equations reduce to

$$x_1^2+x_3^2=2, \quad x_1^2-x_3^2=2r,$$

and the following solution results:

$$\left.\begin{aligned} a_1&=\cdot 25, & x_1&=\sigma_x\sqrt{(1+r)}, & y_1&=\sigma_y\sqrt{(1+r)},\\ a_2&=\cdot 25, & x_2&=-\sigma_x\sqrt{(1+r)}, & y_2&=-\sigma_y\sqrt{(1+r)},\\ a_3&=\cdot 25, & x_3&=\sigma_x\sqrt{(1-r)}, & y_3&=-\sigma_y\sqrt{(1-r)},\\ a_4&=\cdot 25, & x_4&=-\sigma_x\sqrt{(1-r)}, & y_4&=\sigma_y\sqrt{(1-r)}.\end{aligned}\right\} \tag{10}$$

By taking the third additional condition, the equations become

$$x_1^2+x_3^2=2, \quad 2x_1x_3=2r,$$

* It may assist students to note that each formula represents an $n$-term 'pocket' representation of the full distribution $u_{xy}$, all the moments of this 'pocket' distribution up to the order used in the basis of the formula being identical with those of the full distribution.

and the following solution* results:

$$\left.\begin{aligned}
a_1 &= \cdot 25, & x_1 &= \sigma_x\left(\sqrt{[(1+r)/2]}+\sqrt{[(1-r)/2]}\right),\\
& & y_1 &= \sigma_y\left(\sqrt{[(1+r)/2]}-\sqrt{[(1-r)/2]}\right),\\
a_2 &= \cdot 25, & x_2 &= -\sigma_x\left(\sqrt{[(1+r)/2]}+\sqrt{[(1-r)/2]}\right),\\
& & y_2 &= -\sigma_y\left(\sqrt{[(1+r)/2]}-\sqrt{[(1-r)/2]}\right),\\
a_3 &= \cdot 25, & x_3 &= \sigma_x\left(\sqrt{[(1+r)/2]}-\sqrt{[(1-r)/2]}\right),\\
& & y_3 &= \sigma_y\left(\sqrt{[(1+r)/2]}+\sqrt{[(1-r)/2]}\right),\\
a_4 &= \cdot 25, & x_4 &= -\sigma_x\left(\sqrt{[(1+r)/2]}-\sqrt{[(1-r)/2]}\right),\\
& & y_4 &= -\sigma_y\left(\sqrt{[(1+r)/2]}+\sqrt{[(1-r)/2]}\right).
\end{aligned}\right\} \qquad (11)$$

It is interesting to note that, while formula (10) puts one point in each of the four quadrants of the figure, formula (11) puts two points in the first quadrant and two in the third if $r$ is positive and in the other pair of opposite quadrants if $r$ is negative. Referring to the figure, two of the points of formula (10) are equidistant from the origin on the diagonal AC and the other two are equidistant on the diagonal BD. Taking the case where $r$ is positive, it may be seen that formula (9) has been converted into formula (10) by pushing the points A and C outwards, at the same time reducing the weights from $\cdot 25\,(1+r)$ to $\cdot 25$, and by pulling the points B and D inwards towards the origin, at the same time increasing the weights from $\cdot 25\,(1-r)$ to $\cdot 25$. When $r$ is negative, the reverse process applies. It is evident that there is an unlimited number of symmetrical unequally weighted solutions represented by points on these two diagonals. When $r=0$ formula (10) becomes formula (7).

The co-ordinates of the four points in formula (11) may be put in the form

$$\pm\sigma_x\sqrt{[1\pm\sqrt{(1-r^2)}]}, \quad \pm\sigma_y\sqrt{[1\pm\sqrt{(1-r^2)}]}.$$

The distance from the origin of the corresponding points in the figure is, therefore, in each case $\sqrt{2}$, i.e. they all lie on the circumference of the circle, two being in opposite quadrants on one straight line through the origin and the other two being in the same pair of opposite quadrants on another straight line through the origin and inclined to the $y$-axis at the same angle as the first line is inclined to the $x$-axis. Thus it seems that formula (8), where $r=0$, represented by the points E, G, H and F, transforms into formula (5), where $r=1$, by means of formula (11), through the process of the points E and F moving at equal speed round the circumference of the circle towards A (and G and H towards C) as $r$ increases until they finally coincide with A (and C) when $r=1$. When $r$ is negative, the points F and G move correspondingly round the circumference towards B, and H and E towards D.

* Prior to the developments in this paper, R. E. Beard reached formula (11) as an '*n*-ages' solution of the problem of the group valuation of joint and last survivor annuities on two lives, recently discussed by the Faculty (*T.F.A.* Vol. XVII, p. 39), and obtained close results for various test distributions, but his consideration of the two-variable aspect of the '*n*-ages' method in the practical field was interrupted by the transfer of his activities outside the life assurance sphere.

For the sake of completeness, it may be noted that various three-term second-order formulae are possible, e.g. when $r$ is positive

$$\left.\begin{aligned} &a_1=(1-r)/(1+r), && x_1=\sigma_x\sqrt{[r(1+r)/(1-r)]}, && y_1=\sigma_y\sqrt{[r(1+r)/(1-r)]},\\ &a_2=r/(1+r), && x_2=-\sigma_x\sqrt{[(1-r^2)/r]}, && y_2=0,\\ &a_3=r/(1+r), && x_3=0, && y_3=-\sigma_y\sqrt{[(1-r^2)/r]}.\end{aligned}\right\}\quad(12)$$

There is an alternative solution in which the signs of $x_1$ and $x_2$ and of $y_1$ and $y_3$ respectively are reversed.

When $r$ is negative, the solution is as above, substituting $-r$ for $r$ and reversing the signs of $y_1$ and $y_3$.

## ARITHMETICAL EXAMPLES BASED ON THE WHOLE-LIFE MODEL OFFICE

Before considering formulae for higher orders of $f(x, y)$, it is convenient to illustrate arithmetically the degree of accuracy of the second-order formulae already given. The moments and coefficients of correlation have been calculated for King's model office and various examples have been worked out. The statistical constants are given in Table 1, and approximate valuation results by formulae (9) and (10) for the 10-, 30- and 50-year model offices are given in Tables 2 and 3. In applying the formulae, $u_{xy}$ represents the model office distribution of business according to ages at entry ($x$) and duration ($y$), and $f(x, y)$ represents the whole-life policy value for age at entry $x$ and duration $y$.

Table 1. *Statistical constants for King's model office*

| Age of office (years) | $\bar{x}$ (mean entry age) | $\sigma_x$ | $\frac{\mu_3^x}{\sigma_x^3}$ | $\bar{y}$ (mean duration) | $\sigma_y$ | $\frac{\mu_3^y}{\sigma_y^3}$ | $r$ | $r_{x^2y}$ | $r_{xy^2}$ |
|---|---|---|---|---|---|---|---|---|---|
| 5 | 35·31 | 9·88 | ·65 | 2·880 | 1·416 | ·11 | ·006 | −·01 | 0 |
| 10 | 35·29 | 9·77 | ·65 | 5·105 | 2·875 | ·17 | −·002 | −·03 | −·01 |
| 15 | 35·19 | 9·64 | ·65 | 7·225 | 4·322 | ·21 | −·016 | −·05 | −·01 |
| 20 | 35·04 | 9·52 | ·65 | 9·221 | 5·740 | ·26 | −·035 | −·08 | −·02 |
| 25 | 34·86 | 9·38 | ·66 | 11·068 | 7·119 | ·32 | −·058 | −·10 | −·04 |
| 30 | 34·64 | 9·26 | ·67 | 12·729 | 8·432 | ·38 | −·085 | −·13 | −·06 |
| 35 | 34·44 | 9·16 | ·68 | 14·164 | 9·645 | ·44 | −·115 | −·16 | −·09 |
| 40 | 34·24 | 9·08 | ·70 | 15·343 | 10·727 | ·52 | −·145 | −·18 | −·12 |
| 45 | 34·06 | 9·04 | ·72 | 16·255 | 11·642 | ·60 | −·174 | −·18 | −·16 |
| 50 | 33·94 | 9·02 | ·74 | 16·892 | 12·351 | ·67 | −·198 | −·19 | −·20 |

The results by formula (9) are quite close, bearing in mind that this formula takes no account of skewness and that the model office figures show considerable skewness for both variables. The errors nowhere exceed ·6 %; they are all positive, and for the same aged office they are all closely similar. In *J.I.A.* Vol. LV, p. 236, Elderton suggested using a 'pocket' model office instead of a full model office for estimating the cost of a change of valuation basis (see also Elderton and Rowell, *J.I.A.* Vol. LVI, p. 285). If the results of Table 2 are expressed in comparative form, as in Table 4, the power of a simple 'pocket' model office is made apparent. For any practical problem of estimating the cost of a change of valuation basis, the full model office is usually a poor representation of the distribution of the actual business, and it is obvious, therefore, that a judicious choice of ages and durations (both in complete integers, for arithmetical convenience) should give better results than the four ages and four durations of formulae (9) or (10) computed from the model office. In this connexion it may be mentioned that in these days of large numbers of endowment assurances at the young ages, the mean age at entry of whole-life assurances may be expected to be considerably higher than the mean age at entry for the model office, and the skewness in $x$ should therefore be considerably less.

Table 2. *Results of the application of formula* (9) *to the model office valuation*

| Valuation basis | | 10-year office | 30-year office | 50-year office |
|---|---|---|---|---|
| $O^M$ $2\frac{1}{2}$% | Exact | 85,810 | 464,449 | 720,035 |
| | Approx. | 86,200 | 466,500 | 722,700 |
| | Error | + 390 | + 2,051 | + 2,665 |
| $O^M$ 3% | Exact | 80,278 | 440,973 | 688,999 |
| | Approx. | 80,600 | 443,200 | 691,800 |
| | Error | + 322 | + 2,227 | + 2,801 |
| $O^M$ $3\frac{1}{2}$% | Exact | 75,121 | 418,739 | 659,362 |
| | Approx. | 75,300 | 420,900 | 662,900 |
| | Error | + 179 | + 2,161 | + 3,538 |
| A 1924–29 ultimate $2\frac{1}{2}$% | Exact | 85,491 | 469,402 | 729,497 |
| | Approx. | 85,800 | 471,600 | 732,000 |
| | Error | + 309 | + 2,198 | + 2,503 |
| A 1924–29 ultimate 3% | Exact | 79,379 | 443,753 | 695,819 |
| | Approx. | 79,700 | 446,300 | 699,100 |
| | Error | + 321 | + 2,547 | + 3,281 |
| A 1924–29 ultimate $3\frac{1}{2}$% | Exact | 73,830 | 419,750 | 664,020 |
| | Approx. | 74,200 | 422,300 | 667,600 |
| | Error | + 370 | + 2,550 | + 3,580 |

Table 3. *Results of the application of formula* (10) *to the model office valuation*

| Valuation basis | | 10-year office | 30-year office | 50-year office |
|---|---|---|---|---|
| $O^M$ 3% | Exact | 80,278 | 440,973 | 688,999 |
| | Approx. | 80,600 | 443,400 | 694,100 |
| | Error | + 322 | + 2,427 | + 5,101 |
| A 1924–29 ultimate 3% | Exact | 79,379 | 443,753 | 695,819 |
| | Approx. | 79,700 | 446,700 | 702,400 |
| | Error | + 321 | + 2,947 | + 6,581 |

Table 4. *Results in Table 2 expressed in comparative form, with the reserves by* A 1924–29 3% *taken as* 10,000

| Valuation basis | 10-year office | | 30-year office | | 50-year office | |
|---|---|---|---|---|---|---|
| | Exact | Approx. | Exact | Approx. | Exact | Approx. |
| $O^M$ $2\frac{1}{2}$% | 10,810 | 10,816 | 10,466 | 10,453 | 10,348 | 10,338 |
| $O^M$ 3% | 10,113 | 10,113 | 9,937 | 9,931 | 9,902 | 9,896 |
| $O^M$ $3\frac{1}{2}$% | 9,464 | 9,448 | 9,436 | 9,431 | 9,476 | 9,482 |
| A 1924–29 ultimate $2\frac{1}{2}$% | 10,770 | 10,765 | 10,578 | 10,567 | 10,484 | 10,471 |
| A 1924–29 ultimate 3% | 10,000 | 10,000 | 10,000 | 10,000 | 10,000 | 10,000 |
| A 1924–29 ultimate $3\frac{1}{2}$% | 9,301 | 9,310 | 9,459 | 9,462 | 9,543 | 9,549 |

*Note.* As the results in Table 2 were obtained by first difference interpolation in the official table of policy values (to two places of decimals), the last figure in the above approximate values is not significant.

## HIGHER ORDERS OF $f(x, y)$

If $D_{x^s y^t}$ is written for $\dfrac{d^s}{dx^s}\dfrac{d^t}{dy^t}f(0, 0)$ (i.e. differentiating $f(0, 0)$ $t$ times with regard to $y$, keeping $x$ constant, and then differentiating $s$ times with regard to $x$, keeping $y$ constant), and if $\mu_{x^s y^t}$ is written for the product-moment $\Sigma x^s y^t u_{xy}/\Sigma u_{xy}$, the general expression for the two-variable product-sum can be expanded by Taylor's theorem as follows:

$$\begin{aligned}\Sigma u_{xy} f(x, y)/\Sigma u_{xy} &= \Sigma u_{xy} e^{xD_x} e^{yD_y} f(0, 0)/\Sigma u_{xy}\\ &= f(0, 0) + \tfrac{1}{2}(\mu_{x^2} D_{x^2} + 2\mu_{xy} D_{xy} + \mu_{y^2} D_{y^2})\\ &\quad + \tfrac{1}{6}(\mu_{x^3} D_{x^3} + 3\mu_{x^2y} D_{x^2y} + 3\mu_{xy^2} D_{xy^2} + \mu_{y^3} D_{y^3})\\ &\quad + \tfrac{1}{24}(\mu_{x^4} D_{x^4} + 4\mu_{x^3y} D_{x^3y} + 6\mu_{x^2y^2} D_{x^2y^2} + 4\mu_{xy^3} D_{xy^3} + \mu_{y^4} D_{y^4}) + \dots\end{aligned}$$

The corresponding expansions for formulae (9) and (10) are as follows:

*Formula* (9):

$$\begin{aligned}&f(0, 0) + \tfrac{1}{2}(\mu_{x^2} D_{x^2} + 2\mu_{xy} D_{xy} + \mu_{y^2} D_{y^2})\\ &\quad + \tfrac{1}{24}(\mu_{x^2}^2 D_{x^4} + 4\mu_{xy}\mu_{x^2} D_{x^3y} + 6\mu_{x^2}\mu_{y^2} D_{x^2y^2} + 4\mu_{xy}\mu_{y^2} D_{xy^3} + \mu_{y^2}^2 D_{y^4}) + \dots\end{aligned}$$

*Formula* (10):

$$\begin{aligned}&f(0, 0) + \tfrac{1}{2}(\mu_{x^2} D_{x^2} + 2\mu_{xy} D_{xy} + \mu_{y^2} D_{y^2})\\ &\quad + \tfrac{1}{24}[(\mu_{x^2}^2 + \mu_{xy}^2\mu_{x^2}/\mu_{y^2}) D_{x^4} + 8\mu_{xy}\mu_{x^2} D_{x^3y}\\ &\quad + 6(\mu_{x^2}\mu_{y^2} + \mu_{xy}^2) D_{x^2y^2} + 8\mu_{xy}\mu_{y^2} D_{xy^3} + (\mu_{y^2}^2 + \mu_{xy}^2\mu_{y^2}/\mu_{x^2}) D_{y^4}] + \dots\end{aligned}$$

A comparison of these expansions for formulae (9) and (10) with the true expansion for the product-sum shows the constituent elements of the errors produced by these formulae. As $\mu_{x^3}$, $\mu_{y^3}$ and all the third- and fourth-order product-moments except $\mu_{x^2y^2}$ can be either positive or negative, and as the various differential coefficients may vary in amount and sign, it is apparent that the error in each case represents a balance of various positive and negative constituent errors. In much the same way as two-variable interpolation formulae, involving some of the differences of higher order than the second, do not necessarily provide improved results over second-order formulae, so it may be expected that more elaborate $n$-ages formulae satisfying some only of the third- and fourth-order elements will not necessarily produce better results than formulae (9) and (10). It is easy to produce formulae involving various combinations of some of these higher elements, and some of these formulae have been tried on the model office data, but, as expected, less satisfactory results have usually materialized. The model office data represent very intractable distributions; the correlation between age at entry and duration is negative, there is substantial positive skewness both by age at entry and by duration, and the value of $\mu_{x^2y^2}/\mu_{x^2}\mu_{y^2}$ (i.e. $r_{x^2y^2}$), for the 50-year office at least, is less than unity. As already indicated, actual distributions of business to-day are not likely to be so troublesome. Further, it is possible to work with the second variable as the attained age instead of the duration and so produce less awkward distributions (see later).

Proceeding to consider formulae involving higher order moments, we may first consider possible formulae which might be suitable when the distribution $u_{xy}$ is symmetrical in $x$ and $y$, or sufficiently so for practical purposes. In such a case all the third-order moments become nil. Of the fourth-order moments $\mu_{x^2y^2}$ may be considered generally to be the most influential, since in the expansion of the product-sum this moment is multiplied by 6 and there is no reason in general to expect $D_{x^2y^2}$ to be greater or smaller than either $D_{x^4}$ or $D_{y^4}$, although, of course, in a particular case all three might differ greatly.

Reverting to the three equations leading to formulae (10) and (11), we can, instead of using the additional symmetry condition, introduce a condition based on $\mu_{x^2y^2}$, viz.

$$x_1^2y_1^2 + x_3^2y_3^2 = 2r_{x^2y^2} = 2r_2 \text{ (say).}$$

The resulting formula is as follows:

$$\left.\begin{aligned}
a_1 &= \cdot 25, \quad x_1 = \frac{\sigma_x}{\sqrt{2}} \left(\sqrt{[(1+r)(1+\sqrt{(r_2-r^2)})]} + \sqrt{[(1-r)(1-\sqrt{(r_2-r^2)})]}\right), \\
& \qquad\quad y_1 = \frac{\sigma_y}{\sqrt{2}} \left(\sqrt{[(1+r)(1+\sqrt{(r_2-r^2)})]} - \sqrt{[(1-r)(1-\sqrt{(r_2-r^2)})]}\right), \\
a_2 &= \cdot 25, \quad x_2 = -x_1, \quad y_2 = -y_1, \\
a_3 &= \cdot 25, \quad x_3 = \frac{\sigma_x}{\sqrt{2}} \left(\sqrt{[(1+r)(1-\sqrt{(r_2-r^2)})]} - \sqrt{[(1-r)(1+\sqrt{(r_2-r^2)})]}\right), \\
& \qquad\quad y_3 = \frac{\sigma_y}{\sqrt{2}} \left(\sqrt{[(1+r)(1-\sqrt{(r_2-r^2)})]} + \sqrt{[(1-r)(1+\sqrt{(r_2-r^2)})]}\right), \\
a_4 &= \cdot 25, \quad x_4 = -x_3, \quad y_4 = -y_3.
\end{aligned}\right\} \quad (13)$$

It is clear that this formula really represents two solutions, since we are at liberty to make either of our actual variables equal to $x$ and the other equal to $y$.

This formula reduces to formula (10) if we put $r_2 = 1 + r^2$ and to formula (11) if we put $r_2 = r^2$, thus throwing interesting light on these two formulae and on the assumptions upon which they are based. Formula (13) gives imaginary results for $r_2 > 1 + r^2$ or $< r^2$. For the normal frequency surface $r_2 = 1 + 2r^2$ and the formula breaks down. However, when $r_2 > 1 + r^2$ the difficulty may be overcome and at the same time a better approximation to the $x^4$ or $y^4$ element may be secured by introducing a weighted term at $x = 0$, $y = 0$.

The resulting formula is the same as formula (13), with the expression $(k \pm \sqrt{[kr_2 - k^2r^2]})$ substituted for $(1 \pm \sqrt{[r_2 - r^2]})$ and the weights $a_1$, $a_2$, $a_3$ and $a_4$ changed to $\cdot 25/k$, while the weight of the term at $x = 0$, $y = 0$ is $1 - 1/k$, where $k$ is arbitrarily chosen to avoid the difficulty mentioned.

For the 50-year model office, $r_{x^2y^2} = \cdot 91$, and the A 1924–29 3% value by formula (13) works out at 700,900 (taking $y$ = age and $x$ = duration) and at 706,400 (taking $x$ = age and $y$ = duration) compared with the true value of 695,819. These results do not represent any improvement over those obtained by the simpler formulae (9) and (10), and it is clear that the third-order moments and the other fourth-order moments must be brought into account if an improvement is to be obtained.

It is possible, knowing the true valuation of the model office on any given basis, so to choose a value of $k$ in the modified formula (13) that the approximate value is practically without error, and thus obtain a five-case pocket model office that can reasonably be expected to produce very close results for other valuation bases. This can also be done with formula (10) by introducing a term at $x=0$, $y=0$, weighted by $1-1/k$, the other values of $x$ and $y$ being multiplied by $\sqrt{k}$ and the weights of these terms being reduced to $\cdot 25/k$.

For the 50-year model office, if we put $k=1\cdot 4$ in formula (10), the A 1924–29 3% value works out at 696,100 compared with the true value of 695,819. The corresponding figures by $O^{M}$ $2\frac{1}{2}$% using the same value of $k$ are 720,500 approximate and 720,035 true.

These close results, of course, arise merely from a process of trial as a result of knowing the true answers, and represent a balancing of the errors due to the neglected third-, fourth- and higher order elements. For actual valuation purposes, if such a process were used to provide a starting point for an approximate method based on this formula, it is to be expected that in succeeding annual valuations the higher order errors would change very slowly, and a close approximation should result year by year, which could, if desired, be tested from time to time. In any case, so long as there is no sharp disturbance of the emerging annual surplus, any small systematic error in the method can quite properly be regarded in principle with as much equanimity as the systematic errors arising from the various simple ways of fixing the ages at entry and valuation ages or of dealing with the distribution of premium income that are used in practice as a matter of convenience. To illustrate the point made about the slow progression of higher order errors, the A 1924–29 3% valuation of the 30-year model office has been computed on the above 5-term modification of formula (10) using the same value of $k$ as produced the close result for the 50-year office. The approximate value is found to be 444,000 compared with the true value of 443,753.

It is possible to develop more elaborate formulae from formula (13) by introducing additional conditions based on some of the other fourth- and third-order moments by postulating unequal weights in the formula and combining the two cases of this formula. However, the solutions become complicated and difficult. Instead, progress has been made by combining formulae (8) and (10) in suitable proportions and postulating unequal weights. It is obvious that, provided a suitable substitution for $r$ is made in formula (10), the combination of formulae (8) and (10) will give an eight-term formula correct to the second order. For example, if we combine them in equal proportions, we must use $2r$ instead of $r$ in formula (10). Now, the eight terms in such a combined formula comprise four pairs of two terms such that, referring to the figure given earlier, the two terms of each pair lie on a straight line (being either one of the axes or one of the two diagonals) passing through the origin represented by the means $\bar{x}$ and $\bar{y}$. For each of these pairs of terms we can substitute three other unequally weighted terms on the same line (one being put at the origin), by applying the principles of Jones's formula for a single variable, correct to the fourth moment, so that the contributions of each group of terms to the first- and second-order moments in the corresponding equations are unaltered while the contributions of each group to the third- and fourth-order moments are fixed at will.

Applying this process to all four pairs of terms, we can write down the following system for a nine-term formula:

$$\left.\begin{array}{lll}
a_0 = 1 - \cdot 25\,(p_1+p_2+p_3+p_4), & x_0 = 0, & y_0 = 0, \\
a_1 = w_2 p_1/4\,(w_1+w_2), & x_1 = w_1\sigma_x, & y_1 = w_1\sigma_y, \\
a_2 = w_1 p_1/4\,(w_1+w_2), & x_2 = -w_2\sigma_x, & y_2 = -w_2\sigma_y, \\
a_3 = w_4 p_2/4\,(w_3+w_4), & x_3 = w_3\sigma_x, & y_3 = -w_3\sigma_y, \\
a_4 = w_3 p_2/4\,(w_3+w_4), & x_4 = -w_4\sigma_x, & y_4 = w_4\sigma_y, \\
a_5 = w_6 p_3/4\,(w_5+w_6), & x_5 = w_5\sigma_x, & y_5 = 0, \\
a_6 = w_5 p_3/4\,(w_5+w_6), & x_6 = -w_6\sigma_x, & y_6 = 0, \\
a_7 = w_8 p_4/4\,(w_7+w_8), & x_7 = 0, & y_7 = w_7\sigma_y, \\
a_8 = w_7 p_4/4\,(w_7+w_8), & x_8 = 0, & y_8 = -w_8\sigma_y.
\end{array}\right\} \quad (14)$$

It remains to obtain expressions for the various $w$'s and $p$'s in terms of the third- and fourth-order moments.

Introducing $(1-c)$ and $c$ to represent the proportions in which formulae (10) and (8) are notionally combined, it is evident from the second-order equations that

$$w_1 w_2 p_1 = 2\,(1+r-c), \qquad w_3 w_4 p_2 = 2\,(1-r-c),$$
$$w_5 w_6 p_3 = 4c \qquad \text{and} \qquad w_7 w_8 p_4 = 4c.$$

The third-order equations are then

$$2\,(1+r-c)\,(w_1-w_2) + 2\,(1-r-c)\,(w_3-w_4) + 4c\,(w_5-w_6) = 4\mu_3^x/\sigma_x^3 \quad = 4\mathrm{B}_1^x,$$
$$2\,(1+r-c)\,(w_1-w_2) - 2\,(1-r-c)\,(w_3-w_4) + 4c\,(w_7-w_8) = 4\mu_3^y/\sigma_y^3 \quad = 4\mathrm{B}_1^y,$$
$$2\,(1+r-c)\,(w_1-w_2) - 2\,(1-r-c)\,(w_3-w_4) = 4\mu_{x^2y}/\sigma_x^2\sigma_y = 4r_{x^2y},$$
$$2\,(1+r-c)\,(w_1-w_2) + 2\,(1-r-c)\,(w_3-w_4) = 4\mu_{xy^2}/\sigma_x\sigma_y^2 = 4r_{xy^2},$$

whence

$$\begin{aligned}
w_1 - w_2 &= (r_{x^2y}+r_{xy^2})/(1+r-c) = 2\mathrm{M}_1 \text{ (say)}, \\
w_3 - w_4 &= (r_{xy^2}-r_{x^2y})/(1-r-c) = 2\mathrm{M}_2 \text{ ( ,, )}, \\
w_5 - w_6 &= (\mathrm{B}_1^x - r_{xy^2})/c \qquad = 2\mathrm{M}_3 \text{ ( ,, )}, \\
w_7 - w_8 &= (\mathrm{B}_1^y - r_{x^2y})/c \qquad = 2\mathrm{M}_4 \text{ ( ,, )}.
\end{aligned}$$

It is clear that the system of the solution provides the same value for the element $x^3y$ as for $xy^3$, and it is necessary to choose as a condition either one of these two or to take a value between them. There are reasons to suppose that $r_{x^3y}$ and $r_{xy^3}$ will often be similar in sign and size. There are thus four fourth-order equations as follows:

$$2\,(1+r-c)\,(w_1^2 - w_1w_2 + w_2^2) + 2\,(1-r-c)\,(w_3^2 - w_3w_4 + w_4^2) + 4c\,(w_5^2 - w_5w_6 + w_6^2)$$
$$= 4\mu_4^x/\sigma_x^4 = 4\beta_2^x,$$
$$2\,(1+r-c)\,(w_1^2 - w_1w_2 + w_2^2) + 2\,(1-r-c)\,(w_3^2 - w_3w_4 + w_4^2) + 4c\,(w_7^2 - w_7w_8 + w_8^2)$$
$$= 4\mu_4^y/\sigma_y^4 = 4\beta_2^y,$$
$$2\,(1+r-c)\,(w_1^2 - w_1w_2 + w_2^2) + 2\,(1-r-c)\,(w_3^2 - w_3w_4 + w_4^2)$$
$$= 4\mu_{x^2y^2}/\sigma_x^2\sigma_y^2 = 4r_{x^2y^2},$$
$$2\,(1+r-c)\,(w_1^2 - w_1w_2 + w_2^2) - 2\,(1-r-c)\,(w_3^2 - w_3w_4 + w_4^2)$$
$$= 2\,(\mu_{x^3y}/\sigma_x^3\sigma_y + \mu_{xy^3}/\sigma_x\sigma_y^3) = 4\mathrm{R} \text{ (say)}.$$

Hence we have

$$\frac{1}{p_1}=\frac{r_{x^2y^2}+R}{2(1+r-c)^2}-\frac{(r_{x^2y}+r_{xy^2})^2}{2(1+r-c)^3},\qquad \frac{1}{p_2}=\frac{r_{x^2y^2}-R}{2(1-r-c)^2}-\frac{(r_{xy^2}-r_{x^2y})^2}{2(1-r-c)^3},$$

$$\frac{1}{p_3}=\frac{\beta_2^x-r_{x^2y^2}}{4c^2}-\frac{(B_1^x-r_{xy^2})^2}{4c^3},\qquad \frac{1}{p_4}=\frac{\beta_2^y-r_{x^2y^2}}{4c^2}-\frac{(B_1^y-r_{x^2y})^2}{4c^3},$$

and

$$\begin{aligned}
w_1&=\sqrt{[M_1^2+2(1+r-c)/p_1]}+M_1, & w_2&=\sqrt{[M_1^2+2(1+r-c)/p_1]}-M_1,\\
w_3&=\sqrt{[M_2^2+2(1-r-c)/p_2]}+M_2, & w_4&=\sqrt{[M_2^2+2(1-r-c)/p_2]}-M_2,\\
w_5&=\sqrt{[M_3^2+4c/p_3]}\qquad\qquad +M_3, & w_6&=\sqrt{[M_3^2+4c/p_3]}\qquad\qquad -M_3,\\
w_7&=\sqrt{[M_4^2+4c/p_4]}\qquad\qquad +M_4, & w_8&=\sqrt{[M_4^2+4c/p_4]}\qquad\qquad -M_4.
\end{aligned}$$

In this solution the constant $c$ is left for arbitrary choice. In actual arithmetical work, choice may be severely limited by the need to avoid imaginary values of $w_1$, $w_2$, etc. It might be possible to fix a condition for the value of $c$ based on the data by satisfying one of the fifth or sixth moments, but no attempt has been made to proceed thus far.

It is interesting to note that if we insert in the solution the expressions for the third- and fourth-order moments of the normal surface

$$(\text{viz. } B_1^x=B_1^y=r_{x^2y}=r_{xy^2}=0,\ \beta_2^x=\beta_2^y=3,\ r_{x^3y}=r_{xy^3}=3r, \text{ and } r_{x^2y^2}=1+2r^2)$$

and put $c=\frac{1}{2}$, the following symmetrical solution correct to the fourth order results for the normal surface:

$$\left.\begin{array}{lll}
a_0=\cdot 5, & x_0=0, & y_0=0,\\
a_1=(1+2r)/16(1+r), & x_1=\sigma_x\sqrt{[2(1+r)]}, & y_1=\sigma_y\sqrt{[2(1+r)]},\\
a_2=(1+2r)/16(1+r), & x_2=-\sigma_x\sqrt{[2(1+r)]}, & y_2=-\sigma_y\sqrt{[2(1+r)]},\\
a_3=(1-2r)/16(1-r), & x_3=\sigma_x\sqrt{[2(1-r)]}, & y_3=-\sigma_y\sqrt{[2(1-r)]},\\
a_4=(1-2r)/16(1-r), & x_4=-\sigma_x\sqrt{[2(1-r)]}, & y_4=\sigma_y\sqrt{[2(1-r)]},\\
a_5=1/16(1-r^2), & x_5=2\sigma_x\sqrt{(1-r^2)}, & y_5=0,\\
a_6=1/16(1-r^2), & x_6=-2\sigma_x\sqrt{(1-r^2)}, & y_6=0,\\
a_7=1/16(1-r^2), & x_7=0, & y_7=2\sigma_y\sqrt{(1-r^2)},\\
a_8=1/16(1-r^2), & x_8=0, & y_8=-2\sigma_y\sqrt{(1-r^2)}.
\end{array}\right\}\qquad (15)$$

When $r=0$, this formula reduces to one point at the means, weighted by $\cdot 5$, and eight points weighted by $1/16$ each, two on the $x$-axis at $\pm 2\sigma_x$ from the mean, two on the $y$-axis at $\pm 2\sigma_y$ from the mean, and four on the diagonals, one in each quadrant, with co-ordinates $\pm\sigma_x\sqrt{2}$, $\pm\sigma_y\sqrt{2}$. Thus, referring to the figure on p. 379, the eight points are distributed equidistantly round a circle with radius 2.

When $r$ is outside $\pm\cdot 5$, formula (15) produces negative weights at two points. This is somewhat unnatural, but, from an arithmetical point of view, need cause no difficulty until $r$ approaches or is equal to $\pm 1$, when some of the weights become very large and some of the distances from the mean become very small. Apart from the difficulties near and at $r=\pm 1$, the negative weights can be regarded with as little alarm as negative coefficients in some quadrature formulae, with which the whole subject is closely related.*

It does not seem possible to avoid negative weights, however we fix $c$; but the other difficulties can be overcome and a uniform formula over the whole range of $r$ can be obtained by putting $c=\frac{1}{2}(1+2r)(1-r)$ when $r$ is positive and $c=\frac{1}{2}(1-2r)(1+r)$ when $r$ is negative, whence $c=\frac{1}{2}$ if $r=0$ and $c=0$ if $r=\pm 1$. When $r=1$ the formula reduces to Jones's three-term formula correct to the fourth moment for the normal curve, viz.

$$\tfrac{1}{6}\left(f(\sigma\sqrt{3})+4f(0)+f(-\sigma\sqrt{3})\right).$$

Further progress can, however, be made by introducing the conception of the ellipse axes of the normal surface (see Yule and Kendall, *An Introduction to the Theory of Statistics*, p. 231). In a normal surface the lines joining the points of equal frequency are a concentric succession of ellipses. The angle $\theta$ at which the axes of these ellipses are inclined to the axes of measurement of $x$ and $y$ is defined by the equation

$$\tan 2\theta=\frac{2r\sigma_x\sigma_y}{\sigma_x^2-\sigma_y^2}.$$

If measurements ($u$ and $v$) are made from these axes, instead of from the $x$ and $y$ axes, the resulting value of $r_{uv}$ is zero. Hence formula (15), with $r=0$, applies, provided that the standard deviations are also measured from the ellipse axes. These standard deviations can be obtained from the transformation equations:

$$u=x\cos\theta+y\sin\theta,$$
$$v=y\cos\theta-x\sin\theta,$$

whence

$$\sigma_u^2=\sigma_x^2\cos^2\theta+\sigma_y^2\sin^2\theta+2r_{xy}\sigma_x\sigma_y\cos\theta\sin\theta,$$
$$\sigma_v^2=\sigma_y^2\cos^2\theta+\sigma_x^2\sin^2\theta-2r_{xy}\sigma_x\sigma_y\cos\theta\sin\theta.$$

Putting $c=\frac{1}{2}$, the values of $u_1, u_2, \ldots, u_8$ and $v_1, v_2, \ldots, v_8$ in formula (15) (with $r_{uv}=0$) result at once, the weights all being equal to $1/16$. It remains to transform these values of $u$ and $v$ to the original axes of measurement $x$ and $y$ by means of the transformation equations

$$x=u\cos\theta-v\sin\theta,\quad y=v\cos\theta+u\sin\theta.$$

* Many quadrature formulae can be obtained as special cases of a corresponding '$n$-ages' formula; for example, Jones's fourth-order formula for $\Sigma u_x f(x)$ applied to $\int_{-1}^{+1} f(x)\,dx$ (the frequencies $u_x$ being treated as constant, so that $\sigma^2=\frac{1}{3}$ and $\beta_2=1{\cdot}8$) produces the quadrature formula $\frac{1}{9}[5f(-\sqrt{\cdot 6})+8f(0)+5f(\sqrt{\cdot 6})]$ correct to the fifth order. See also Whittaker and Robinson, *The Calculus of Observations*, p. 163, where the special case of Jones's fourth-order formula, when $u_x$ is the normal curve, is given as due to A. Berger.

It seems that Tchebycheff, following up a special case by Bronwin, was the originator of the single-variable '$n$-ages' method, reaching a general solution for an $n$-term equally weighted formula, correct up to the $n$th order, including the remainder term (see *The Calculus of Finite Differences*, by L. M. Milne-Thompson, p. 177). When the distribution $u_x$ is rectangular, Tchebycheff's quadrature formulae result and the simple quadrature formulae referred to by Elderton in *J.S.S.* Vol. II, No. 2, are special approximate cases. The fifth-order Tchebycheff quadrature formula given by Whittaker and Robinson (p. 159) is easily obtained by postulating a symmetrical distribution and reaching an '$n$-ages' formula in the form of $x_3=0$, $x_5=-x_1$, $x_4=-x_2$, by solving the equations $2x_1^2+2x_2^2=5\sigma^2$ and $2x_1^4+2x_2^4=5\mu_4$. The quadrature formula results when the moments of the rectangular distribution are inserted. J. E. Kerrich's note *Approximate Integration*, *J.I.A.* Vol. LXIV, p. 545, is also of interest in this connexion.

When $r_{xy} = \pm 1$, either $\sigma_u$ or $\sigma_v$ is zero, and the result of the above process for the normal surface when $r_{xy} = 1$ is the single-variable formula

$$\tfrac{1}{16}[f(2\sigma) + 2f(\sigma\sqrt{2}) + 10f(0) + 2f(-\sigma\sqrt{2}) + f(-2\sigma)],$$

which produces 10 as the value of $\mu_6/\sigma^6$ compared with the true normal curve value of 15 and the value of 9 which results from Jones's fourth-order single-variable formula for the normal curve, viz.

$$\tfrac{1}{6}[f(\sigma\sqrt{3}) + 4f(0) + f(-\sigma\sqrt{3})].$$

In the same way as single-variable quadrature formulae can be obtained from single-variable $n$-ages formulae by using the moments of a rectangular distribution, so can two-variable quadrature formulae be obtained from two-variable $n$-ages formulae by using the moments of a distribution in the form of a rectangular solid, represented by $\int_{-1}^{+1}\int_{-1}^{+1} dx\,dy$. The various moments of the distribution are as follows:

$$\bar{x} = \bar{y} = 0, \qquad \sigma_x = \sigma_y = 1/\sqrt{3}, \quad r_{xy} = 0, \quad \beta_1^x = \beta_1^y = 0,$$
$$r_{x^2y} = r_{xy^2} = 0, \quad \beta_2^x = \beta_2^y = 1\cdot 8, \quad r_{x^2y^2} = 1, \quad r_{x^3y} = r_{xy^3} = 0.$$

If we put $a_0 = 0$ in formula (14), then $c$ must be $4/7$ and the formula gives the eight-term quadrature formula due to Burnside (Whittaker and Robinson, *The Calculus of Observations*, p. 375) which is correct up to and including fifth differences ($p_1 = p_2 = 18/49$; $p_3 = p_4 = 80/49$; $a_1 = a_2 = 9/49$; $a_3 = a_4 = 40/49$; $w_1 = w_2 = w_3 = w_4 = \sqrt{(7/3)}$; $w_5 = w_6 = w_7 = w_8 = \sqrt{(7/15)}$; and the $x$'s and $y$'s follow at once). It is easy to produce various nine-term two-variable quadrature formulae from formula (14), correct to fifth differences, e.g. by putting $c = \cdot 5$, or by making $a_0$ a maximum, instead of nil as in Burnside's formula. If $c = \cdot 5$, $p_1 = p_2 = \cdot 5$, $p_3 = p_4 = 1\cdot 25$, $a_0 = 1/8$, $a_1 = a_2 = a_3 = a_4 = 1/16$, $a_5 = a_6 = a_7 = a_8 = 5/32$, $w_1 = w_2 = w_3 = w_4 = \sqrt{2}$, $w_5 = w_6 = w_7 = w_8 = 2\sqrt{(2/5)}$.

Returning to a general consideration of formula (14), it is clear that the transformation process can be used for any kind of distribution and for any angle of rotation. The particular angle which results in $r_{uv} = 0$ is of little interest when the distribution is not normal, although the odd power moments are likely to be small when $r_{uv}$ is small. It is clear that at some angles $r_{u^3v}$ will be equal to $r_{uv^3}$ (e.g. about $47\frac{1}{2}°$ for the 50-year model office) and if the rotation is made through such an angle, formula (14) becomes correct to all the fourth-order elements. There does not appear to be any simple direct way of finding such an angle in a particular application of the formula, but, with some labour, it can be found by trial by transforming the moments through two or three angles and then interpolating.

A certain amount of arithmetic has been done with formula (14) on the 50-year model office data. The moments up to the third order are given in Table 1. The fourth-order moments are as follows: $\beta_2^x = 3\cdot 28$, $\beta_2^y = 2\cdot 54$, $r_{x^3y} = -\cdot 685$, $r_{xy^3} = -\cdot 59$, $r_{x^2y^2} = \cdot 914$. The work involved in applying the formula is somewhat heavy and tricky, and it is desirable to keep more decimals than those in the moments seem to warrant. Choice of a value for $c$ may be restricted by the need to avoid imaginary values of the $w$'s. In the application to the 50-year model office, large negative weights and small negative durations arise. Neither of these, however, need cause any trouble; the policy value for a negative duration can be computed by the formula $A_{x-t} - P_x a_{x-t}$. The average reserve per 100 sum assured of the 50-year model office by the

A 1924–29 ultimate table with 3% interest is 27·66. Various approximate average reserves by formula (14) (with different values of $c$ and various transformations) give errors ranging up to about $\pm\cdot 10$. The last figure of the approximate average reserves is not very precise owing to the use of first-difference interpolation in the official table of policy values, which are given to two decimal places only. But this does not explain away the small errors that still remain. Some part of them may be due to the fact that the A 1924–29 table is not completely smooth, but there is evidently a significant though very small residual error of the fifth and higher orders.

In Table 5 are given the weights, ages and durations that arise from a direct application of formula (14) with $c = \cdot 385$ and $R = -\cdot 638$ to the 50-year model office data, without transformation.

Table 5. *A nine-term representation of the 50-year model office, based on formula* (14) *with* $c = \cdot 385$

| Weight | Age | Duration |
|---|---|---|
| ·1801 | 33·94 | 16·892 |
| −3·4879 | 30·41 | 12·060 |
| 2·5082 | 29·03 | 10·173 |
| ·1069 | 46·35 | −·096 |
| ·1060 | 21·42 | 34·033 |
| ·0591 | 56·62 | 16·892 |
| 2·0300 | 33·28 | 16·892 |
| ·1533 | 33·94 | 39·253 |
| − ·6557 | 33·94 | 22·120 |

In this case the A 1924–29 3% average value for 100 sum assured works out at 27·62 compared with the true value of 27·66, and the $O^M$ 3% average value works out at 27·39 compared with 27·39.

The following statement gives a comparison of the component parts of the 50-year model office reserve by $O^M$ 3%, together with the amount of the net premiums:

| | Exact value | Approximate value using the ages and durations in Table 5 |
|---|---|---|
| Value of the sums assured | 1,449,006 | 1,448,950 |
| Value of the net premiums | 760,007 | 760,000 |
| Reserve | 688,999 | 688,950 |
| Amount of net premiums | 55,331 | 55,370 |

The value of the bonuses has also been computed on this basis. As the bonuses do not commence until after 5 years, the direct application of the Table 5 weights, ages and durations, with $(t-5)$ bonuses of 1·5 each, assumes negative bonuses up to duration 5. The necessary adjustment is easily made by applying formula (10) to the data and moments applicable to the 5-year model office.

The results are as follows:

| | |
|---|---|
| (*a*) Approximate value of 'bonuses' by formula (14) | 306,950 |
| (*b*) Adjustment by formula (10) for negative bonuses included in (*a*) | 7,710 |
| Approximate value of bonuses | 314,660 |
| Exact value | 314,866 |

The model office data represent a particularly difficult distribution for any approximate treatment owing to the extensive skewness in age at entry. It may be that negative weights and negative durations can be avoided if age at entry ($x$) and attained age ($z$) are used as the two variables instead of age at entry ($x$) and duration ($y$). The moments for $x$ and $z$ are readily obtainable from those for $x$ and $y$ since $x+y=z$. A certain amount of work has been done on this basis with the model office data but it has not been possible in present conditions to pursue this aspect very far.

It is possible to obtain various simplifications of formula (14) by satisfying some only of the third- and fourth-order moments. Two examples are as follows:

(1) If we put $p_1=p_2=p_3=p_4=1$ and $c=\cdot 5$, formula (14) can be reduced to eight terms and $w_1$, $w_2$, etc. are then obtained from the four third-order equations, as follows:

$$\left.\begin{aligned}
w_1&=\sqrt{(M_1^2+1+2r)}+M_1, & w_2&=\sqrt{(M_1^2+1+2r)}-M_1,\\
w_3&=\sqrt{(M_2^2+1-2r)}+M_2, & w_4&=\sqrt{(M_2^2+1-2r)}-M_2,\\
w_5&=\sqrt{(M_3^2+2)}\quad\;\;+M_3, & w_6&=\sqrt{(M_3^2+2)}\quad\;\;-M_3,\\
w_7&=\sqrt{(M_4^2+2)}\quad\;\;+M_4, & w_8&=\sqrt{(M_4^2+2)}\quad\;\;-M_4.
\end{aligned}\right\} \tag{16}$$

This formula is, of course, correct to the third order only and is a two-variable form of Jones's single-variable third-order formula.

(2) If we put $p_3=p_4=0$, $p_1=p_2$ and $c=0$, a five-term formula can be obtained, $p_1$, $w_1$, $w_2$, $w_3$ and $w_4$ being obtainable from any two of the third-order equations and one of the fourth-order equations. Using the $B_1^x$ and $B_1^y$ equations and a mean value of $\beta_2=\frac{1}{2}(\beta_2^x+\beta_2^y)$ the following solution results:

$$\left.\begin{aligned}
a_0&=1-p, & x_0&=0, & y_0&=0,\\
a_1&=w_2p/2\,(w_1+w_2), & x_1&=w_1\sigma_x, & y_1&=w_1\sigma_y,\\
a_2&=w_1p/2\,(w_1+w_2), & x_2&=-w_2\sigma_x, & y_2&=-w_2\sigma_y,\\
a_3&=w_4p/2\,(w_3+w_4), & x_3&=w_3\sigma_x, & y_3&=-w_3\sigma_y,\\
a_4&=w_3p/2\,(w_3+w_4), & x_4&=-w_4\sigma_x, & y_4&=w_4\sigma_y,
\end{aligned}\right\} \tag{17}$$

where
$$\frac{1}{p}=\frac{\beta_2}{1+r^2}-\frac{(B_1^x+B_1^y)^2}{2(1+r)(1+r^2)}-\frac{(B_1^x-B_1^y)^2}{2(1-r)(1+r^2)}$$

and
$$w_1=\sqrt{[M_1^2+(1+r)/p]}+M_1,\quad w_2=\sqrt{[M_1^2+(1+r)/p]}-M_1,$$
$$w_3=\sqrt{[M_2^2+(1-r)/p]}+M_2,\quad w_4=\sqrt{[M_2^2+(1-r)/p]}-M_2,$$

where
$$M_1=\frac{B_1^x+B_1^y}{2(1+r)}\quad\text{and}\quad M_2=\frac{B_1^x-B_1^y}{2(1-r)}.$$

Some experiments with the 50-year model office using this last formula and moments in age at entry and attained age have, despite one large negative duration, produced very close results for the constituent parts of the reserve as well as for the reserve, but details are not worth quoting without further testing, as the close balancing of errors appears to be fortuitous so far as the value of the net premiums (and, consequently, of the reserve) is concerned. The moments taken into account indicate that good results should be obtained for the value of the sums assured and the amount of the net premiums.

## THREE-VARIABLE FORMULAE

Corresponding formulae for three variables can be easily obtained and as an illustration the following eight-term formula corresponding to formula (10) may be of interest:

$$\left.\begin{array}{llll}
a_1=\cdot 125, & x_1=\sigma_x R_1, & y_1=\sigma_y R_1, & z_1=\sigma_z R_1,\\
a_2=\cdot 125, & x_2=\sigma_x R_2, & y_2=\sigma_y R_2, & z_2=-\sigma_z R_2,\\
a_3=\cdot 125, & x_3=\sigma_x R_3, & y_3=-\sigma_y R_3, & z_3=-\sigma_z R_3,\\
a_4=\cdot 125, & x_4=\sigma_x R_4, & y_4=-\sigma_y R_4, & z_4=\sigma_z R_4,\\
a_5=\cdot 125, & x_5=-\sigma_x R_1, & y_5=-\sigma_y R_1, & z_5=-\sigma_z R_1,\\
a_6=\cdot 125, & x_6=-\sigma_x R_2, & y_6=-\sigma_y R_2, & z_6=\sigma_z R_2,\\
a_7=\cdot 125, & x_7=-\sigma_x R_3, & y_7=\sigma_y R_3, & z_7=\sigma_z R_3,\\
a_8=\cdot 125, & x_8=-\sigma_x R_4, & y_8=\sigma_y R_4, & z_8=-\sigma_z R_4,
\end{array}\right\} \qquad (18)$$

where

$$R_1=\sqrt{(1+r_{12}+r_{23}+r_{13})},\quad R_2=\sqrt{(1+r_{12}-r_{23}-r_{13})},$$
$$R_3=\sqrt{(1-r_{12}+r_{23}-r_{13})},\quad R_4=\sqrt{(1-r_{12}-r_{23}+r_{13})},$$

and

$$\left.\begin{array}{l}
r_{12}=\Sigma xyf(x,y,z)/\Sigma f(x,y,z)\,\sigma_x\sigma_y,\\
r_{23}=\Sigma yzf(x,y,z)/\Sigma f(x,y,z)\,\sigma_y\sigma_z,\\
r_{13}=\Sigma xzf(x,y,z)/\Sigma f(x,y,z)\,\sigma_x\sigma_z.
\end{array}\right\}$$

## TWO-VARIABLE FORMULAE FOR ENDOWMENT ASSURANCES

In the paper on valuation methods by the present writer (*J.I.A.* Vol. LXIV, p. 279), a formula was devised for valuing endowment assurances grouped by years of entry, in which half the sums assured at each duration were assigned to one original term and half to another. This formula was based on the assumption that ${}_tV_{M-n:\overline{n|}}$ takes the form $a/n+b+cn$ when $t$ and M are constant. To obtain a two-variable formula by which sums assured at all durations and all terms can be valued together, it is necessary to include elements to cover at least the position and spread of the durations and the correlation between duration and term. By including the elements $t$ and $t^2$ and either $tn$ or $t/n$, it is easy to obtain alternative four-term weighted formulae analogous to formula (9) given earlier in this paper.

The four values of $n$ and $t$ are as follows:

$$\begin{array}{ll}
n_1=\bar{n}\sqrt{(1-1/\bar{n}\underline{n})}, & t_1=\sigma_t,\\
n_2=-\bar{n}\sqrt{(1-1/\bar{n}\underline{n})}, & t_2=-\sigma_t,\\
n_3=\bar{n}\sqrt{(1-1/\bar{n}\underline{n})}, & t_3=-\sigma_t,\\
n_4=-\bar{n}\sqrt{(1-1/\bar{n}\underline{n})}, & t_4=\sigma_t,
\end{array}$$

where the $n_1$, $n_2$, etc. are measured from $\bar{n}$ and the $t_1$, $t_2$, etc. from $\bar{t}$, $\underline{n}=\Sigma\,(\mathrm{S}/n)/\Sigma\mathrm{S}$, and the weights are as follows:

$$\begin{array}{ll} \text{If the element } tn \text{ is used} & \text{If the element } t/n \text{ is used} \\ a_1=\cdot 25\,(1+r_{tn}), & a_1=\cdot 25\,(1-r_{t/n}), \\ a_2=\cdot 25\,(1+r_{tn}), & a_2=\cdot 25\,(1-r_{t/n}), \\ a_3=\cdot 25\,(1-r_{tn}), & a_3=\cdot 25\,(1+r_{t/n}), \\ a_4=\cdot 25\,(1-r_{tn}), & a_4=\cdot 25\,(1+r_{t/n}), \end{array} \qquad (19)$$

where

$$r_{tn}=\frac{\Sigma\,(\mathrm{S}tn)/\Sigma\mathrm{S}-\bar{t}\bar{n}}{\sigma_t\bar{n}\,\sqrt{(1-1/\bar{n}\underline{n})}}$$

and

$$r_{t/n}=\frac{\Sigma\,(\mathrm{S}t/n)/\Sigma\mathrm{S}-\bar{t}\underline{n}}{\sigma_t\underline{n}\,\sqrt{(1-1/\bar{n}\underline{n})}}$$

For the purposes of arithmetical illustrations, the appropriate moments and ratios have been computed for Buchanan's endowment assurance model office and the results are given in Table 6.

Table 6. *Statistical constants for Buchanan's endowment assurance model office*

| Age of office | $\bar{n}$ (mean term) | $\underline{n}$ (mean reciprocal of term) | $\sqrt{(1-1/\bar{n}\underline{n})}$ | $\bar{t}$ (mean duration) | $\sigma_t$ | $r_{tn}$ | $r_{t/n}$ | $r_{t^2/n}$ |
|---|---|---|---|---|---|---|---|---|
| 10 | 24·084 | ·044885 | ·27377 | 5·213 | 2·891 | −·0152 | ·0166 | 3·650 |
| 25 | 24·846 | ·043148 | ·25931 | 10·582 | 6·697 | ·1783 | −·2206 | 3·641 |
| 40 | 25·348 | ·042389 | ·26333 | 11·701 | 7·914 | ·3122 | −·3320 | 3·400 |

The formulae assume a common maturity age and, as I. J. Bunney and W. J. Falconer (*J.I.A.* Vol. LXVI, p. 433) have given the valuations of the model office for maturity age 55 by $\mathrm{O^M}$ 3%, this maturity age and valuation basis have been used for illustration.

The results using the two forms of formula (19) are as follows:

| Age of office | True value | Approximate value using $r_{tn}$ | Error | Approximate value using $r_{t/n}$ | Error |
|---|---|---|---|---|---|
| 10 | 383,100 | 383,100 | — | 383,100 | — |
| 25 | 1,490,600 | 1,522,300 | 31,700 | 1,505,000 | 14,400 |
| 40 | 1,700,600 | 1,745,000 | 44,400 | 1,734,100 | 33,500 |

The approximations for fractional durations have been made by first differences, and those for fractional terms $(n+k)$ by first-difference interpolation between $n\times{}_t\mathrm{V}_{\mathrm{M}-n\,\overline{n}|}$ and $(n+1)\times{}_t\mathrm{V}_{\mathrm{M}-n-1\,\overline{n+1}|}$ and then dividing by $n+k$. For the 40-year model office, one of the durations exceeds the corresponding term. The corresponding policy value has been obtained by expressing ${}_t\mathrm{V}_{x\overline{n}|}$ in terms of the commutation columns, thus obtaining an appropriate policy value greater than 1. It may be worth mentioning that this feature occasionally arises in using the two-terms method for valuing the business of each year of entry separately.

It is clear that the formulae (19) do not give close enough results for the 25-year and 40-year offices, and similarly poor results have been obtained with office data. It has therefore been necessary to seek a more elaborate formula. It will be recalled that the original two-term formula received support from a consideration of the expansion of the sinking fund policy reserve, viz.

$$ {}_tV_{\overline{n}|} = \bar{s}_{\overline{t}|}\left(\frac{1}{n}-\frac{\delta}{2}+\frac{n\delta^2}{12}-\frac{n^3\delta^4}{720}+\dots\right). $$

Omitting terms after $n\delta^2/12$ and expanding $\bar{s}_{\overline{t}|}$ we have

$$ {}_tV_{\overline{n}|} = \left(t+\frac{t^2\delta}{2}+\frac{t^3\delta^2}{6}+\dots\right)\left(\frac{1}{n}-\frac{\delta}{2}+\frac{n\delta^2}{12}-\dots\right) $$

Consideration of the various product terms from the point of view of numerical size for normal values of $t$, $n$ and $\delta$ indicates that the element $t/n$ is much more important than $tn$ and that after the elements $t$, $t^2$ and $t/n$ the most important is $t^2/n$. It will be observed also that the elements $n$ and $1/n$ do not arise. The purpose of any formula devised is to apply it in practice to the valuation of endowment assurances, where the ages at entry will be assumed to be $M-n$, and in conditions where the policy value will take the form $\frac{1}{2}P_{x\overline{n}|}+{}_tV_{x\overline{n}|}$, and where the component values of the sums assured and of the net premiums and the amount of the net premiums will be required. It is necessary, therefore, to retain the elements $n$ and $1/n$. Although simple formulae without these elements could easily be devised and might be useful for some purposes, this line has not been pursued. Instead a formula has been developed on the basis of the six elements $t$, $t^2$, $n$, $1/n$, $t/n$ and $t^2/n$, and this formula is as follows:

$$ a_1=\cdot 25,\quad n_1=-\bar{n}\sqrt{(1-1/\overline{nn})},\quad t_1=\quad r_{t/n}\sigma_t+\sigma_t\sqrt{\left(1+r_{t^2/n}-r_{t/n}^2-\frac{1}{\sqrt{(1-1/\overline{nn})}}\right)}, $$

$$ a_2=\cdot 25,\quad n_2=-\bar{n}\sqrt{(1-1/\overline{nn})},\quad t_2=\quad r_{t/n}\sigma_t-\sigma_t\sqrt{\left(1+r_{t^2/n}-r_{t/n}^2-\frac{1}{\sqrt{(1-1/\overline{nn})}}\right)}, $$

$$ a_3=\cdot 25,\quad n_3=\quad\bar{n}\sqrt{(1-1/\overline{nn})},\quad t_3=-r_{t/n}\sigma_t+\sigma_t\sqrt{\left(1-r_{t^2/n}-r_{t/n}^2+\frac{1}{\sqrt{(1-1/\overline{nn})}}\right)}, $$

$$ a_4=\cdot 25,\quad n_4=\quad\bar{n}\sqrt{(1-1/\overline{nn})},\quad t_4=-r_{t/n}\sigma_t-\sigma_t\sqrt{\left(1-r_{t^2/n}-r_{t/n}^2+\frac{1}{\sqrt{(1-1/\overline{nn})}}\right)}, $$

where $\quad r_{t^2/n}=[\Sigma\,(St^2/n)/\Sigma S-2\bar{t}\,\Sigma\,(St/n)/\Sigma S+\bar{t}^2\underline{n}]/\sigma_t^2\underline{n}\sqrt{(1-1/\overline{nn})}.$ (20)

The results of applying formula (20) to the model office valuation by $O^M$ 3% with M = 55 are as follows:

| Age of office | True value | Approximate value | Error |
|---|---|---|---|
| 10 | 383,100 | 383,100 | — |
| 25 | 1,490,600 | 1,489,700 | − 900 |
| 40 | 1,700,600 | 1,696,100 | −4500 |

The 40-year model office is, of course, an extreme case, having regard to the new business assumptions involved, but even so the error is less than −·3%. To illustrate the degree of approximation in the component parts of the reserve, the true and approximate values of the sums assured for the 40-year model office have been computed by $O^M$ 3% with $M = 55$, and the result, together with the value of the net premiums (by subtracting the reserve) and the amount of the net premiums, is as follows:

| | True | Approximate | Error |
|---|---|---|---|
| Value of the sums assured | 3,038,100 | 3,033,000 | −5100 |
| Reserve | 1,700,600 | 1,696,100 | −4500 |
| Value of the net premiums | 1,337,500 | 1,336,900 | − 600 |
| Amount of net premiums | 145,500 | 145,500 | — |

It may perhaps be of interest to mention that, using office data, close results have also been obtained for the component parts as well as for the reserve.

Imaginary values of $t$ could arise in using formula (20) for a distribution in which $t$ and $n$ are very highly correlated, a situation unlikely to arise in practice.

## PRACTICAL APPLICATION IN ACTUAL VALUATION

In a large office where the classification system is directed to the use of the two-ages and two-terms method of the previous paper, thereby achieving the complete emancipation of the classification from the valuation basis and permitting easy and frequent change of valuation basis when desired, the punched valuation cards contain constants based on $xS$ and $x^2S$ for whole-life assurances and $nS$ and $S/n$ for endowment assurances. The classification is maintained in two separate ways (by year of entry and by year of birth or year of maturity), the class totals being carried on summary cards prepared automatically from the tabulator totals by means of the summary card punch machine. The double classification which is quite automatic has permitted the valuation of any class to be made both by years of entry and by years of birth (or maturity), thereby providing an accurate value of the sums assured and bonuses and a valuable check in respect of very large blocks of business, without any important increase in work. The war has, however, made it necessary to seek the shorter methods set out in this paper, at least for the smaller classes of business. The year of entry classification of the whole-life assurances (single-life, joint-life, with and without profits) has enabled the duration moments and any combination of powers of $t$ with $x$ and $x^2$ to be readily obtained. For most purposes one of the formulae involving only $\bar{x}$, $\sigma_x$ (obtained directly from the classification totals) and $\bar{t}$, $\sigma_t$ and $r_{xt}$ (obtained with speed and facility from the year of entry classification) is all that is required.

From the examples given in relation to the model office, it will be appreciated that the error arising in the second-order formulae varies according to the conditions, the most important being the nature of the distribution of the business. The choice of formula in any particular case needs to be made with some discretion, but the device of the insertion of an extra term at the mean age and mean duration is available to rectify the error to any degree thought necessary.

It is obvious that all the moments up to the fourth order, except $\mu_3^x$, $\mu_4^x$ and $\mu_{x^3t}$, can be obtained from the year of entry class totals. From the year of birth classification it is possible to obtain $m_{x^2(x+t)}$, $m_{x^2(x+t)^2}$ and $m_{x(x+t)^3}$, and hence by suitable combination with the other known moments $\mu_3^x$, $\mu_4^x$ and $\mu_{x^3t}$ can be obtained. Thus, all the moments up to the fourth order are available if it is desired at any time to apply formula (14) as a test of the continued closeness of the results by a simple formula such as formula (9) or by a modified five-term version of formula (9).

In the case of endowment assurances $\bar{n}$ and $\underline{n}$ are obtained directly from the class totals and $\bar{t}$, $\sigma_t$, $r_{t/n}$ and $r_{t^2/n}$ are similarly obtained with speed and facility from the year of entry classification. Further, it is found that changes in the items $\sqrt{\left(1+r_{t^2/n}-r_{t/n}^2-\frac{1}{\sqrt{(1-1/\bar{n}\underline{n})}}\right)}$ and $\sqrt{\left(1-r_{t^2/n}-r_{t/n}^2+\frac{1}{\sqrt{(1-1/\bar{n}\underline{n})}}\right)}$ are slow and unimportant and can be fixed approximately in the light of the previous year's values calculated at leisure. Apart from the production of ancillary tables of valuation factors for steps of ·1 in the appropriate ranges of ages and terms, the valuation work is as easily effected on one valuation basis as on another and the calculation of the effect of a change of basis is a simple matter. Moreover, in these days of the growing use of a common maturity age in the valuation of endowment assurances, it is of some advantage to be able to fix this at will from time to time instead of being tied by the fact that the net premiums in the class totals are based on a particular maturity age.

## CONCLUSION

Apart from the usefulness of the $n$-ages method in practical actuarial work, the extensive use of moments and measures of correlation provides a bridge between actuarial work and the methods of descriptive statistics. It is a remarkable fact that so little application has so far been found for general statistical method in actuarial work. The consequence has been that actuarial students find little solid ground on which to base their statistical studies, which it is suggested have an educational value that cannot be overrated, and few practical applications on which to build a sound judgment between arithmetical sufficiency and the mathematical refinements in which the subject of statistics has become so steeped. The use of the $n$-ages method in the actuarial department goes some way to fill this gap.

This paper could never have been completed in present conditions without the invaluable assistance of Messrs G. Rowland and E. G. Neville, A.C.I.I., of the Valuation Department of the Pearl Assurance Company, Ltd., particularly in the calculation of the model office moments. Acknowledgments and thanks are also due to Mr A .E. Lacey, F.I.A., who not only has checked all the algebra and some of the arithmetic but, prior to the preparation of this paper and throughout the development of the subject matter, gave invaluable assistance and showed the utmost patience in testing on office data many of the formulae in the paper as well as a number of abortive formulae (e.g. formulae up to the third order) that appear to be the inevitable accompaniment of work of this kind.

## ABSTRACT OF THE DISCUSSION

**Mr Wilfred Perks,** in introducing the paper, said that Mr Lidstone had asked him to mention two points. One concerned the name of the method. Mr Lidstone objected to the suggestion 'weighted quadrature', which he regarded as a terminological inexactitude, and he (the author) agreed with him. Mr Lidstone suggested that the process should be called the '*n*-point method'.

Mr Lidstone had also asked him to mention that, if an actuary used the *n*-ages method to obtain an estimate of the cost of a change of valuation basis, better results might be obtained if the *n*-ages method were applied to the differences between the factors, or—what came to the same thing—if the same approximate method were used in both cases and the difference between the two taken as the estimate of the cost of the change.

**Mr H. Tetley,** in opening the discussion, said that the original paper in which Elderton and Rowell had introduced the *n*-ages method dealt primarily with the reform of the Assurance Companies Act, 1909, and particularly with the Schedules. The author's earlier paper had dealt with a modification of the net premium method of valuation, as well as with the *n*-ages method and with the calculation of isolated values of actuarial functions. In comparing the three papers, it was interesting to observe how the emphasis had gradually shifted from the practical to the theoretical. The original method was of limited theoretical interest but was of the utmost practical value, because it was adaptable and, used with judgment, it produced results which were not only sufficiently accurate for practical purposes but were often uncannily close to those brought out by detailed valuations. For its successful application, however, it was necessary to have an existing valuation of the data involving all the functions needed for the approximate method. For instance, starting from a net premium valuation he had been unable to make a satisfactory approximation to a bonus reserve valuation because many of the functions needed did not appear in the former.

The author's earlier method did not presuppose any existing valuation, but in other respects it made considerably greater demands. The data had to be grouped according to year of entry, and it was necessary to have mechanical sorting of punched cards, which included subsidiary constants $x$S, $x^2$S or $n$S, S/$n$.

Preoccupation with the practical problems of everyday work should not be allowed to obscure the fact that the author had produced a general method of the widest application outside, as well as inside, the actuarial world by extending to functions of two variables the work previously done by Tchebycheff for a single-variable function. He had in fact shown how to find with the greatest accuracy the sum of a set of products $\Sigma uf$ by multiplying $\Sigma u$ by the weighted mean of a few select values of $f$, although for practical purposes he had confined his attention to the particular case where $f$ was a polynomial in $x$ and $y$.

Before dealing further with the theoretical side of the subject, there were one or two small points in the paper to which he wished to refer, not because of their intrinsic importance, which was negligible, but because they had confused him and might confuse others. The letters A to H were included in the author's diagram on p. 379 and thereafter appeared throughout the text, and it was rather unfortunate that the same letters, in the same type, appeared as the coefficients of $f(x, y)$. It would perhaps have been wiser to have used a completely different set of letters for the diagram. The change of notation in the middle of p. 380 could also with advantage have been omitted for the sake of clarity. Standardized variates, i.e. variates measured in terms of the standard deviation as unity, had great advantages in theoretical work, but in the paper solutions were always given by the author in the original notation. For instance, the conditions given in the middle of p. 380 were in standardized variates and solution (10) was in the original notation. In the next line there was a change back to standardized variates, and solution (11) was again in the original notation. That persisted throughout the paper and there was a similar example on p. 385. Some of the confusion could have been avoided by a change in the symbols used to denote the standardized variates.

The next point was the type of difficulty which arose in solving the various equations. The number of constants exceeded the number of conditions to be fulfilled in order to secure a satisfactory fit, and the author therefore imposed several reasonable but largely arbitrary conditions in order to obtain a solution—for instance, that the formulae should be equally weighted and that the solutions should be symmetrical between $x$ and $y$ as well as in $x$ and $y$. Other conditions might be tested and possibly even better solutions obtained, but there was a warning to be given in that respect. A few trials would, he thought, convince anyone that the author had probably taken the only conditions which led to fairly simple and manageable solutions. It was surprising that nearly all the formulae were symmetrical between $x$ and $y$. In the practical application of the formulae to policy values it would seem to be more logical to attempt to get a closer fit for the variable duration at the expense of a rougher approximation with regard to age. He understood that the author had made some attempts in that direction but without any great success, and had actually found that it was better to ensure a fit with regard to all the terms up to, say, the second order, or all the terms up to the third order, rather than to mix some of the second- with some of the third-order terms. An example appeared on p. 385, where the author tried to improve his second-order formula by discarding the condition of symmetry between $x$ and $y$ and replacing it by a condition relating to the terms in $x^2y^2$, and it would be noticed that the result was no improvement on the simpler formula.

There was an interesting generalization at the foot of p. 385 which introduced a degree of elasticity to the formula, the introduction of a term at $x=0$, $y=0$. He felt, however, that the author spoilt a good case by laying himself open to the charge of jobbing backwards. The author found his constant $k$ by reference to a 50-year model office, and showed that the same value gave a very close approximation for a 30-year office. It would be more convincing to find $k$ for the 30-year office and to show that it still gave a satisfactory answer twenty years later.

With regard to the neat theoretical work on pp. 388 and 389, he wished to point out that the device of measuring the variables from the axes of the ellipses of equal frequency, thus cutting out any possibility of correlation between them, was of value only for the normal surface which, unfortunately, seldom applied to the type of data handled in actuarial work.

In dealing with endowment assurances, the author was obliged to consider the relative importance of the various terms, $t$, the duration, being of much more importance than $n$, the original term. He was interested in the formulae for $r_{tn}$ and $r_{t/n}$ given at the top of p. 394. Clearly those were closely akin to coefficients of correlation, but the denominators were not the usual products of standard deviations.

As a practical weapon the method described in the paper was of restricted application because of the demands which it made on the office arrangements. Punched cards and mechanical sorting were needed, but, given those conditions, the work was not prohibitive, particularly in its demands on the skilled staff at a time when a valuation had to be made. He was particularly impressed by the neat way in which the cross-moments of the fourth order were obtained, as outlined at the top of p. 397, from the grouping according to attained age or year of birth. The work could be still further reduced by some of the most up-to-date machines, such as the multiplying punch.

There was one note of caution which he would like to give. A formula which seemed complicated and involved heavy analysis in its development might actually be less laborious to apply than a simpler formula. For instance, formula (15) seemed more involved than formula (13); actually it involved less labour in its application because it included only $\sigma_x$, $\sigma_y$ and $r$, whereas formula (13) involved in addition the function which the author called $r_2$, a sort of coefficient of correlation based on $x^2$ and $y^2$ instead of $x$ and $y$.

**Mr W. G. Bailey** said that the opener had suggested that the author's choice of formulae was arbitrary. It should be said in defence of the author that, considering the six-term assumption for $f(x,y)$, there was no two-point solution unless $r=\pm 1$, and those values for $r$ resulted in the unique equally weighted solutions (5) and (6). The author therefore tried to solve the problem by choosing two points on each of the two regression lines. That was not possible unless $r$ was zero, in which case formula (8) resulted, but a method of approach, since they were equally interested in $x$ and $t$, was to try to fix the points on two lines lying in the same pair of opposite quadrants and equally inclined to their respective axes, measurements being made in standardized units. On that basis, for an equally weighted solution, formula (11) was unique and, as would be seen, it reduced to (8) as a special case. A further method which the author tried was similar, except that the lines were in all four quadrants yielding formula (10) as an equally weighted solution, with formula (7) as a special case.

With regard to the opener's point as to the limited use of the transformations referred to on p. 389, the author admittedly invited criticism by the way in which he introduced the idea of the transformations. In fact, the mention of the normal distribution was not really necessary. What the author said was that if there were two correlated variables $x$ and $y$, it was possible to find by means of the formulae on p. 389 two uncorrelated variables $u$ and $v$. Conversely, by starting with two uncorrelated variables, $u$ and $v$, it was possible to derive two correlated variables, $x$ and $y$, whose correlation was determined by the formula on p. 389. If $x$ and $y$ were measured in terms of their standard error, the standard error of $u$ was $\sqrt{(1+r)}$ and the standard error of $v$ was $\sqrt{(1-r)}$, with the result that, if $x$ and $y$ were replaced in formulae (7) and (8) by $u$ and $v$ (those formulae holding true for uncorrelated variables) and if the expressions on p. 389 were then substituted for $u$ and $v$, formulae holding good for correlated variables were obtained producing formulae (11) and (10) in that order. The author's method was not as arbitrary as it seemed.

It seemed worth while to ask whether the idea of using the moments was purely algebraic, or whether it had some fundamental basis. The conception of moments was central to statistical theory because it produced the same results as the least-squares method if the curve fitted was a polynomial. The study of moments had been such that it could be said that the first four moments of a distribution gave most of the information that would be of use. The author did not begin with a distribution of policy values. He had a frequency distribution in $x$ and $t$ and he sought to replace that stereogram by $n$ isolated points, not in order to reproduce the original distribution but so that the sum of the products of the weights and the ${}_tV_x$ factors at the isolated points would produce the same result as if the process had been carried out over the whole original stereogram. Moments were used only because ${}_tV_x$ had been expanded in terms of ascending powers of $x$ and $t$.

By that device, which was a very respectable one, the author transferred attention from that rather complicated problem to the one merely of reproducing the original $x$, $t$ stereogram. It was tempting to say that it was of no consequence what ${}_tV_x$ was because the problem was one of replacing the $x$, $t$ distribution, and some support was lent to that view by the fact that A, B, C, etc., did not appear in the final result. That, of course, was not true; ${}_tV_x$ had to bear some relation to $x$ and $t$, and, if moments were to be used, ${}_tV_x$ had to be expansile in $x$ and $t$ or at any rate capable of graduation by a formula so expansile.

**Mr A. W. Joseph** said that the advantages of the methods of valuation given in the paper were so obvious that it was a matter of some importance to overcome any technical imperfections.

There was a powerful and natural way of obtaining H. G. Jones's one-variable formulae, which was given on pp. 292–6 of his own paper in *J.I.A.* Vol. LXV. The principle was fairly simple. The valuation function $f(x)$ (he was referring to one-variable functions) was expanded as a Lagrange polynomial in terms of the values of $f(x)$ at a number of points $x_1$, $x_2$, etc. The expression was multiplied by $u_x$ and summed over the range considered. The result was an expansion for $\Sigma u_x f(x)$ in terms of the values of $f(x)$ at $x_1$, $x_2$, etc. The two-variable analogue of the Lagrange interpolation formula was not so general. For one variable, $f(x)$ might be expressed in terms of $f(x_1)$, or of $f(x_1)$ and $f(x_2)$, or of $f(x_1)$, $f(x_2)$ and $f(x_3)$, and so on. For two variables, $f(x, y)$ had to be expressed in a kind of square matrix of terms, e.g. in one term $f(x_1, y_1)$, or in four terms $f(x_1, y_1)$, $f(x_1, y_2)$, $f(x_2, y_1)$ and $f(x_2, y_2)$, or in nine terms $f(x_1, y_1)$, $f(x_1, y_2)$, $f(x_1, y_3)$, $f(x_2, y_1)$, etc., and so on. The four-term formula was

$$f(x, y) = \frac{(x-x_2)(y-y_2)}{(x_1-x_2)(y_1-y_2)} f(x_1, y_1) + \text{similar expressions involving } f(x_1, y_2), \text{ etc.}$$
$$+R(x, y),$$

where $R(x, y)$ was the remainder. If that expression were multiplied by $u_{xy}$ and summed for all values of $x$ and $y$ in the range considered, the following formula was obtained, the origin being moved to the mean and $x_1$, $x_2$ being expressed in units of $\sigma_x$ and $y_1$, $y_2$ in units of $\sigma_y$:

$$\Sigma u_{xy} f(x, y) = \Sigma \mathrm{H}_{x_1y_1} f(x_1, y_1) + \Sigma u_{xy} \mathrm{R}(x, y),$$

where

$$\mathrm{H}_{x_1y_1} = \Sigma u_{xy} \frac{(x-x_2)(y-y_2)}{(x_1-x_2)(y_1-y_2)} = \frac{\mu_{xy} - y_2\mu_x - x_2\mu_y + x_2y_2}{(x_1-x_2)(y_1-y_2)} \Sigma u_{xy} = \frac{r + x_2y_2}{(x_1-x_2)(y_1-y_2)} \Sigma u_{xy}.$$

If $f(x, y)$ was of the form $\mathrm{A}+\mathrm{B}x+\mathrm{C}y+\mathrm{D}xy$, $\mathrm{R}(x, y)$ vanished and a four-term expression of the type developed by the author was obtained with $x_1$, $x_2$, $y_1$, $y_2$ completely at their disposal. In order to fix $x_1$, $x_2$, $y_1$, $y_2$, it was natural to try to satisfy higher orders of $f(x, y)$. As there were four variables, the natural orders to satisfy were $x^2$, $x^2y$, $y^2$, $xy^2$. The resulting equations were linear in $x_1+x_2$, $x_1x_2$, $y_1+y_2$ and $y_1y_2$, and thus in theory could easily be solved. A complication might occur, however, which was not present in the one-variable case. In his paper in *J.I.A.* Vol. LXV, he had shown that the points $x_1$, $x_2$, etc. were all real and distinct and comprised in the range over which a summation was required. In the two-variable case that did not necessarily hold true. In fact, the solution of the equations for the 10-year model office gave points outside the range. For the 50-year model office, however, there was a solution within the range but the resulting approximate valuations showed no improvement in accuracy on the author's formula (9). If the $x^2y$ and $xy^2$ moments were dispensed with, there were two degrees of freedom which could be used by fixing $x_1$ and $x_2$ at equal distances from the mean $\bar{x}$ and $y_1$ and $y_2$ at equal distances from the mean $\bar{y}$. The resulting formula was the author's formula (9).

Turning to the nine-term formula

$$\Sigma u_{xy} f(x, y) = \Sigma \mathrm{H}_{x_1y_1} f(x_1, y_1) + \Sigma u_{xy} \mathrm{R}(x, y),$$

where
$$\mathrm{H}_{x_1y_1} = \Sigma u_{xy} \frac{(x-x_2)(x-x_3)(y-y_2)(y-y_3)}{(x_1-x_2)(x_1-x_3)(y_1-y_2)(y_1-y_3)},$$

the remainder vanished absolutely if $f(x, y)$ was of the form

$$\mathrm{A}+\mathrm{B}x+\mathrm{C}y+\mathrm{D}xy+\mathrm{E}x^2+\mathrm{F}y^2+\mathrm{G}x^2y+\mathrm{H}xy^2+\mathrm{J}x^2y^2.$$

A nine-term formula of the square type was thus obtained. The six variables $x_1$, $x_2$, $x_3$, $y_1$, $y_2$ and $y_3$ were completely at their disposal and in theory the further orders $x^3$, $y^3$, $x^4$, $x^3y$, $xy^3$, $y^4$ could be satisfied so that the formula would be correct to the fourth order. The resulting $x_1$, $x_2$, $x_3$, $y_1$, $y_2$, $y_3$ would frequently come outside the range considered.

It had occurred to him that a more practical solution could be obtained by satisfying as nearly as possible the orders $x^3$, $y^3$ and choosing integral values for $x_1$, $x_2$, $x_3$, $y_1$, $y_2$, $y_3$. The results were quite good, and a nine-term formula in integral $x_1$, etc. was almost as easy to apply as a four-term formula with fractional $x_1$, etc.

As to the author's treatment of endowment assurances, if it were assumed that V was of the form $a/n+b+cn$ the natural method of development was to note that the product $u\mathrm{V}$ was the same as $u/n \times n\mathrm{V}$. All the formulae developed in the earlier part of the paper and those he had just suggested could be applied to the distribution $u/n$ valued by the function $n\mathrm{V}$. He had experimented on those lines with the 40-year model office. Unfortunately, the distribution was rather intractable and the points for the author's formula (9) came outside the range. It was possible to obtain a four-term formula of the square type in integral points within the range, but the accuracy was not as great as for the author's formula (20), perhaps because the latter involved an extra moment, viz. $t^2$. But with the addition of the orders $tn$ and $t^2n$ it was possible to find quite satisfactory and accurate nine-term formulae of the square type in integral points. The coefficients of some of the terms were negative.

He thought a warning was necessary about the use of four-term formulae. He had experimented with a number of four-term formulae of the square type, which in theory should be just as good as the author's formula (9). Some of them gave better results than formula (9) but most of them gave worse results. It was not necessary to go beyond formula (9) to see the danger. Supposing the variables were changed to $\xi=x+y$ and $\eta=x-y$, then if the valuation factor was a second-degree surface in $x$ and $y$ it would likewise be a second-degree surface in $\xi$ and $\eta$, whence formula (9) ought to apply equally well when the valuation factor was expressed in terms of $\xi$ and $\eta$ as when expressed in terms of $x$ and $y$. It would be found, however, that King's 50-year model office valued by the $O^M$ 3 % table, using $\xi$ and $\eta$ as the variables, produced an error of £12,300, i.e. more than four times as much as when $x$ and $y$ were the variables. The fact was that the choice of terms might be unlucky. If the valuation factor was a second-degree surface in $t$ and $n$, both ways of applying formula (9) would give exact results. But if the formula happened to be based on one or two values where the valuation factor differed rather widely from a second-degree surface, the four terms were not sufficient.

There were two distinct uses of the author's principles, namely (1) as a method of rapidly obtaining the cost of a change of valuation basis, and (2) as a substitute for a detailed valuation. The author's formulae were admirable for the former but he was not satisfied that they were sufficiently dependable for the latter. There was another way of dealing with the information which would be available if the author's tabulation methods were adopted. Suppose it was assumed that the valuation factor $f(x, y)$ was of the form $a+bx+cy+dx^2+exy+fy^2$. The author only used $a$, $b$, $c$, $d$, $e$ and $f$ as a means of expressing the value of $\Sigma u_{xy} f(x, y)$ in terms of particular values of $f(x, y)$. But it was possible actually to calculate $a$, $b$, $c$, $d$, $e$ and $f$ by equating various moments of $f(x, y)$ to like moments of $a+bx+cy+dx^2+exy+fy^2$ over the valuation range. The range would be a kind of triangle or half-rectangle. It was possible to simplify the computation by the use of a set of orthogonal polynomials similar to the Tchebycheff polynomials of the single variable. For the 50-year model office he computed the constants by using every fifth age of entry from 20 to 85 inclusive, and every fifth duration from 1 to 66 inclusive. The labour of calculating the constants was by no means prohibitive and it could be done between valuations.

The results of his computations gave some indication of the relative accuracy of the various methods to which he had alluded (see below).

*King's 50-year model office*

| Formula | | Error | Moments used |
|---|---|---|---|
| Author's formula (9) | | +·41 % | $x, x^2, y, y^2, xy$ |
| An integral age nine-term formula | $\begin{pmatrix} x=22, 33, 51 \\ y=\ \ 2, 16, 39 \end{pmatrix}$ | −·27 % | $x, x^2, x^3, y, y^2, y^3, xy, xy^2$ $x^2y^2, x^2y$ |
| An orthogonal polynomial formula | | +·33 % | $x, x^2, y, y^2, xy$ |

*Buchanan's 40-year model office*

| Formula | | Error | Moments used |
|---|---|---|---|
| Author's formula (20) | | −·26% | $1/n, n, t/n, t^2/n, t, t^2$ |
| An integral age four-term formula | $\begin{pmatrix} n=18, 31 \\ t=\ \ 3, 18 \end{pmatrix}$ | +·53 % | $1/n, n, t/n, t^2/n, t$ |
| An integral age nine-term formula | $\begin{pmatrix} n=19, 24, 29 \\ t=\ \ 3, 11, 19 \end{pmatrix}$ | +·13 % | $1/n, n, t/n, t^2/n, t, t^2, tn, t^2n$ |
| An orthogonal polynomial formula | | −·08 % | $1/n, n, t/n, t^2/n, t$ |

He was interested to note the ingenious way in which the author obtained the higher moments by separate classifications according to year of entry and year of birth. But why not a third classification by age at entry, i.e. year of entry minus year of birth? If that were done, it would be possible to obtain the moments $x$, $x^2$, $y$, $y^2$ and $xy$ without the tabulation of any special function whatever, and those moments were sufficient for the use of formula (9) or an integral age four-term formula or an orthogonal polynomial formula. If it were thought advisable to use a nine-term formula, all the necessary moments could be found by tabulating $x$S only.

For endowment assurances, the three separate classifications would be by year of entry, year of maturity and original term, but in that case the function S/$n$ would have to be tabulated.

A valuation of endowment assurances by the author's method need not be restricted by the assumption of a fixed maturity age if the Z-factors were tabulated for the classification by year of maturity. The average age at maturity was a fairly regular function of the period to run to maturity (i.e. $n-t$) and it could be fitted with good accuracy by a second-degree curve in $(n-t)$. If $n({}_tV_{M-n:\overline{n}|})$ could be expressed approximately as a second-degree surface in $t$ and $n$ when M remained constant, the same could be assumed to be true when M varied according to a second-degree curve. Thus any of the author's formulae could be applied. It was only necessary to find, by means of the fitted curve in $(n-t)$, the appropriate M corresponding to each $t$ and $n$ of the terms of the author's formulae.

**Mr A. E. Lacey** remarked that, in connexion with the application of formula (17) to the 50-year whole-life model office, the author referred to the fact that the close balance of errors appeared to be fortuitous. He had pursued that point and, as an experiment, had applied formula (17) to a large block of whole-life business. The result at least added strength to the author's suspicion: the reserve brought out for the block of business was about 2 % in defect, although the values of the sums assured and of the net premiums were exceedingly close to the true values.

Turning to endowment assurances, he was interested in an alternative expression for the endowment assurance policy reserve ${}_tV_{x\overline{n}|}$. The author did not refer to the assumption of ${}_tV_{x\overline{n}|}$ taking the form of a polynomial in $n$ and $t$. It was true that in his previous paper in *J.I.A.* Vol. LXIV, he had given them some comparisons between the results obtained on the assumptions that ${}_tV_{x\overline{n}|}$ took the forms $A+Bn+Cn^2$ and $A/n+B+cn$. As was only to be expected from the comparisons in that paper, formulae such as (9) and (10) gave very poor results with endowment assurance data, but in formula (14) they had a much more powerful instrument. As an experiment he had applied that formula to a very large block of endowment assurance business, obtaining exceedingly good results, not only for the reserve but also for the actual components of the reserve.

The application of formula (17) to the same data, using moments in $n$, the original term, and $(n-t)$, the unexpired term, gave quite a good result for the value of the sums assured and for the total of the net premiums, but, as he expected, the result for the reserve was not at all good. Incidentally, taking the form of classification described by the author on p. 396, where valuation cards for endowment assurances carried a constant in the form of $n$S and S/$n$, it was possible to obtain all the moments in $t$ and $n$ up to the fourth order by the simple device of two systems of classification, one by year of entry and the other by year of maturity.

A further point of interest was that formula (20) had been found, by experiment, to produce very good results when applied to endowment assurances which carried guaranteed bonuses.

The author's formulae (19) and (20) presupposed a classification by year of entry, and no comment was made in the paper as to the suitability or otherwise of those formulae when applied to an alternative classification by year of maturity. At the top of p. 395 the author gave an interesting analysis which was readily adaptable to endowment assurances classified by years of maturity. An examination of the numerical size of the elements involved in the expression obtained had led him to think that good results would be obtained by applying formula (20) to the data tabulated according to the year of maturity. In one interesting case the value of $(n-t)$ became imaginary but in all other cases he had been more fortunate, and in several cases a reasonably good result was obtained. But there had also been one or two very disturbing results, and a reason why formula (20) was unsatisfactory when applied to endowment assurance data in that form had to be sought.

The reason why formula (20) broke down became obvious when the numerical work was examined. When the four values of $n$ and the four values of $(n-t)$ had been obtained, it was simple to obtain the value of the standard deviation $\sigma_t$, and to compare that with the value of $\sigma_t$ obtained by a direct computation from the alternative classification by year of entry. It was found that the two values of $\sigma_t$ did not agree, and that was in itself sufficient, he thought, to account for the failure of formula (20). Where the difference between the values of $\sigma_t$ obtained by the two systems of classification was small there was a fairly close agreement between the reserves obtained by applying formula (20) to each method of classification, and what had led him to pursue the matter further were those cases where the differences between the two values of $\sigma_t$ were really significant. He had made a number of experiments with various other formulae in the hope of producing something which could be applied with equal confidence to endowment assurance data classified by year of maturity or by year of entry but unfortunately nothing satisfactory had yet resulted.

**Mr R. E. Beard** said that, as mentioned in the footnote on p. 381, he had arrived at certain formulae prior to the author's developments but for various reasons his researches had not been developed or published. It might appear strange that the relatively complicated formula (11) had been reached. The reason was, however, quite simple, as in forming a mental picture of the mathematical problem he had followed the same line of thought as Mr Bailey and had had in mind the shape of the two-variable surface and the position of the lines of regression, the original object being to find a solution which gave the points on the lines of regression and related to the density of the distribution.

Members might be interested to hear the results of valuing two blocks of two-life last-survivor annuity business. Two companies which he thought would be a fairly good test for the method were selected from the Board of Trade returns and the business was valued by taking the ages as recorded in the returns. The first example consisted of 28 groups with a total annual rent of £4728. The range was from £8 to £1000, representing a difficult test for approximate methods. The true liability was £76,993 and the approximate figure from formula (11) was £154 less, an error of about two per mille. The second example consisted of 138 groups comprising a total annual rent of £12,877 and a range of from £5 to £1850, so that again it was an awkward distribution. The true liability emerged at £174,392 and the approximate figure was £349 larger, once more an error of two per mille, but of opposite sign. The annuity-values used in the calculation were obtained by interpolation from the $a\ (f)$ and $a\ (m)$ 3 % tables and the last digits were not significant.

The method thus appeared to be very powerful and could, of course, be easily developed for use with a punched card system. It would be necessary to carry five classification constants, but they would not generally occupy a field of more than 25 columns and would thus be within easy reach of the 90-column card.

During the investigations trials had been made of a modified formula which was capable of wider application. The author developed the functions in ascending or descending powers of the variables. That method was convenient and, in general, sufficient, but there was no need to restrict consideration to the polynomial form. With actuarial functions, particularly whole-life functions, a low order polynomial would not represent the function satisfactorily over a long range and expansion in terms of exponentials might give better results.

That method had been applied to some whole-life valuation data, the basic expansion being $\mathrm{A}+\mathrm{B}c^x+\mathrm{D}c^{2x}$. Various values of $c$ were tried but, although some slight improvement could be obtained, the limitation implied by the extra constant in the classification did not seem to justify the gain in accuracy. The method had also been applied to the last-survivor annuity valuation, but only one value of $c$ was tried and it produced slightly worse results than the polynomial expansion.

Returning to the formulae as developed by the author and considering the single-variable case, it would be appreciated that the basic underlying assumption was that one of the two functions was defined by its successive moments and the other was assumed to be expanded in ascending powers of the variable. In so far as analytical functions normally had unique sets of moments, solutions expressed in terms of the moments readily provided a source of approximate integration formulae by substitution of the appropriate values of the moments. That process was exemplified by the author on pp. 389 and 390 of the paper.

During recent years he had solved in detail the equations up to the seventh order of moments, and had developed from those general solutions many well-known approximate integration formulae of the Gaussian and Tchebycheff form as special cases. The author had suggested in conversation with him that the time was long overdue when the form of the error term in such formulae should be determined. That he had done, and the whole investigation of the single-variable formulae was nearly complete.

In connexion with the general problem of expressing actuarial functions by simple analytical expressions, a process necessary for the efficient application of the formulae under discussion, he wished to mention an investigation carried out some years ago in connexion with the calculation of office premiums. The original problem had been to see whether it was possible to fit a convenient surface to a scale of office premiums given at quinquennial ages and terms, to include whole-life, endowment assurance and limited-payment policies. The more limited case of endowment assurances had been successfully solved and it had been found possible to represent the office premiums by a formula consisting of the sum of two components, one a term of the form $(A/n+B+Cn)$ and the other a product of the form $\dfrac{a(n-1)}{n+2}\times\dfrac{r^n.c^{x+n}}{1+dc^{x+n}}$. That formula applied from age 15 to the upper limit of age met in practice and for all values of $n$ from 5 upwards. A moderately successful scale of whole-life premiums could be made by appropriate selection of the term $n$. The experiment was stopped before full study had been made of the limited-payment premiums.

**Mr H. G. Jones** said that he had not tried the effect in practice of the formulae in question, and without having done so it was difficult to make a valuable contribution to the discussion.

Attention had already been drawn to what he would call a certain untidiness about the basic conditions. When dealing with the method in a single-variable form, the use of $n$ terms required $2n$ conditions to be fulfilled. The first $2n$ moments could be taken, or some other criterion, and it was quite simple to take the approximation to the desired order. If two variables were being dealt with and it was decided to use $n$ points, there were $3n$ unknowns to find, but the number of moments up to the order $m$ was the sum of an arithmetical progression and equal to $m(m+1)/2$, so that the conditions available never seemed to fit the number of unknowns to be found. That gave rise to the necessity of introducing some arbitrary conditions. Perhaps it was not altogether a bad thing: even with the single-variable formulae it was found that the only practicable way of getting results was to throw overboard some of the desirable conditions and to introduce arbitrary conditions to make the algebra possible.

He thought there was much to be said for the author's comments in the first paragraph of his conclusion; the value of work on the lines of the paper lay in securing a mental picture of statistical principles. The method, in the case of a single variable, could be regarded as wrapping up an array of data into three or four separate packets, for each of which was found what was roughly a centre of gravity. The three or four centres of gravity were taken as a representation of the whole array. Incidentally, that method had certain similar characteristics to graduation. The distribution was to be represented, not by a smooth curve as in graduation but by a limited number of points. There was, however, a distinct difference. In graduation, the distribution was represented by a smooth curve and any point on that smooth curve should be a fair representation of the original or the ideal underlying the original data relating to that point. In the $n$-ages method, on the other hand, a limited number of points were selected with the object of taking those points together as a representation of the whole of the data. The method was of very little use, if any, as an indication of the value of the data at some particular point in the range.

He was interested to note Mr Joseph's suggestion that some of the arbitrary conditions that had to be used might well be in the form of choosing integral values for $x$ and $t$, because that did in some circumstances undoubtedly simplify the practical work. He had found at an early stage in dealing with single-variable formulae that, if he limited himself to three ages which were found very accurately, he had to interpolate for the values at those ages and in effect had to take six points. If four or five points could be found at integral ages to a similar degree of approximation, the final calculation based on those results would probably be very much easier.

**Mr N. J. Carter** remarked that the office with which he was concerned had had to make a quinquennial valuation at the end of 1941. A 3% net premium valuation had been made and it was desired to reduce the valuation rate of interest to strengthen the reserves. It was not possible to record new net premiums and the $n$-ages method was used for the six main classes—whole-life, limited-payment and endowment assurance, with and without profits.

As no constant of the nature suggested by the author was recorded on the cards and a double classification was not available, the ages had to be found by inspection of the distribution. The new net premiums were tabulated subsequently on the cards and a new valuation made. The result was that for those six main classes, with total reserves of nearly £4,500,000, the error by the $n$-ages method was an overstatement of less than £9000. An overstatement in the reserve had been expected owing to the application of the $n$-ages method to certain altered policies of the nature of children's deferred assurances which were included in the whole-life or endowment assurance classes at artificially reduced net premiums.

The valuation by the $n$-ages method had been accepted for the Board of Trade returns.

**Mr H. W. Haycocks** referred to Mr Lidstone's suggestion that the method might be called the '$n$-point method'. That was very similar to the name given to it by Tchebycheff. A book on mathematical probability by a Russian mathematician, J. V. Uspensky, contained an interesting appendix on the method of moments. According to Uspensky, Tchebycheff devised the $n$-point method in order to prove the famous Fundamental Limit Theorem of mathematical statistics. He considered a mass distributed over a given interval $(a, b)$. Then, he said, if the whole mass could be concentrated in a finite number of points so as to produce the same first $k$ moments as the given distribution, they had an 'equivalent point distribution' in respect of the first $k$ moments. Historically, therefore, some such name as that suggested by Mr Lidstone would be appropriate.

Reference had been made to the location of the points in relation to the elementary regression lines. It might be thought that the points could be on either or both of those lines. That was impossible if the polynomial contained the $x^2$ and $y^2$ terms. For instance, in the case of formula (4), there was no $y^2$ term and the points were located on one of the regression lines.

In statistical method it was difficult to define the ideal regression line when both variables were subject to error. In a sense the same problem would arise in the author's more general cases. Statisticians had solved the problem in practice by taking the line the slope of which was the mean of the slopes of the two elementary regression lines or the line which minimized the sum of the squares of the perpendiculars from the points to the line. The first of those lines had a slope equal to $\sigma_y/\sigma_x$ and it was interesting to note that formulae (2), (3), (5), (7), (9) and (10) had points on that line. The second of the lines was called the 'orthogonal regression line' and it was one of the axes of the ellipse mentioned on p. 389. If those axes were taken as the frame of reference, $r=0$ and all the four-term formulae reduced to the simple types (7) and (8). If the measurements were expressed in units of the standard deviations, the orthogonal regression line became a line through the origin with a slope of 45°, and thus became one of the lines AC and BD in the diagram on p. 379.

Some speakers had indicated that the essence of the method was not 'moments' but 'means'. That fact was clearly illustrated in the case of endowment assurances where the variable $1/n$ was used. The initial data were a set of weights $S_{x,t}$ and a corresponding set of variables ${}_tV_x$. The author required a set of $n$ weights at $n$ points such that both that set and the original set gave approximately the same mean value of ${}_tV_x$. In order to solve the problem, certain variables connected with ${}_tV_x$ were chosen. The author chose $x$, $x^2$, $t$, $t^2$ and $xt$ and a set of $n$ weights which reproduced the means of those variables. Since the weights reproduced exactly those mean values, it was argued that they would produce the mean value of ${}_tV_x$ approximately. It was clear that other variables might be used, e.g. $\log x$, $e^x$ or $1/x$, and then the conception of moments would hardly be used.

The method was strikingly similar to that of a sampling technique devised several years ago by Italian statisticians. As the next census date was approaching, it was desired to clear out the old data, in order to make room for the new, keeping a representative sample of the old data; the problem was the selection of such a sample. The data were subdivided into about 300 regions and it was thought that the best plan would be to choose 30 representative regions. What, however, constituted a representative region? Certain control variables were chosen, e.g. age of males, age of females, proportion occupied, etc. of which the mean for the population as a whole was known. A region was said to be representative if it had approximately the same set of means. The method failed, for, when tests were made on variables other than the controls, it was found that the sample mean was often significantly different from the population mean. They would be very suspicious of the author's results if he had used $n$ ages determined for valuation purposes in order to estimate, say, the total income of the policyholders!

It followed that there was some relevant connexion between the mean variable and the control variables. It was also possible to select an unsuitable set of weights for the problem in hand. An example was found in the author's first paper in which the approximate values of certain actuarial functions were obtained according to the $H^M$ table. The weights used to calculate $A_{\overset{1}{30:45:60}}$ were the deaths from age 30 to the end of the table. As the variable became zero at about age 64, the correct set of weights should have been the deaths from age 30 to age 64. Using a two-term formula with those weights, an approximate premium per death could be calculated which, when multiplied by $\sum_{30}^{64} d_x/l_{30}$, would give an estimate of the single premium required. The result obtained was much better than those obtained by Messrs Perks and Jones using third and higher moments. Further, over that range of deaths the standard deviation was very nearly equal to that of a rectangular distribution over the same range, so that a good approximate result could be obtained in a few minutes.

In conclusion, he wished to emphasize the descriptive nature of the method. Most problems of correlation and regression were concerned with the establishment of scientific hypotheses about the relationship between certain variables or factors, and standard deviations were used in the statistical testing of the hypotheses. The author's problem was not one of that nature.

**Mr R. H. Daw** pointed out that Table 2 showed formula (9) to give too high a value in every case when applied to King's model office. That suggested that a slight change in the statistical constants derived from the model office might improve the results. In the formula one of the constants used was the correlation coefficient, which involved the assumption of linear regression. Unless the regressions happened to be linear, the correlation ratio would appear to be more appropriate. Of course, its use would mean that the expression assumed for $f(x, y)$ would not be quite the same as in formula (9), but that did not seem to him to matter if better results were obtained.

The formulae for both the correlation coefficient and the correlation ratio could be put into the same form, that for the correlation coefficient ($r$) being

$$r^2 = 1 - \frac{S_y^2}{\sigma_y^2} = 1 - \frac{S_x^2}{\sigma_x^2},$$

where $S_x^2$ was the mean square of deviations from the regression line of $x$ on $y$. For the correlation ratio ($\eta$) the formulae were

$$\eta_{yx}^2 = 1 - \frac{S_y'^2}{\sigma_y^2} \quad \text{and} \quad \eta_{xy}^2 = 1 - \frac{S_x'^2}{\sigma_x^2},$$

where $S_x'^2$ was the mean square of deviations calculated from the mean ages $x$ corresponding to each duration $y$. In other words, the correlation coefficient was based on assumed mean ages for each duration (i.e. the regression line), while the correlation ratio was based on the actual mean ages for each duration.

In making an approximate valuation it was the actual distribution of policies that mattered (as in the correlation ratio) and not whether the mean ages lay on a straight line (as in the correlation coefficient). Both $r$ and $\eta$ varied in numerical value from zero to unity, and $\eta = | r |$ if the regression was linear; otherwise $\eta > | r |$. Also there were two values of $\eta$, based respectively on the mean ages and the mean durations.

Some calculations he had made showed the improvements when $r$ in formula (9) was replaced by $\eta$ and applied to King's model office, using the A 1924–29 ultimate 3 % table and calculating to the nearest hundred (see table below). The mean of the two values of $\eta$ was used and given the same sign as the corresponding value of $r$.

| Age of office | Exact valuation by the A 1924–29 ultimate table at 3 % | Error of formula (9) | |
|---|---|---|---|
| | | Using $r$ | Using $\eta$ |
| 10 | 79,379 | + 321 | + 121 |
| 30 | 443,753 | +2,547 | +1,547 |
| 50 | 695,819 | +3,281 | +2,681 |

In view of those results and the nature of the correlation ratio he would expect formula (9) always to give better results if $r$ were replaced by $\eta$. Possibly the same change would improve the results of other formulae, but he had not considered that question.

**Mr Hosking Tayler,** who closed the discussion in the absence of Mr Redington owing to indisposition, said he gathered that actuaries of his own generation found the paper difficult, and he thought it would be of some value to try to discern where the difficulty lay. By their training and practice they had formed the habit of thinking in terms of series and of their differences and differential coefficients, rather than in terms of frequency distributions and their moments, and perhaps unconsciously they sought for the point at which the new path of inquiry diverged from their habitual road. The author did not offer to take them that way. In statistical practice and analysis the method of moments was the habitual road.

Comment had been made on the title '$n$-ages method' and he thought they would agree that it was not very appropriate. To the actuary it conveyed no hint of the underlying principles of the method, and to others it was quite meaningless. The problem, as he saw it, was to establish relations between product-summations. That problem recurred again and again in their work. If product-summations were used to obtain an isolated value of a function, it was called 'interpolation'. If an ungraduated series of values was replaced by a complete set of interpolations it was called 'graduation by summation'. If the problem was to approximate to the area bounded by a curve, it was called 'quadrature', and if to approximate to other product-summations it was called the '$n$-ages method'. The principle involved in relating product-summation by moments was given in the footnote on p. 380. If the function $f(x)$ could be expanded by Taylor's theorem, the product-sum $u_x f(x)$ was equal to the product-sum $v_x f(x)$ if the sum and moments of $u_x$ were respectively equal to the sum and moments of $v_x$ about the same origin.

Should $f(x)$ not be linear, the product sums would be approximately equal if the equation of sums and moments extended to a given order of moments only. The relation would be true for more than one variable if the equation extended to the product moments. The ranges of the two summations, the numbers of terms and the intervals could all be unequal. Thus, if the expression 'product-sum' was extended to include the case where one factor of the products was unity throughout and also the case where a single term only was summed, the method supplied a kind of compendium of formulae for interpolation, graduation by summation, quadrature and approximate product-summation. In each case the problem was, given the sum and moments of one series or distribution, to find another series or distribution with the same sum and moments, which was suitable for the required purpose. It was usually possible to find an indefinite number of series or distributions with the same sums and moments up to a given order, and choice would be made by imposing conditions additional to the equation of sums and moments.

It might help to relate the finite-difference method with the method of moments if the finite-difference method were regarded as a means of selecting suitable series which did give equality of sums and moments. Suitability would not usually be unique, and choice of the most suitable would be made by consideration of the form or assumed form of $f(x)$ and of the characteristics of the distribution. It looked as if it ought to be possible to systematize the generation of series giving equality of sums and moments and to devise criteria to guide choice for particular purposes. The characteristics of a distribution were best expressed by functions of the moments, but it might happen that something more than was expressed in the moments available was known, at least approximately, about the characteristics of the distribution. For example, in a single-variable distribution, if the second moment only were known, there would be no measure of the skewness of the distribution. But if the range were known approximately, the magnitudes of the partial ranges on either side of the mean would give a measure of skewness, and that measure, although imperfect, could be introduced into the conditions added to the condition of equation of sums and moments and might lead to a better approximation than the use of conditions implicitly assuming a symmetrical distribution.

Additional information about the characteristics of the distribution could also be used in the form of an educated guess at the values of functions involving a higher order of moments. He had impelled the author to do some work on the former idea, and he hoped that the author would find opportunity and inclination to give publicity to what he had done. Actuaries had long been familiar with a simple but important example of product-summation by an auxiliary function in Lidstone's Z-method for valuing endowment assurances in groups. The author's method for valuing endowment assurances was a combination of product-summation by moments with product-summation by auxiliary functions, and he thought they should not miss the significance of what he had done.

To the statistician and actuary it might afford a reminder that failure in the search for the simplest law might indicate, not the need for one more parameter but for one different parameter, or, in other words, for a re-consideration of form. They were in an age of quickened tempo of change, and there could be little doubt that that would extend to valuation bases and methods. Many years ago the problem of valuing policies in groups was solved by including functions derived from the valuation basis in the group totals. It seemed not unlikely that that solution would fail them for the problems which lay ahead. The author had, he thought, performed a valuable service by showing them that still wider groupings could be established which, without involving any particular valuation basis, adequately described the groups for valuation and other purposes. In his earlier paper, the author had remarked that it was an unmitigated nuisance that groups which were adequately described for the purposes of valuation were not adequately described for the purpose of preparing the returns under the Fifth Schedule to the 1909 Act. It would be unfortunate if that Act were to block improvements in the technique of valuation, and he suggested that, when the Act was amended, the form of the valuation returns should be made a matter of agreement between the Board of Trade and each individual office, and that statutory forms should become obligatory only on failure to reach agreement.

**Mr Redington,** in a written communication, observed that the general $n$-ages problem had two distinct parts.

With a complicated, amorphous mass of data, the first part of the problem was to reduce a shapeless mass into a smaller number—say 6 or 10—of typical 'aspects', or attributes, or elements (as the author called them). That part of the problem he proposed to call 'reduction'.

The second part of the problem was, given those aspects, to make use of them in the practical work of valuation and so on. In general, they were not of direct use, and so the device was adopted of reconstructing from those aspects a skeleton body of data—a pocket model office—which had the same typical aspects as the original data. That part of the problem he proposed to call 'reconstruction'.

Thus the logical steps in the problem were (*a*) to reduce the data into compact typical aspects, and (*b*) to reconstruct from those aspects a skeleton working model.

Dealing first with reduction, the typical aspects which the author employed were mostly the common statistical constants, such as the first, second and third moments and product moments, but it removed a good deal of the mystery if it were pointed out that what the author did in fact was to value the original data, not with the usual factors such as ${}_tV_x$ and $A_{x+t}$, but with the much simpler factors $x$, $x^2$, $t$, $t^2$ and $xt$. Indeed, the typical aspects into which the author reduced his whole-life data in order to get formulae (9) to (12) were the six results obtained from valuing the data by the six factors

$$\text{unity}, \quad x \text{ (age at entry)}, \quad t \text{ (duration)}, \quad x^2, \quad t^2, \quad xt.$$

The question leaped to mind, 'Are these the six best aspects?' The answer was, 'Certainly not', if they were concerned solely with accuracy. He had tried experiments on the lines of Elderton and Rowell's original paper in which the aspects chosen were fewer than six in number; more accurate results were often produced than by the author's methods. Elderton and Rowell's aspects were the result of valuing the data on some normal valuation basis and were thus more typical aspects.

Nevertheless, the author's statistical aspects were very powerful—surprisingly so in view of their remoteness from the usual actuarial factors, which tended to be geometric in shape. They had great practical advantages. They were impersonal, as it were, being independent of any particular valuation basis. They were easily manipulated and they were supported by a great body of literature which facilitated developments. Finally, they led to relatively simple solutions of the second part of the problem, reconstruction.

He thought the author would agree with him that it was unwise lightly to dismiss other types of aspects. Indeed, the author showed his own independence of thought by using the factor $1/n$ for endowment assurances, which was outside the usual statistical line of succession.

Turning to the problem of reconstruction, they were faced with an unusual amount of arbitrariness. There was always an infinite number of $n$-age solutions which faithfully reproduced the given aspects or elements to which the original data had been reduced. It was worth emphasizing that what had been lost in reduction was lost for ever and could not be replaced during reconstruction. Formulae (9), (10), (11) and (12) all exactly reproduced the six given elements and only differed in accuracy to the extent that they accidentally happened to reproduce (or failed to reproduce) the unknown higher moments of the data which they were used to represent.

The whole subject of reconstruction was highly controversial and he thought the author had paid too much attention to it. His own attitude would be to concentrate on finding the most powerful reductions and to be content with any reconstruction which was faithful to that reduction.

He could not, however, dismiss the subject quite so lightly as that because many reductions did not lend themselves to any reconstruction at all, except by lengthy trial-and-error processes which had to be ruled out for routine work. For example, if the author were to tell him the pure premiums and reserves for a section of his business on three bases, say A 1924–29 $2\frac{1}{2}$%, A 1924–29 3% and $O^M$ 3%, he would have six tremendously powerful aspects of that business, but it would be extremely difficult to reconstruct a 'pocket' model office which reproduced those aspects. One of the great advantages of the author's statistical approach was that it led to fairly easy reconstruction.

He thought it worth mentioning to anyone who was dissatisfied with the polynomial statistical-moment approach and who wished to try geometrical elements that they might find the solution of some of their awkward geometrical equations facilitated by the use of nomograms.

The practical problem of operating the method was itself a subject for an interesting paper. The two extreme procedures were (*a*) to use a large number of constants and bulk classification, or (*b*) to use elaborate sub-classification and no constant. Between those formidable extremes there were more efficient compromises such as those which the author had used. The main advantage of the method was, however, not to be sought in simplicity of operation, nor was it to be found in its accuracy. The inaccuracies of the simpler formulae were very considerable and the improvement in accuracy by employing more complicated formulae had to be measured against the greater labour entailed. Nevertheless, he was inclined to agree with the author that the variation in the error from year to year was slight and was no more important than other small systematic errors which arose under well-established approximations.

There was one great advantage in the method—it was absolutely independent of the valuation basis. The author could value his business as easily on any one basis as another. The appraisal of the value of the method depended on the weight attached to that factor.

Summing up, he felt that the method had great advantages and still more possibilities. The reduction of masses of data into compact, orderly form was a gain in comprehension and technical mastery. The 'pocket' model office was another gain. The systematization of those two ideas in the *n*-ages method, as developed by the author and H. G. Jones, was a further step forward.

**The President (Mr R. C. Simmonds),** in proposing a vote of thanks to the author, recalled G. K. Chesterton's remark that a great many people who said that they agreed with Bernard Shaw did not understand him, whereas he (Chesterton) was the only one who understood Shaw and did not agree with him. Personally, not understanding much of Mr Perks's paper, he was quite prepared to say, for what it might be worth, that he agreed with it. The author had succeeded, with what might be termed malicious modesty, in knocking valuation formulae into such shape that even their best friends could not recognize them. He, the President, knew another actuary who could do that supremely well, using only ordinary algebra and the formula for the office premium for a whole-life policy!

Seriously, he wished to emphasize two points made by the author. It was wholly admirable that he laid emphasis on the need for informed judgment as one of the chief functions of the actuary; the judge had to be efficiently judicious. That was a professional responsibility and one which should be discharged to the utmost of their power; they hoped to do it even better in the future than in the past. The other point was the great importance of emancipating the valuation process from any clog imposed by classification. It was greatly to be desired that the actuary who needed or wished to have alternative valuation results for consideration should not be deterred by any question relating to consequential changes of classification.

It was not surprising that the author had not rested content with his earlier achievement. How could he rest content? Animated by the spirit of research, he would surely go on towards the unattained and unattainable summit. They wished the author well and hoped that he would not be alone in his quest.

**Mr Perks,** in reply, said that he would confine his remarks to three points which had arisen during the discussion.

The opener referred to a process on p. 386 where a formula for the 50-year model office had been used in which a particular constant, $k$, was chosen to obtain an exact result, and that formula had been applied with the same value of $k$ to the 30-year model office.

It was suggested that that course had the appearance of jobbing backwards. In an arithmetical problem such as that, he (the speaker) recognized no time direction and he hoped that he was absolved from the suggestion.

Mr Joseph approached the problem from a different viewpoint, and, as he understood it, suggested the use of the Henry process in two dimensions. That, he thought, would involve the two-variable graduation of every factor entering into valuation. When Mr Joseph compared the results it was not satisfactory to say that mathematically the same order was being used in the second-order Henry process as in the second-order *n*-ages process. In fact, in the Henry process moments were taken both of the factors and of the distribution of business, whereas in the *n*-ages method moments were taken only of the distribution of business. He considered that the second-order Henry process should properly be compared with the fourth-order *n*-ages method.

With reference to Mr Carter's remarks, he considered that they should distinguish between the *n*-ages method of Elderton and Rowell and what would, he hoped, be called either 'approximate product-summation', as suggested by Mr Hosking Tayler, or the '*n*-point method', as suggested by Mr Lidstone and Mr Haycocks. There was a great deal of difference between them. The original method assumed that the results on one valuation basis were known and those results were used to pass to the results on another valuation basis. The *n*-point method did not require knowledge of the results on any basis. The results on any desired basis were computed direct from the classification data arranged in the appropriate form.

**Mr Perks** has subsequently written as follows:

The changes in notation in the paper between original units and standard units, to which Mr Tetley referred, were not made entirely without warning. The initial work is in original units. The change to standard units is specifically referred to in the middle of p. 380 and the return to original units in setting out formula (10) is so obvious and consistent with previous formulae that reference to it seemed hardly necessary. However, Mr Tetley's warning sufficiently repairs the fault. His references to the need for a punched-card system in using the method in practice go too far. The method can quite well be used with a written-card system but punched cards are a great facility. He is probably right in suggesting that more up-to-date machines, such as the multiplying punch and, possibly, tabulators with grand totalling and transfer devices, would facilitate the use of the method. The whole subject of the mechanical computation of moments would repay study.

If Mr Tetley had been in a large office in which the completion of the annual valuation by the end of January was the peace-time custom, he would have appreciated the need in war-time for attention to every means by which the work at the time of the actual valuation could be reduced, without the classification work being increased throughout the year. Experience had shown that the original year-of-entry method did not attain the advantage of flexibility without some increase in valuation work compared with the orthodox methods. The two-variable method has more than offset this increase.

The denominators of the expressions for $r_{tn}$ and $r_{t/n}$ in the paper are, as Mr Tetley said, of considerable interest and it may be worth while to refer to some of the properties of the 'statistics' $\bar{n}\sqrt{(1-1/\bar{n}\underline{n})}$ and $\underline{n}\sqrt{(1-1/\bar{n}\underline{n})}$. Provided that the origin of measurement of $n$ is outside the range of variation, the harmonic mean of the $n$-distribution can readily be expressed in series in terms of the successive moments in $n$, namely,

$$\underline{n}=\{1+\mu_2/\bar{n}^2-\mu_3/\bar{n}^3+\mu_4/\bar{n}^4-\ldots\}/\bar{n}.$$

By inversion

$$1/\bar{n}\underline{n}=1-\mu_2/\bar{n}^2+\mu_3/\bar{n}^3-\mu_4/\bar{n}^4+\mu_2^2/\bar{n}^4-\ldots,$$

and from this may be obtained the following interesting expansions:

$$\bar{n}\sqrt{(1-1/\bar{n}\underline{n})}=\sigma\sqrt{(1-\mu_3/\sigma^2\bar{n}+\mu_4/\sigma^2\bar{n}^2-\sigma^2/\bar{n}^2+\ldots)} \quad \text{and}$$

$$\underline{n}\sqrt{(1-1/\bar{n}\underline{n})}=\frac{\sigma}{\bar{n}^2}\sqrt{\left(1-\frac{\mu_3}{\sigma^2\bar{n}}+\frac{\mu_4}{\sigma^3\bar{n}^2}+\frac{\sigma^2}{\bar{n}^2}-\ldots\right)}.$$

If the variations round $\bar{n}$ are small in relation to $\bar{n}$, the harmonic mean $1/\underline{n} \doteq \bar{n}(1-\sigma^2/\bar{n}^2)$, as given in *Advanced Statistics* by Kendall, p. 47. In the same conditions, $\bar{n}\sqrt{(1-/1\bar{n}\underline{n})}$ is a close approximation to $\sigma$ and $\sqrt{(1-1/\bar{n}\underline{n})}$ is a kind of coefficient of variation.

If $x$ is measured from the origin of a Type III distribution, the expression $\bar{x}\sqrt{(1-1/\bar{x}\underline{x})}$ is actually equal to $\sigma$. In the case of a Type I distribution (i.e. $x^m(1-x)^n$), we have

$$\bar{x}\sqrt{(1-1/\bar{x}\underline{x})}=\sigma\sqrt{[(m+n+3)/(m+n+1)]}.$$

I am grateful to Mr Haycocks for drawing my attention to the appendix in Uspensky's book. The expression in the appendix 'equivalent $n$-point distribution' clearly indicates that a very apt short name for the method described in the paper would be '$n$-point method', as suggested by Mr Lidstone. The appendix also shows that the method of moments is much more fundamental to statistical theory than would appear from Mr Bailey's reference to least squares. The limit theorems, characteristic functions and, in particular, Fisher's conception that the successive moments (or cumulants) extract successively diminishing 'amounts of information' from statistical data are all far more significant for the method of moments than for the method of least squares.

While appreciating, perhaps more clearly than others, the imperfections of the paper, I do not think that Mr Joseph's very valuable comments 'overcome technical imperfections' of the paper. In bringing to the fore the Lagrange type of formulae, he has performed a very useful service. Although I have given a little attention to these in an elementary way, I have the same dislike for them as for Lagrange interpolation. In practical valuation work the freedom to choose integral ages would actually be an embarrassment. Complete systematization is invaluable and this is also the answer to that part of Mr Redington's remarks about 'aspects' other than moments and to his being content with 'any reconstruction'. In their pure form Lagrange $n$-point formulae always require more terms, for the same order, than the formulae which we may, by analogy, call the Gaussian type. Mr Joseph's illustration of the nine-term formula must have involved a certain amount of trial to achieve a near agreement for the $x^3$ and $y^3$ elements. Even then his result is not greatly superior to the simple second-order four-term formula (9). It may be supposed that, without this 'rigging' to satisfy $x^3$ and $y^3$, his nine-term pure Lagrange type would have produced results justifying the remark in the paper that formulae involving some but not all of the third-order elements often do not show improvement over the second-order formulae. I agree with him that the second-order formulae do not work so well with age attained as the variable in place of duration. Mr Lacey's work provides confirmation of this feature as well as valuable contributions to a number of points left undeveloped in the paper. Consideration of the neglected higher-order elements in the Taylor expansions throws a certain light on the relative value of using duration or attained age, and also throws doubt on the logic of Mr Joseph's comment that, because the transformation of a second-degree function in $x$ and $y$ to a function in $(x+y)$ and $(x-y)$ produces a function still of the second degree, formula (9) 'ought to apply equally well' to functions in $(x+y)$ and $(x-y)$ as to functions in $x$ and $y$ when neither are of the second degree.

His remarks that, in a particular application, four terms might not be sufficient are particularly appropriate if the table on which the factors are based is not completely smooth, e.g. the A 1924–29 table, but are less relevant to a smooth table like the $O^M$. On the general question of the errors of the method, it is important to distinguish between the absolute error and the change in error from valuation to valuation. The former can be overcome on the lines indicated in the paper by inserting a term at the mean age and mean duration, to secure as close an agreement as may be desired between the reserve by the full method hitherto used and the reserve by the approximate method at the date of its adoption. It is thought that the paper gives sufficient reason and evidence for supposing that the changes in the error thereafter would be small enough to be neglected. For those who are concerned about the size of the absolute error, it is sufficient to refer them to Fraser's paper, *J.I.A.* Vol. XXXVIII, p. 385.

What Mr Joseph refers to as the 'orthogonal polynomial formulae' are really the Henry process applied to the two-variable problem. If I understand the matter correctly, the object of the use of orthogonal polynomials is to facilitate the analysis in producing appropriate formulae which could, more laboriously, be produced by ordinary algebraic process, and there appears to be nothing fundamental about the use of this technique. The additional work in graduating all the factors which is required in this Henry method is hardly justified by the small improvement in the error which Mr Joseph's illustrations show. Moreover, his reference to the moments used should have made it clear that he has used the moments of the factors as well as the moments of the distribution.

Mr Joseph's reference to the possibilities of a triple classification is of considerable interest. In this way the use of valuation constants is reduced to a minimum and an office with a punched-card system for its renewals and other office records might well find it convenient to use these cards for its valuation. A triple classification on a punched-card system is by no means as cumbersome as it might appear.

The substitution of the mean of the two correlation ratios for the correlation coefficient, as suggested by Mr Daw, seems to be quite an arbitrary process. At least, Mr Daw does not provide any analytic support for it. Since the ratio is never numerically less than the coefficient, there seems to be no reason to suppose that it would generally produce better results. In valuation work the calculation of the two correlation ratios would be a very laborious matter.

Mr Hosking Tayler referred to the possible use of the partial ranges to fix approximately the ratio of $\sigma_1$ to $\sigma_2$ in Jones's third-order formula, when the third moment is not known. This is a useful idea and a study of the skew triangle and of Type I distributions can be made to yield useful approximate rules on these lines. These rules and the wider subject of approximations to the higher moments in using $n$-point formulae merit more attention than can be given to them here.

As Mr Redington says, all kinds of 'aspects' could be used, but in practical work we are conditioned by the need for reasonable systematic 'reconstructions'. For example, we could work with $y=c^x$ and $z=v^t$, taking moments in $y$ and $z$ and using the ordinary formulae; or we could write $n=w-x$ and apply the endowment assurance formulae to whole-life assurances. The choice of $c$, $v$ and $w$ would be arbitrary and experiments so far have not yielded improved results. Further investigation on these lines would, however, seem to be well worth while.

The remarks made by a number of speakers to whom I have made no reference and by others to whom my reference has been brief throw much light on the method. My silence is explained by the fact that there is no matter on which I desire to differ from them.