TFA 40 (1985-1987) 247-248



# ACTUARIAL NOTE

*A clarification of the definition of "duration", or "length", in traditional immunisation theory*

*by*

A. D. SHEDDEN, B.Sc., F.F.A., F.S.A.

A recent paper by Fen (1) distinguishes two measures of "duration" which occur in immunisation theory but which could be confused. These he denotes as Macaulay Duration and Modified Duration, defined* as follows:

Let P($i$) be the present value of an asset at rate of interest $i$, so that

$$P(i) = \Sigma B_t (1 + i)^{-t}$$

where $B_t$ is the cash flow to be received in $t$ years.

Then, Macaulay Duration is

$$\frac{(1+i)}{P} \cdot \frac{dP}{di} = \frac{\Sigma t B_t (1+i)^{-t}}{\Sigma B_t (1+i)^{-t}}$$

and Modified Duration is

$$\frac{1}{P} \cdot \frac{dP}{di} = \frac{\Sigma t B_t (1+i)^{-t-1}}{\Sigma B_t (1+i)^{-t}}$$

Obviously,

$$\text{Macaulay Duration} = (1 + i) \times \text{Modified Duration.}$$

As Fen points out, Macaulay Duration is a measure of the price sensitivity to proportional changes in $(1 + i)$ whereas Modified Duration deals with absolute changes in $i$ or $(1 + i)$.

The formula given above for the Macaulay Duration, so-called after Frederick Macaulay (2), is familiar to U.K. actuaries as being the weighted mean present value expression derived by Redington (3). On the other hand, the Modified Duration formula appears more commonly in the literature of Finance, where price sensitivity is usually measured in terms of proportional changes in price for a unit change in interest rate. It is this latter measure of price sensitivity that is referred to in my paper on immunisation (4) as denoting the "length" or "volatility" of an asset or liability. The theory in this paper was developed in terms of $\delta$, (i.e. on a continuous basis), although it was applied in terms of $i$ when preparing the worked examples contained therein. The extension of the Redingtonian theory which

*For convenience I have taken interest and cash flow to be on an annual basis. Fen developed the formulae on an nthly basis, where n is a fraction of a year.*

the paper dealt with did not constrain the elements of asset income or liability outgo to be invariant with changes in $\delta$ or $i$, but it was pointed out that if these constraints did apply then the usual immunisation relationships emerged: in particular, P′/P would measure the mean term of the value of the asset proceeds or liability outgo.

For clarification, it should be pointed out that this statement is only true where differentiation is with respect to $\delta$, because Macaulay Duration and Modified Duration are then identical. Where differentiation is with respect to $i$, however, P′/P represents the Modified Duration rather than the more familiar Macaulay or Redingtonian Duration. Indeed, later on in the paper the "length" of a single premium pure endowment is given as $n/(1 + i)$, where $n$ is the term, whereas the weighted mean present value of the endowment's outgo is $n$. This discrepancy was noted by Wilkie in his discussion of the paper, wherein he argued for differentiation with respect to $\delta$ rather than $i$.

## BIBLIOGRAPHY

1. Fen, A. M. "Interest Rate Futures: An Alternative to Traditional Immunisation in the Financial Management of Guaranteed Investment Contracts". *T.S.A.*, **37**, 65.
2. Macaulay, F. R. "Some Theoretical Problems Suggested by the Movements of Interest Rates, Bond Yields, and Stock Prices in the United States since 1856." New York: National Bureau of Economic Research, 1938.
3. Redington, F. M. "Review of the Principles of Life Office Valuation." *J.I.A.*, **78**, 296.
4. Shedden, A. D. "A Practical Approach to Applying Immunisation Theory." *T.F.A.*, **35**, 313.