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THE VALUATION OF LAST-SURVIVOR ANNUITIES

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In recent years the number of last-survivor annuities on two lives, male and female, in some offices has grown so large that their valuation individually has become impracticable. It is the purpose of this note to record the results of some experiments in a simple method of approximate valuation which permits the annuities to be grouped according to the age of one of the partners or, possibly, according to their mean age. The annuities in each age-group are then valued by the factor appropriate to that age and the average age of the other life. The problem may be considered quite simply from first principles.

2. The valuation factor for last-survivor annuities on two lives assumed to be subject to different mortality experiences is a function $f(x, y)$ of the two variables x and y, the ages of the male and female annuitants respectively. Let $y=x+t$, so that t is the age-disparity between the two lives and let $u_{x:x+t}$ be the corresponding amount of annuity to be valued. Then, assuming $f(x, x+t)$ can be expanded by Taylor's theorem in a convergent series, with x constant and t varying, and writing $f'(x, x+k)$ for $\left[\frac{\partial}{\partial t}f(x, x+t)\right]_{t=k}$ and similarly for higher derivatives,

$$\begin{aligned}\sum_t u_{x:x+t}f(x, x+t)=&\left[\sum_t u_{x:x+t}\right]f(x, x+k)+\left[\sum_t (t-k)u_{x:x+t}\right]f'(x, x+k)\\ &+\frac{1}{2!}\left[\sum_t (t-k)^2\, u_{x:x+t}\right]f''(x, x+k)\\ &+\frac{1}{3!}\left[\sum_t (t-k)^3\, u_{x:x+t}\right]f'''(x, x+k)\\ &+\ldots\ldots\ldots\ldots, \end{aligned}\qquad (1)$$

where k is an arbitrary constant. If we put $k=\sum_t tu_{x:x+t}\Big/\sum_t u_{x:x+t}=\bar{t}$, say, the weighted age-disparity, the second term of the expansion is zero and we can write

$$\begin{aligned}\sum_t u_{x:x+t}f(x, x+t)=&\left[\sum_t u_{x:x+t}\right][f(x, x+\bar{t})+m_2 f''(x, x+\bar{t})/2!\\ &+m_3 f'''(x, x+\bar{t})/3!+m_4 f^{\text{iv}}(x, x+\bar{t})/4!+\ldots],\end{aligned}\qquad (2)$$

where m_2, m_3, etc., are the second, third, etc., moments, measured from the mean, of the distribution $u_{x:x+t}$ with respect to t.

3. If all terms after the first on the right-hand side of (2) could be ignored a valuation of the annuities could be made by grouping according to the age of one of the lives, e.g. the male, and valuing the total annuities in each group

by the valuation factor $f(x, x+\bar{t})$. The factor would depend upon the age of the male and the mean age-disparity in the group.

4. The error introduced into the valuation by this procedure naturally depends upon the magnitude of the successive derivatives of $f(x, x+t)$, for $t=\bar{t}$, after the first and upon the moments of $u_{x:x+t}$.

5. The values of the derivatives of the valuation factor can be calculated by methods of finite differences. The present investigation has been confined to the $a(f)$ and $a(m)$ ultimate tables and $f(x, x+t)=a_{\overline{x:x+t}}$ (male and female) at $3\frac{1}{2}\%$ has been examined. Table 1 gives the estimated values of $f''(x, x-3)/2!$ and $f'''(x, x-3)/3!$ when the age of the male is fixed whilst the age-disparity, and therefore the age of the female, varies. A mean age-disparity of three years has been assumed, the female on average being the younger. Values of $f^{\mathrm{iv}}(x, x-3)/4!$ calculated from tabulated annuity values were too small to be measured accurately, but seem to be of the order of 10^{-6}.

Table 1. Coefficients of the moments in formula (2), age of male fixed

Age of male x	$a_{\overline{x:x-3}}$	$f''(x, x-3)/2!$	$f'''(x, x-3)/3!$
52	18·458	−·0001	+·00007
61	15·517	+·0008	+·00011
70	11·992	+·0028	+·00015
79	8·194	+·0061	+·00015
88	4·887	+·0084	−·00013

6. Table 2 gives the corresponding values from the same table when the age of the female is fixed whilst the age-disparity varies. The same mean age-disparity has been assumed. Here also values of $f^{\mathrm{iv}}(y+3, y)/4!$ were too small to be measured accurately.

Table 2. Coefficients of the moments in formula (2), age of female fixed

Age of female y	$a_{\overline{y+3:y}}$	$f''(y+3, y)/2!$	$f'''(y+3, y)/3!$
49	18·458	+·0021	−·00004
58	15·517	+·0029	−·00001
67	11·992	+·0039	−·00001
76	8·194	+·0052	+·00012
85	4·887	+·0042	+·00006

7. Finally, formula (2) could be used, with appropriate modification, with the mean age of male and female fixed and the age-disparity varying as before. Some care has to be taken with the variable if a valid comparison is to be made with the figures in Tables 1 and 2. Since both ages are now varying Table 3 gives the values of $f''(z+\frac{3}{2}, z-\frac{3}{2})/2!$ and $f'''(z+\frac{3}{2}, z-\frac{3}{2})/3!$ with the age-disparity as the variable. The mean age-disparity is the same as for Tables 1 and 2, the male again being the older.

8. The numerical values in the last two columns of Tables 1–3 are, of course, derivatives of the curves formed by the respective intersections of three planes with the surface $f(x, y)$, multiplied by $1/2!$ and $1/3!$ respectively.

9. If an approximate valuation is performed using only the first term of formula (2), each of the three applications under consideration will give different errors depending upon the distribution of $u_{x:y}$ according to the ages of male and female. The next step, therefore, is to consider the likely magnitude of the moments of the age-disparity of the following distributions of amounts of annuity, $u_{x:y}$:

(i) age of male fixed, age-disparity varying (classification M);
(ii) age of female fixed, age-disparity varying (classification F);
(iii) mean age of male and female fixed, age-disparity varying (classification M+F).

Table 3. Coefficients of the moments in formula (2), mean age of male and female fixed

Mean age z	$a_{\overline{z+\frac{3}{2}:z-\frac{3}{2}}}$	$f''(z+\frac{3}{2}, z-\frac{3}{2})/2!$	$f'''(z+\frac{3}{2}, z-\frac{3}{2})/3!$
$50\frac{1}{2}$	18·458	+·0019	+·00003
$59\frac{1}{2}$	15·517	+·0028	+·00004
$68\frac{1}{2}$	11·992	+·0039	+·00005
$77\frac{1}{2}$	8·194	+·0048	+·00007
$86\frac{1}{2}$	4·887	+·0045	−·00000

Table 4. Summary of valuation data showing the number and amounts of annuities in force on the valuation date

Age-group (ages last birthday)	Primary classification according to					
	Age of male (M)		Age of female (F)		Mean age (M+F)	
	Number of annuities	Amount of annuities	Number of annuities	Amount of annuities	Number of annuities	Amount of annuities
		£		£		£
−59	24	4,644	56	8,162	36	6,079
60–69	94	10,922	70	9,304	92	11,690
70–79	70	11,753	65	9,861	66	10,072
80–89	18	3,037	15	3,029	12	2,514
All ages	206	30,355	206	30,355	206	30,355

The valuation data of a particular office were investigated and consisted of 206 annuities, each payable during the joint lifetime of a male and a female life and during the lifetime of the survivor. There were no 'duplicates' and the individual annuities ranged in amount from £11 4*s.* to £1132 10*s.* Ages last birthday were recorded; the youngest annuitant was a female aged 34 (paired with a male aged 42) and the oldest a female aged 89 (paired with a male aged 83). It is known that a large proportion of the annuitants were in fact husband and wife. A summary of the valuation data in groups and according to primary classification is given in Table 4. Table 5 shows the distributions of numbers and amounts of annuities by age-disparity for all groups combined. Because of the somewhat limited extent of the data the moments of age-disparity were calculated according to age-groups of the primary classifications. For example, in classification M, males were combined in four groups 40–59, 60–69, 70–79 and 80–89, and the statistics of age-disparity obtained for each group. The results are shown in Table 6, in which the weights are numbers and amounts of annuities respectively.

Table 5. Distributions of numbers and amounts of annuities by age-disparity for all groups combined

Age-disparity	Number of annuities	Amount of annuities £ s. d.	Age-disparity	Number of annuities	Amount of annuities £ s. d.
−30	1	49 16 0	−3	12	1461 19 4
−27	1	47 13 0	−2	16	2217 7 0
−23	1	95 4 0	−1	14	2114 5 8
−22	1	78 0 0	0	15	2620 1 10
−20	1	140 0 0	1	16	2376 16 0
−17	2	147 3 0	2	14	3393 5 0
−16	1	27 0 0	3	6	754 1 0
−14	2	317 2 0	4	9	1710 14 0
−13	1	112 0 6	5	5	621 4 0
−12	6	480 6 8	6	3	702 8 0
−11	3	540 0 0	7	1	109 10 0
−10	3	374 5 0	8	1	29 15 0
−9	7	1404 5 0	9	2	353 14 0
−8	11	1077 9 8	10	1	171 14 0
−7	11	1499 1 8	11	1	249 0 0
−6	5	609 7 0	14	2	148 14 0
−5	16	2035 12 0	24	1	81 2 0
−4	14	2205 12 4			

10. The data analysed are insufficient to draw any firm conclusions about the statistics summarized in Table 6. But these statistics are not inconsistent with what would be expected of a sample consisting mainly of married couples at the older ages. Because of the effect of mortality upon older partners and of the tendency of marriages to couple males with wives who are younger than themselves—particularly if they marry late in life—the general trends of the mean age-disparity (m_1') in the three classifications are as would be expected. The usual statistical tests, in fact, support the view that the correlation of age-disparity with age is significant for each classification. The expected trends of m_2 are less certain but might well increase with age in classification M and decrease in classification F, for the same reasons. There is a suggestion that the values of m_2 are smaller in classification F, the average value being about three-quarters of the average value in either of the other classifications. Tests support the view that this difference is significant.

11. The values of the moments for a particular body of data could be associated with the corresponding coefficients in formula (2), and it would be a simple matter to estimate on theoretical grounds the error of the approximate method of valuation suggested. However, an inspection of Tables 1–5 suggests that the contribution to error of the third and fourth terms in formula (2) is not likely to be of much importance; the main error is obviously caused by ignoring the term $m_2 f''(x, x+\bar{t})/2!$—as is frequently found in other methods of approximate valuation.

Table 6. Moments of the distributions of age-disparity classified according to age of male, age of female and mean age, and weighted by numbers and amounts of annuities respectively

Age-group (ages last birthday)	Weighted by numbers of annuities						Weighted by amounts of annuities					
	m_1'	m_2	m_3	m_4	b_1	b_2	m_1'	m_2	m_3	m_4	b_1	b_2
						Primary classification M						
-59	−1·0	24	91	2,616	·62	4·7	−1·4	14	17	1,029	·10	5·0
60–69	−3·1	47	76	11,531	·06	5·3	−2·6	39	11	8,729	·00	5·8
70–79	−1·7	30	− 27	2,693	·03	3·0	−1·2	34	− 69	3,074	·12	2·7
80–89	−7·5	84	−996	27,155	1·67	3·8	−3·8	44	−542	16,052	3·45	8·3
All ages	−2·8	44	−137	11,630	·22	5·9	−2·0	34	− 89	6,567	·19	5·6
						Primary classification F						
-59	−8·1	55	−296	10,551	·53	3·5	−5·8	41	−268	7,048	1·06	4·3
60–69	−2·7	15	45	1,459	·55	6·2	−3·1	17	1	1,064	·00	3·7
70–79	− ·1	20	25	1,290	·08	3·2	·2	17	3	921	·00	3·0
80–89	4·9	48	343	10,255	1·06	4·4	4·3	28	185	4,722	1·50	5·9
All ages	−2·8	44	−137	11,630	·22	5·9	−2·0	34	− 89	6,567	·19	5·6
						Primary classification M+F						
-59	−4·9	51	− 90	9,590	·06	3·6	−3·6	32	−179	5,583	·96	5·4
60–69	−4·0	37	−156	7,130	·48	5·2	−3·6	34	− 67	4,733	·11	4·0
70–79	− ·6	43	−152	18,008	·29	9·6	− ·0	28	− 22	7,727	·02	9·5
80–89	·0	24	56	1,312	·22	2·2	1·5	18	1	834	·00	2·6
All ages	−2·8	44	−137	11,630	·22	5·9	−2·0	34	− 89	6,567	·19	5·6

Notes. (1) Age-disparity in all groups has been taken as age of female *minus* age of male.
(2) $b_1 = m_3^2/m_2^3$; $b_2 = m_4/m_2^2$.

12. The sets of values of m_2 according to age differ markedly between the three different classifications under discussion for the particular data examined, and this will usually be so in practice. Partly because the sets of coefficients of m_2 in Tables 1–3 are all different and partly because the values of m_2 will, in general, differ according to classification, one of the three classifications will be superior for a particular body of valuation data.

13. It has been suggested in § 10 that the average value and trend of m_2 in Table 6 might be different according to classification. Now the number of lives in each individual age-group is not likely to be very large in practice. Moreover, an unknown proportion of annuities are payable during the lifetimes of other than married couples. This heterogeneity, the weighting by amounts of annuity and the limited extent of the data at each age, will result in sampling fluctuations of m_2 which will usually be considerable compared with any differences in the average value and trend of m_2 according to classification. It needs to be stressed that we are, of course, concerned, not with underlying 'parametric' values of m_2, but with the actual values found in the sample of annuities which have to be valued. Having regard to the magnitude of these fluctuations in m_2 it is doubtful whether much more can be said than that the actual values of m_2 at each age are of roughly the same order in each classification, but that there is reason to believe that, on average, the values are rather smaller in classification F.

14. The next step is to compare the coefficients of m_2 in Tables 1, 2 and 3. There is little to choose between the sets of values in Tables 2 and 3. There would be a marked advantage in the coefficients of Table 1 (by adopting classification M) if the bulk of the annuities were confined to ages below 75 (male)—as is common in practice.

15. The average values of m_2, weighted by annuities, for the various age-groups in Table 6 are (M) 34, (F) 25 and (M+F) 31. Now if m_2 had a value of 50 at each age in the primary classification—a severe assumption—the total error would be about $1\frac{1}{2}\%$ of the valuation liability for each classification, assuming an even distribution of annuities between ages 50 and 90. If most of the annuities were at the younger ages there would be a marked reduction in the error for classification M but little change for the others. It is indeed unlikely that classification M would, in practice, lead to an error of as much as $1\frac{1}{4}\%$.

16. On the other hand, if m_2 itself is smaller on average for classification F there would be a correspondingly smaller total error by adopting that classification. It seems unlikely that the total error here would, in practice, be as much as $1\frac{1}{4}\%$.

17. The conclusion is that either classification M or F is likely to give the best result, in the one case because of the more favourable coefficients and in the other because of the rather smaller values of m_2. Both appear superior to classification M+F.

18. An objection to the method of approximation is that it undervalues the true liability. This is because the coefficients of m_2 in Tables 1, 2 and 3 are nearly all positive and m_2 itself is necessarily positive. It is interesting to recall that in A. E. King's method of approximate valuation (*J.I.A.* XLVIII, 121), which resembles the method under discussion, a second difference correction was introduced to deal with this sort of difficulty. More recently, the n-point method was directed to the same end by putting $n=2$.

19. An obvious means of reducing the error of approximation would be to reduce the values of the second central moments by splitting the data into two parts, according to whether the older life is male or female, and valuing the two parts separately. Strictly the greatest reduction in m_2 can be effected by splitting the data about the mean age-disparity; it would involve a division according to whether the female was younger or older than the age of the male less, say, three years. But a division depending on the sex of the older life has the appeal of practical simplicity and the difference in the effect on m_2 is trivial. Pairs of equal ages can be regarded as falling in the 'older female' group. Table 7 shows the effect upon the moments of the distributions of age-disparity of classifying the annuities in two parts in this way.

Table 7. Moments of the distributions of age-disparity (weighted by amounts of annuities) classified according to age of male, when the data are subdivided according to the sex of the older life

Age-group (ages last birthday)	m_1'	m_2	m_3	m_4	b_1	b_2
			OLDER MALES			
-59	−4·1	6	− 9	78	·28	1·9
60–69	−5·9	21	−142	2,408	2·11	5·3
70–79	−6·1	16	− 33	650	·27	2·6
80–89	−6·8	38	−630	13,509	7·07	9·2
All ages	−5·8	20	−158	3,132	3·19	8·0
			OLDER FEMALES			
-59	1·8	5	51	621	20·22	24·5
60–69	3·1	18	217	4,236	8·74	13·8
70–79	3·3	7	27	249	1·69	4·4
80–89	1·7	7	18	99	·93	2·0
All ages	2·9	11	91	1,562	7·08	14·1
			ALL POLICIES			
-59	−1·4	14	17	1,029	·10	5·0
60–69	−2·6	39	11	8,729	·00	5·8
70–79	−1·2	34	− 69	3,074	·13	2·7
80–89	−3·8	44	−542	16,052	3·45	8·3
All ages	−2·0	34	− 89	6,567	·20	5·6

Note. Age-disparity in all groups has been taken as age of female *minus* age of male.

20. The average values of m_2 in Table 7 are 'older males' 19, 'older females' 10, 'all policies' 34. The marked skewness of the two classifications can be seen in the values of m_3, which are consistently negative and positive respectively, and in the values of b_1.

21. The data having been split in this way it is necessary to decide whether each body of data should be valued by classifying by age of male, age of female, or mean age. It is interesting therefore to compare the appropriate coefficients of m_2 in formula (2). The figures are given in Table 8; the first column of coefficients in each table is reproduced from Tables 1, 2 and 3. The second and third columns give the coefficients of m_2 for 'older males'

and 'older females' respectively. For 'older males' a mean age-disparity of six years has been assumed, and for 'older females' three years. For 'older males' (column 2) the most favourable set of coefficients appears under classification M, bearing in mind that the bulk of annuities fall between ages 60 and 80. For 'older females' (column 3) the most favourable set appears under classification F. The suggestion is that if the data are split

Table 8. Coefficients of the second moment in formula (2)

CLASSIFICATION M

Age of male x	$f''(x, x-3)/2!$	$f''(x, x-6)/2!$	$f''(x, x+3)/2!$
	(1)	(2)	(3)
52	$-\cdot 0001$	$-\cdot 0008$	$+\cdot 0013$
61	$+\cdot 0008$	$-\cdot 0003$	$+\cdot 0028$
70	$+\cdot 0028$	$+\cdot 0013$	$+\cdot 0054$
79	$+\cdot 0061$	$+\cdot 0044$	$+\cdot 0084$
88	$+\cdot 0084$	$+\cdot 0091$	$+\cdot 0055$

CLASSIFICATION F

Age of female y	$f''(y+3, y)/2!$	$f''(y+6, y)/2!$	$f''(y-3, y)/2!$
	(1)	(2)	(3)
49	$+\cdot 0021$	$+\cdot 0021$	$+\cdot 0011$
58	$+\cdot 0029$	$+\cdot 0029$	$+\cdot 0027$
67	$+\cdot 0039$	$+\cdot 0041$	$+\cdot 0036$
76	$+\cdot 0052$	$+\cdot 0040$	$+\cdot 0054$
85	$+\cdot 0042$	$+\cdot 0037$	$+\cdot 0050$

CLASSIFICATION M+F

Mean age z	$f''(z+\frac{3}{2}, z-\frac{3}{2})/2!$	$f''(z+3, z-3)/2!$	$f''(z-\frac{3}{2}, z+\frac{3}{2})/2!$
	(1)	(2)	(3)
$50\frac{1}{2}$	$+\cdot 0019$	$+\cdot 0015$	$+\cdot 0022$
$59\frac{1}{2}$	$+\cdot 0028$	$+\cdot 0023$	$+\cdot 0034$
$68\frac{1}{2}$	$+\cdot 0039$	$+\cdot 0033$	$+\cdot 0046$
$77\frac{1}{2}$	$+\cdot 0048$	$+\cdot 0041$	$+\cdot 0058$
$86\frac{1}{2}$	$+\cdot 0045$	$+\cdot 0044$	$+\cdot 0042$

according to the sex of the older life, it would be best to adopt classification M for the 'older males' and classification F for the 'older females'; compared with a single classification this procedure effectively reduces the coefficients of the second moments.

22. When considering the single classifications it was concluded that the contribution to error of the third and higher moments could be ignored. Splitting the data into two classifications normally produces positive and negative values of m_3 for the two distributions respectively—as may be seen in Table 7. If these values of m_3 are combined with the appropriate coefficients in formula (2) the contribution to error from this source is no longer negligible and may increase or decrease the total error depending upon the classifications used for the 'older males' and 'older females' respectively. Coefficients of m_3 are shown in Table 9. It so happens that if classification M is used for

the 'older males' the contribution to error from m_3 offsets the contribution from m_2 and might even extinguish it altogether. The same happens if classification F is used for the 'older females'. The combined effect of this fortunate result, the different coefficients of m_2 and the reduction in the values of m_2 brought about by classifying in two parts is to reduce considerably the expected total errors of approximation. These become about one-third or less of the errors that can be expected from the single classifications M or F; it is therefore unlikely in practice that a classification in two parts would lead to an error in the total liability of more than ·4% and, as will be seen later, the actual figure is likely to be much less than that.

Table 9. Coefficients of the third moment in formula (2)

CLASSIFICATION M

Age of male x	$f'''(x, x-3)/3!$	$f'''(x, x-6)/3!$	$f'''(x, x+3)/3!$
	(1)	(2)	(3)
52	+·00007	+·00006	+·00008
61	+·00011	+·00010	+·00010
70	+·00015	+·00016	+·00010
79	+·00015	+·00020	−·00001
88	−·00013	+·00008	−·00016

CLASSIFICATION F

Age of female y	$f'''(y+3, y)/3!$	$f'''(y+6, y)/3!$	$f'''(y-3, y)/3!$
	(1)	(2)	(3)
49	−·00004	+·00000	−·00005
58	−·00001	+·00001	−·00004
67	−·00001	−·00001	−·00004
76	+·00012	+·00010	−·00012
85	+·00006	+·00006	+·00009

CLASSIFICATION M+F

Mean age z	$f'''(z+\frac{3}{2}, z-\frac{3}{2})/3!$	$f'''(z+3, z-3)/3!$	$f'''(z-\frac{3}{2}, z+\frac{3}{2})/3!$
	(1)	(2)	(3)
$50\frac{1}{2}$	+·00003	+·00003	+·00001
$59\frac{1}{2}$	+·00004	+·00005	+·00002
$68\frac{1}{2}$	+·00005	+·00006	+·00001
$77\frac{1}{2}$	+·00007	+·00007	+·00001
$86\frac{1}{2}$	−·00000	+·00002	−·00003

23. The method of approximation using a single classification only can, of course, be regarded as a single-point application of the n-point method. To classify the annuities in two parts and to value 'older males' and 'older females' separately is little different in principle from a two-point application. Considering the single classification M, for example, good results could be expected by valuing the annuities at each age of male by the mean of the two factors $a_{\overline{x:x+\bar{t}_x-s_x}}$ and $a_{\overline{x:x+\bar{t}_x+s_x}}$, where $\bar{t}_x$ and s_x are the mean and standard deviation of the age-disparity for age of male x. This procedure is obviously

not greatly different from splitting the annuities in that group according to whether the age-disparity is more or less than $\bar{t}_x$ and valuing the two sub-groups separately. As was pointed out in § 19 it is more convenient to split the annuities according to whether the female is younger or older than the male but the principle is the same. It seems to follow that the improvement to be expected from a classification in two parts over a single classification method will represent a large proportion of the improvement of a two-point over a single-point application of the n-point method.

24. The comparison between the coefficients in Tables 1, 2, 3, 8 and 9 by the $a(f)$ and $a(m)$ ultimate table at the other rates of interest is similar and the conclusions reached are much the same. The percentage error in the total liability increases slightly as the rate of interest decreases; for example, at a rate of interest of $2\frac{1}{2}\%$ the total error of $1\frac{1}{2}\%$ mentioned at the beginning of § 15 would be increased to about $1\frac{3}{4}\%$ of the liability, the other estimates being correspondingly affected.

25. The foregoing discussion will now be illustrated by a number of valuations of the data referred to in § 9. First, a 'true' valuation of the annuities was made by the $a(f)$ and $a(m)$ ultimate tables at $3\frac{1}{2}\%$. Values of $a_{\overline{xy}}$ not tabulated were interpolated from three values using second differences.* The annuities were each multiplied by $a_{\overline{xy}}$ and the result expressed to the nearest pound. The total liability amounted to £379,169.

26. Each annuity was recorded on a card together with a constant equal to the product of the age-disparity and the annuity. The cards were then sorted and the data tabulated for the primary classifications M and F, at each year of age, and M+F, at each half year of age. A mean age-disparity was calculated for each age-group to three decimal places and the total annuities of the age-group valued by the corresponding annuity factor obtained by a process of linear interpolation in the published table of $a_{\overline{xy}}$. The valuation results are given in Table 10.

27. In general, the value of m_2 at a particular age in the primary classification tends to reduce as the number of annuities at that age becomes smaller and it becomes zero if there is only one annuity in the group. The approximate valuations were made by classifying the data according to individual ages in the primary classification and since there were only 206 annuities altogether the number at each age was sometimes quite small, there being only one annuity at some ages. The result has been to produce values of m_2 smaller, on average, than the values found by grouping several ages together as in Tables 5 and 6. The smallness of the errors is therefore partly due to the limited number of annuities valued. Such good results could not be expected with more extensive valuation data because, for that reason alone, the values of m_2 would tend to be greater. The unexpectedly favourable result produced by classification M+F is due to the effect of considerable subdivision of limited data upon the values of m_2. There were very nearly twice as many valuation age-groups for this classification as for either of the others. If the data had been more extensive or if the grouping had been carried out to the nearest integral mean year of age, instead of the nearest half year of age, the error for classification M+F would not have been so flattering compared with the others.

* Linear interpolation for intermediate values of $a_{\overline{xy}}$ would have led to an error comparable with some of those under discussion (see Table 10).

28. Another contribution to the favourable results is the method of linear interpolation used in the approximate valuations to obtain $a_{\overline{x:x+t}}$. This tends to overstate the annuity values and therefore to offset slightly the principal error of approximation.

29. The results in Table 10 for the classification in two parts are remarkably good even allowing for the favourable effect created solely by the limited quantity of data valued. The total error of £6 is comparable with the total of rounding-off errors and is less than one-fortieth of the error that would have been obtained by a valuation of the individual annuities if untabulated values of $a_{\overline{xy}}$ had been obtained by linear interpolation between pairs of the published values.

Table 10. Valuation results by different methods

True valuation liability on the basis of $a(f)$ and $a(m)$ $3\frac{1}{2}\%$ ult. £379,169

Method	Classification	Approximate valuation liability £	Error compared with true liability £	%
Individual valuation using linear interpolation for untabulated values of $a_{\overline{xy}}$	—	379,441	+ 272	+·07
Single classification	M	377,179	−1990	−·52
Single classification	F	377,816	−1353	−·36
Single classification	M+F	377,716	−1453	−·38
Classification in two parts	'Older males' M	*230,337*	+ *39*	+·*02*
	'Older females' F	*148,826*	− *45*	−·*03*
		379,163	− 6	−·00
'Elphinstone and Lindsay'	—	379,073	− 96	−·03

30. In using these approximate methods there is no doubt that there would be an appreciable saving in work if interpolation for the values of $a_{\overline{x:y+r}}$, r fractional, could be avoided. The error introduced by using nearest integral ages for each value is not likely to be important in practice because errors would tend to compensate. The standard error in age due to rounding off the mean age-disparity to the nearest integer would evidently be about ·29 year. Tests show that the effect on the total liability expressed as a standard error would be of the order of ·15% for usual distributions of annuities.

31. An alternative method of avoiding the interpolations would be to make two valuations using the respective factors $a_{\overline{x:x+k}}$ and $a_{\overline{x:x+k+1}}$, where k is an integer such that $k<t<k+1$ and t is the mean age-disparity for all annuities (−2·00440 years for the single classifications). This method gives a total error for classification M of −·65% (compared with −·52% in Table 10). Applied to a classification in two parts the method gives a total error of −·37% (compared with −·00% in Table 10) which is not entirely satisfactory.

32. To complete the experiments a valuation was made of the same annuities by one of the methods given by M. D. W. Elphinstone and W. G. P. Lindsay ('The valuation of joint life and survivorship annuities', *T.F.A.* XVII, 39). By this method, each annuity is divided into two parts by multiplying by the factors α and β, where α and β depend upon the age-disparity. The first part is valued by the factor $a_{\overline{xx}}$ and the second part by $a_{\overline{yy}}$ where x and y are the ages of male and female respectively. The method therefore involves two classifications, one for each of the respective parts of the annuities. There are practical advantages in the authors' alternative suggestion of using the factors α' and $1-\alpha'$ because the two parts are then equal in total to the original annuity. Specimen values of α' by $a(f)$ and $a(m)$ ultimate at $3\frac{1}{2}\%$ are as follows:

Age-disparity (age of female *minus* age of male)	α'
−30	·214
−20	·265
−10	·343
0	·447
10	·565
20	·662
30	·740

The total error shown in Table 10 for this particular valuation is also smaller than that for a valuation of the individual annuities if untabulated values of $a_{\overline{xy}}$ are obtained by linear interpolation between pairs of the published values.

33. In conclusion, some of the practical considerations in valuation by the average age-disparity method may be mentioned. Classification M+F has about twice as many natural age-groups as either classification M or F. They can, of course, be reduced to about the same number but there is reason to believe that the total error will then be rather larger.

34. The method introduces a valuation constant, but it is a very simple one and no prepared tables are necessary. Moreover, the mean age-disparity for age-groups of several annuities changes little from year to year and this helps in checking the calculations. The method is quite independent of the valuation basis except that the error in the total liability alters slightly if the basis changes. It might in practice be worth while to make a small addition to the estimated total liability to correct for the known bias of the method, e.g. $\frac{1}{2}\%$ if a single classification is adopted.

35. Results which are very nearly accurate can be obtained by adopting separate classifications according to the sex of the older life and the additional work seems well worth while. The error in the total valuation liability can be regarded as negligible for all practical purposes.

36. No consideration has been given in this note to the valuation of last-survivor annuities on the lives of two persons of the same sex or on more than two lives. The number of these annuities on the books is usually so small that individual calculations are suitable.

37. The experimental valuations were made before the publication of the last-survivor annuity values on the basis of the *a*(55) mortality tables. These values, at $3\frac{1}{2}\%$, have since been seen and the figures in Tables 1, 2, 3, 8 and 9 have been compared with the corresponding figures on that basis. The *a*(55) coefficients of m_2 are usually a little greater but the differences are negligible: the trends of the figures are remarkably similar. The coefficients of m_3 are almost identical on the two bases. The conclusions reached in this note are not affected in any way and the estimates of error would be little affected. There is no reason to believe that the errors of the actual valuations made would be materially different if the *a*(55) tables had been used.

38. No originality can be claimed for the methods of approximation discussed in this note; they are simple applications of familiar actuarial technique. Mr W. Perks, F.I.A., first suggested to me the n-point method of solving this problem using a single-point application. K. K. Weatherhead, M.A., F.F.A., and R. G. Deas ('An approximation to the value of the reversionary annuity $a_{x+t}-a_{x:x+t}$ with consequent simplifications in the valuation of last-survivor annuities', *T.F.A.* XV, 422) suggested a direct application of a formula similar to formula (1) using three terms and therefore two valuation constants. Their formula introduces $\Delta f(x, x+k)$ and $\Delta^2 f(x, x+k)$ which must therefore be extracted and tabulated. The method suggested in the present note may be regarded as a modification of 'Weatherhead and Deas' which avoids the use of the tabulated differences. I am indebted to Messrs B. P. Pain, M.A., F.I.A., W. T. L. Barnard, F.I.A., and W. J. Goshawke, F.I.A., for carrying out the valuations of the annuities, and to Mr G. I. White, B.A., for his help with the valuation statistics and coefficients.