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THE VALUATION OF SICKNESS BENEFITS FOR NON-STANDARD AND STANDARD PERIODS

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1. *Introduction*

SICKNESS tables for non-standard periods are often encountered in Friendly Society practice—more often than may generally be expected. For example, in a substantial Order recently valued by the authors, the first fifty-six lodges were found to be using altogether forty-two such tables.

How to treat these benefits actuarially is quite a formidable problem, for no valuation factors are available for non-standard periods and none of the substitutes in general use is entirely satisfactory. For instance, the actuary can either use valuation factors for suitably chosen standard periods and make a rough over-all adjustment or he can turn to tables, such as those prepared by Rhodes,* and toil through the construction of the appropriate commutation columns. The first method is of doubtful accuracy and is very sensitive to variations in the age distribution, while the second involves an amount of labour which is generally quite unacceptable.

The method suggested in this paper combines the advantages of the two alternatives. It depends upon the use of linear combinations of standard sickness rates and valuation factors combined with an over-all adjustment which is calculated mathematically. The method is equally applicable to non-standard and standard periods. For the former it provides probably the most satisfactory method available and it is suggested as a suitable standard method. For the latter, since a simple and exact method of valuation is already available, it provides a method of approximate valuation which is sufficiently accurate for use in much Friendly Society work, and a technique which may sometimes be found valuable for other purposes.

2. *The underlying idea*

The idea behind the method is simple enough. It is that sickness claims may be regarded as of two kinds: short-term claims for acute sickness from which recovery is rapid, and long-term claims for sickness that gets better slowly or not at all. The former are predominant at the younger ages; the latter become more important with advancing age.

The paper demonstrates how sickness rates can be represented in mathematical terms by a linear combination of two decreasing exponential functions of duration. One exponential represents the acute short-term sickness; the other, having a longer time-constant, typifies the chronic or semi-chronic sickness.

After investigating how the time-constants of the exponentials vary with age when the law is fitted exactly to the tabular rates at four durations at each age, the paper then shows that two values of the time-constants may be selected and

* *Percentage Table for the Estimation of Sickness Rates for Special Periods of Sickness*, by Francis Rhodes, M.A., F.I.A., *J.I.A.* Vol. LXXII, pp. 455–69.

applied to all ages in such a way that the loss of accuracy in a practical valuation is small.

This latter assumption, of time-constants which do not vary with age, facilitates the development of a method which is the same for both non-standard and standard benefit periods and which applies alike to the experience investigation and the valuation calculations. The method entails, first, a multiplication by factors which vary only with the valuation basis and, secondly, the transformation of the resulting figures to the benefit periods required. The factors are either tabulated factors or easily obtained linear combinations of them derived directly from the published valuation tables. The transformation to the required benefit periods implies, theoretically the evaluation of a determinant, but in practice only the extraction of a few figures from prepared tables similar to the samples included in this paper.

The exponential law is not regarded by the authors as having any profound biological significance. All that is claimed for it is that it fits the facts reasonably well and results in a useful and powerful actuarial technique.

3. *Some definitions*

It is necessary in this paper to have the basic sickness symbols carefully defined, and it is convenient to adapt the accepted notation slightly for the special purposes in view. Hence:

${}_r\mathbf{z}_x$ is defined as the central sickness rate in weeks per year for all sickness of more than r weeks' duration claimed by persons between the xth and $x+1$th birthdays.

The usual notation is related to the special notation of this paper by relations of which the following are typical:

$$z_x^{13} = {}_0\mathbf{z}_x - {}_{13}\mathbf{z}_x \text{ (first thirteen weeks' sickness)},$$

$$z_x^{13/13} = {}_{13}\mathbf{z}_x - {}_{26}\mathbf{z}_x \text{ (second thirteen weeks' sickness)}.$$

Some of the formulae and ideas which follow are more simply expressed in terms of the ratios of the sickness rates for different durations. Hence we define

$${}_r\zeta_x = {}_r\mathbf{z}_x / {}_0\mathbf{z}_x, \quad = Z^{r/all} / Z^{all} \qquad (1)$$

and for convenience we shall sometimes refer to these quantities ${}_r\zeta_x$ as 'normalized sickness rates'. They represent the proportion of the total sickness at each age which occurs after duration r; as a result ${}_0\zeta_x = 1$ for all values of x and for all sickness tables.

4. *The observed variation with r*

Suppose the normalized rates ${}_{13}\zeta$ are found for all ages for the occupation group A.H.J. of the Manchester Unity 1893–97 tables and plotted against age, and a smooth curve drawn through the points. Suppose, further, that the process is repeated in turn for the other occupation groups B.C.D. and E.F. Three distinct curves will result. If ${}_{26}\zeta$, ${}_{52}\zeta$ and ${}_{104}\zeta$ are similarly treated, a further three sets of three distinct curves will be produced, one set for each benefit period.

Now, for the value of ${}_{26}\zeta$ at each age in the A.H.J. table there is a corresponding value of ${}_{13}\zeta$. If each of these pairs of values is plotted as a point $({}_{26}\zeta, {}_{13}\zeta)$ the result will be a graph of ${}_{13}\zeta$ against ${}_{26}\zeta$. If the same is done for the other two occupation groups, it will be found that the three graphs lie very close together. In fact, it is found that all the points $({}_{26}\zeta, {}_{13}\zeta)$ lie so closely about a curve drawn through the middle of them that, if a value of ${}_{26}\zeta$ is chosen for any age and occupation group, the corresponding value of ${}_{13}\zeta$ can be estimated from the one curve with little loss of accuracy.

The same thing will be found to be true when ${}_{52}\zeta$ and ${}_{104}\zeta$ are plotted against ${}_{26}\zeta$. The three curves are shown in Fig. 1.

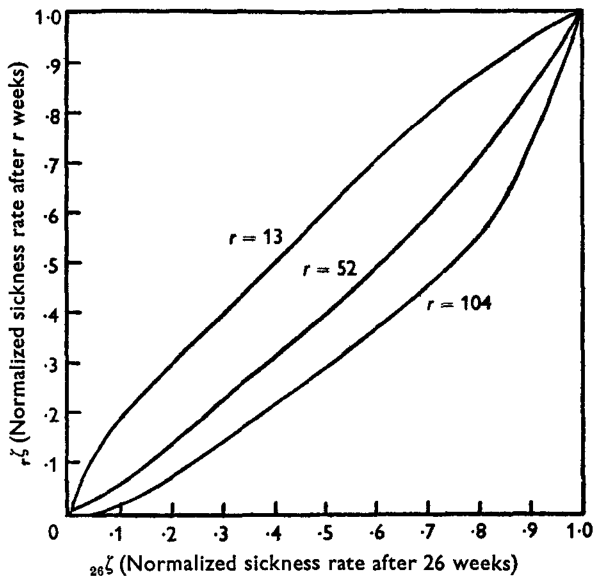

Fig 1. ${}_r\zeta$ against ${}_{26}\zeta$ (Manchester Unity, all groups)

By preparing the graph in this way we have represented the other normalized rates as curves with ${}_{26}\zeta$ as argument, and in the graph neither the age nor the occupation group occurs explicitly.

It is, of course, not possible to plot the corresponding curves for other values of r, since the figures are not tabulated. It is, however, reasonable to suppose that, if they were, the curves would fall into their proper places in a family of which those plotted are particular examples.

It follows that once ${}_{26}\zeta$ is known for a given age and table, other values of ${}_r\zeta$ are determinable from the same curves irrespective of age or sickness table. Hence once ${}_0\mathbf{z}$ and ${}_{26}\mathbf{z}$ are known all other sickness rates may be determined.

This may be conveniently illustrated by extracting from Fig. 1 the values of ${}_{13}\zeta$, ${}_{52}\zeta$ and ${}_{104}\zeta$ corresponding to ${}_{26}\zeta = \cdot 25$, $\cdot 50$ and $\cdot 75$, and then plotting them as graphs of ${}_r\zeta$ against r. The results when smooth curves are drawn through the plotted points are shown in Fig. 2.

It is, of course, impossible to provide more than the five points on each curve, but it is to be noted that the curves 'look like' decreasing exponential curves and that, so far, is encouraging.

We may notice, in passing, that Rhodes's tables, already referred to, and the curves of Fig. 2, do exactly the same thing and are really equivalent. The discussion in this section might in fact be applied with equal force to Rhodes's work.

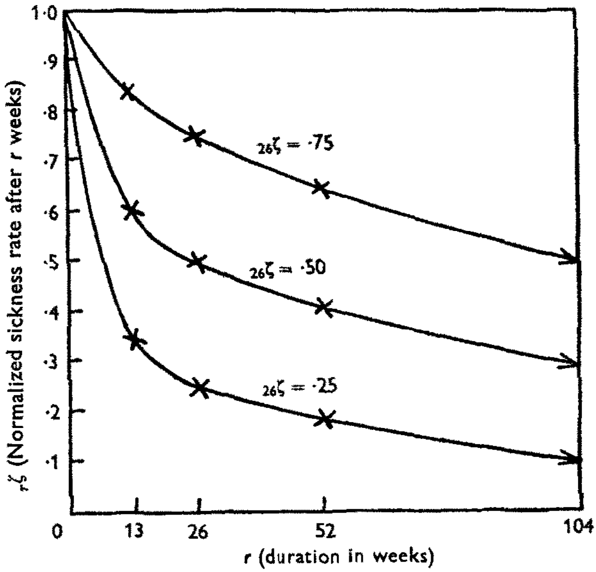

Fig. 2. $_r\zeta$ against r (Manchester Unity, all groups)

5. *Mathematical law*

Let it now be assumed that

$$_r\mathbf{z}_x = F_x e^{-k_1 r} + G_x e^{-k_2 r}, \tag{2}$$

where $_r\mathbf{z}_x$ has already been defined (§3),

F_x and G_x vary with age x but are independent of r,

k_1 and k_2 may (in the first instance) vary with age x but are independent of r.

This equation, which appears later in a slightly different form, is the fundamental equation on which most of the discussion in this paper is based.

The problem is now one of seeing how closely this formula can be fitted to the facts, first with full freedom of variation of F_x, G_x, k_1 and k_2 with age, and secondly with k_1 and k_2 suitably chosen but independent of age.

The immediate problem is thus to determine for each age values of k_1, k_2, F_x and G_x such that the law fits the data as well as may be. The sickness rates available at each age—after suitable linear combination—are $_0\mathbf{z}_x$, $_{13}\mathbf{z}_x$, $_{26}\mathbf{z}_x$, $_{52}\mathbf{z}_x$ and $_{104}\mathbf{z}_x$, and it is to these that the law must be fitted.

6. *First evaluation of the constants*

For the first fitting $_0\mathbf{z}_x$, $_{13}\mathbf{z}_x$, $_{26}\mathbf{z}_x$ and $_{52}\mathbf{z}_x$ will be used, and it will be shown how the four constants can be obtained on the basis that the law fits these four points exactly at each age.

The four equations, dropping the suffix, to be solved at each age are:

$$ {}_0\mathbf{z}=F+G,\ {}_{13}\mathbf{z}=Fe^{-13k_1}+Ge^{-13k_2},\ {}_{26}\mathbf{z}=Fe^{-26k_1}+Ge^{-26k_2},\ {}_{52}\mathbf{z}=Fe^{-52k_1}+Ge^{-52k_2}. \quad (3)$$

Writing $\alpha=e^{-k_1}$ and $\beta=e^{-k_2}$ we have

$$\left.\begin{aligned} {}_0\mathbf{z}&=F+G, & {}_{13}\mathbf{z}&=F\alpha^{13}+G\beta^{13},\\ {}_{26}\mathbf{z}&=F\alpha^{26}+G\beta^{26}, & {}_{52}\mathbf{z}&=F\alpha^{52}+G\beta^{52}, \end{aligned}\right\} \quad (4)$$

whence by simple substitution and algebraic transformation

$$\begin{aligned} {}_0\mathbf{z}\,{}_{26}\mathbf{z}-{}_{13}\mathbf{z}^2&=(F+G)(F\alpha^{26}+G\beta^{26})-(F\alpha^{13}+G\beta^{13})^2\\ &=FG(\alpha^{13}-\beta^{13})^2. \end{aligned} \quad (5)$$

Similarly $${}_0\mathbf{z}\,{}_{52}\mathbf{z}-{}_{26}\mathbf{z}^2=FG(\alpha^{26}-\beta^{26})^2. \quad (6)$$

Dividing (6) by (5) and transposing

$$(\alpha^{13}+\beta^{13})^2=\frac{{}_0\mathbf{z}\,{}_{52}\mathbf{z}-{}_{26}\mathbf{z}^2}{{}_0\mathbf{z}\,{}_{26}\mathbf{z}-{}_{13}\mathbf{z}^2}, \quad (7)$$

also $$\begin{aligned}(\alpha^{13}+\beta^{13})\,{}_{13}\mathbf{z}&=(\alpha^{13}+\beta^{13})(F\alpha^{13}+G\beta^{13})\\ &=F\alpha^{26}+G\beta^{26}+\alpha^{13}\beta^{13}(F+G)\\ &={}_{26}\mathbf{z}+\alpha^{13}\beta^{13}\,{}_0\mathbf{z},\end{aligned}$$

whence $$\begin{aligned}\alpha^{13}\beta^{13}&=\frac{{}_{13}\mathbf{z}}{{}_0\mathbf{z}}(\alpha^{13}+\beta^{13})-\frac{{}_{26}\mathbf{z}}{{}_0\mathbf{z}}\\ &=\pm\frac{{}_{13}\mathbf{z}}{{}_0\mathbf{z}}\left\{\frac{{}_0\mathbf{z}\,{}_{52}\mathbf{z}-{}_{26}\mathbf{z}^2}{{}_0\mathbf{z}\,{}_{26}\mathbf{z}-{}_{13}\mathbf{z}^2}\right\}^{\frac{1}{2}}-\frac{{}_{26}\mathbf{z}}{{}_0\mathbf{z}}.\end{aligned} \quad (8)$$

The identity $$(\alpha^{13}-\beta^{13})^2\equiv(\alpha^{13}+\beta^{13})^2-4\alpha^{13}\beta^{13}$$

combined with equations (7) and (8) gives

$$\alpha^{13}-\beta^{13}=\pm\left\{\frac{{}_0\mathbf{z}\,{}_{52}\mathbf{z}-{}_{26}\mathbf{z}^2}{{}_0\mathbf{z}\,{}_{26}\mathbf{z}-{}_{13}\mathbf{z}^2}-4\left[\pm\frac{{}_{13}\mathbf{z}}{{}_0\mathbf{z}}\left(\frac{{}_0\mathbf{z}\,{}_{52}\mathbf{z}-{}_{26}\mathbf{z}^2}{{}_0\mathbf{z}\,{}_{26}\mathbf{z}-{}_{13}\mathbf{z}^2}\right)^{\frac{1}{2}}-\frac{{}_{26}\mathbf{z}}{{}_0\mathbf{z}}\right]\right\}^{\frac{1}{2}} \quad (9)$$

and $$\alpha^{13}+\beta^{13}=\pm\left\{\frac{{}_0\mathbf{z}\,{}_{52}\mathbf{z}-{}_{26}\mathbf{z}^2}{{}_0\mathbf{z}\,{}_{26}\mathbf{z}-{}_{13}\mathbf{z}^2}\right\}^{\frac{1}{2}}, \quad (10)$$

from which α^{13} and β^{13} are obtainable and F and G follow immediately by substitution in any two of the equations.

Transformed into 'normalized' sickness rates, ${}_r\zeta$, the solutions to these equations simplify slightly to the following form:

$$\alpha^{13}-\beta^{13}=\pm\left\{\frac{{}_{52}\zeta-{}_{26}\zeta^2}{{}_{26}\zeta-{}_{13}\zeta^2}-4\left[\pm{}_{13}\zeta\left(\frac{{}_{52}\zeta-{}_{26}\zeta^2}{{}_{26}\zeta-{}_{13}\zeta^2}\right)^{\frac{1}{2}}-{}_{26}\zeta\right]\right\}^{\frac{1}{2}}, \quad (11)$$

$$\alpha^{13}+\beta^{13}=\pm\left(\frac{{}_{52}\zeta-{}_{26}\zeta^2}{{}_{26}\zeta-{}_{13}\zeta^2}\right)^{\frac{1}{2}}. \quad (12)$$

Since α and β are both positive, and on the convention that $\alpha>\beta$, only the positive signs in the above equations will apply.

If the fundamental equations for ${}_0\mathbf{z}$, ${}_{26}\mathbf{z}$, ${}_{52}\mathbf{z}$ and ${}_{104}\mathbf{z}$ are used instead of those for ${}_0\mathbf{z}$, ${}_{13}\mathbf{z}$, ${}_{26}\mathbf{z}$ and ${}_{52}\mathbf{z}$, the same procedure will give values α^{26} and β^{26} instead of α^{13} and β^{13}.

7. *Variation of α and β with age*

Using equations (11) and (12) values of α^{13} and β^{13} are obtained which give an exact fit for the four values of ${}_r\mathbf{z}$ chosen, and using this procedure we will now examine the variations of α^{13} and β^{13} with age.

For this purpose we tabulated ${}_0\mathbf{z}$, ${}_{13}\mathbf{z}$, ${}_{26}\mathbf{z}$, ${}_{52}\mathbf{z}$ and ${}_{104}\mathbf{z}$ for selected ages in the Manchester Unity (Whole Society) table and found values of α^{13} and β^{13} from the above equations, working first from ${}_0\mathbf{z}$, ${}_{13}\mathbf{z}$, ${}_{26}\mathbf{z}$ and ${}_{52}\mathbf{z}$ to give columns (2) and (3) in Table 1 and then from ${}_0\mathbf{z}$, ${}_{26}\mathbf{z}$, ${}_{52}\mathbf{z}$ and ${}_{104}\mathbf{z}$ to give columns (4) and (5).

It may be noticed from an examination of Table 1 that, except at the extreme ages, α^{13} and β^{13} do not vary much with age. The values derived from ${}_0\mathbf{z}$, ${}_{13}\mathbf{z}$, ${}_{26}\mathbf{z}$ and ${}_{52}\mathbf{z}$, however, tend to be less than the values derived from ${}_0\mathbf{z}$, ${}_{26}\mathbf{z}$, ${}_{52}\mathbf{z}$ and ${}_{104}\mathbf{z}$, and this suggests, what is later confirmed, that the law when operated with values of α and β derived in this way tends to underestimate the sickness at the shorter durations. For many purposes this is a serious shortcoming for which compensating adjustment of α and β must be made. An example of this will be given later.

Table 1

Age	Based on ${}_0\mathbf{z}$, ${}_{13}\mathbf{z}$, ${}_{26}\mathbf{z}$, ${}_{52}\mathbf{z}$		Based on ${}_0\mathbf{z}$, ${}_{26}\mathbf{z}$, ${}_{52}\mathbf{z}$, ${}_{104}\mathbf{z}$	
	α^{13}	β^{13}	α^{13}	β^{13}
(1)	(2)	(3)	(4)	(5)
20	·60	·05	·65	·12
30	·81	·09	·87	·20
40	·86	·11	·92	·26
50	·90	·17	·93	·30
60	·91	·22	·94	·38
70	·94	·29	·95	·41
80	·97	·46	·97	·35
90	·99	·83	·98	·52

8. *Values of α and β independent of age*

The hope of using the law with reasonable arithmetical facility rests chiefly on the possibility of obtaining useful results with values of α and β, suitably chosen but independent of age. We proceed now to the discussion of such possibilities.

In the fundamental equations (4) the rates are completely defined by four quantities at each age. If therefore α and β are fixed independently of age all the rates are completely determined (according to the law) by two rates at each age. In other words, given α and β, we obtain a two-point fit at each age.

For our present purposes we have used five pairs of values of α and β, one pair a reasonable choice and four pairs somewhat extreme. We have then fitted the law using ${}_0\zeta$ (identically equal to 1) and ${}_{26}\zeta$ equal in turn to ·25, ·50 and ·75. We have chosen these values arbitrarily to represent the sickness at three ages well separated in the standard sickness table.

The results are shown in Table 2. The calculated values of ${}_r\zeta$ for $r = 13$, 52 and 104 are shown for each value of ${}_{26}\zeta$ and each pair of values of α^{13} and β^{13}, and compared with the tabular values obtained by entering the standard tables with the ages (shown in the table) at which ${}_{26}\zeta = $ ·25, ·50 and ·75 respectively. As we shall subsequently confirm, the variations when α and β are reasonably chosen are not so great as to prevent the results from being useful.

Table 2

${}_{26}\zeta$	Manchester Unity (Whole Society) Age	r	Manchester Unity (Whole Society) tabular values of ${}_r\zeta$	$\alpha^{13}=\cdot 99$ $\beta^{13}=\cdot 01$	$\cdot 99$ $\cdot 49$	$\cdot 97$ $\cdot 37$	$\cdot 75$ $\cdot 01$	$\cdot 75$ $\cdot 49$
				Calculated values of ${}_r\zeta$				
·25	35	13	·36	·26	·50	·46	·34	·50
		26	·25	·25	·25	·25	·25	·25
		52	·17	·25	·07	·14	·14	·07
		104	·11	·24	·02	·11	·05	·01
·50	54	13	·61	·51	·67	·64	·67	·70
		26	·50	·50	·50	·50	·50	·50
		52	·40	·49	·38	·41	·28	·27
		104	·30	·47	·32	·34	·09	·08
·75	67	13	·84	·76	·84	·83	1·00	·90
		26	·75	·75	·75	·75	·75	·75
		52	·64	·74	·68	·68	·42	·47
		104	·49	·71	·63	·60	·13	·16

9. *Sickness rates for non-standard periods*

The advantage of the mathematical law lies very largely in the fact that it provides a simple, convenient method of dealing precisely with periods of sickness not tabulated in the standard tables. It is important, therefore, to see how rates for non-standard periods (particularly at the shorter durations), derived from the application of the law, compare with the results of other work on the subject.

There appear to be two important methods of estimating sickness rates at non-standard periods. They are the method described in the White Paper Cd. 6907,* and the extensive tables in Rhodes's paper already referred to.

In Figs. 3–5 we have plotted against duration the sickness rates obtained by these different methods using the tabulated rates at ages 25, 50 and 70 from the Manchester Unity (Whole Society) table. Ages 25 and 70 have been chosen to give a large contrast in the shapes of the curves, while age 50 has been included to show the comparison between the methods at an intermediate age.

In all three diagrams the curve representing the method of this paper is based on values of α and β derived from ${}_0\mathbf{z}$, ${}_{13}\mathbf{z}$, ${}_{26}\mathbf{z}$ and ${}_{52}\mathbf{z}$ for the relevant age, that is to say the fit is exact at four points. The rates derived from the law in this way are significantly lower than those derived by other methods. In Fig. 3, therefore, an additional curve has been drawn, based on values of α and β chosen to emphasize the short-term sickness and to agree closely with these other methods.

[*Note.* These graphs have been presented in the form of ${}_0\mathbf{z} - {}_r\mathbf{z}$ against $\log(1+r)$. The difference ${}_0\mathbf{z} - {}_r\mathbf{z}$ has been used instead of ${}_r\mathbf{z}$ because it represents directly the sickness in the first r weeks (on which we are at present concentrating) instead of the sickness after r weeks. The function $\log(1+r)$ has been used in order to spread out the early durations so that they are more clearly seen and at the same time to retain on the graphs the origin which, if $\log r$ had been used, would have been at minus infinity.]

* *Memorandum on Rates of Sickness and Disablement: Report for* 1912–13 *on the administration of the National Insurance Act, Part I* (*Health Insurance*) (Cd. 6907). Appendices V and VI, pp. 590–94. (See Reprints, 1946, pp. 99–103.)

It seems abundantly clear that the law gives results which are too low for short durations of sickness unless α and β are suitably chosen, in which case it gives results in close agreement with the other methods. It is not clear, however, that the other methods are so soundly based on observation or theory that they must be accepted as authoritative.

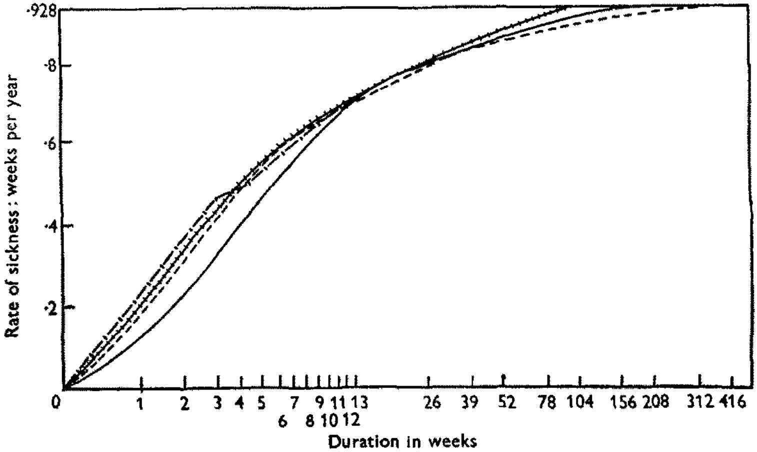

Fig. 3. Manchester Unity (Whole Society) sickness rates. Age 25. —— Method of the paper fitted at $r=0$, 13, 26, 52. —ı— Rhodes's method. - - - - Method of Cd. 6907. +++ Method of the paper fitted at the shorter durations

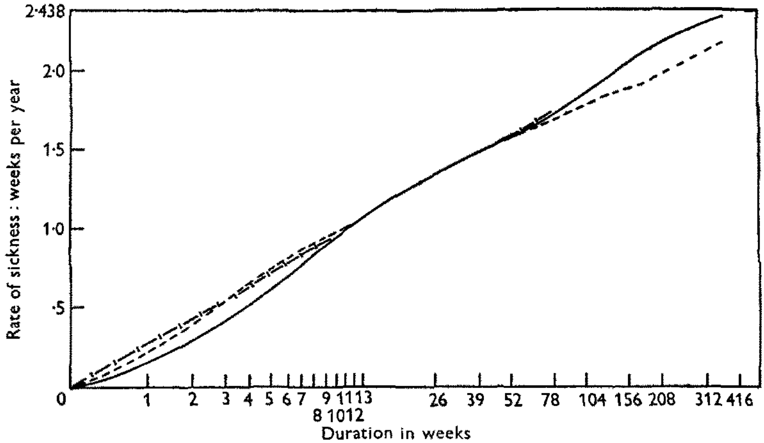

Fig. 4. Manchester Unity (Whole Society) sickness rates. Age 50. —— Method of the paper fitted at $r=0$, 13, 26, 52. —ı— Rhodes's method. - - - - Method of Cd. 6907

Rhodes, beyond mentioning the use of graphical and exponential extrapolation (and interpolation), gives insufficient detail on which to base a critical appreciation. He does, however, suggest that his figures may be too high at the shorter durations. The merit can be claimed for the figures from Cd. 6907 that they make use of the additional information, the 'proportion claiming'. In order to do so, however, it is necessary to take the difference of two quantities, one of which has to be adjusted. It is a rational thing to do, but it is doubtful whether the results are necessarily superior to others.

Even for short-term sickness it is thus suggested that the mathematical law is reasonably applicable provided the values of α and β are chosen with proper regard to the ultimate purpose of the particular investigation.

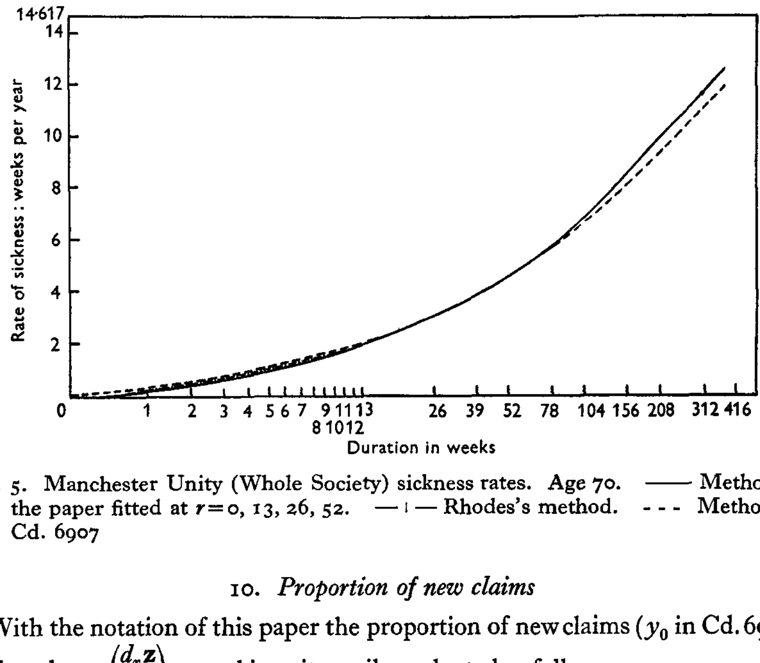

Fig. 5. Manchester Unity (Whole Society) sickness rates. Age 70. —— Method of the paper fitted at $r=0, 13, 26, 52$. —ı— Rhodes's method. - - - Method of Cd. 6907

10. *Proportion of new claims*

With the notation of this paper the proportion of new claims (y_0 in Cd. 6907) is given by $-\left(\frac{d_r\mathbf{z}}{dr}\right)_{r=0}$ and is quite easily evaluated as follows:

$$y_0 = -\left(\frac{d_r\mathbf{z}}{dr}\right)_{r=0} = F_x(-\log\alpha) + G_x(-\log\beta), \tag{13}$$

where every quantity on the right-hand side of the equation is known and is positive.

The values of y_0 given in Cd. 6907 are obtained, on the other hand, by subtracting the 'force of sickness' (with a small correction) from the 'proportion sick'. The values obtained in these two ways are compared in Table 3.

The figures in column (3) have been obtained by inserting in equation (13) the values of α and β from columns (2) and (3) in Table 1 and the values of F_x and G_x found from them and from the underlying values of ${}_0\mathbf{z}$, ${}_{13}\mathbf{z}$, ${}_{26}\mathbf{z}$ and ${}_{52}\mathbf{z}$ for these ages.

As the figures of Table 1 would lead us to expect, the values for the proportion claiming obtained from the law using the values of α and β indicated are appreciably lower than the estimates of Cd. 6907.

We have no direct knowledge of the sickness rates at the very short durations, because by their nature the shortest durations of sickness are not the subject of financial claims and are not therefore observed.

It seems probable that the observed ${}_r\mathbf{z}$ will not continue to increase at an increasing rate right down to $r=0$, but will flatten out slightly at some period within the first three days. In general this is not important, but there may be times when very short durations are being considered, and at such times the double exponential law which always increases in gradient as r decreases will not be suitable. A further term or an alternative function might then be required.

Summarizing this section, and the last, a week-by-week comparison of the available methods shows that when the method of this paper is applied age by age to the tabular rates (giving an exact fit at four points at each age), the sickness is lower at durations under thirteen weeks, higher at durations over 104 weeks, and in good agreement in between. When values of α and β are suitably chosen, very close agreement can be obtained at durations down to (say) one week.

Table 3. Proportion of new claims (Manchester Unity, Whole Society)

Age (1)	Cd. 6907 (2)	From α and β in Table 1 (3)
20	·25	·18
30	·22	·14
40	·22	·15
50	·24	·17
60	·27	·20
70	·29	·22

11. *Applications of the law*

So far, having selected the law we have first fitted it closely to the data available on an age-for-age basis. The agreement or otherwise of the law with all available information has then been considered, but in its application to practical problems the law has done no more than Rhodes's tables have already done. The law may provide a mathematical background for such tables but it will not make it any easier to use them.

It has already been noticed that the variation with age of α and β when fitted in this way is not great. It will now be shown how the assumption that α and β are independent of age permits the easy transformation of the various 'experience' and valuation functions. Later we shall return to the determination of suitable values of α and β and to the investigation of the errors that arise in practice.

For our immediate purpose, then, the fundamental equation becomes

$$ {}_r\mathbf{z}_x = F_x\alpha^r + G_x\beta^r, \qquad (14)$$

where α and β are known constants determined once and for all for the practical purpose in view.

Once F_x and G_x are known for each age the equations enable the sickness rates for any durations to be obtained. The values of F_x and G_x can be found in practice by selecting any two of the known tabular functions or any two different combinations of them and solving the equations.

For example, if ${}_0\mathbf{z}_x$ and ${}_{26}\mathbf{z}_x$ are selected as the two tabular functions to be used, we simply solve for each age

$${}_0\mathbf{z}_x = F_x + G_x, \quad {}_{26}\mathbf{z}_x = F_x\alpha^{26} + G_x\beta^{26}.$$

In practice, therefore, once the values of F_x and G_x have been determined it is only necessary to use them as 'experience factors' to be applied to numbers exposed to risk to obtain an 'F' and a 'G' element for the expected sickness, and by combining these two elements in the appropriate proportions to obtain the experience for the periods in which we may be interested.

It is worth remarking that this was originally one of the approaches of the authors to the problem, but it will now be shown that there is never any need to determine the values of F and G explicitly. The tabular functions can be used as published and the combination of any two, suitably chosen, achieves the same effect.

12. *Linear combinations of ${}_r\mathbf{z}_x$*

In effect the procedure consists in selecting a variable γ_x which varies only with age and another variable ${}_r\delta$ depending only on the amounts and durations of the rates of benefit to be considered. The normal processes of obtaining the expected sickness from the exposed to risk, and of valuation, consist (when γ and δ are suitably chosen) in finding $\sum\limits_x \sum\limits_r \gamma_x \cdot {}_r\delta \cdot {}_r\mathbf{z}_x$ and this is, by our basic equation (14), equivalent to

$$\sum_x \sum_r \gamma_x \cdot {}_r\delta \cdot {}_r\mathbf{z}_x = \left(\sum_x \gamma_x F_x\right)\left(\sum_r {}_r\delta \cdot \alpha^r\right) + \left(\sum_x \gamma_x G_x\right)\left(\sum_r {}_r\delta \cdot \beta^r\right), \tag{15}$$

i.e. the sum of two products in each of which one factor depends only on age and the other only on duration. Let it be supposed, for example, that it is decided to estimate the expected claims in a Society offering sickness benefits of £1 per week from the fourth week to the eleventh, and 10*s*. per week from the twelfth week to the twentieth (in all cases inclusive).

The values of γ_x simply become the numbers E_x exposed to risk at each age, and the values of ${}_r\delta$ become zero except for ${}_3\delta = 1$, ${}_{11}\delta = -\frac{1}{2}$, ${}_{20}\delta = -\frac{1}{2}$, because the estimate of the sickness benefits required at age x is

$$({}_3\mathbf{z}_x - {}_{11}\mathbf{z}_x) + \tfrac{1}{2}({}_{11}\mathbf{z}_x - {}_{20}\mathbf{z}_x) = {}_3\mathbf{z}_x - \tfrac{1}{2}\cdot{}_{11}\mathbf{z}_x - \tfrac{1}{2}\cdot{}_{20}\mathbf{z}_x,$$

and the values of ${}_r\delta$ given above are equal to the coefficients of ${}_r\mathbf{z}_x$ on the right-hand side of the equation.

The general equation (15) then becomes

$$\begin{aligned}\sum_x E_x\{{}_3\mathbf{z}_x - \tfrac{1}{2}\cdot{}_{11}\mathbf{z}_x - \tfrac{1}{2}\cdot{}_{20}\mathbf{z}_x\} &= \left(\sum_x E_x F_x\right)(\alpha^3 - \tfrac{1}{2}\alpha^{11} - \tfrac{1}{2}\alpha^{20}) \\ &\quad + \left(\sum_x E_x G_x\right)(\beta^3 - \tfrac{1}{2}\beta^{11} - \tfrac{1}{2}\beta^{20}).\end{aligned}$$

Again, it would be necessary to derive the F_x's and G_x's and, having applied them as valuation factors to the E_x's, to combine them suitably to get the particular estimate that is required.

It is convenient at this stage to introduce a further shorthand. Call for simplicity the scale benefits above written [U] and write for short

$$\mathbf{z}[U] = \mathbf{z}\begin{bmatrix} 3 & 8 & 9 & R \\ 0 & 1 & \frac{1}{2} & 0 \end{bmatrix},$$

Note. Notwithstanding the matrix notation these are not matrices.

meaning thereby sickness payments of nil for three weeks, 1 per week for eight weeks, $\frac{1}{2}$ per week for nine weeks, and nil for the remainder. Then

$$\mathbf{z}\,[\mathrm{U}] = {}_3\mathbf{z}_x - \tfrac{1}{2}\cdot{}_{11}\mathbf{z}_x - \tfrac{1}{2}\cdot{}_{20}\mathbf{z}_x$$

as before. Further, in pursuance of this notation, define

$$\begin{aligned}\boldsymbol{\alpha}\,[\mathrm{U}] &= \boldsymbol{\alpha}\begin{bmatrix} 3 & 8 & 9 & \mathrm{R} \\ 0 & 1 & \frac{1}{2} & 0 \end{bmatrix} \\ &= \alpha^3 - \tfrac{1}{2}\alpha^{11} - \tfrac{1}{2}\alpha^{20},\end{aligned}$$

and similarly

$$\begin{aligned}\boldsymbol{\beta}\,[\mathrm{U}] &= \boldsymbol{\beta}\begin{bmatrix} 3 & 8 & 9 & \mathrm{R} \\ 0 & 1 & \frac{1}{2} & 0 \end{bmatrix} \\ &= \beta^3 - \tfrac{1}{2}\beta^{11} - \tfrac{1}{2}\beta^{20}.\end{aligned}$$

The symbol [U] then becomes, as it were, a linear operator-cum-function with the above specialized meaning. Further, let us write $\boldsymbol{\Theta}\,[\mathrm{U}]$ for the expected sickness, for the particular group, on the sickness benefits $\mathbf{z}\,[\mathrm{U}]$. In this notation equation (15) becomes

$$\boldsymbol{\Theta}\,[\mathrm{U}] = \{\sum_x \mathrm{E}_x.\mathbf{z}_x\,[\mathrm{U}]\} = \{\sum_x \mathrm{E}_x\mathrm{F}_x\}\,\boldsymbol{\alpha}\,[\mathrm{U}] + \{\sum_x \mathrm{E}_x\mathrm{G}_x\}\,\boldsymbol{\beta}\,[\mathrm{U}]. \tag{16}$$

But now the ordinary tabular values of z_x are no more than special cases of $\mathbf{z}\,[\mathrm{U}]$, for example, $z^{13/13} = \mathbf{z}\begin{bmatrix} 13 & 13 & \mathrm{R} \\ 0 & 1 & 0 \end{bmatrix}$. If, therefore, we select any two of these factors (or, if for a special reason we prefer it, a linear combination of them) and call them $\mathbf{z}\,[\mathrm{X}]$ and $\mathbf{z}\,[\mathrm{Y}]$ we have also

$$\boldsymbol{\Theta}\,[\mathrm{X}] = \sum_x \{\mathrm{E}_x\mathbf{z}_x\,[\mathrm{X}]\} = \{\sum_x \mathrm{E}_x\mathrm{F}_x\}\,\boldsymbol{\alpha}\,[\mathrm{X}] + \{\sum_x \mathrm{E}_x\mathrm{G}_x\}\,\boldsymbol{\beta}\,[\mathrm{X}], \tag{17}$$

$$\boldsymbol{\Theta}\,[\mathrm{Y}] = \sum_x \{\mathrm{E}_x\mathbf{z}_x\,[\mathrm{Y}]\} = \{\sum_x \mathrm{E}_x\mathrm{F}_x\}\,\boldsymbol{\alpha}\,[\mathrm{Y}] + \{\sum_x \mathrm{E}_x\mathrm{G}_x\}\,\boldsymbol{\beta}\,[\mathrm{Y}], \tag{18}$$

where the quantities on the left-hand side, $\boldsymbol{\Theta}\,[\mathrm{X}]$ and $\boldsymbol{\Theta}\,[\mathrm{Y}]$, are found directly by the usual technique of scheduling and multiplying by the appropriate sickness rates or combinations thereof.

From equations (16), (17) and (18) the quantities $\sum_x \mathrm{E}_x\mathrm{F}_x$ and $\sum_x \mathrm{E}_x\mathrm{G}_x$ are readily eliminated to give an equation for $\boldsymbol{\Theta}\,[\mathrm{U}]$ (which is required) in terms of $\boldsymbol{\Theta}\,[\mathrm{X}]$ and $\boldsymbol{\Theta}\,[\mathrm{Y}]$ (which are calculated from the tabular functions).

The equation is

$$\begin{vmatrix} \boldsymbol{\Theta}\,[\mathrm{U}] & \boldsymbol{\alpha}\,[\mathrm{U}] & \boldsymbol{\beta}\,[\mathrm{U}] \\ \boldsymbol{\Theta}\,[\mathrm{X}] & \boldsymbol{\alpha}\,[\mathrm{X}] & \boldsymbol{\beta}\,[\mathrm{X}] \\ \boldsymbol{\Theta}\,[\mathrm{Y}] & \boldsymbol{\alpha}\,[\mathrm{Y}] & \boldsymbol{\beta}\,[\mathrm{Y}] \end{vmatrix} = 0, \tag{19}$$

which when expanded gives

$$\boldsymbol{\Theta}\,[\mathrm{U}] = \left\{\frac{\boldsymbol{\alpha}\,[\mathrm{U}]\,\boldsymbol{\beta}\,[\mathrm{Y}] - \boldsymbol{\alpha}\,[\mathrm{Y}]\,\boldsymbol{\beta}\,[\mathrm{U}]}{\boldsymbol{\alpha}\,[\mathrm{X}]\,\boldsymbol{\beta}\,[\mathrm{Y}] - \boldsymbol{\alpha}\,[\mathrm{Y}]\,\boldsymbol{\beta}\,[\mathrm{X}]}\right\}\boldsymbol{\Theta}\,[\mathrm{X}] + \left\{\frac{\boldsymbol{\alpha}\,[\mathrm{X}]\,\boldsymbol{\beta}\,[\mathrm{U}] - \boldsymbol{\alpha}\,[\mathrm{U}]\,\boldsymbol{\beta}\,[\mathrm{X}]}{\boldsymbol{\alpha}\,[\mathrm{X}]\,\boldsymbol{\beta}\,[\mathrm{Y}] - \boldsymbol{\alpha}\,[\mathrm{Y}]\,\boldsymbol{\beta}\,[\mathrm{X}]}\right\}\boldsymbol{\Theta}\,[\mathrm{Y}]. \tag{20}$$

Because of the generality of the notation the results may look a little complicated but they are absolutely symmetrical and it will subsequently be shown that in particular cases the values of the expressions in the curly brackets can be easily determined and incorporated.

13. *Valuations from a linear combination of* ${}_r\mathbf{z}$

So far attention has been focused on the direct estimate of the sickness claims, the quantities such as $\Theta\,[\mathrm{U}]$. In practice, however, we require at least as frequently to value the liabilities of a fund which consists of a series of contracts to pay benefits characterized by $\mathbf{z}\,[\mathrm{U}]$. In this problem the schedules will be the numbers of the members of the society existing on the valuation date, L_x, instead of the exposed to risk, E_x.

If now we were valuing £1 throughout the remainder of sickness in the rth week and after, we should have for one person aged x a value of

$$\sum_{y=x}^{w} \frac{\overline{\mathrm{D}}_y}{\mathrm{D}_x}\,{}_r\mathbf{z}_y.$$

Similarly, if it is a given linear combination of sickness $\mathbf{z}\,[\mathrm{U}]$ in which we are interested, the value for one person aged x is

$$\sum_{y=x}^{w} \frac{\overline{\mathrm{D}}_y}{\mathrm{D}_x}\,\mathbf{z}_y\,[\mathrm{U}],$$

and for a series of groups of persons numbering L_x at age x it is

$$\sum_x \mathrm{L}_x \sum_{y=x}^{w} \frac{\overline{\mathrm{D}}_y}{\mathrm{D}_x}\,\mathbf{z}_y\,[\mathrm{U}], \qquad = \sum_{x=a}^{\omega} \frac{L_x}{\bar{D}_x} \times \bar{D}_x\, Z_x\,[U]$$

for which we shall write $\mathbf{V}\,[\mathrm{U}]$ just as we have previously written $\Theta\,[\mathrm{U}]$.

In passing it may be remarked that it is sometimes useful to extend the previous notation to $\mathbf{V}$ in special cases writing, for example, $\mathbf{V}\begin{bmatrix} 3 & 8 & 9 & \mathrm{R} \\ 0 & 1 & \tfrac{1}{2} & 0 \end{bmatrix}$ as we wrote similar expressions for $\mathbf{z}$, α and β.

Applying the law we now have

$$\begin{aligned} \mathbf{V}\,[\mathrm{U}] &= \Sigma \mathrm{L}_x \left(\sum_{y=x}^{w} \frac{\overline{\mathrm{D}}_y}{\mathrm{D}_x}\,\mathbf{z}_y\,[\mathrm{U}] \right) \\ &= \left\{ \Sigma \mathrm{L}_x \left(\sum_{y=x}^{w} \frac{\overline{\mathrm{D}}_y}{\mathrm{D}_x}\,\mathrm{F}_y \right) \right\} \alpha\,[\mathrm{U}] + \left\{ \Sigma \mathrm{L}_x \left(\sum_{y=x}^{w} \frac{\overline{\mathrm{D}}_y}{\mathrm{D}_x}\,\mathrm{G}_y \right) \right\} \beta\,[\mathrm{U}]. \qquad (21) \end{aligned}$$

Just as before we now select two arbitrary combinations [X] and [Y] which are convenient or specially suitable for our purpose, and we have

$$\mathbf{V}\,[\mathrm{X}] = \left\{ \Sigma \mathrm{L}_x \Sigma \frac{\overline{\mathrm{D}}_y}{\mathrm{D}_x}\,\mathrm{F}_y \right\} \alpha\,[\mathrm{X}] + \left\{ \Sigma \mathrm{L}_x \Sigma \frac{\overline{\mathrm{D}}_y}{\mathrm{D}_x}\,\mathrm{G}_y \right\} \beta\,[\mathrm{X}], \qquad (22)$$

$$\mathbf{V}\,[\mathrm{Y}] = \left\{ \Sigma \mathrm{L}_x \Sigma \frac{\overline{\mathrm{D}}_y}{\mathrm{D}_x}\,\mathrm{F}_y \right\} \alpha\,[\mathrm{Y}] + \left\{ \Sigma \mathrm{L}_x \Sigma \frac{\overline{\mathrm{D}}_y}{\mathrm{D}_x}\,\mathrm{G}_y \right\} \beta\,[\mathrm{Y}], \qquad (23)$$

from which the quantities in the curly brackets can be eliminated to give the equation for $\mathbf{V}\,[\mathrm{U}]$ in terms of $\mathbf{V}\,[\mathrm{X}]$ and $\mathbf{V}\,[\mathrm{Y}]$ which are evaluated in the usual way in a form identical with that for Θ already found, namely:

$$\begin{vmatrix} \mathbf{V}\,[\mathrm{U}] & \alpha\,[\mathrm{U}] & \beta\,[\mathrm{U}] \\ \mathbf{V}\,[\mathrm{X}] & \alpha\,[\mathrm{X}] & \beta\,[\mathrm{X}] \\ \mathbf{V}\,[\mathrm{Y}] & \alpha\,[\mathrm{Y}] & \beta\,[\mathrm{Y}] \end{vmatrix} = 0, \qquad (24)$$

from which the identical expanded form can also be reached.

The application of the law, therefore, with constant α and β for all ages leads to the same simple equations for both the experience determination and the valuation in terms of the two selected valuation combinations.

14. *The second evaluation of α and β*

The outcome of the preceding sections is a series of formulae which follow from the assumption that α and β do not vary with age and which depend for their practical usefulness upon whether values of α and β exist which give reasonable agreement between the facts—as tabulated in the standard table—and the law.

It has been noted that the values of α and β fitted to the sickness rates at each age do not vary widely, and it has been indicated that the practical choice of α and β may be influenced by the particular purpose in view. We say this not to suggest that many different values of α and β are necessary but to emphasize the flexibility of the method. In practice it will be found that the values of α and β computed by the method now to be described will be adequate for most purposes.

We now turn to the estimation of suitable values of α and β which will apply over a wide range of ages, and to the examination of the errors which are likely to arise in practice from their use for the valuation of a Friendly Society insuring sickness benefits.

The most obvious approach is to find mean values, weighted or otherwise, of the values of α^{13} and β^{13} found by the method of § 6. Thus we might take as α^{13} and β^{13} the mean values of columns (2) and (3) (or of columns (4) and (5)) respectively in Table 1.

These values will be found to give undue weight (for valuation purposes) to the values of α^{13} and β^{13} for the younger ages, and a more satisfactory solution is to find means of ${}_0\mathbf{z}$, ${}_{13}\mathbf{z}$, ${}_{26}\mathbf{z}$ and ${}_{52}\mathbf{z}$ (or of ${}_0\mathbf{z}$, ${}_{26}\mathbf{z}$, ${}_{52}\mathbf{z}$ and ${}_{104}\mathbf{z}$) and to calculate α^{13} and β^{13} from them.

Another approach is to use the sum for all ages of the present values of sickness benefits. This is in principle equivalent to the method of using the sickness rates, but it weights the higher ages more heavily and, for valuation purposes when the future of every member is projected many years ahead, present values give a more appropriate result for even the young ages.

Table 4 shows the values of α^{13} and β^{13} found for the Manchester Unity (Whole Society) table on these three methods.

Table 4

Method	Based on ${}_0\mathbf{z}, {}_{13}\mathbf{z}, {}_{26}\mathbf{z}, {}_{52}\mathbf{z}$		Based on ${}_0\mathbf{z}, {}_{26}\mathbf{z}, {}_{52}\mathbf{z}, {}_{104}\mathbf{z}$	
	α^{13}	β^{13}	α^{13}	β^{13}
Mean α^{13} and β^{13}	·87	·28	·90	·32
Mean ${}_r\mathbf{z}$	·96	·21	·97	·36
Mean valuation factor (3 %)	·95	·21	·96	·35

There are, however, other methods which may be more suitable and a particular application of one will now be considered.

Let it be supposed that a large number of Friendly Societies granting many non-standard scales of sickness benefit are to be valued using only two experience factors and two valuation factors for each age-group. Then it is first necessary to select the two standard linear combinations, $\mathbf{z}[X]$ and $\mathbf{z}[Y]$, by reference to which the fixed experience and valuation factors are to be determined. For this purpose we have chosen

$$\mathbf{z}[X]=\mathbf{z}\begin{bmatrix} 13 & 13 & 26 & R \\ 1 & \frac{1}{2} & \frac{1}{4} & 0 \end{bmatrix}$$

$$= {}_0\mathbf{z}-\frac{{}_{13}\mathbf{z}}{2}-\frac{{}_{26}\mathbf{z}}{4}-\frac{{}_{52}\mathbf{z}}{4}, \tag{25}$$

and

$$\mathbf{z}[Y]=\mathbf{z}\begin{bmatrix} 52 & R \\ 0 & \frac{1}{4} \end{bmatrix}$$

$$=\frac{{}_{52}\mathbf{z}}{4}. \tag{26}$$

These have been chosen because $\mathbf{z}[X]$ is very similar to the kind of benefits insured by many societies and $\mathbf{z}[Y]$ gives due weight to the 'Remainder' benefit, if any.

Then if α and β are known, any required combination of sickness benefits, $\mathbf{z}[U]$, can be obtained from an equation of the form

$$\mathbf{z}[U]=A\mathbf{z}[X]+B\mathbf{z}[Y], \tag{27}$$

where A and B are written for the expressions in the curly brackets in equation (20). Since both A and B are functions of α and β, they may be evaluated either from their analytic form given α, β and $\mathbf{z}[U]$, or the values of A and B corresponding to the chosen $\mathbf{z}[U]$ may be found by some other method and corresponding values of α and β found subsequently.

As an example of the first alternative, the values of A and B corresponding to ${}_0\mathbf{z}$, found from the values of α^{13} and β^{13} in Table 4, are shown in Table 5.

Table 5

	Based on ${}_0\mathbf{z}, {}_{13}\mathbf{z}, {}_{26}\mathbf{z}, {}_{52}\mathbf{z}$		Based on ${}_0\mathbf{z}, {}_{26}\mathbf{z}, {}_{52}\mathbf{z}, {}_{104}\mathbf{z}$	
	A	B	A	B
Mean α^{13} and β^{13}	1·175	5·042	1·220	4·758
Mean ${}_r\mathbf{z}$	1·128	4·293	1·254	4·185
Mean valuation factor	1·128	4·378	1·244	4·250

Alternatively, having chosen $\mathbf{z}[U]$, we may find the best values of A and B according to some criterion which depends on the purpose we have in view. For example, satisfactory results are obtained below by fitting ${}_0\mathbf{z}$ to $\mathbf{z}[X]$ and $\mathbf{z}[Y]$ for the ages of 18, 38, 58 and 78 by the method of least squares using unweighted present values based on Eastern Counties mortality, 3 % interest and A.H.J. sickness rates.

There is not much difference between the Manchester Unity 'A.H.J.' and 'Whole Society' rates, and we should therefore expect values of A and B calculated from the former to be very close to those calculated from the latter. In the earlier part of this paper it was necessary to compare our results with the only available figures which were 'Whole Society'. The rest of this section of the paper is based on the A.H.J. rates, because they have been combined with the Central Counties and Eastern Counties mortality tables to give us the most up-to-date valuation tables at present published, and are more suitable in many respects for the purposes of the rest of this paper.

Values of A and B are found to make the expression

$$\sum_x ({}_0\mathbf{z}_x - A\mathbf{z}_x[X] - B\mathbf{z}_x[Y])^2 = \Delta^2 \quad \text{(say)},$$

a minimum, where the summation is over all the ages selected (in this case the ages 18, 38, 58 and 78). The method is the standard one consisting in solving

$$\frac{\partial \Delta^2}{\partial A} = 0 \quad \text{and} \quad \frac{\partial \Delta^2}{\partial B} = 0$$

for A and B. The solution of the equations gives

$$A = 1{\cdot}262, \quad B = 4{\cdot}221,$$

whence $${}_0\mathbf{z} = 1{\cdot}262\,\mathbf{z}[X] + 4{\cdot}221\,\mathbf{z}[Y].$$

A and B found above relate to a particular [U], viz. $\begin{bmatrix} R \\ 1 \end{bmatrix}$. Substituting the general expanded forms of A and B shown in equation (20) we have

$$A\,\alpha[X] + B\,\alpha[Y]$$

$$\equiv \left\{\frac{\alpha[U]\,\beta[Y] - \alpha[Y]\,\beta[U]}{\alpha[X]\,\beta[Y] - \alpha[Y]\,\beta[X]}\right\}\alpha[X] + \left\{\frac{\alpha[X]\,\beta[U] - \alpha[U]\,\beta[X]}{\alpha[X]\,\beta[Y] - \alpha[Y]\,\beta[X]}\right\}\alpha[Y]$$

$$\equiv \alpha[U]. \tag{28}$$

Inserting the particular values appropriate to the use of ${}_0\mathbf{z}$ and expanding $\alpha[X]$ and $\alpha[Y]$ we have

$$1{\cdot}262\left[1 - \frac{\alpha^{13}}{2} - \frac{\alpha^{26}}{4} - \frac{\alpha^{52}}{4}\right] + 4{\cdot}221\,\frac{\alpha^{52}}{4} = 1. \tag{29}$$

From symmetry we see that the equation holds when β is substituted for α. That is to say,

$$A\beta[X] + B\beta[Y] \equiv \beta[U]. \tag{30}$$

Since equations (28) and (30) are identical in form, it follows that if in solving the first of them two roots are found between 0 and 1, the one will be α and the other β.

Approximate solutions of equation (29) were first found graphically, and more accurate solutions were then found using an iterative method. The result was $\alpha^{13} = {\cdot}9645$ and $\beta^{13} = {\cdot}3689$, while the other two roots were found to be imaginary. These values have been adopted for the illustrative examples shown in the following sections.

15. *The method in practice: Evaluation of factors*

Now, from equation (19), we have

$$\begin{vmatrix} {}_r\mathbf{z} & \alpha^r & \beta^r \\ \mathbf{z}[X] & \alpha[X] & \beta[X] \\ \mathbf{z}[Y] & \alpha[Y] & \beta[Y] \end{vmatrix} = 0, \tag{31}$$

or
$$\begin{aligned} {}_r\mathbf{z} = {} & \left[\left(\frac{\alpha[Y]}{\alpha[Y]\beta[X]-\alpha[X]\beta[Y]}\right)\beta^r - \left(\frac{\beta[Y]}{\alpha[Y]\beta[X]-\alpha[X]\beta[Y]}\right)\alpha^r\right]\mathbf{z}[X] \\ & + \left[\left(\frac{\alpha[X]}{\alpha[X]\beta[Y]-\alpha[Y]\beta[X]}\right)\beta^r - \left(\frac{\beta[X]}{\alpha[X]\beta[Y]-\alpha[Y]\beta[X]}\right)\alpha^r\right]\mathbf{z}[Y]. \end{aligned} \tag{32}$$

From the above equation we may evaluate the coefficients of $\mathbf{z}[X]$ and $\mathbf{z}[Y]$ for any required values of r. Appendix I gives these coefficients for most of the values of r encountered in practice. They are, of course, also the coefficients for use with $\mathbf{\Theta}$ and $\mathbf{V}$.

Table 6

r	Coefficients of	
	[X]	[Y]
0	1·262	4·221
13	·449	4·316
26	·150	4·253
52	·000	4·000
104	−·020	3·467

An abridged table is shown (Table 6) for those values of r for which tabular rates are already available. Thus

$$\begin{aligned} {}_0\mathbf{z} &= 1{\cdot}262\,\mathbf{z}[X] + 4{\cdot}221\,\mathbf{z}[Y], \\ z^{13} &= {}_0\mathbf{z} - {}_{13}\mathbf{z} \\ &= {\cdot}813\,\mathbf{z}[X] - {\cdot}095\,\mathbf{z}[Y], \end{aligned}$$

and so on.

16. *Example of procedure*

Suppose we have a society in which the sickness benefits are:

£1 for first twenty-six weeks,

10*s*. for next twenty-six weeks,

5*s*. for remainder.

We assume for the sake of example that at the valuation date there are ten persons in each quinary age-group. Further, purely for example we assume that the exposed to risk for sickness for the previous five years was exactly five times the number in each valuation group.

It is next assumed that we have, already prepared, a table of valuation factors, on the basis of Eastern Counties (Rural Districts) male mortality, with

Manchester Unity Group A.H.J. sickness rates and 3 % interest, for each valuation age-group relating to Θ [X], Θ [Y], V [X] and V [Y]. The totals of the working schedules will then be found to be:

$$\Theta[X] = 1189{\cdot}81, \quad \Theta[Y] = 1410{\cdot}14,$$

$$V[X] = 3713{\cdot}0, \quad V[Y] = 3484{\cdot}5.$$

We now form a table for the expected sickness as shown in Table 7.

Table 7

Period	Coefficient [X] (1)	Coefficient [Y] (2)	Rate (3)	(1) × (3) (4)
First 26 weeks	1·112	−·032	£1	£1·112
Next 26 weeks	·150	·253	£0·5	£0·075
Remainder	—	4·000	£0·25	—
Totals	1·262	4·221	—	£1·187

Period	(2) × (3) (5)	Θ [X] × (4) (6)	Θ [Y] × (5) (7)	Expected sickness (6) + (7) (8)
First 26 weeks	−£0·032	£1323·1	−£ 45·1	£1278·0
Next 26 weeks	£0·126	£ 89·2	£ 177·7	£ 266·9
Remainder	£1·000	—	£1410·1	£1410·1
Totals	£1·094	£1412·3	£1542·7	£2955·0 (£2918·9)

Thus the valuation liability is given by

$$= £3713{\cdot}0 \times 1{\cdot}187 + £3484{\cdot}5 \times 1{\cdot}094$$

$$= £8219 \ (£8173).$$

The figures in brackets are those obtained by the orthodox method of computation.

17. *Comparative accuracy*

The next step is to investigate the degree of accuracy obtainable by the method for normal valuation work. Three different and completely artificial types of membership distribution were built up to show the effect of level numbers and weighting at the extremes. In the first example a society has been assumed to have equal numbers in each age-group, in the second the numbers in successive age-groups increase in arithmetical progression with age, and in the third example the numbers decrease with age. In all these examples the benefits have been calculated on two bases: throughout life, and ceasing at age 65. The valuation basis throughout is Eastern Counties (Rural Districts) male mortality, with Manchester Unity Group A.H.J. sickness rates and 3 % interest. The same values of α and β have been used throughout, the values selected being those used in §§ 15 and 16 above. The results are shown in Tables 8 and 9.

Table 8

Basis I: (*a*) Benefits throughout life.
(*b*) Society with total full sick pay of £100.
(*c*) Data grouped at quinquennial ages 18, 23, ..., 88.
(*d*) Exposed to risk for sickness assumed to be exactly the same as the valuation data.

Membership: (i) Equal numbers in each group.
(ii) Number in age-group $(5n+13)$ is proportional to n $(n=1, 2, \ldots, 15)$.
(iii) Number in age-group $(5n+13)$ is proportional to $16-n$ $(n=1, 2, \ldots, 15)$.

	Membership								
	(i)			(ii)			(iii)		
	Actual	Calculated	C/A %	Actual	Calculated	C/A %	Actual	Calculated	C/A %
Experience:	£	£		£	£		£	£	
Θ [X]	158·64	—	—	197·96	—	—	119·32	—	—
Θ [Y]	188·02	—	—	312·36	—	—	63·68	—	—
£1 for first 13 weeks	116	111	95·7	136	131	96·3	96	91	94·8
£1 for first 26 weeks	167	170	101·8	208	210	101·0	126	131	104·0
£1 for first 52 weeks	235	242	103·0	312	319	102·2	159	165	103·8
£1 for first 104 weeks	335	345	103·0	468	489	104·5	202	201	99·5
£1 for all periods	987	994	100·7	1,561	1,568	100·4	414	419	101·2
£1 for first 26 weeks, 10*s*. for next 26 weeks, 5*s*. for next 52 weeks	226	232	102·7	299	307	102·7	153	157	102·6
£1 for first 26 weeks, 10*s*. for next 26 weeks, 5*s*. after 52 weeks	389	394	101·3	572	577	100·9	206	211	102·4
Valuation:									
V [X]	2,475·3	—	—	2,161·4	—	—	2,789·2	—	—
V [Y]	2,323·0	—	—	2,794·7	—	—	1,851·3	—	—
£1 for first 13 weeks	1,827	1,792	98·1	1,512	1,492	98·7	2,143	2,092	97·6
£1 for first 26 weeks	2,615	2,678	102·4	2,281	2,314	101·4	2,949	3,042	103·2
£1 for first 52 weeks	3,632	3,637	100·1	3,339	3,345	100·2	3,925	3,929	100·1
£1 for first 104 weeks	5,051	4,925	97·5	4,893	4,878	99·7	5,209	4,972	95·4
£1 for all periods	12,924	12,929	100·0	14,518	14,524	100·0	11,330	11,334	100·0
£1 for first 26 weeks, 10*s*. for next 26 weeks, 5*s*. for next 52 weeks	3,478	3,479	100·0	3,199	3,213	100·4	3,758	3,746	99·7
£1 for first 26 weeks, 10*s*. for next 26 weeks, 5*s*. after 52 weeks	5,447	5,480	100·6	5,605	5,624	100·3	5,288	5,337	100·9

Table 9

Basis II: (*a*) Benefits to age 65.
(*b*) Society with total full sick pay of £100.
(*c*) Data grouped at quinquennial ages 18, 23, ..., 63.
(*d*) Exposed to risk for sickness assumed to be exactly the same as the valuation data.

Membership : (i) Equal numbers in each group.
(ii) Number in age group $(5n+13)$ is proportional to n $(n=1, 2, \ldots, 10)$.
(iii) Number in age group $(5n+13)$ is proportional to $11-n$ $(n=1, 2, \ldots, 10)$.

	Membership								
	(i)			(ii)			(iii)		
	Actual	Calculated	C/A %	Actual	Calculated	C/A %	Actual	Calculated	C/A %
Experience:	£			£	£		£	£	
Θ [X]	108·17	—	—	128·72	—	—	87·62	—	—
Θ [Y]	20·11	—	—	31·76	—	—	8·46	—	—
£1 for first 13 weeks	92	86	93·5	106	102	96·2	78	70	89·7
£1 for first 26 weeks	114	120	105·3	136	142	104·4	92	97	105·4
£1 for first 52 weeks	135	141	104·4	167	169	101·2	103	112	108·7
£1 for first 104 weeks	155	154	99·4	198	189	95·5	112	119	106·2
£1 for all periods	215	221	102·8	294	296	100·7	137	146	107·4
£1 for first 26 weeks, 10*s*. for next 26 weeks, 5*s*. for next 52 weeks	130	134	103·1	159	161	101·3	100	106	106·0
£1 for first 26 weeks, 10*s*. for next 26 weeks, 5*s*. after 52 weeks	145	150	103·4	183	187	102·2	106	113	106·6
Valuation:									
V [X]	1754·25	—	—	1518·4	—	—	1990·1	—	—
V [Y]	372·50	—	—	385·1	—	—	359·9	—	—
£1 for first 13 weeks	1463	1391	95·1	1240	1198	96·6	1686	1584	94·0
£1 for first 26 weeks	1856	1939	104·5	1611	1676	104·0	2101	2202	104·8
£1 for first 52 weeks	2235	2296	102·7	1984	2001	100·9	2486	2591	104·2
£1 for first 104 weeks	2598	2530	97·4	2353	2231	95·1	2843	2823	99·3
£1 for all periods	3725	3786	101·6	3524	3542	100·5	3926	4031	102·7
£1 for first 26 weeks, 10*s*. for next 26 weeks, 5*s*. for next 52 weeks	2136	2176	101·9	1890	1897	100·4	2383	2454	103·0
£1 for first 26 weeks, 10*s*. for next 26 weeks, 5*s*. after 52 weeks	2418	2490	103·0	2183	2224	101·9	2654	2756	103·8

The 'actual' figures quoted in these tables are the results of multiplying the number in each age-group by the tabular rate, and the 'calculated' figures are those obtained by the method of this paper.

There is close agreement between the 'actual' and the 'calculated' over most of the periods quoted in the tables. The 'calculated' values for the first thirteen weeks and for the first 104 weeks are a little below the 'actual' in nearly all the examples. For all the other periods the 'calculated' are equal to or slightly greater than the 'actual'. What matters is that the ratios C/A for corresponding periods are reasonably similar for the experience and for the valuation.

For normal distributions and normal schemes of benefit the system formed by $\alpha^{13} = \cdot 9645$, $\beta^{13} = \cdot 3689$ gives an accuracy which is well within the accuracy of estimation.

18. *Negative values*

Negative values need present no great difficulties. Specimen values can be calculated by the method of this paper and will indicate whether any further investigation is required. The method is applied in the same way whether for all the members insured under a sickness table, or for any one of these members when an individual valuation is required.

19. *Revision of benefits*

Quite frequently as the result of a surplus or deficiency being disclosed by a valuation, changes in the benefit periods are suggested. When this happens it is more often than not a change to non-standard periods. The problem then arises of how to estimate the effect of this change on the valuation. The method of this paper makes the solution of the problem not only exact (according to the law) but also simple. Since **V** [X] and **V** [Y] will already have been calculated all that is necessary is to determine from the prepared table the coefficients corresponding to the new periods, to find the products and to sum. The calculation will be similar to that in Table 7 but will apply to present values instead of expected sickness.

Even if the original valuation has not been made by reference to the two standard linear combinations used in §§ 14–16, the valuation will nearly always provide two basic valuation figures from which the required results can be obtained. In the absence of prepared tables the problem then reduces in effect to little more than the evaluation of a third-order determinant.

20. *Cyclic benefits*

Normally cyclic benefits are valued by assuming them to be equivalent to an average remainder value. This method suffers, however, from the disadvantage that the remainder rate must somehow be 'unscrambled' into the various rates of pay if the analysed comparison of 'actual' and 'expected' in the form F 40 is to have any significance. The new method does this automatically. The procedure is discussed in Appendix II.

21. *Extension to other tables*

The method can be extended to other tables, such as the rates in National Health Insurance, Valuation Regulations (S.R. and O. No. 281 of 1938), where the rates for only two periods are given. An illustration of the procedure is given as Appendix III to this paper.

22. *Extensions of the law*

It may be argued that the fit of the exponential curves described in this paper is not sufficiently close. It should, however, be appreciated that the equations in §§ 12 and 13 are of more general application. Many functions other than exponentials could be chosen without in any way disturbing the essentials of the method. If other functions were in any particular case found to give better agreement with the facts, the results in §§ 12 and 13 which depend upon the linearity would remain unaffected.

In addition, there may be occasions on which a closer fit would be obtained by defining the **z**'s as the sum of three exponential (or other) functions. No practical work has been done along these lines as for normal valuation work such refinements would be unjustified and would detract considerably from the arithmetical simplicity of the method. On the other hand, the theory would hardly be complicated, as everything would turn upon a determinant of the fourth order instead of the third. As an illustration of this, examination of Table 1 makes it clear that, if closer agreement were required in practice, it could readily be obtained by a suitable choice of α, β and γ.

23. *Conclusion*

In conclusion we should like to record our gratitude to certain fellow actuaries who prefer to remain anonymous for their help and searching criticisms of the paper in draft, and to Mr J. A. Cox for checking the extensive calculations on which the tables in this paper are based.

APPENDIX I

Table of the coefficients of equation (32) using the values of α^{13} and β^{13} found from equation (29), namely, to five significant figures, ·96447 and ·36890 respectively.

n	α^n	β^n	Coefficient of [X]*	Coefficient of [Y]*
0	1·000	1·000	1·262	4·221
½	·999	·962	1·214	4·230
1	·997	·926	1·167	4·239
2	·994	·858	1·079	4·254
3	·992	·794	·997	4·267
4	·989	·736	·922	4·278
5	·986	·681	·852	4·288
6	·983	·631	·787	4·296
7	·981	·585	·727	4·303
8	·978	·541	·671	4·308
9	·975	·501	·620	4·312
10	·973	·464	·572	4·314
11	·970	·430	·528	4·316
12	·967	·398	·487	4·316
13	·964	·369	·449	4·316
14	·962	·342	·414	4·315
15	·959	·316	·382	4·313
16	·956	·293	·352	4·310
17	·954	·271	·324	4·306
18	·951	·251	·298	4·302
19	·948	·233	·274	4·298
20	·946	·216	·252	4·293
21	·943	·200	·232	4·287
22	·941	·185	·213	4·281
23	·938	·171	·195	4·274
24	·935	·159	·179	4·268
25	·933	·147	·164	4·260
26	·930	·136	·150	4·253
27	·928	·126	·137	4·245
28	·925	·117	·125	4·237
29	·922	·108	·114	4·228
30	·920	·100	·104	4·220
31	·917	·093	·094	4·211
32	·915	·086	·086	4·202
33	·912	·080	·077	4·193
34	·910	·074	·070	4·184
35	·907	·068	·063	4·174
36	·905	·063	·057	4·164
37	·902	·059	·051	4·155
38	·900	·054	·045	4·145
39	·897	·050	·040	4·135

* $[X]=\begin{bmatrix} 13 & 13 & 26 & R \\ 1 & \frac{1}{2} & \frac{1}{4} & 0 \end{bmatrix}$; $[Y]=\begin{bmatrix} 52 & R \\ 0 & \frac{1}{4} \end{bmatrix}$.

APPENDIX I (*continued*)

n	α^n	β^n	Coefficient of [X]*	Coefficient of [Y]*
40	·895	·046	·035	4·125
41	·892	·043	·031	4·115
42	·890	·040	·027	4·105
43	·887	·037	·023	4·094
44	·885	·034	·020	4·084
45	·882	·032	·017	4·074
46	·880	·029	·014	4·063
47	·877	·027	·011	4·053
48	·875	·025	·008	4·042
49	·872	·023	·006	4·032
50	·870	·022	·004	4·021
51	·868	·020	·002	4·011
52	·865	·019	·000	4·000
55	·858	·015	−·005	3·969
60	·846	·010	−·010	3·915
65	·834	·007	−·014	3·863
70	·823	·005	−·017	3·810
74	·814	·003	−·018	3·768
78	·805	·003	−·019	3·727
80	·800	·002	−·019	3·706
85	·789	·001	−·020	3·655
90	·778	·001	−·020	3·605
95	·768	·001	−·020	3·555
100	·757	—	−·020	3·506
104	·749	—	−·020	3·467
110	·736	—	−·020	3·410
120	·716	—	−·020	3·316
130	·696	—	−·019	3·225
140	·677	—	−·019	3·137
150	·659	—	−·018	3·051
156	·648	—	−·018	3·000
208	·560	—	−·016	2·596
260	·485	—	−·013	2·246

$$* \ [X]=\begin{bmatrix} 13 & 13 & 26 & R \\ 1 & \tfrac{1}{2} & \tfrac{1}{4} & 0 \end{bmatrix}; \ [Y]=\begin{bmatrix} 52 & R \\ 0 & \tfrac{1}{4} \end{bmatrix}.$$

APPENDIX II

The evaluation of cyclic benefits by the method of the paper

Suppose that the scheme of benefits is

£r_1 per week for n_1 weeks,
£r_2 per week for n_2 weeks,
£r_3 per week for n_3 weeks,
o per week for n_4 weeks,
£r_1 per week for n_1 weeks, and so on,

or more shortly in the notation introduced in the paper

$$\begin{bmatrix} \dot{n}_1 & n_2 & n_3 & \dot{n}_4 \\ r_1 & r_2 & r_3 & 0 \end{bmatrix},$$

where the dots signify the recurrent nature of the benefits.

The problem is then to find

$$\mathbf{z}[\mathrm{U}] \equiv \mathbf{z}\begin{bmatrix} \dot{n}_1 & n_2 & n_3 & \dot{n}_4 \\ r_1 & r_2 & r_3 & 0 \end{bmatrix}. \tag{1}$$

Let

$$n = n_1 + n_2 + n_3 + n_4, \tag{2}$$

and

$$\mathbf{z}[\mathrm{U}'] \equiv \mathbf{z}\begin{bmatrix} n_1 & n_2 & n_3 & \mathrm{R} \\ r_1 & r_2 & r_3 & 0 \end{bmatrix}, \tag{3}$$

that is, $[\mathrm{U}']$ is the same scheme without the cyclic part. From (1)

$$\begin{aligned} \boldsymbol{\alpha}[\mathrm{U}] &\equiv \boldsymbol{\alpha}\begin{bmatrix} \dot{n}_1 & n_2 & n_3 & \dot{n}_4 \\ r_1 & r_2 & r_3 & 0 \end{bmatrix} \\ &= [(1-\alpha^{n_1})r_1 + (\alpha^{n_1} - \alpha^{n_1+n_2})r_2 + (\alpha^{n_1+n_2} - \alpha^{n_1+n_2+n_3})r_3](1 + \alpha^n + \alpha^{2n} + \ldots) \\ &= \boldsymbol{\alpha}\begin{bmatrix} n_1 & n_2 & n_3 & \mathrm{R} \\ r_1 & r_2 & r_3 & 0 \end{bmatrix}\left(1 + \frac{\alpha^n}{1-\alpha^n}\right), \end{aligned}$$

or

$$\boldsymbol{\alpha}[\mathrm{U}] = \boldsymbol{\alpha}[\mathrm{U}'] + \left(\frac{\alpha^n}{1-\alpha^n}\right)\boldsymbol{\alpha}[\mathrm{U}']. \tag{4}$$

Similarly

$$\boldsymbol{\beta}[\mathrm{U}] = \boldsymbol{\beta}[\mathrm{U}'] + \left(\frac{\beta^n}{1-\beta^n}\right)\boldsymbol{\beta}[\mathrm{U}']. \tag{5}$$

The value of $\mathbf{z}[\mathrm{U}]$ is found from the determinant

$$\begin{vmatrix} \mathbf{z}[\mathrm{U}] & \mathbf{z}[\mathrm{X}] & \mathbf{z}[\mathrm{Y}] \\ \boldsymbol{\alpha}[\mathrm{U}] & \boldsymbol{\alpha}[\mathrm{X}] & \boldsymbol{\alpha}[\mathrm{Y}] \\ \boldsymbol{\beta}[\mathrm{U}] & \boldsymbol{\beta}[\mathrm{X}] & \boldsymbol{\beta}[\mathrm{Y}] \end{vmatrix} = 0, \tag{6}$$

which may be written

$$\begin{vmatrix} \mathbf{z}[\mathrm{U}] & \mathbf{z}[\mathrm{X}] & \mathbf{z}[\mathrm{Y}] \\ \boldsymbol{\alpha}[\mathrm{U}'] + \left(\frac{\alpha^n}{1-\alpha^n}\right)\boldsymbol{\alpha}[\mathrm{U}'] & \boldsymbol{\alpha}[\mathrm{X}] & \boldsymbol{\alpha}[\mathrm{Y}] \\ \boldsymbol{\beta}[\mathrm{U}'] + \left(\frac{\beta^n}{1-\beta^n}\right)\boldsymbol{\beta}[\mathrm{U}'] & \boldsymbol{\beta}[\mathrm{X}] & \boldsymbol{\beta}[\mathrm{Y}] \end{vmatrix} = 0, \tag{7}$$

and deducting the equation

$$\begin{vmatrix} \mathbf{z}[\mathrm{U}'] & \mathbf{z}[\mathrm{X}] & \mathbf{z}[\mathrm{Y}] \\ \boldsymbol{\alpha}[\mathrm{U}'] & \boldsymbol{\alpha}[\mathrm{X}] & \boldsymbol{\alpha}[\mathrm{Y}] \\ \boldsymbol{\beta}[\mathrm{U}'] & \boldsymbol{\beta}[\mathrm{X}] & \boldsymbol{\beta}[\mathrm{Y}] \end{vmatrix} = 0 \tag{8}$$

from equation (7), the remainder is

$$\begin{vmatrix} \mathbf{z}[\mathrm{U}]-\mathbf{z}[\mathrm{U}'] & \mathbf{z}[\mathrm{X}] & \mathbf{z}[\mathrm{Y}] \\ \left(\dfrac{\alpha^n}{1-\alpha^n}\right)\boldsymbol{\alpha}[\mathrm{U}'] & \boldsymbol{\alpha}[\mathrm{X}] & \boldsymbol{\alpha}[\mathrm{Y}] \\ \left(\dfrac{\beta^n}{1-\beta^n}\right)\boldsymbol{\beta}[\mathrm{U}'] & \boldsymbol{\beta}[\mathrm{X}] & \boldsymbol{\beta}[\mathrm{Y}] \end{vmatrix} = 0. \tag{9}$$

Equation (9) is the equation from which the cyclic correction term $\mathbf{z}[\mathrm{U}]-\mathbf{z}[\mathrm{U}']$ is found.

When n is (as it usually is) 104 or more, the term in $\beta^n/(1-\beta^n)$ becomes negligible, while $\boldsymbol{\beta}[\mathrm{Y}]$ is very small. Very little accuracy will be lost by taking both these quantities equal to zero. The determinant (9) then becomes

$$\begin{vmatrix} \mathbf{z}[\mathrm{U}]-\mathbf{z}[\mathrm{U}'] & \mathbf{z}[\mathrm{X}] & \mathbf{z}[\mathrm{Y}] \\ \left(\dfrac{\alpha^n}{1-\alpha^n}\right)\boldsymbol{\alpha}[\mathrm{U}'] & \boldsymbol{\alpha}[\mathrm{X}] & \boldsymbol{\alpha}[\mathrm{Y}] \\ 0 & \boldsymbol{\beta}[\mathrm{X}] & 0 \end{vmatrix} = 0, \tag{10}$$

and dividing by $\boldsymbol{\beta}[\mathrm{X}]$ and multiplying out

$$\mathbf{z}[\mathrm{U}]-\mathbf{z}[\mathrm{U}'] = \left(\frac{\alpha^n}{1-\alpha^n}\right)\frac{\boldsymbol{\alpha}[\mathrm{U}']}{\boldsymbol{\alpha}[\mathrm{Y}]}\mathbf{z}[\mathrm{Y}], \tag{11}$$

and this is the addition to be made for the cyclic part of the benefit.

$\boldsymbol{\alpha}[\mathrm{U}']$ when expanded is $(1-\alpha^{n_1})r_1+(\alpha^{n_1}-\alpha^{n_1+n_2})r_2+(\alpha^{n_1+n_2}-\alpha^{n_1+n_2+n_3})r_3$, each term referring to a benefit period at a different rate of pay, and the additions to be made to the expected sickness calculated for the first $n_1+n_2+n_3$ weeks are:

to the first rate, $r_1(1-\alpha^{n_1})\left\{\left(\dfrac{\alpha^n}{1-\alpha^n}\right)\Big/\boldsymbol{\alpha}[\mathrm{Y}]\right\}\boldsymbol{\Theta}[\mathrm{Y}]$,

to the second rate, $r_2(\alpha^{n_1}-\alpha^{n_1+n_2})\left\{\left(\dfrac{\alpha^n}{1-\alpha^n}\right)\Big/\boldsymbol{\alpha}[\mathrm{Y}]\right\}\boldsymbol{\Theta}[\mathrm{Y}]$,

to the third rate, $r_3(\alpha^{n_1+n_2}-\alpha^{n_1+n_2+n_3})\left\{\left(\dfrac{\alpha^n}{1-\alpha^n}\right)\Big/\boldsymbol{\alpha}[\mathrm{Y}]\right\}\boldsymbol{\Theta}[\mathrm{Y}]$.

These can be quickly calculated with the aid of Appendix I.

APPENDIX III

Application to another sickness experience

In this Appendix an attempt is made to apply the method to the sickness rates for married women given in the National Health Insurance, Valuation Regulations (S.R. and O. No. 281 of 1938). In these tables only two independent rates are given. They are the sickness rates for the first twenty-six weeks excluding the first three days and the sickness rates for after the first twenty-six weeks. The information is clearly inadequate to permit treatment in the way the Manchester Unity experience has been discussed in the paper. Further assumptions are necessary, and it will be found that for a satisfactory solution two must be made.

We first postulate that the double exponential law still holds. By the nature of the information available this cannot possibly be more than an assumption. The second assumption is different in the two methods shown in this Appendix. In one method we simply take over the values of α and β (found in § 14 of the paper) for the Manchester Unity (A.H.J.) experience. In the other we assume that the shape of the sickness curve for married women is the same as the shape of a curve in the Manchester Unity experience—not at the same age but at an 'equivalent' age obtained by selecting the age at which the same value of $\frac{{}_{26}\mathbf{z}}{{}_{\frac{1}{2}}\mathbf{z}}$ is obtained. Having done that we proceed to obtain values of α and β which do not vary with age, using the method of § 14.

Method I

The assumption is made that α and β for which the table in Appendix I has been calculated are equally applicable to the table now under discussion. This method has the merit of simplicity and the advantage of values of α^r and β^r ready calculated. The values of ${}_r\mathbf{z}$ for all values of r are given by the equation

$$\begin{vmatrix} {}_r\mathbf{z}_x & {}_{\frac{1}{2}}\mathbf{z}_x & {}_{26}\mathbf{z}_x \\ \alpha^r & \alpha^{\frac{1}{2}} & \alpha^{26} \\ \beta^r & \beta^{\frac{1}{2}} & \beta^{26} \end{vmatrix} = 0, \tag{1}$$

and inserting the numerical values the equation becomes

$$\begin{vmatrix} {}_r\mathbf{z}_x & {}_{\frac{1}{2}}\mathbf{z}_x & {}_{26}\mathbf{z}_x \\ (\cdot 997)^r & \cdot 999 & \cdot 930 \\ (\cdot 926)^r & \cdot 962 & \cdot 136 \end{vmatrix} = 0. \tag{2}$$

When this equation is solved for standard periods the following values are obtained:

r	α^r	β^r	Coefficients of	
			${}_{\frac{1}{2}}\mathbf{z}_x$	${}_{26}\mathbf{z}_x$
0	1·000	1·000	+1·046	− ·048
$\frac{1}{2}$	·999	·962	+1·000	·000
13	·964	·369	+ ·279	+ ·736
26	·926	·136	·000	+1·000
52	·865	·019	− ·132	+1·071
104	·749	·000	− ·134	+ ·949

Since we are using predetermined values of α and β these coefficients are independent of the mortality and interest basis chosen. Once that is fixed, all that remains to be done is to form commutation columns of $K_x^{\frac{1}{2}|25\frac{1}{2}}$ and $K_x^{26|26}$, to prepare valuation factors by dividing by D_x and to proceed with the valuation on similar lines to that in § 16, except that ${}_{\frac{1}{2}}\mathbf{z}$ and ${}_{26}\mathbf{z}$ are used, instead of the $\mathbf{z}[X]$ and $\mathbf{z}[Y]$ defined by equations (25) and (26).

Method II

It may be argued that Method I is based on an unwarrantable assumption, and that there is no reason at all why the A.H.J. values of α and β should apply to married women's sickness rates. In other words, it may be felt that Method I does not make the best use of admittedly inadequate information. The second method indicated above is therefore proposed as one which may prove to be more acceptable.

This method follows the least square technique outlined on p. 102. The evaluation of α^{13} and β^{13} requires three sickness rates (or combinations of rates) at each of a series of selected ages. In the N.H.I. table only two rates are given, that is to say, ${}_{\frac{1}{2}}\mathbf{z}-{}_{26}\mathbf{z}$ and ${}_{26}\mathbf{z}$. Since under normal circumstances we shall require rates *including* the first three days, an obvious choice of the third rate would be ${}_{0}\mathbf{z}$ if such a quantity were available. To estimate ${}_{0}\mathbf{z}$ we must make some assumption about the way it is related, age by age, to the rates already available.

In the method now to be described, we found this relationship by reference to the Manchester Unity table. §4 of the paper discusses the variation with duration of the Manchester Unity (Whole Society) rates. In columns (2) and (3) of Table 1 are given the values of α^{13} and β^{13} which we found by fitting the fundamental equation (2) to the tabular values of ${}_{0}\mathbf{z}$, ${}_{13}\mathbf{z}$, ${}_{26}\mathbf{z}$ and ${}_{52}\mathbf{z}$ at ages 20, 30, 40, ..., 90. From these values of α^{13} and β^{13}, and using the method of § 8, we evaluated ${}_{\frac{1}{2}}\mathbf{z}$ and hence the ratios ${}_{\frac{1}{2}}\mathbf{z}/{}_{0}\mathbf{z}\,(\equiv{}_{\frac{1}{2}}\zeta)$ and ${}_{26}\mathbf{z}/{}_{\frac{1}{2}}\mathbf{z}\,(\equiv{}_{26}\zeta/{}_{\frac{1}{2}}\zeta)$ at those ages. We then plotted ${}_{\frac{1}{2}}\zeta$ against ${}_{26}\zeta/{}_{\frac{1}{2}}\zeta$, and from the graph read off values of ${}_{\frac{1}{2}}\zeta$ for values of ${}_{26}\zeta/{}_{\frac{1}{2}}\zeta$ from ·10 to ·80 at intervals of ·05. The results are shown in Table A.

The assumption which was made to evaluate ${}_{0}\mathbf{z}$, and which appears to be the most reasonable available within the limits of the method, is that the relationship which ${}_{r}\zeta$ (where r has a selected value) bears to ${}_{26}\zeta/{}_{\frac{1}{2}}\zeta$ for the Manchester Unity (Whole Society) rates applies also (at the 'equivalent' age) to the N.H.I. rates. The values of ${}_{0}\mathbf{z}\,(={}_{\frac{1}{2}}\mathbf{z}/{}_{\frac{1}{2}}\zeta)$ were thus found from Table A and are tabulated for ages 20, 30, 40, 50 and 60 in Table B.

The next step was to select periods or combinations of periods for $\mathbf{z}[X]$ and $\mathbf{z}[Y]$. The first choice was ${}_{\frac{1}{2}}\mathbf{z}$ and ${}_{26}\mathbf{z}$, but the similarity between ${}_{0}\mathbf{z}$ and ${}_{\frac{1}{2}}\mathbf{z}$ made the fitting of ${}_{0}\mathbf{z}$ by least squares rather unsatisfactory. Another choice was therefore necessary, and $\mathbf{z}[X]$ and $\mathbf{z}[Y]$ were chosen as defined in equations (25) and (26) of the paper because they seemed to offer a reasonable solution and because α and β are fairly easy to evaluate from the resulting quartic.

It was then necessary to estimate the values of $\mathbf{z}[X]$ and $\mathbf{z}[Y]$ at the selected ages (20 ... 60), and we again assumed that the relationship between ${}_{r}\zeta$ and ${}_{26}\zeta/{}_{\frac{1}{2}}\zeta$ is the same (at equivalent ages) for the Manchester Unity and N.H.I. rates. The parallel procedure was followed as for ${}_{0}\mathbf{z}$. The numerical results are shown in Table A, where $\mathbf{z}[X]/{}_{0}\mathbf{z}$ and $\mathbf{z}[Y]/{}_{0}\mathbf{z}$ are tabulated for the Manchester Unity tables, and in Table B, where $\mathbf{z}[X]$ and $\mathbf{z}[Y]$ are shown for the married women's table at the relevant ages.

When $\mathbf{z}[U] = {}_0\mathbf{z}$, equation (27) of the paper may be written

$$ {}_0\mathbf{z} = A\mathbf{z}[X] + B\mathbf{z}[Y]. \tag{3}$$

We show in § 14 the procedure for solving for A and B by the method of least squares.

The resulting quartic in α^{13} is

$$1 = 1\cdot129\left[1 - \frac{\alpha^{13}}{2} - \frac{\alpha^{26}}{4} - \frac{\alpha^{52}}{4}\right] + 4\cdot588\frac{\alpha^{52}}{4} \tag{4}$$

of which the relevant solutions are $\alpha^{13} = \cdot926$ and $\beta^{13} = \cdot210$.

Table A. Manchester Unity (all groups)

${}_{26}\zeta/{}_{\frac{1}{3}}\zeta$	${}_{\frac{1}{3}}\zeta$	$\mathbf{z}[X]/{}_0\mathbf{z}$	$\mathbf{z}[Y]/{}_0\mathbf{z}$
·10	·9138	·875	·011
·15	·9255	·823	·020
·20	·9340	·780	·030
·25	·9408	·732	·040
·30	·9470	·685	·051
·35	·9534	·635	·062
·40	·9597	·585	·073
·45	·9655	·535	·084
·50	·9705	·485	·096
·55	·9750	·436	·107
·60	·9795	·388	·119
·65	·9832	·339	·132
·70	·9867	·290	·145
·75	·9900	·242	·159
·80	·9928	·195	·173

Table B. Married women's sickness rates

Age	${}_{26}\zeta/{}_{\frac{1}{3}}\zeta$	${}_{\frac{1}{3}}\mathbf{z}$	${}_{\frac{1}{3}}\zeta$	${}_0\mathbf{z}$	$\mathbf{z}[X]$	$\mathbf{z}[Y]$
20	·184	3·85	·9315	4·13	3·283	·110
30	·394	3·60	·9590	3·75	2·213	·267
40	·530	3·87	·9733	3·98	1·851	·410
50	·659	5·42	·9843	5·51	1·818	·733
60	·767	9·23	·9908	9·32	2·125	1·519

When α and β are fixed, values of ${}_r\mathbf{z}$ for all values of r can be found from the equation:

$$\begin{vmatrix} {}_r\mathbf{z} & \mathbf{z}[X] & \mathbf{z}[Y] \\ \alpha^r & \boldsymbol{\alpha}[X] & \boldsymbol{\alpha}[Y] \\ \beta^r & \boldsymbol{\beta}[X] & \boldsymbol{\beta}[Y] \end{vmatrix} = 0. \tag{5}$$

Since we have only found the values of $\mathbf{z}[X]$ and $\mathbf{z}[Y]$ (as defined by equations (25) and (26) of the paper) for the five ages in Table B, we propose to use ${}_{\frac{1}{3}}\mathbf{z}$ and ${}_{26}\mathbf{z}$ instead, and the analytic form of the equation becomes (as in Method I)

$$\begin{vmatrix} {}_r\mathbf{z} & {}_{\frac{1}{3}}\mathbf{z} & {}_{26}\mathbf{z} \\ \alpha^r & \alpha^{\frac{1}{3}} & \alpha^{26} \\ \beta^r & \beta^{\frac{1}{3}} & \beta^{26} \end{vmatrix} = 0. \tag{6}$$

When numerical values are inserted it becomes

$$\begin{vmatrix} {}_{r}\mathbf{z} & {}_{\frac{1}{2}}\mathbf{z} & {}_{26}\mathbf{z} \\ (\cdot 926)^{r/13} & \cdot 9970 & \cdot 8575 \\ (\cdot 210)^{r/13} & \cdot 9416 & \cdot 0439 \end{vmatrix} = 0. \qquad (7)$$

The values of the coefficients of ${}_{\frac{1}{2}}\mathbf{z}$ and ${}_{26}\mathbf{z}$ for standard periods calculated from this determinant are shown below:

r	α^r	β^r	Coefficients of	
			${}_{\frac{1}{2}}\mathbf{z}$	${}_{26}\mathbf{z}$
0	1·0000	1·0000	+1·066	− ·073
$\frac{1}{2}$	·9970	·9416	+1·000	·000
13	·9260	·2095	+ ·182	+ ·868
26	·8575	·0439	·000	+1·000
52	·7353	·0019	− ·040	+ ·904
104	·5407	·0000	− ·031	+ ·667

Equations (5) and (6) may give slightly different numerical results because $\mathbf{z}$ [X] and $\mathbf{z}$ [Y] were obtained by reference to Manchester Unity curves for which values of α and β independent of age were not assumed.

The coefficients in the above table will apply to ${}_{\frac{1}{2}}\mathbf{z}$ and ${}_{26}\mathbf{z}$ when combined with any suitable mortality table and rate of interest.

Comparison of methods

The rates for standard periods on these two methods are compared below:

Function ...	${}_{0}\mathbf{z}$		${}_{13}\mathbf{z}$		${}_{52}\mathbf{z}$		${}_{104}\mathbf{z}$	
Method ...	I	II	I	II	I	II	I	II
Age								
20	4·00	4·05	1·60	1·32	·25	·49	·16	·35
30	3·70	3·73	2·05	1·89	1·05	1·14	·87	·84
40	3·95	3·98	2·59	2·48	1·69	1·70	1·43	1·25
50	5·50	5·52	4·14	4·08	3·11	3·01	2·66	2·21
60	9·31	9·32	7·79	7·82	6·36	6·03	5·48	4·44

In the absence of more complete observed data it must remain a matter of opinion which is the better solution.

ABSTRACT OF THE DISCUSSION

Mr J. C. S. Hymans, in presenting the paper, said that the really important part of the paper was contained in §§ 12 and 13, where it was shown that the system put forward did not demand the use of exponentials nor, indeed, of any other particular mathematical functions, so long as they were functions of duration only.

The possibility of further developments along the same lines was a matter to which the authors had not, so far, been able to give any real attention, but he still had hopes that further researches would produce functions which applied with acceptable accuracy to any sickness table. Possibly that was being a little optimistic.

The later sections of the paper mainly described the method which the authors had evolved for valuation purposes. They had found it simple to use and invaluable in their work.

Mr J. K. Scholey, in opening the discussion, said that by the term 'standard periods' were understood periods for which the rates of sickness were tabulated in the sickness table being used for valuation. The valuation functions at certain rates of interest corresponding to such periods would also be available so that the process of valuation in such instances presented no difficulty. By the term 'non-standard periods' were understood periods for which certainly the valuation factors were not available and probably even the basic sickness rates were not tabulated. The normal procedure, therefore, when valuing sickness benefits for non-standard periods, was for the actuary to decide what sickness rates to use; as a guide, he would probably take into consideration the rates for the nearer standard periods, or he might choose to have recourse to tables such as those prepared by Rhodes.

Having decided the rates of sickness, the actuary next constructed any commutation functions that might be required. The authors said that that was a formidable problem, but it was, of course, formidable only because it was long and not because it was complicated. Alternatively, the valuer could easily—and perhaps legitimately—calculate factors at every fifth age by the construction of shortened forms of commutation functions. In contrast to those two methods, which were both based upon the assumption that the absolute rates of sickness had first of all been settled at every age or at any rate at every fifth age, the authors suggested valuing with the aid of a mathematical formula which enabled the results of the valuation of certain standard period benefits to be adjusted so as to give the values of benefits for non-standard periods.

In Tables 8 and 9 the authors had given the values obtained by the use of the formula for various benefits and various membership distributions, so that it could be seen whether the formula gave the correct values for examples where the calculations could be readily made by standard methods. The authors claimed that the comparisons made in those tables showed that over most of the normal standard periods there was close agreement between the true values which they referred to as 'actual' and their approximate values which they referred to as 'calculated'. They further claimed that the ratios of calculated to actual for corresponding periods were reasonably similar for both experience and valuation. To examine whether those results were fortuitous it was necessary to compare figures at individual ages.

The table on p. 118 gives the results for certain ages.

For 'first 7 weeks' sickness the calculated sickness rates were substantially lower than those given by Cd. 6907, the deficit being 42 % at age 18, 37 % at age 38, 27 % at age 53 and 7 % at age 68.

If the aim was to value by the Manchester Unity sickness table those figures showed that the authors' method might well be as inaccurate and just as sensitive to variations in age distribution as the first method which they criticized in their opening section. Moreover, was it certain that the method always involved so much less labour than the previous methods? At the end of § 9 the authors suggested that the mathematical formula was reasonably applicable provided that the values of α and β were chosen with proper regard to the ultimate purpose of the particular investigation, but later on, in § 14, they stated that the values of α and β calculated in that section would be adequate for most purposes. Perhaps at some stage the authors would deal in rather more detail with the circumstances under which they suggested that α and β would need recalculation. It did seem that if fixed values of α and β were used for all purposes the accuracy of the results would be fortuitous, whereas if different values of α and β were used for different valuations then the labour involved in the method might be considerable.

When a valuation was being made the first stage was to compare the expected sickness on the valuation basis with the actual experience and to make any necessary adjustment, whether positive or negative, to the sickness rates. But if a method such as the authors' was being used, there might be a 5 % difference between the percentage of the calculated to the true expected sickness and the percentage of the calculated to the true value of the sickness liability (see, for example, the 'first 104 weeks' figures in Table 8). The value of benefits might be 5 % out and, since the value of contributions was not affected by the use of the authors' method, there would be an error of more than 5 % in the reserve. In fact, the error might be much more than that for certain distributions of membership. Furthermore, at a valuation the actual claims could be compared with the expected, and so some of the errors involved in making an approximate valuation could perhaps be corrected, but a similar check could not be made when the rates of contribution for a new table of benefits were being calculated; nor was there a check when the cost of altering a period of benefit from, perhaps, 26 weeks to 20 weeks was being estimated.

Percentage of 'calculated' to 'actual'

Age	First 13 weeks			First 104 weeks			All periods	
	Sickness rate %	Value throughout life %	Value to age 65 %	Sickness rate %	Value throughout life %	Value to age 65 %	Sickness rate %	Value throughout life %
18	85	93	92	120	100	102	125	103
38	90	96	96	104	95	95	108	100
53	95	101	100	98	81	88	102	99
68	105	105	—	85	92	—	98	99

It seemed then that, if an approximate method for the valuation of non-standard periods were to be adopted, it should keep much closer to the tabulated rates; it was possible to find such a method which fitted the Cd. 6907 figures and might perhaps be used for other tables. The rate of sickness z^n (when n was less than 13 weeks) could be represented by the formula $a+bz^{13}+cz^{13/13}$, where a, b and c were functions of n but were independent of age and could be obtained once and for all by the method of least squares. The procedure was to value first the life or temporary annuity, depending upon when the sickness ceased; secondly the 'first 13 weeks' sickness; and thirdly the 'second 13 weeks' sickness. The totals could then be multiplied by a, b and c respectively. As a test he had calculated the necessary values when $n=1$, 4, 7 and 10 weeks (separate values of a, b and c being obtained, of course, for each value of n) and had checked the results against the Cd. 6907 figures. When n was 10 the differences at each age between the Cd. 6907 figures and those derived from the formula were no more than 1 % and the maximum difference was $1\frac{1}{2}$ % at age 70, which was the highest age tabulated. Where n was 7 the differences were usually below 1 %, although there was a difference of $1\frac{1}{2}$ % at age 16, the youngest age. Where n was 4 the differences in rates were not more than 1 % at any age except at age 65 where there was an excess of 2 % and at age 70 where there was a deficit of 3 %. Even taking the extreme case when n was unity, the difference in rates was not more than $3\frac{1}{2}$ % except at age 16 where it was 7 % and age 70 where it was 5 %. In dealing with the 'second 13 weeks' sickness the formula $az^{13}+bz^{13/13}+cz^{26/26}$ could be used, though for most purposes a two-term function would probably be satisfactory. The three-term function had been used for $n=16$, and the resulting rates of sickness were only 1/1000th of a week away from the actual at any age except at ages 16 and 25, there being a deficit of 2/1000ths of a week at the former age and a similar excess at the latter age. Such a method did give both ease of calculation and sufficient accuracy, and at the same time it enabled the calculator to check the resulting values more easily than could be done when a more powerful mathematical formula was used.

There was one point of principle which might, perhaps, be overlooked when considering the authors' paper because it fell outside the original ambit of the paper; that was to understand what was meant by, for example, the 'first six weeks' sickness rate. By that was not meant some figure which was abstracted blindly from a standard table but the average number of weeks of payment which it was expected would fall within the first six weeks of incapacity in the society being valued. The beginning of the incapacity was determined by the rules of the particular society, and the question whether a member would or would not claim depended not only upon whether he was sick but also to some extent on such factors as whether he was in employment, the general state of employment at the time, his wages, the rate of sick pay to which he would be entitled if he claimed, and also, no doubt, on his own conscience. In face of all those factors, was there not a danger of trying to be too precise in the methods by which the rate of sickness to be valued at a single age was estimated, particularly where short-duration sickness was under consideration? When benefits for non-standard periods at the lower durations of sickness were to be valued, a rough approximation by reference to the nearest standard tables might be as proper a method as the computation of a progression of rates for the non-standard periods by a complicated mathematical formula.

On that same question of taking figures direct from a table, it seemed that the method proposed for the valuation of cyclic benefits might be open to criticism. If the valuation table was the Manchester Unity experience, the authors' method presupposed that the sickness in, say, the third year of sickness would be the same as that in the standard table. Was that likely? In the Manchester Unity experience all sickness was paid for and would therefore be declared, and where a claim came within 52 weeks of the previous claim the claims were linked up. But where cyclic benefits were paid a member would be unlikely to register a period of sickness for which he received no benefit. The result might be that even though the total weeks of sickness benefit paid for were the same as those expected on the standard table, their division into weeks remunerated at full, half and quarter pay might well differ. On that score, there seemed reason to doubt whether the accuracy pretended by the authors' method for cyclic benefits would necessarily be obtained.

Another question of principle was that of the off-period, which was not mentioned in the paper. That was a pity, because a different off-period was bound to affect profoundly the swing of the sickness curve. Supposing that the same values of α and β could be adopted for all societies with the same off-period, different values would still be needed for societies with different off-periods. After all, though the total sickness rates were, by and large, unaffected by the length of the off-period, the bulk of the claims would lie in the first six months of sickness, perhaps at all ages, if there were no off-period; but if there were a society with the unusually long off-period of, say, 40 years, then there would be practically no sickness in the first six months except that due to new entrants.

When dealing with the application of their method in Appendix III, the authors remarked that two different sets of results were obtained from separate assumptions and that neither could be judged better than the other. Their method, however, when applied to the Manchester Unity tables, seemed to give too low values for sickness at the early durations, and the same fault might lie in the figures of the Appendix. As an experiment he himself had obtained other figures by a third method in which the proportions of the total sickness at each age falling within the periods 'first 13 weeks', 'next 13 weeks', and so on, had been taken as those obtained from the Manchester Unity experience at equivalent ages, the equivalent age being obtained by considering the relationship $z^{26/\text{all}}$ to $z^{\frac{1}{2}/\text{all}}$. That method seemed to have as much in its favour as the two methods given by the authors and was much simpler. The resulting figures for 'first 13 weeks' sickness were higher at nearly all ages than those obtained by the authors' method II, the excess at age 20 being 10%; for 'after 104 weeks' sickness the rates were lower at most ages than those given by either method.

He had criticized a number of points in the authors' paper, but the method had been used in practice, and he had to admit that in practice many of those criticisms might be found to be insignificant; but at least they were there, and perhaps another actuary would feel unable to use the method without further research. Quite apart, however, from the practical questions of valuation, the theoretical consideration given to the problems involved was bound to be of help to anyone pursuing the matter further, and in particular to anyone who was considering the graduation of the rates of sickness for the different periods at each age by a mathematical formula, whether that formula was the authors' or another.

Miss P. Merriman quoted from the cover of a recent number of *J.S.S.*:

> I maintain that the actuary should not abandon himself to mere empiricism. I maintain that he should be the person most suited to develop actuarial technique towards a practical end. In an actuarial department it is not a question of practical men and theoreticians: it is a question of practical men who do not know theory and practical men who do know it. E. P. CANTELLI

That was what the authors had done in their paper. They had presented a new practical technique which was both simpler to apply and theoretically more justifiable than the methods used hitherto.

She wished to compare the authors' method with two other approximate methods based on Cd. 6907. Both those methods depended on the construction of ratios of non-standard to standard periods of sickness from Cd. 6907. For the experience the ratios were obtainable directly, and for the valuation they were obtained by dividing the sum of the sickness rates for the desired period from valuation age to age 70 by the corresponding sum for the standard period, thus making some allowance for the increasing proportion of later-period sickness as age increased.

The first method, which she called Cd. I, consisted of applying to the standard experience and valuation factors at each age the appropriate ratios to obtain experience and valuation factors applicable to the special periods. That method was an accurate application of Cd. 6907 for the experience factors, but it avoided the construction of special commutation columns for the valuation factors. Though the results were suitable for practical purposes, the method was open to the theoretical objection that the experience and valuation bases were not strictly consistent.

The second method, which she called Cd. II, consisted of using the ratios as calculated above as over-all adjustments in total to the experience and valuation calculations, using the ratios according to the average age. The practical results obtainable by that method differed very little from those obtained by Cd. I. The inconsistency of experience and valuation bases remained, and it might be magnified if the average age of the exposed to risk differed from the average age of the members at the valuation date.

Those two methods both, in effect, expressed factors for non-standard periods as linear combinations of the five standard Manchester Unity factors. The new method of the paper, by suitably combining the five Manchester Unity factors, expressed factors for non-standard periods as linear combinations of the two factors, [X] and [Y], and at the same time showed that the multipliers could be assumed to be independent of age without the inconsistency she had mentioned. Thus the work of calculation was at least halved, and at the same time the method was theoretically reasonable. Furthermore, whereas any alteration in the rate of interest or mortality basis rendered useless the factors calculated for Cd. I, the constants calculated by the new method served for the experience, and for the valuation on any basis.

As a numerical illustration, she had selected at random a lodge from an affiliated order and had valued one of the whole-life tables by the three methods. The sickness benefits were 10*s.* first 21 weeks; 5*s.* next 22 weeks; and 2*s.* 6*d.* next 22 weeks.

There were 155 years of exposure to risk with an age-range 43–98 and average age 66, and 28 members at the valuation date with an age-range 53–98 and average age 69. The absence of continuous pay under the scheme made a good test of the new method, though it was rather unfortunate in the light of the opener's remarks that there were no young members. The basis was: sickness, Manchester Unity A.H.J.; mortality, Eastern Counties; and interest, $3\frac{1}{4}\%$ p.a. The numerical results were as follows:

EXPERIENCE

Method	Full pay	Reduced pay	Further reduced pay	Total
	£	£	£	£
Hymans and Lane	170	32	10	212
Cd. I	163	40	12	215
Cd. II	164	37	14	215

VALUATION

Method	Full pay	Reduced pay	Further reduced pay	Total
	£	£	£	£
Hymans and Lane	304	77	30	411
Cd. I	298	85	35	418
Cd. II	298	85	35	418

The close agreement of the three methods was a striking argument in favour of the adoption of the simplest of them, the one proposed in the paper.

Mr M. T. L. Bizley proposed to confine his remarks to one application of the authors' method, viz. to the valuation of a type of benefit which was not dealt with at all in the paper.

He had recently had occasion to value a Friendly Society which offered a lump sum as a sickness benefit—£5 payable after six weeks' continuous sickness, with no payment in the event of sickness of any shorter duration. That was, of course, a non-standard type of benefit as distinct from a standard benefit for a non-standard period, and the difficulty in dealing with it was that it could not be directly expressed in terms of weeks of sickness per annum for any periods whether standard or not. He had, however, he thought, succeeded in finding a way of adapting the method of the paper to enable the benefit to be valued with very little trouble.

The adaptation he had in mind was extremely simple. The proportion of claims in a year of age appeared to be given by the differential coefficient with respect to r of the authors' function ${}_r\mathbf{z}_x$ and in the particular case $r=6$, corresponding to six weeks. In practice it would be a wise precaution to put $r=5$, say, instead of 6 to allow for the very human tendency not to recover after about five weeks' sickness when a few days more would bring in the lump sum benefit. With $r=5$, it was found that the differential coefficient took the simple form, in the authors' notation,

$$\underset{(r=5)}{\frac{d}{dr}}\,{}_r\mathbf{z}_x = \mathrm{F}_x\,\alpha^5 \operatorname{colog}\alpha + \mathrm{G}_x\,\beta^5 \operatorname{colog}\beta.$$

But that was in the same form as the original expression for ${}_5\mathbf{z}_x$, except for the introduction of colog α and colog β. The lump sum benefit could, it seemed, be valued by the method of the paper as if the benefit were a weekly sum payable for every week's sickness after the fifth, slightly different coefficients being used. The precise amendment needed to the coefficients was obvious from the result of differentiating equation (32) of the paper. The values of $\text{colog}_e\,\alpha$ and $\text{colog}_e\,\beta$ were ·003 and ·077 respectively.

A complication which arose in practice was that under the rules the lump sum benefit was payable to a member once only in his lifetime. It might, however, be possible to overcome that difficulty by simply incorporating in the basic l_x column a series of decremental rates, each term in the series being in exact correspondence to the proportion of claims in the appropriate year of age. By that method, a life who had claimed was automatically removed from the exposure.

While the authors' method would undoubtedly be of use in valuing the normal benefit for a non-standard period, it might prove in practice to be equally useful for valuing a benefit of a non-standard type.

Mr G. Heywood welcomed a paper on special periods of sickness rates. There had been very little work along those lines, and the subject was one of great importance to the actuary who was engaged in Friendly Society practice.

The paper fell naturally into two parts—the mathematical theory of the method and the practical application of that theory. He had little comment to make on the first part except that he thought the method was ingenious and that its ingenuity had the advantage of producing a practical working method. It was natural to compare the authors' method with other methods of dealing with the problem and he had been particularly interested to compare it with the method developed by the late Francis Rhodes in *J.I.A.* Vol. LXXII. He did not think that the authors' theory was so readily obvious as the theory behind Rhodes's method, because in the latter method it was possible to see the objective at every stage of the calculations. In the authors' method that was rather more difficult. For example, it was not easy to estimate at the start of the calculations the probable size or even the sign of the coefficients derived from Appendix I. The first result which could be readily checked by general reasoning was the final answer itself.

It was in the practical application of the method that his main interest lay because during the past few years he had encountered the problem on many occasions. Many of the previous valuations had actually been made by Rhodes, using his own method, and, in making the next valuation, he (the speaker) had followed, in almost every case, Rhodes's procedure. Considerable labour was sometimes involved because it was necessary to go back to the individual sickness rates and to construct fresh commutation columns and valuation factors. On other occasions he had been more fortunate; a change of basis was not necessary and he could simply use Rhodes's original factors. He had therefore been very anxious, on reading the authors' paper, to compare the results which their method might bring out with some of those which he already had available on the basis of Rhodes's method.

Accordingly, he had selected four districts of an Affiliated Order, all of which had benefits for special periods. That Order was valued on the basis of its own experience, but he had used the values of α and β and the coefficients of [X] and [Y] from Appendix I of the paper. Taking each of the four districts in turn the results by the authors' method were respectively 100·0%, 100·0%, 100·8% and 99·6% of the results by Rhodes's method. The figures showed an extremely close agreement in the results produced by the two methods.

In applying the method, he had been struck by the small amount of arithmetic which was entailed when once the values of α and β and the coefficients of [X] and [Y] had been found. That was in marked contrast to the lengthy arithmetic of Rhodes's method, a fact which would, indeed, weigh against the latter. He was, therefore, led to consider whether Rhodes's method could be shortened and it appeared that it might be possible to do so by using the same principle but applying it to valuation totals instead of to sickness rates at individual ages. That process would eliminate the construction of commutation columns and valuation factors and would reduce the calculations to a valuation by standard factors of 'first 26 weeks', 'second 26 weeks', and the remainder of sickness. The value of the benefits for special periods would be found by taking percentages of the total present values, the appropriate percentages being obtained from Rhodes's tables in the ordinary way. The arithmetic of his method was thus reduced to that of a normal valuation.

Trying that modification on the four districts of the Affiliated Order to which he had already referred, he had obtained results respectively of 98·7%, 99·3%, 99·2% and 100·4% of the results obtained by the use of Rhodes's method in full, which was a sufficiently close agreement. The figures for the four districts of the Order by the three methods were shown in the following summary.

District (1)	Benefits (2)	Present value of benefits: Rhodes's method (3)	Authors' method (4)	Rhodes's modified (5)	Percentage of (4) to (3) (6)	Percentage of (5) to (3) (7)
		£	£	£	%	%
A	$\begin{bmatrix} 10 & 10 & R \\ 1 & \frac{1}{2} & \frac{1}{4} \end{bmatrix}$	29,653	29,639	29,275	100·0	98·7
B	$\begin{bmatrix} 13 & 13 & R \\ 1 & \frac{1}{2} & \frac{1}{4} \end{bmatrix}$	28,588	28,595	28,398	100·0	99·3
C	$\begin{bmatrix} 16 & 16 & R \\ 1 & \frac{1}{2} & \frac{1}{4} \end{bmatrix}$	11,708	11,800	11,615	100·8	99·2
D	$\begin{bmatrix} 20 & 20 & R \\ 1 & \frac{1}{2} & \frac{1}{4} \end{bmatrix}$	5,019	4,998	5,039	99·6	100·4

As a further test of the modification of Rhodes's method, he had calculated the present value of sickness benefit for the 'first 13 weeks', as given in Tables 8 and 9 of the paper. He had used, in each case, the three hypothetical distributions of the authors. The percentages of calculated to actual (which compared directly with the corresponding figures given by the authors in Tables 8 and 9) were, for Table 8, 97·3, 98·7 and 97·7 and, for Table 9, 100·1, 98·5 and 99·6. These percentages were about the same as the corresponding percentages in Table 8 and showed a closer agreement of calculated with actual than the percentages in Table 9. The authors' method and the modification of Rhodes's method seemed to be closely related because they both proceeded to obtain the value of benefits for special periods by a linear combination of valuation totals. But in one method the coefficients by which the totals were multiplied were found by mathematical processes, while in the other method they were found by the use of an empirical table. It appeared, however, from the few calculations which he had made, that similar results were produced by both methods.

Mr L. G. K. Starke, in closing the discussion, remarked that it had centred very much on the practical side. A number of people had done a number of sums to find out for themselves whether the authors had really produced a useful working instrument which gave good results in practice.

He was not a Friendly Society consultant, but he had always been interested in attempts to put actuarial problems on a mathematical basis. He congratulated the authors on having had the courage to bring mathematics to bear, in a new way, on a subject about which very little was really known. He said 'courage', because he could not help feeling that at least among the not-quite-so-young members of the profession those who retained an interest in the mathematical basis underlying actuarial science were perhaps in a minority. He was old enough to remember the days when man was making his first clumsy attempts to take to the air in a heavier-than-air contraption, and he could remember the large numbers of otherwise quite sensible people who made the remark that if God had meant man to fly He would have given him wings. Perhaps they were right; but he would invert the remark that evening and say that he, at least, felt quite sure that God would not have given man mathematical abilities unless He had wanted man to use them.

There were, he supposed, two essentially different ways of bringing mathematics to bear on an actuarial problem or, indeed, on any sort of problem. There was what might be called the philosophical approach, which relied upon processes of pure thought aided by such information of a social, economic, biological, psychological or other character about, for instance, the rate of sickness as was available. A fairly strong conviction might conceivably emerge that the nature of sickness—or whatever other term might be employed to describe the complex of phenomena which went to make up a successful claim to sickness benefit—was such as to demand expression by means of the sum of two declining geometrical progressions. He did not think it was likely, but it might happen. If it did, and if the actuary's convictions were strong enough, he must abide by them and not discard them in favour of, say, arithmetical progressions merely because the latter might be easier to handle. Further, he would want to know the starting points of the two geometrical progressions and the rates at which they ran down—in the same way as, when man at length persuaded himself that the world was round, he was not satisfied with that but had to go on to find ways and means of estimating its diameter.

The other approach to a problem from a mathematical point of view could, he supposed, be called purely empirical. If little was known of the general nature of a subject but calculations had to be made, a tool would be needed to enable the sums to be done easily, systematically, and fairly accurately, the general shape or nature of the tool not being pre-determined by any strong convictions of a philosophical character. Observation and experiment would impose limits which should not be transgressed but within those limits there was freedom, he suggested, to adopt whatever way was most convenient of setting about the particular problem.

In the paper before the meeting that evening, the authors did not claim that they had arrived at a philosophical view of the nature of sickness. It was true that in one passage they had gone so far as to identify the two terms in their double Gompertz formula with long-term and short-term sickness respectively; but it was stated quite clearly elsewhere that the exponential law was not regarded by the authors as having any profound biological significance. That being so, he was inclined to quarrel a little with the use of the word 'law' as being too definite for the purpose the authors had in mind. He would rather have said 'hypothesis'; indeed, he was not sure whether the circumstances required or permitted any stronger word than 'device'.

The paper was not concerned with forecasting changes in the rate of sickness over time, or anything of that nature. It was not really very much concerned with extrapolation of any kind, except that the authors would no doubt claim that the method enabled them to compute, if need be, reasonable figures for 'after 3 years' sickness, 'after 4 years' sickness and so on. The main object was interpolation—to find a way of getting in between the fixed points represented by 0, 13, 26, 52 and 104 weeks, so as to value sickness benefits for non-standard periods. That being so, he wondered whether the authors could not have found something simpler to manipulate than a double Gompertz. He was taking the view, all along, that the approach was empirical rather than philosophical, and that what was wanted was a practical working instrument which did not involve much mathematics and which could be fairly easily applied in practice.

The authors were unable to derive one single 'best' set of solutions by the simultaneous use of all the five points given by the data. They had, it would be remembered, to take a particular set of four points which enabled them to solve their equations and another set of four points to get an alternative set of results. Three other combinations could have been employed; perhaps the authors had experimented with those also, but if so they had not given the results in the paper, and he suspected that they were not quite so easy to handle.

The real trouble, of course, was that the fitting of a double Gompertz was very difficult unless the values of the independent variable were equidistant. For the equidistant case there was to be found in Whittaker and Robinson's *Calculus of Observations* the method known as Prony's method of interpolation by exponentials. The process was laborious and, he thought, to some extent artificial; but it was systematic and it did produce results based on the consistent use of all the data. So far as he could see, however, it was of no avail in the problem under discussion.

It had occurred to him to wonder whether the whole mathematical approach to the problem in the paper might not have been simplified by equating r (the duration) to 13×2^n. Values of 0, 13, 26, 52 and 104 for r would thus correspond to values $-\infty$, 0, 1, 2 and 3 for n. It would then have been possible to look at the sickness rates in relation to equidistant values of the new independent variable, and something simpler than the double Gompertz assumption might have resulted.

Another feature he had noticed was that each of the three little sets of tabular values of ζ running down the middle of Table 2 gave a close approximation to a geometrical progression for $\frac{1-\zeta}{\zeta}$, the ratio of pre-r to post-r sickness. Moreover, the rates of progression of those series for the three quite different ages—35, 54 and 67—did not differ greatly from one another.

Those stray ideas might possibly provide a simpler empirical treatment of the problem which the authors had set themselves. The method they had chosen led them to the galaxy of **z**'s and ζ's on p. 91. He felt sure that the authors would agree with him that p. 91 did not consist of anything more than the simplest algebra; nevertheless, it presented rather an intimidating appearance, and he suspected that many of the readers of the paper would have been quite content to take it as read.

Later on the paper blossomed into a perfect rash of new symbols, and his own first reaction had been to murmur 'Why on earth can't they say what they want to say without putting me to the trouble of learning their own private language?' But that was mere peevishness; anyone who propounded a new idea had the right to invent his own way of expressing it if he made a neat job of it as the authors had done. As far as the editors and the printers were concerned, what they lost on the swings they gained on the roundabouts. If the authors had set out the full contents of their determinants, square brackets and so on every time, they would probably still be waiting for the paper to be printed.

He was unable to add anything to what had already been said about the quality of the results achieved by the authors in their practical examples, about the comparisons with the devices employed in Cd. 6907, or even about Appendix III, dealing with the National Health Insurance Valuation Regulations for the valuation of married women's sickness benefits. There again the authors deserved a pat on the back for their courage; why they had chosen to experiment with such intractable data as the sickness of married women he did not know.

It was fashionable to believe that, since the introduction of centralized sickness benefit arrangements for the whole nation, statistical investigators were about to enter a gold-mine which was full of every conceivable rich vein of raw material. It was not for him to express any thoughts on how much of that potential material would eventually see the light of day in a useful form for actuarial research. As a civil servant, he knew that considerations of finance and manpower were apt to impinge heavily on any programme for the extraction of statistics from official sources; and as an individual with an appetite for figures he knew how rarely he found himself possessed of all he would like to have. However, he could at least suggest that the presentation and discussion of the paper that evening were sure signs that there was among the members of the Institute no lack of energy and initiative to make use of experimental material wherever it could be found. He ended as he began, by congratulating the authors on the ingenuity which they had shown in bringing mathematics to bear on a neglected subject, and doing it in a novel fashion.

The President (Sir George H. Maddex, K.B.E.) said it was perhaps rather surprising that so little attention had been given to the question of breaking down sickness rates so that they could be more easily used in practical problems. One of the main reasons, he presumed, was that it had been exceedingly difficult with the means at command. A vast amount of manpower had to be directed upon a complex analysis of records in order to test against the actual experience the results of particular mathematical or empirical means of breaking down the rates. Secondly, those actuaries who had worked in that field had always been much impressed by the variability of rates of sickness and their dependence upon the special circumstances of each individual society. Consequently, rough and ready methods had been used for the valuation of special types of benefit and actuaries had largely relied upon those few indications of method which had been given in the work of Hardy and Watson.

He wondered why the authors, at the end of their paper, had tackled the analysis of the married women's sickness rates given in the National Health Insurance valuations. He did not know whether anybody had had the curiosity to turn up those rates, but they were, indeed, an extraordinary set of figures and, though they were closely related to the National Health Insurance experience at a certain point of time, he would be most surprised if a valuer came across such an experience more than once in a lifetime in ordinary valuations.

Mr Scholey's remarks on the age twists brought out by the authors' method made him regret they had not given some figures showing their results broken down into the main age-groups, because, of course, the age distribution was of importance in the consideration of a method such as the authors'.

He asked the members to pass a very hearty vote of thanks to the authors for their paper.

Mr R. C. B. Lane, in replying to the discussion, said that it had pointed to the weaknesses—and perhaps the strengths—of the method. The weaknesses did need underlining; they were quite definitely there. Mr Scholey's remarks in that respect were particularly valuable. How they affected the method depended largely upon the use to which the method was being put.

The authors had tried to find a workable method. That, perhaps, was why they had not tried a transformation of the co-ordinates and had come to a double Gompertz instead. The figures varied more than they had hoped and the choice of suitable values of α and β was difficult. They could hardly be blamed for adopting values that served the particular cases that cropped up in practice.

The figures given by one of the speakers showed that the method gave good results for the age distributions, weighted heavily in favour of the higher ages, which occurred in friendly society practice. The authors had made a number of valuations by their method and by the more usual type of method and in that way they had gained confidence in their procedure.

Mr Scholey had remarked that the younger ages had to be watched; if the membership were concentrated at the younger ages the constants would have to be altered or another method used. Personally, he would alter the constants, because the problem was likely to recur. The values of α and β given in the paper were calculated in a way which weighted heavily the higher ages. That was right, because the higher ages were those which occurred in practice. In special circumstances when the weight was predominantly at the lower ages, different values of α and β were clearly indicated.

The illustrations given had all been based on a certain [X] and a certain [Y] as the special linear combinations. Again, they were chosen because they occurred in practice and were similar to the sort of thing that was continually being met. By the use of a standard that was not too far removed from what was wanted the correction was of less importance.

If short-term sickness only was being considered—for example a full rate of benefit for six weeks and half-rate for nine weeks—it would be better, perhaps, to use as the standard valuation factors the functions for the 'first 13 weeks' sickness and the 'second 13 weeks' sickness and to employ the different linear combinations appropriate to that choice. It would be necessary to go beyond the table of Appendix I to a determinant, but it would be a simple one and if the problem was a recurring one, it would be easy to make up another table for the particular [X] and [Y] being used.

The object of the paper had been quite definitely the mathematical analysis of sickness rates, leading to the approximate valuation of benefits for special periods. In practice, the *arithmetic* of valuation was only the beginning. Having made the valuation by some standard table it was necessary to consider how it should be adjusted so as to fit the peculiarities of the particular society. The arithmetic was the simplest and relatively the least important part of the valuation. The important part was the work calling for professional judgment in dealing correctly with those special features. There he agreed entirely with Mr Scholey.

Some of the figures Mr Scholey had given were based upon a three-constant approximation. If the authors had used three constants, instead of two, and had had an α, β, and γ, the mathematical analysis would have been very little changed. It would have ended in a fourth order determinant instead of a third. The actual work in practical valuation would be very little more. There would have been three valuation columns—three things to combine linearly. Though he had not tested the question arithmetically, he felt sure that the use of three well-chosen values for α, β, and γ respectively would tremendously improve the results given in Tables 8 and 9.

He had found the symbolism useful, and he thought that others would find it so.

Finally, he stressed that, in his view, one of the merits of the paper was that all that horrific mathematical appearance dropped right away from it in any sort of routine application.

Mr J. C. S. Hymans writes: In §§ 14–17 Mr Lane and I have put the emphasis on the double Gompertz and have chosen values of α and β for a specific purpose, that is, the everyday valuation of small friendly societies which provide sickness benefits for non-standard periods. For those somewhat restricted uses we consider that the way in which we have applied the method leads to remarkably accurate results. The remarks of Miss Merriman and Mr Heywood corroborate this opinion. We have, moreover, found the method particularly suitable for untrained staff, and we have also found it useful in conjunction with mechanical tabulation.

Mr Scholey made a number of criticisms to which I wish to reply in some detail.

He said that the agreement in Table 8 was fortuitous. That is not so, because it is the inevitable result of the method of fitting by which the values of α and β were produced. If accurate rates or values are required at the younger ages or shorter durations, other values of α and β must be used.

His remarks on cyclic benefits may be justified, but it should be noticed that the text-book method is to take (in the notation of Appendix II) the rate of benefit after n weeks as $(r_1n_1+r_2n_2+r_3n_3)/n$ and to value it accordingly. In comparison with our method this understates the liability as would be expected from *a priori* reasoning.

We did not mention the off-period because we were concerned with fitting functions to the Manchester Unity tables. Changes in the off-period raise questions of re-graduation, or alternatively of loading the existing rates in some suitable manner.

Mr Scholey has shown that the application of our method as described in §§ 14–17 does not apply accurately at individual ages, but it should not be thought that this is a weakness in the method. It is a weakness only of a particular application of the method—one in which we have put ${}_r\mathbf{z}_x=\mathrm{F}_x\,(\cdot 9645)^{r/13}+\mathrm{G}_x\,(\cdot 3689)^{r/13}$.

The real value of the paper, to my mind, lies in the possibility of further developments which may come from our exposition in §§ 12 and 13, using functions which are not exponentials, and the best results may indeed come from the use of three functions, as we have indicated in § 22.

We chose the married women's rates for our graduation in Appendix III merely because we wished to show how our method could be applied and because we required in practice a graduation of the married women's rates. I think that the illustration shows the power and flexibility of the method and it should be emphasized that the valuations by our method are exact on the basis of the graduated sickness rates derived from our method whatever values of α and β may be chosen.

Regarding the use of 2^r as argument, Mr Starke's comment is true—it may reveal more useful functions than the double Gompertz, but the first abscissa being $-\infty$

may cause some difficulties. It was used in Cd. 6907 for graduating the later periods of sickness.

The accuracy mentioned by Mr Heywood and Miss Merriman is in agreement with our own everyday results and I was particularly interested in the use which Mr Bizley made of our method to evaluate lump-sum benefits.

The new way of using Rhodes's method is interesting and we are not surprised to see that it gives similar results to our own. However, it does not share the consistency between experience and valuation which, as Miss Merriman remarked, is a feature of our method.

Finally, it is impossible at this stage to prepare detailed comments on the connexion between Mr Scholey's suggestions for a three-constant formula, Mr Heywood's modification of Rhodes's method, and the method of the paper, but Mr Lane and I hope to do so in an actuarial note to the *Journal* in the near future.

Mr J. K. Scholey writes: In the discussion on the paper I suggested that formulae could be devised which would enable valuations of sickness for non-standard periods to be made when the M.U. (Whole Society) Table, the rates for which are given in Cd. 6907, was employed. At the end of this note I give the necessary formulae and constants which have been derived from these rates at quinquennial ages from 20 to 70. The rates resulting from the formulae at each of these ages have been compared with the actual rates, and the totals of the positive and negative deviations and maximum positive and negative deviations are also given. If the valuation is being made by the M.U. (Whole Society) Table, then the formulae need not be employed for calculating 'expected sickness' since rates at quinquennial ages are available; in evaluating the liability the small deviations shown will usually be immaterial since positives and negatives will tend to cancel each other. The same formulae and constants may also be found suitable for use even when a valuation is being made by another Table.

The rates for z^{13}, z^{26} and z^{52} were those given in the standard table (and not those given in Cd. 6907); the rates employed for $z^{13/n}$ and $z^{26/n}$ were obtained by subtracting z^{13} and z^{26} as given in Cd. 6907 from z^{13+n} and z^{26+n} as given therein.

Weeks 1–12

$$z^n = a + bz^{13} + cz^{13/13}$$

n	a	b	c	Total deviations (calculated-actual)		Maximum deviations	
				−	+	−	+
1	·1309	·0777	·0093	·018	·019	·009	·004
2	·1605	·2372	−·0721	·022	·022	·009	·005
3	·1592	·3906	−·1415	·028	·026	·009	·010
4	·1508	·5128	−·1738	·030	·029	·009	·012
5	·1304	·6202	−·1973	·029	·030	·011	·013
6	·1127	·6997	−·1948	·031	·032	·012	·013
7	·0974	·7595	−·1758	·031	·032	·013	·012
8	·0829	·8078	−·1492	·029	·028	·011	·010
9	·0705	·8462	−·1133	·024	·025	·009	·009
10	·0578	·8804	−·0743	·024	·025	·010	·009
11	·0399	·9192	−·0426	·023	·024	·012	·010
12	·0219	·9555	−·0076	·022	·021	·014	·010

Weeks 14–25

$$z^{13/n-13} = az^{13} + bz^{13/13} + cz^{26/26}$$

n	a	b	c	Total deviations		Maximum deviations*	
				−	+	−	+
14	·0023	·1104	−·0213	·002	·004	·001	·002
15	·0026	·2278	−·0458	·008	·006	·002	·004
16	·0033	·3366	−·0657	·009	·009	·003	·005
17	·0035	·4393	−·0818	·011	·011	·004	·006
18	·0035	·5356	−·0945	·012	·013	·005	·007
19	·0020	·6336	−·1075	·013	·015	·005	·008
20	·0029	·6976	−·0996	·017	·017	·006	·010
21	·0025	·7647	−·0938	·018	·018	·007	·012
22	·0032	·8158	−·0791	·019	·019	·008	·014
23	·0007	·8951	−·0830	·019	·019	·009	·013
24	·0004	·9472	−·0707	·019	·017	·009	·013
25	−·0022	1·0118	−·0663	·017	·017	·010	·013

* These maximum deviations occurred at either age 65 or age 70 at all durations.

Weeks 27–51

$$z^{26/n-26} = az^{26/26} + bz^{52/52}$$

(*Note.* A two-term formula appeared adequate at these durations.)

n	a	b	Total deviations		Maximum deviations*	
			−	+	−	+
30	·2666	−·0588	·007	·008	·002	·003
34	·4717	−·0861	·011	·010	·003	·003
39	·6691	−·0880	·013	·015	·004	·006
43	·7979	−·0761	·010	·011	·003	·005
47	·9019	−·0506	·006	·008	·002	·004

* These maximum deviations occurred at age 40 or over at all durations.