JOURNAL

OF THE

INSTITUTE OF ACTUARIES.

On the Valuation of Staff Pension Funds. Part 2.—Widows' and Children's Pensions. *By* HENRY WILLIAM MANLY, *Actuary of The Equitable Life Assurance Society, and Past-President of the Institute of Actuaries. With Tables by* HERBERT FOOT, B.A., *of The Northern Assurance Company, Fellow of the Institute of Actuaries.*

[Read before the Institute, 27 April 1903.]

I HAVE been asked to explain how the benefits of a pension to widows and an allowance to children can be valued when they are included in a pension scheme; and as I have had no experience whatever of a fund providing such benefits, I am, of course, eminently qualified to lecture on the subject. At least, I approach the subject with an unprejudiced mind, and am not affected by anything I may have said or done.

The problem I am about to investigate is different from the ordinary Widows' Funds; because I am assuming that it is compulsory on every member of the staff, whether bachelor, husband, or widower, to contribute towards the benefits, either by a fixed annual sum for the whole of life or by a percentage of his salary, or of his salary and pension.

I have made several enquiries as to the nature of the benefits offered, and I find that the pension generally takes the form of a fixed annuity to the widow, and, on the death of the widow, a continuation of the annuity until the youngest surviving child reaches a certain age, say 14, 16, or 21. There is generally a provision that the annuity shall cease on the

re-marriage of the widow, but even in that case it is sometimes continued for the benefit of the children. Sometimes an allowance is made for every child left by a married man until the attainment of a certain age, say 14 or 16, in addition to the pension to the widow. This additional benefit is of the form of Mr. G. King's "Family Annuities", see *J.I.A.*, vol. xxx, p. 291. I have come across one fund which provides a pension to the widow based in some way on the retiring allowance which the husband would have received if he had retired at the date of his death.

It not infrequently happens that a consolation is given to the persistent bachelor, and the widower leaving no children, in the shape of the payment at death of a fixed sum or the return of contributions without interest.

As I propose to use, when suitable, Mr. A. Hewat's Tables in his "Widows' and Pension Funds", based on the experience of the Widows' Funds of the Scottish Banks, for which he has kindly given his permission, I may at once state that no provision will be made for forfeiture of the widow's pension in the event of re-marriage, and that second marriages are not provided for. (See Hewat, p. 80, "Widows' and Pension Funds.")

We are now entering on a new and wide field of investigation, and shall require a new notation, in which the symbols shall be so distinctive that they cannot be mistaken for others we are in the habit of using. In this I have found it necessary to break away from the signs hitherto adopted for reasons which will appear obvious. For instance, w which has hitherto been used to represent a widow, I have already appropriated for withdrawals, and $wrmq_x$, which has been used to represent the probability of a widower marrying between the ages x and $x+1$, is much too complex when a simpler symbol can be found.

Primary Symbols.

As p_x is used to represent the probability of a person aged x living from age x to age $x+1$, and q_x to represent the probability of a person aged x dying between the ages x and $x+1$, I propose that m_x shall represent the probability of a person aged x marrying between the ages of x and $x+1$. This has a certain advantage, because m without the suffix will represent the probability of marrying. The following is the scheme of notation I propose to use :—

B = bachelor.

H = husband.

K = widower.

M = married man.

W = widow.

O = orphans, that is, all the children left by a husband or widower.

YC = youngest child.

$\mathrm{B}l_x$ = number of bachelors living at the age x.

$\mathrm{H}l_x$ = „ husbands „ „ x.

$\mathrm{K}l_x$ = „ widowers „ „ x.

$\mathrm{M}l_x$ = „ married men „ „ $x = \mathrm{H}l_x + \mathrm{K}l_x$.

$\mathrm{B}d_x$ = number of bachelors dying between the ages of x and $x+1$.

$\mathrm{H}d_x$ = number of husbands dying between the ages of x and $x+1$.

$\mathrm{K}d_x$ = number of widowers dying between the ages of x and $x+1$.

$\mathrm{M}d_x$ = number of married men dying between the ages of x and $x+1 = \mathrm{H}d_x + \mathrm{K}d_x$.

$\mathrm{B}p_x$ = the probability of a bachelor aged x living from age x to age $x+1$.

$\mathrm{H}p_x$ = the probability of a husband aged x living from age x to age $x+1$.

$\mathrm{K}p_x$ = the probability of a widower aged x living from age x to age $x+1$.

$\mathrm{M}p_x$ = the probability of a married man aged x living from age x to age $x+1$.

$\mathrm{B}q_x$ = the probability of a bachelor aged x dying between the ages of x and $x+1$.

$\mathrm{H}q_x$ = the probability of a husband aged x dying between the ages of x and $x+1$.

$\mathrm{K}q_x$ = the probability of a widower aged x dying between the ages of x and $x+1$.

$\mathrm{M}q_x$ = the probability of a married man aged x dying between the ages of x and $x+1$.

m_x, or $\mathrm{B}m_x$ = the probability of a bachelor aged x marrying between the ages x and $x+1$.

m^2_x, or $\mathsf{K}m_x$ = the probability of a widower aged x marrying between the ages of x and $x+1$.

$\mathsf{B}_x m$ = the probability of a bachelor aged x marrying.

$\mathsf{K}_x m$ = „ „ widower „ „

x is used to denote the age of a male.

y „ „ age of a female, but generally the age of the widow at the death of her husband.

$\mathsf{W}a$ = the present value of an annuity of 1 to a widow.

$\mathsf{W}_y a$ = the present value of an annuity of 1 to a widow aged y.

$\mathsf{O}a(n)$ = present value, at the death of a husband or widower, of an annuity of 1 to each of the children, until they reach the age of n.

$\mathsf{YC}a(n)$ = present value, at the death of a married man, of an annuity of 1 until the youngest child attains the age of n.

$\mathsf{K.YC}a(n)$ = present value, at the death of a widower, of an annuity of 1 until the youngest child attains the age of n.

$.(n)$, or $\mathsf{E}a(n)$ = the addition, at the death of a husband, to the present value of an annuity of 1 to a widow to provide for the continuance of the annuity until the youngest child attains the age of n.

$\mathsf{K\text{-}C}$ = a childless widower.

$\mathsf{K\text{-}C}(n)$ = a widower without children under the age of n.

$\overline{\mathsf{K\text{-}C}(n)}d_x$ = the number of widowers without children under the age of (n) dying between the ages x and $x+1$.

If it be desired to represent the probability of a person dying a bachelor, or a husband, or a widower, we can use b, h, and k respectively for such probabilities, so that

$\mathsf{B}_x b$ = probability of a bachelor aged x dying a bachelor.

$\mathsf{B}_x h$ = „ „ „ „ husband.

$\mathsf{B}_x k$ = „ „ „ „ widower.

$\mathsf{H}_x h$ = „ „ husband aged x dying a husband.

$\mathsf{H}_x k$ = „ „ „ „ widower.

$\mathsf{K}_x h$ = „ „ widower aged x dying a husband.

$\mathsf{K}_x k$ = „ „ „ „ widower.

It will sometimes be necessary to use a symbol to represent the number of bachelors marrying in a year, or marrying and dying the same year—similarly with husbands becoming widowers and widowers marrying. These will be represented thus:—

$^{B}hl_x$ = the number of bachelors becoming husbands between the ages of x and $x+1$.

$^{H}kl_x$ = the number of husbands becoming widowers between the ages of x and $x+1$.

$^{K}hl_x$ = the number of widowers becoming husbands between the ages of x and $x+1$.

$^{B}hd_x$ = number of bachelors marrying and dying as husbands between the ages x and $x+1$.

$^{H}kd_x$ = number of husbands becoming widowers and dying as widowers between the ages x and $x+1$.

$^{K}hd_x$ = number of widowers marrying and dying as husbands between the ages x and $x+1$.

Other symbols will be developed as the work proceeds.

Construction of Table showing the number of Bachelors, Husbands and Widowers Dying at each Age.

We have now to construct a working Table. It must in the first place be a Table showing the numbers remaining, withdrawing and dying each year out of a certain number entering at a given age. That we have in my Table No. 3 (*J.I.A.*, vol. xxxvi, p. 261). It is not necessary to separate those who retire, because the liability we have to deal with now, arises only on death, whether before or after retirement. In the second place we have to carry the deaths to the extreme limit of life; and for this purpose I have adopted the English Life No. 3 Experience. In the third place, we have to find, out of all those who die at each age, how many are bachelors, how many are husbands, and how many widowers. Fourthly, we have to determine the average age of the widow at the death of the husband; and, finally, we require to know how many of the husbands and widowers will leave children, and the average age of the youngest child at the death of the father.

Statistics on these points are very limited, and are not always given in a form convenient for use.

Marriage Rates.

The number of husbands dying at each age must of necessity depend largely upon the marriage rate. Now the marriage rate differs according to occupation; and this has been clearly shown by Messrs. Deuchar, Hewat, Huie, Meikle, and Sprague, in the experiences of the different Widows' Funds in Scotland, and the Peerage Families. The Ministers and Schoolmasters, who attain their maximum income early in life, marry earlier than the bankers and advocates who wait until their incomes will enable them to marry comfortably. I imagine that the marriage rate in the class for whom these Funds are created will correspond more closely to that prevailing in a bank staff than in any other; and, consequently, I decided to adopt Mr. Archibald Hewat's experience of Scottish Bankers.

On page 63 of his work on "Widows' and Pension Funds" he gives a Marriage and Mortality Table, based on the experience of the Widows' Funds of the Scottish Banks, showing the numbers living separated into bachelors and married, and the numbers dying separated into bachelors and married, as well as the number of bachelors marrying. The numbers dying in my extended Table were divided into bachelors dying and married dying, in the same proportion as in Mr. Hewat's Table. Having thus obtained the numbers dying married, it became necessary to sub-divide them into husbands and widowers. This was done, up to age 65, in the proportion of husbands dying to widowers dying by means of Mr. G. King's Table C in his valuable paper "On Family Annuities." The number of bachelors, husbands and widowers living at the age of 65 and over, are not sub-divided according to ages in the Census Returns, and, in consequence, we had to have recourse to another method of separating our married men at death, after that age, into husbands and widowers. This was done by means of Dr. Farr's English Life Tables. It was assumed that all the married men of the age of 65 would have wives of the average age of 10 years younger than themselves, and we ascertained how many couples, out of a given number of couples of husband and wife living at age 65, would survive every year to the end of the Table. By this means we were able to extend Mr. King's curve for husbands to the end of life, and it was particularly gratifying to find that the total bachelors, husbands and widowers, which we thus obtained, corresponded very closely

with the Census Returns. The question of marriage or re-marriage after 65 was excluded, because in these Funds marriage after 65 is either not recognized, or, if it is, a large fine is imposed.

The result of this distribution of the deaths is shown in Table 45 on pp. 138–9; and, until more trustworthy statistics are forthcoming, I venture to think that this Table is the best that can be produced. The mortality amongst husbands is greater after age 47 than in Mr. Hewat's experience of married men in the class of Scottish Bankers, so that the liability, according to my Table, will be on the safe side.

Widows' Age.

Having found the number of husbands who die at each age we had next to find the ages of the widows which they leave on the Fund. On page 81 of his work Mr. Hewat gives a Table of the difference of age between husband and wife at marriage, based on the actual experience of the Scottish Bankers, of which the following is an extract:

Table J.

Husband's Age at Marriage	Average Age of Wife	Difference
20	22	−2
25	24	+1
30	26	4
35	27	8
40	29	11
45	29	16
50	33	17
55	37	18
60	40	20

There will not, of course, be so much difference of age between the wife and husband at the death of the husband, especially as the majority of marriages takes place between the ages of 27 and 40. After due consideration of the probable survivors, I fixed upon the following scale:

TABLE K.

Average Age of Husband at Death	Average Age of Widow at Death of Husband	Difference
$25\frac{1}{2}$	24	$1\frac{1}{2}$
$30\frac{1}{2}$	28	$2\frac{1}{2}$
$35\frac{1}{2}$	$31\frac{1}{2}$	4
$40\frac{1}{2}$	$34\frac{1}{2}$	6
$45\frac{1}{2}$	38	$7\frac{1}{2}$
$50\frac{1}{2}$	42	$8\frac{1}{2}$
$55\frac{1}{2}$	$46\frac{1}{2}$	9
$60\frac{1}{2}$	51	$9\frac{1}{2}$
$65\frac{1}{2}$	$55\frac{1}{2}$	10

The difference between the ages of husband at death and widow for the intermediate ages of husband at death was determined by first differences (see Table 46).

If these figures err at all, they err upon the side of making the difference of age, especially after age 40, too large; but if that is so, then the error is upon the safe side, because in my Tables I shall be making a larger provision for the annuity to the widow than is necessary.

The present value of an annuity of 1 payable to the widow during the remainder of her life was taken from Mr. Hewat's Table II (page 87).

CHILDREN'S BENEFITS.

In order to ascertain the ages and the number of children left by a married man at his death, I have had to go to the same source as Mr. King did when he constructed his Table of "Family Annuities", namely, the "Statistics of the Colony of New Zealand." In vol. xxx of the *Journal*, on page 300, Mr. King gave a copy of the "Table showing, for the year 1890, "the total number of males who died at each year of age from "20 to 65; the number of married males who died (*a*) leaving "no children, and (*b*) leaving children; and the number and "ages of the children living at the time of the father's death." In the statistics of the Colony for the year 1899 a similar Table is given for the five years 1895 to 1899 inclusive, for all ages of the father up to 65, and also a Table for the year 1899, for all ages of males at death from 20 to 99. As I shall make very considerable use of this Table, I will give the figures, grouping five ages together.

Table L.

Orphanhood of Children.

Table showing the Total number of Married Men stated in the Registers of the Colony of New Zealand, as having died (from age 20 to 65, during the five years 1895 to 1899, and from age 65 to the end of life, for the year 1899) (a) Childless; (b) leaving Children; and the number and ages of the Children left.

Ages of Married Men at Death	Central Age	Childless	Leaving Children	Ages of Children																					Age not specified
				0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21 and over	
Under 23	—	4	5	4	2	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
23–27	25	33	71	40	19	21	8	13	9	5	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
28–32	30	53	170	46	53	58	51	51	45	38	23	19	11	8	0	1	1	...	...	...	...	...	...	...	...	7
33–37	35	54	300	84	54	87	91	85	87	73	89	65	65	51	39	24	21	14	9	1	4	1	...	...	...	21
38–42	40	70	360	59	47	84	84	80	91	98	98	104	94	100	93	95	77	74	59	48	38	26	16	7	4	62
43–47	45	73	452	46	49	60	65	92	71	107	111	113	119	107	106	125	101	132	78	125	82	82	67	59	159	69
48–52	50	85	548	32	31	57	62	59	85	98	96	102	122	129	130	150	126	161	150	149	153	124	143	123	591	87
53–57	55	140	755	25	30	39	49	56	71	82	98	101	108	115	129	152	164	187	173	183	214	183	211	204	1482	143
58–62	60	130	840	9	18	19	36	24	35	40	55	59	79	71	104	99	134	114	166	152	181	179	203	203	2736	127
63–67	65	37	223	2	...	3	2	2	3	2	5	6	6	5	12	9	12	17	20	18	29	19	43	30	957	33
68–72	70	23	196	...	...	1	3	...	1	1	1	2	2	2	1	3	2	6	8	6	10	15	17	15	895	49
73–77	75	25	155	...	...	...	...	.	...	...	1	1	1	1	1	2	2	4	2	2	8	2	8	...	720	29
78–82	80	13	107	...	...	. .	1	...	.	...	...	...	...	...	1	...	...	...	1	1	...	1	3	1	498	35
83–87	85	7	58	...	...	...	...	. .	..	...	...	...	...	...	...	...	1	1	...	...	1	...	1	...	259	19
88–92	90	3	19	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	1	...	1	...	1	...	107	2

No attempt has been made to distribute the numbers in the column headed "Age not specified", consequently that column has been ignored.

The marriage rate in the Colony is no doubt different from that prevailing amongst the class we are dealing with, but I agree with Mr. King that no great error will be committed if it be assumed that the number of children to each deceased married man is much the same in New Zealand as in Great Britain.

There are two ways in practice of providing for orphans; one by continuing the widow's annuity until the youngest child reaches a certain age, the other by granting an annuity to each child until the attainment of a certain age.

Continuation of Widow's Annuity to Youngest Child.

This plan of providing for the children presents two distinct problems: (1) The value, on the death of a husband, of a deferred annuity to the youngest child to commence at the death of the widow, and to continue until the youngest surviving child attains a certain age; (2) the value on the death of a widower of an annuity until the youngest child reaches the assigned age. The former is a term survivorship-annuity, the latter is, practically, an immediate term-annuity-certain.

Our statistics do not distinguish between husbands' and widowers' children, nor do they give any information as to the youngest child in the family. These defects prevent us from arriving at anything like extreme accuracy in our estimates, but we shall produce results which, if not exact, will at least be on the safe side. The better to explain the methods pursued, I here give a Table showing the different stages by which the values were produced.

TABLE M.—*Showing the Method of Ascertaining the Value, on the Death of a Husband or Widower, of the continuance of the Widow's Annuity until the Youngest Child reaches a certain Age* (14).

Central Age of Married Men at Death	Number of Married Men who died		Number of Assumed Youngest Children, under 14 Years of Age, Living at each Age														Average Age of Children	Value of Curtate Annuity-Certain at 4 per-cent	Age of Widow at Death of Husband	Value of Term Annuity on Life of Widow for $14-x'-\frac{1}{2}$ Years at 4 per-cent	Extra for Youngest Child till 14 per Husband leaving Children under 14	Extra per Husband leaving Children	Extra per Husband Dying	Value of Continuous Annuity-Certain at 4 per-cent	Value of Annuity to Youngest Child till 14 per Widower leaving Children	Value of Annuity to Youngest Child till 14 per Widower Dying
			Ages of Children (4)																							
	Childless	Leaving Children	0	1	2	3	4	5	6	7	8	9	10	11	12	13	Total	x'	$a_{\overline{14-x'-\frac{1}{2}}}$	y	$a_{\overline{14-x'-\frac{1}{2}}}a_y$			$E(14)_x$	$\frac{1}{2}+a_{\overline{14-x'-\frac{1}{2}}}$		$K \cdot YC a(14)$
(1)	(2)	(3)															(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
25	33	71	40	19	12	...	...	...	...	...	...	...	...	...	...	...	71	1·1	9·625	24	8·977	·648	·648	·551	10·125	10·125	8·606
30	53	170	46	53	58	13	...	...	...	...	...	...	...	...	...	...	170	1·7	9·260	28	8·663	·627	·627	·533	9·760	9·760	8·296
35	54	300	84	54	87	75	...	...	...	...	...	...	...	..	...	...	300	2·0	9·073	$31\frac{1}{2}$	8·480	·593	·593	·504	9·573	9·573	8·137
40	70	360	59	47	84	84	80	6	...	...	...	...	...	...	..	...	360	2·8	8·566	$34\frac{1}{2}$	8·022	·544	·544	·462	9·066	9·066	7·706
45	73	452	46	49	60	65	92	71	69	...	...	...	...	...	...	...	452	3·8	7·908	38	7·418	·490	·490	·417	8·408	8·408	7·147
50	85	548	32	31	57	62	59	85	98	96	28	...	...	...	...	...	548	5·0	7·084	42	6·667	·417	·417	·354	7·584	7·584	6·446
55	140	755	25	30	39	49	56	71	82	98	101	108	96	...	...	...	755	6·8	5·774	$46\frac{1}{2}$	5·485	·289	·289	·246	6·274	6·274	5·333
60	130	840	9	18	19	36	24	35	40	55	59	79	71	104	99	134	782	9·5	3·630	51	3·493	·137	·128	·109	4·130	3·845	3·268
65	37	223	2	...	3	2	2	3	2	5	6	6	5	12	9	12	69	9·6	3·544	$55\frac{1}{2}$	3·385	·159	·049	·042	4·044	1·251	1·063
70	23	196	...	..	1	3	...	1	1	1	2	2	2	1	3	2	19	8·7	4·287	$60\frac{1}{2}$	3·966	·321	·031	·026	4·787	·464	·394
75	25	155	...	...	...	...	...	...	...	1	1	1	1	1	2	2	9	11·1	2·242	$65\frac{1}{2}$	2·104	·138	·008	·007	2·742	·159	·135
80	13	107	...	...	...	1	...	...	...	...	...	...	...	1	...	...	2	7·5	5·242	$70\frac{1}{2}$	4·135	1·107	·021	·018	5·742	·107	·091
85	7	58	...	...	...	...	...	...	...	..	...	...	...	...	...	1	1	13·5	0·	$75\frac{1}{2}$	0·000	·000	·000	·000	0·500	·009	·007
90	3	19	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
	746	4,254	343	301	420	390	313	272	292	256	197	196	175	119	113	151	3,538	...	...	...	...	...	...	...	...	...	...

Now there can only be one youngest child for each deceased husband leaving a family; consequently, the 170 married men (which for the present we will assume to be husbands) leaving children, who died at ages within the group of which 30 is the centre, would leave 170 youngest children. If then we assume that the youngest 170 children are the youngest children of the 170 different families, we shall certainly be placing our estimate on the safe side, because some of these will undoubtedly be first and second from the youngest in the same family, and the youngest of some families will be older. On the other hand the fact that widowers' children are included would have the tendency, when widowers are treated as husbands, to make the youngest child somewhat older than the husbands' youngest children alone; but the proportion of widowers who die to husbands who die is very small, so that our assumption will still keep us on the side of safety.

The youngest 170 children opposite central age 30 are, 46 aged 0, 53 aged 1, 58 aged 2, and to make up the 170 we take 13 aged 3. On the principle that the ages are scattered equally throughout the year, we shall assume that all those entered under age 0 are, on the average, $\frac{1}{2}$ a year old; all those entered under age 1 are, on the average, $1\frac{1}{2}$ years' old, and so on. The average age of the youngest child will, therefore, be $\{(46\times\frac{1}{2})+(53\times1\frac{1}{2})+(58\times2\frac{1}{2})+(13\times3\frac{1}{2})\}\div170=\frac{293}{170}=1{\cdot}7$.

Having ascertained the average age of the youngest child, which we will call x', our first object will be to find the addition to be made to the value of the widow's annuity at the death of the husband to provide for the continuance of the annuity to the youngest child until the age of, say 14, in the event of the widow dying before such time. We have not to provide for the youngest child alone, for if the youngest dies the next youngest in the family takes his place, and so on, until the youngest surviving reaches the age of 14. The value of the annuity for the youngest child, therefore, is not much short of an annuity-certain; so that by assuming, as I have done, that the value of the annuity to the youngest child is the value of an annuity-certain for the term between the age of the youngest child at the death of a husband, and the age when the annuity is to cease, we are arranging that the estimated value shall be on the safe side. Huie assumes that every husband leaves a child just born, and that such child will certainly live 21 years, which is an exaggerated view to take of the case.

The term during which a provision has to be made for the youngest child, until the age of 14, is $14-x'$, and the value of a continuous annuity-certain for that period is approximately $\frac{1}{2}+a_{\overline{14-x'-\frac{1}{2}}|}$. The value of a continuous annuity to the widow on the death of the husband for the same term is approximately $\frac{1}{2}+{}_{|14-x'-\frac{1}{2}}a_y$. The addition, therefore, to be made to the value of the widow's annuity to provide for the youngest child until 14, is

$$a_{\overline{14-x'-\frac{1}{2}}|}-{}_{|14-x'-\frac{1}{2}}a_y,$$

consequently, we only tabulate $a_{\overline{14-x'-\frac{1}{2}}|}$.

The values of the term annuities on the lives of widows were based on the H^F Table, because the rate of mortality in that Experience corresponded fairly well with Mr. Hewat's rate of mortality amongst widows, and the values could be easily obtained from the Institute H^F Tables. As Mr. Hewat does not publish the D and N columns in his Tables we had no ready means of calculating the term values on his experience.

The difference between the annuity-certain for the youngest child and the term annuity on the life of the widow, will give the addition to be made to the widow's annuity *per husband leaving children under* 14 *years of age*. It will be seen, on reference to the Table, that there are sometimes more married men leaving children than there are children under 14, and this is the case at central age 60 at death and over. In order to find the addition to the widow's annuity per husband leaving children, we require to multiply the value already obtained by the total number of children under 14, and divide by the number of married men leaving children; that is to say, for central age 60, the value in col. (10), which is ·137, must be multiplied by $\frac{782}{840}$, which produces ·128; for central age 65, the value in col. (10) must be multiplied by $\frac{69}{223}$, and so on. In this way col. (11) was produced, showing the addition to the widow's annuity *per husband leaving children*. But then it is not every husband who leaves a child; and on comparing cols. (2) and (3) it will be seen that 15 per-cent of the total married men die childless, and that the ratio is nearly the same at all ages. By multiplying, then, the values in col. (11) by ·85, we obtain the addition to the widow's annuity *per husband dying*, which values are inserted in col. (12). These values form a fairly regular series, and the values for each age (see Table 48) were obtained by means of a curve drawn graphically through the values at the central ages.

PROVISION FOR YOUNGEST CHILD ON DEATH OF A WIDOWER.

I have already explained that, owing to the New Zealand statistics not distinguishing between husbands and widowers in the married men who die, it is quite impossible to separate their children; consequently, we have no choice between assuming that the ages of the children left by widowers are the same as those left by husbands and widowers together, or, making an empirical assumption as to the possible increase in the ages of the children on the death of a widower compared with the ages on the death of a husband. I prefer to assume that the ages of the children are exactly the same, for the reason that, by doing so, the provision I make for the children of husbands and widowers taken together will correspond exactly with the provision for the children left by married men at death according to the statistics. The number of widowers dying at ages when the children are young is very small, even when only one marriage is allowed for; but, if second and third marriages are taken into consideration, it will be subsequently shown that the number of widowers dying leaving young children is very small indeed.

At present, we do not take second marriages into consideration, and we accordingly assume that the values in col. (7) in the above Table, with the addition of ·5, will represent the value of the continuous annuity to the youngest child of a widower leaving children under 14. These values reduced, first to the values per widower leaving children, and then to the values per widower dying, in the same way as the additions to the widow's annuity were treated, will produce the values in col. (15), which were also graduated graphically in order to obtain the values for each age of widower at death.

ANNUITY TO EACH OF THE CHILDREN OF A HUSBAND OR WIDOWER ON THE DEATH OF THE FATHER; OTHERWISE CALLED ORPHANS' ANNUITIES.

I have adopted the term "orphans' annuities" in preference to "family annuities" used by Mr. King, because although, in a narrow sense, the children of the same parents considered collectively apart from their parents may be called the family, it in no way implies that the liability arises only on the death of the father. On the other hand, although, I suppose, in its strict sense, an orphan means a child bereft of both parents, it is now most commonly used to signify a child whose father is dead, and therefore more clearly expresses the status of the child who is to receive the annuity.

We have not now to distinguish between husbands' children and widowers' children, consequently we have to calculate the value of the annuities as at the moment of death of a married man.

Mr. King based his annuities on the English Life No. 3 Experience, and I think wisely so, because he was dealing with a population problem, but that Table does not represent the mortality experience of the class with which we are dealing. Mr. Hewat in his work on "An Investigation of the Marriage "and Mortality Experience of a Scottish Ministers' Widows' "and Orphans' Fund, &c.", gives a Table of the mortality experience amongst children, taken out to four places of decimals. I am content to use only three places of decimals to represent the probability of dying in a year. The following Tables show the rate of mortality I have used, and the values of the temporary annuities deduced therefrom. I have not used the elaborate formula of Mr. King for a continuous term annuity, being satisfied with the approximate formula, $\frac{1}{2} + {}_{|n-x'-\frac{1}{2}}a_{x'}$, as a slight difference in the rate of mortality would neutralize any difference produced by extreme accuracy in calculating benefits.

The rate of children's mortality which I have assumed is lighter than that used by Mr. King, and, consequently, my temporary annuities are larger than his.

TABLE N.

Table of the Mortality Experience amongst Children.

Age x	q_x	l_x	d_x
0	·062	10,000	620
1	·016	9,380	150
2	·010	9,230	92
3	·009	9,138	83
4	·008	9,055	72
5	·007	8,983	63
6	·006	8,920	53
7	·005	8,867	45
8	·004	8,822	35
9	·004	8,787	35
10	·004	8,752	35
11	·004	8,717	35
12	·003	8,682	26
13	·002	8,656	17
14	·002	8,639	18
15	·003	8,621	25
16	·004	8,596	34
17	·004	8,562	35
18	·005	8,527	43
19	·006	8,484	50
20	·006	8,434	51
21	...	8,383	...

TABLE O.

Value of Continuous Temporary Annuity.

Year of Age	To Age 14		To Age 16		To Age 21	
	3 per-cent	4 per-cent	3 per-cent	4 per-cent	3 per-cent	4 per-cent
0– 1	10·281	9·690	11·440	10·697	14·018	12·865
1– 2	9·995	9·457	11·237	10·548	14·003	12·898
2– 3	9·409	8·939	10·706	10·089	13·593	12·564
3– 4	8·764	8·361	10·114	9·568	13·116	12·167
4– 5	8·085	7·746	9·487	9·012	12·605	11·738
5– 6	7·372	7·093	8·826	8·420	12·063	11·276
6– 7	6·624	6·401	8·132	7·791	11·488	10·781
7– 8	5·843	5·671	7·405	7·124	10·880	10·251
8– 9	5·029	4·902	6·644	6·420	10·240	9·687
9–10	4·183	4·097	5·854	5·681	9·572	9·092
10–11	3·309	3·256	5·037	4·910	8·882	8·472
11–12	2·405	2·377	4·192	4·105	8·168	7·824
12–13	1·468	1·459	3·316	3·262	7·426	7·144
13–14	·500	·500	2·407	2·380	6·652	6·427
14–15	...	...	1·469	1·459	5·849	5·676
15–16	...	...	·500	·500	5·023	4·897
16–17	...	...	...	...	4·175	4·089
17–18	...	...	...	...	3·300	3·248
18–19	...	...	...	...	2·397	2·370
19–20	...	...	...	...	1·465	1·456
20–21	...	...	...	...	·500	·500

My Tables of orphans' annuities were constructed in the way described by Mr. King in his paper on "Family Annuities" (*J.I.A.*, xxx, p. 303), but using only the numbers for the groups of five ages of married men at death (Table L). The curve drawn through the values for central ages was almost exactly of the same form as Mr. King's. These Tables will be found on pages 146–7.

If my values "to age 14" at 3 per-cent be compared with Mr. King's values for the same limiting age and rate of interest, it will be found that my values are less than his, particularly from ages 30 to 40, notwithstanding that my annuities are higher, which shows that the number of children in a family in New Zealand had decreased in the ten years from 1890 to 1900.

Having given the particulars of the Tables upon which all our calculations are to be based, I think the best way of showing the construction of the Valuation Tables will be to follow the same course as before; that is, to give solutions of the values of the different benefits in the form of problems. By this means every investigation is kept clear and distinct.

Problem Ic.—Assuming that the members of a staff consist of bachelors, husbands, and widowers, distributed in the pro-

portions according to the Tables, what would be the present value in respect of a person aged x of an annuity of 1 to a widow for the remainder of her life to commence on the death of a husband? No marriage after age 65 and no second marriage to be recognized. What annual premium payable for the whole of life by each member of the staff, entering at age x, would be required to provide such benefit?

Out of l_x persons living at age x, $\mathrm{H}d_x$ husbands will die between the ages x and $x+1$; say, on the average, at age $x+\frac{1}{2}$. $\mathrm{H}d_{x+1}$ husbands will die between the ages of $x+1$ and $x+2$; say, on the average, at age $x+1\frac{1}{2}$; and so on. If the average age of the widow at the death of a husband dying at the average age of $x+\frac{1}{2}$ is y, and if we assume that $\bar{a}_y=\frac{1}{2}+a_y$ approximately, then we shall have to provide for $\mathrm{H}d_x\times(\frac{1}{2}+a_y)$ at age $x+\frac{1}{2}$, $\mathrm{H}d_{x+1}\times(\frac{1}{2}+a_{y+1})$* at age $x+1\frac{1}{2}$, &c., and the present value of the annuity in respect of a person living at age x will be

$$\frac{\mathrm{H}d_x(\frac{1}{2}+a_y)\,v^{x+\frac{1}{2}}+\mathrm{H}d_{x+1}(\frac{1}{2}+a_{y+1})v^{x+1\frac{1}{2}}+\ldots.}{l_x v^x}$$

If now we call $\mathrm{H}d_x(\frac{1}{2}+a_y)v^{x+\frac{1}{2}}={}^{wa}\mathrm{C}_x$, and sum the values of ${}^{wa}\mathrm{C}_x$ so that ${}^{wa}\mathrm{M}_x={}^{wa}\mathrm{C}_x+{}^{wa}\mathrm{C}_{x+1}+{}^{wa}\mathrm{C}_{x+2}+\ldots$, then the present value of the widows' annuity will be represented by

$$\frac{{}^{wa}\mathrm{M}_x}{\mathrm{D}_x^{(3)}}$$

The annual premium, in the ordinary way, would be found by substituting $\mathrm{N}_{x-1}^{(3)}$ for D_x; but if the premium is to be payable monthly, and the funds invested quarterly, then we must apply similar adjustments as in our other problems, and use

$$(v\mathrm{N}_{x-1}^{(3)}-\tfrac{1}{2}\mathrm{M}_x^{(3)}-\tfrac{1}{2}{}^{w}\mathrm{M}_x^{(4)})\ (1+\tfrac{3}{8}i)$$

for the denominator.

Problem IIc.—What is the present value, in respect of a person aged x, of the additional payment required to provide for the continuance of the widow's annuity of 1 from her death until the youngest surviving child reaches the age of (n), and the corresponding annual premium payable for the whole of life?

We have already ascertained (page 113) the value of the addition at the death of the husband, and agreed to call such value $\mathrm{E}(n)_x$. Out of l_x persons, therefore, living at age x, we

* a_{y+1} does not mean that the annuity is for the age 1 year older than y, but $y+1$ is the age of the widow on the death of a husband at the age of $x+1\frac{1}{2}$.

shall have to provide in respect of $\mathsf{H}d_x$ husbands dying between the ages x and $x+1$, $\mathsf{H}d_x.\mathsf{E}(n)_x$; and in respect of $\mathsf{H}d_{x+1}$ husbands dying between the ages $x+1$ and $x+2$, $\mathsf{H}d_{x+1}.\mathsf{E}(n)_{x+1}$; and so on. The present value of the addition in respect of a person aged x will therefore be

$$\frac{\mathsf{H}d_x.\mathsf{E}(n)_x.v^{x+\frac{1}{2}}+\mathsf{H}d_{x+1}.\mathsf{E}(n)_{x+1}.v^{x+1\frac{1}{2}}+\ldots.}{l_xv^x}$$

If then we call $\mathsf{H}d_x.\mathsf{E}(n)_x.v^{x+\frac{1}{2}}={}^{\mathrm{E}(n)}\mathrm{C}_x$, and sum the values of ${}^{\mathrm{E}(n)}\mathrm{C}_x$ so that ${}^{\mathrm{E}(n)}\mathrm{M}_x={}^{\mathrm{E}(n)}\mathrm{C}_x+{}^{\mathrm{E}(n)}\mathrm{C}_{x+1}+{}^{\mathrm{E}(n)}\mathrm{C}_{x+2}+\ldots$, the present value of the addition to provide for the continuance of the widow's annuity until the youngest surviving child reaches the age of n, will be represented by

$$\frac{{}^{\mathrm{E}(n)}\mathrm{M}_x}{\mathrm{D}_x^{(3)}},$$

and the annual premium will be either

$$\frac{{}^{\mathrm{E}(n)}\mathrm{M}_x}{\mathrm{N}_{x-1}^{(3)}} \text{ or } \frac{{}^{\mathrm{E}(n)}\mathrm{M}_x}{(v\mathrm{N}_{x-1}^{(3)}-\frac{1}{2}\mathrm{M}_x^{(3)}-\frac{1}{2}{}^{w}\mathrm{M}_x^{(4)})(1+\frac{3}{8}i)},$$

according as the premium is to be paid yearly in advance, or deducted from salary.

Problem IIIc.—What is the present value, in respect of a person aged x, of an annuity of 1 to the youngest surviving child of a widower until the age of (n), from the moment of death of the widower; and what the corresponding annual premium?

We have already ascertained (page 113) the present value, at the death of a married man, of an annuity of 1 until the youngest surviving child reaches the age of n, and agreed to call the value $\mathsf{YC}a(n)_x$.

Out of l_x persons living at the age of x, $\mathsf{K}d_x$ will die as widowers between the ages of x and $x+1$, and the annuities to be provided will be $\mathsf{K}d_x.\mathsf{YC}a(n)_x$. $\mathsf{K}d_{x+1}$ will die as widowers between the ages $x+1$ and $x+2$, and the annuities to be provided for them will be $\mathsf{K}d_{x+1}.\mathsf{YC}a(n)_{x+1}$; and so on. So that the present value in respect of a person aged x will be

$$\frac{\mathsf{K}d_x.\mathsf{YC}a(n)_x.v^{x+\frac{1}{2}}+\mathsf{K}d_{x+1}.\mathsf{YC}a(n)_{x+1}.v^{x+1\frac{1}{2}}+\ldots}{l_xv^x}.$$

If now we call $\mathsf{K}d_x.\mathsf{YC}a(n)_x.v^{x+\frac{1}{2}}={}^{\mathrm{K.YC}a(n)}\mathrm{C}_x$ and sum the values so that ${}^{\mathrm{K.YC}a(n)}\mathrm{M}_x={}^{\mathrm{K.YC}a(n)}\mathrm{C}_x+{}^{\mathrm{K.YC}a(n)}\mathrm{C}_{x+1}+\ldots$, the

present value of an annuity of 1 to the youngest surviving child of a widower until the age of n will be

$$\frac{{}^{\text{K.YC}a(n)}\text{M}_x}{\text{D}_x^{(3)}}$$

and to find the annual premium we use the same denominator as in the two previous Problems.

Problem IVc.—What portion of salary payable during active service, equated to 1 of salary at age x, would be required to provide for an annuity of 1 to a widow and an annuity of 1 from the death of a widow or widower until the youngest child reaches the age of n?

The value of the benefits have been ascertained in Problems Ic, IIc, and IIIc, and, equating the payments to the benefits, we have

$$\text{P}\{\mathbb{N}_x^s - \tfrac{1}{2}({}^d\text{M}_x^{ls} + {}^w\text{M}_x^{ls} + {}^r\text{M}_x^{ls})\}(1+\tfrac{3}{8}i) \div \text{D}_x^s = ({}^{wa}\text{M}_x + {}^{\text{E}(n)}\text{M}_x + {}^{\text{K.YC}a(n)}\text{M}_x) \div \text{D}_x^{(3)}$$

and

$$\text{P} = \frac{{}^{wa}\text{M}_x + {}^{\text{E}(n)}\text{M}_x + {}^{\text{K.YC}a(n)}\text{M}_x}{(1+\tfrac{3}{8}i)\{\mathbb{N}_x^s - \tfrac{1}{2}({}^d\text{M}_x^{ls} + {}^w\text{M}_x^{ls} + {}^r\text{M}_x^{ls})\}} \times \frac{\text{D}_x^s}{\text{D}_x^{(3)}}$$

Problem Vc.—What is the present value, in respect of a person aged x, of an annuity of 1 to each of the children of a family, from the moment of death of the father, until they respectively attain the age of (n)?

Here we must remember that both husbands and widowers leave children which are treated alike, and consequently we have to base our calculations on the number of children left by married men.

Out of l_x persons living at age x, $\text{M}d_x$ married men will die between the ages x and $x+1$, say, on the average, at age $x+\frac{1}{2}$. $\text{M}d_{x+1}$ married men will die between the ages $x+1$ and $x+2$, say, on the average at age $x+1\frac{1}{2}$, and so on. The present value of an annuity to the orphans left by a married man dying between the ages x and $x+1$, until they respectively reach the age of (n) is $\text{O}a(n)_x$; consequently the present value of the orphans' annuities in respect of a person living at age x will be

$$\frac{\text{M}d_x \cdot \text{O}a(n)_x \cdot v^{x+\frac{1}{2}} + \text{M}d_{x+1} \cdot \text{O}a(n)_{x+1} \cdot v^{x+1\frac{1}{2}} + \ldots}{l_x v^x}.$$

If now we call $\text{M}d_x \cdot \text{O}a(n)_x \cdot v^{x+\frac{1}{2}} = {}^{\text{O}a(n)}\text{C}_x$, and sum the values of ${}^{\text{O}a(n)}\text{C}_x$, so that ${}^{\text{O}a(n)}\text{M}_x = {}^{\text{O}a(n)}\text{C}_x + {}^{\text{O}a(n)}\text{C}_{x+1} + {}^{\text{O}a(n)}\text{C}_{x+2} + \ldots$

then the present value of the orphans' annuities of 1 each until they respectively reach the age of (n) will be represented by

$$\frac{{}^{oa(n)}\mathrm{M}_x}{\mathrm{D}_x^{(3)}}.$$

Problem VIc.—What is the present value, in respect of each person living at age x, whether bachelor, husband or widower, of 1 payable on the death of a bachelor?

Here out of l_x persons living at age x, $\mathrm{B}d_x$ bachelors will die between the ages of x and $x+1$; $\mathrm{B}d_{x+1}$ will die between the ages $x+1$ and $x+2$, and so on: so that the present value of 1 on the death of a bachelor will be

$$\frac{\mathrm{B}d_x.v^{x+\frac{1}{2}}+\mathrm{B}d_{x+1}.v^{x+1\frac{1}{2}}+\mathrm{B}d_{x+2}.v^{x+2\frac{1}{2}}+\ldots}{l_x v^x}$$

If then we call $\mathrm{B}d_x.v^{x+\frac{1}{2}}={}^{\mathrm{B}d}\mathrm{C}_x$, and sum these values so that ${}^{\mathrm{B}d}\mathrm{C}_x+{}^{\mathrm{B}d}\mathrm{C}_{x+1}+{}^{\mathrm{B}d}\mathrm{C}_{x+2}+\ldots={}^{\mathrm{B}d}\mathrm{M}_x$, the present value required will be represented by

$$\frac{{}^{\mathrm{B}d}\mathrm{M}_x}{\mathrm{D}_x^{(3)}}.$$

It is not, I believe, usual in these funds to provide for a return of contributions on withdrawal from the staff; but if such a rule is introduced it is only necessary to add to the benefit side of the equation of payments and benefits the value $\mathrm{P}.{}^{w}\mathrm{R}_x$, or $\mathrm{P}.{}^{w}\mathrm{R}_x^s$ according as P is a fixed annual payment, or a proportion of salary; or, to be more exact,

$$\mathrm{P}({}^{w}\mathrm{R}_x-\tfrac{1}{2}{}^{w}\mathrm{M}_x)(1+\tfrac{1}{2}i),\text{ or }\mathrm{P}({}^{w}\mathrm{R}_x^s-\tfrac{1}{2}{}^{w}\mathrm{M}_x^{ls})(1+\tfrac{1}{2}i),$$

using, of course, the proper denominator of D_x or D_x^s. (*See* Problems Ia and Vb.)

Problem VIIc.—What is the present value, in respect of a person aged x, of the assurance of 1 on the death of a widower leaving no children under the age of (n)?

This seems to be a benefit which is rapidly coming to the front; and in the absence of statistics, anything like an exact calculation is out of the question. It will be seen from Table L that the proportion of married men dying childless is 33 per-cent at age 25, and 25 per-cent at age 30, and after that about 15 per-cent. If we assume that, as a rule, the last child we should have to deal with would be born when the husband's age was 55, we might with safety assume that the number of widowers who would become claims, if the children's benefits ceased at

age 16, would be 20 per-cent of $\mathsf{K}d_x$ up to age 30, the percentage increasing by 1 each year up to age 45, when the number would be 25 per-cent of $\mathsf{K}d_x$; then increasing by 2 each year to age 55, then increasing by 3 each year up to age 65, when the number would be 75 per-cent of $\mathsf{K}d_x$; and then increasing by 5 each year up to age 70, when the claims for that age and over would amount to 100 per-cent of $\mathsf{K}d_x$.

If the children's benefits are continued till age 21, then I think the percentage might be deferred 5 years; that is to say, take 20 per-cent of $\mathsf{K}d_x$ up to age 35, increase the percentage by 1 each year to age 45, and so on.

The symbol for a childless widower is $(\mathsf{K}-\mathsf{C})$, and for a widower without children under the age of (n), $\overline{\mathsf{K}-\mathsf{C}(n)}$. If, now, the column we are supposed to have just constructed be called $\overline{\mathsf{K}-\mathsf{C}(n)}d_x$, then out of l_x persons living at age x, $\overline{\mathsf{K}-\mathsf{C}(n)}d_x$ will die between the ages x and $x+1$ as widowers leaving no children under the age of (n). Then, following the same reasoning as in Problem VIc; if $\overline{\mathsf{K}-\mathsf{C}(n)}d_x \times v^{x+\frac{1}{2}} = {}^{\overline{\mathsf{K}-\mathsf{C}(n)}d}\mathrm{C}_x$, and ${}^{\overline{\mathsf{K}-\mathsf{C}(n)}d}\mathrm{M}_x = {}^{\overline{\mathsf{K}-\mathsf{C}(n)}d}\mathrm{C}_x + {}^{\overline{\mathsf{K}-\mathsf{C}(n)}d}\mathrm{C}_{x+1} + \ldots$, the present value of the assurance of 1 on the death of a widower leaving no children under age (n) will be represented by

$$\frac{{}^{\overline{\mathsf{K}-\mathsf{C}(n)}d}\mathrm{M}_x}{\mathrm{D}_x^{(3)}}.$$

If the contributions are to be returned on the death of a widower leaving no child on the Funds, then Tables would have to be formed of ${}^{\overline{\mathsf{K}-\mathsf{C}(n)}d}\mathrm{R}_x$, or ${}^{\overline{\mathsf{K}-\mathsf{C}(n)}d}\mathrm{M}_x^s$, and ${}^{\overline{\mathsf{KC}-(n)}d}\mathrm{R}_x^s$. (*See* Problems IA and VB.)

Sometimes the contribution is a fixed percentage of the salary and pension, in which case it is not quite so easy to find the value of the contributions.

Problem VIIIc.—What is the present value, at age x, of future salary and pension to the end of life, equated to 1 of salary at age x?

We have to deal with this problem in three sections, namely:

(i) The present value of the salary during active service, equated to 1 of salary at age x;

(ii) The present value of the pension in respect of those who retire at the pension age, say 65, from the date of their retirement to the end of life, equated to 1 of salary at age x;

(iii) The present value of the pension in respect of those who retire before the pension age, from the date of their retirement to the end of life, equated to one of salary at age x;

and for these we must use the values based upon my Table 4. The value of (i) we already know, namely,

$$\frac{\{\mathbb{N}^s_x - \frac{1}{2}({}^d\mathrm{M}^{ls}_x + {}^w\mathrm{M}^{ls}_x + {}^r\mathrm{M}^{ls}_x)\}(1+\frac{3}{8}i)}{\mathrm{D}^s_x}$$

but (ii) and (iii) must depend upon the mode in which the pension is calculated.

Let us suppose, first, that the retiring allowance is a fixed percentage of average salary for every year of service, say π per cent.; then the value of (ii) will be

$$\frac{\pi}{100} \cdot \frac{\Sigma s_x(\mathrm{N}_{65} + \frac{1}{2}\mathrm{D}_{65})}{\mathrm{D}^s_x}$$

But this is not now a liability payable at the moment of the occurrence of the event, but an asset dependent upon the accumulation of capital; and so, according to the rule upon which I have based the adjustments, namely, that the investments are not made immediately, but, on the average, every quarter, the above value should be multiplied by $v^{\frac{3}{8}}$.

The value of (iii) will be (see Problem X B.)

$$\frac{\pi}{100} \frac{({}^{ra}\mathrm{R}^s_x - \frac{1}{2}{}^{ra}\mathrm{M}^{ls}_x)(1+\frac{3}{8}i)}{\mathrm{D}^s_x}$$

and the total value of the future salary and pension to the end of life, equated to one of salary at age x, will be

$$\Big[\{\mathbb{N}^s_x - \tfrac{1}{2}({}^d\mathrm{M}^{ls}_x + {}^w\mathrm{M}^{ls}_x + {}^r\mathrm{M}^{ls}_x)\}(1+\tfrac{3}{8}i) + \frac{\pi}{100} \cdot \Sigma s_x(\mathrm{N}_{65} + \tfrac{1}{2}\mathrm{D}_{65}) \cdot v^{\frac{3}{8}} + \frac{\pi}{100}({}^{ra}\mathrm{R}^s_x - \tfrac{1}{2}{}^{ra}\mathrm{M}^{ls}_x)(1+\tfrac{3}{8}i)\Big] \div \mathrm{D}^s_x.$$

If past service has to be taken into account in calculating the pension on retirement, then the complete value of (ii) based on past and present salary,—(not equated to 1 of salary)—will be

$$\frac{\pi}{100}\left\{\frac{(\text{total past salary}) \times (\mathrm{N}_{65} + \frac{1}{2}\mathrm{D}_{65})}{\mathrm{D}_x} + \frac{(\text{present salary}) \times \Sigma s_x(\mathrm{N}_{65} + \frac{1}{2}\mathrm{D}_{65})}{\mathrm{D}^s_x}\right\} v^{\frac{3}{8}}$$

and the complete value of (iii) will be

$$\frac{\pi}{100}\left\{(\text{total past salary})\times\frac{{}^{ra}\mathrm{M}_x}{\mathrm{D}_x}+(\text{present salary})\times\frac{{}^{ra}\mathrm{R}^s_x-\frac{1}{2}{}^{ra}\mathrm{M}^{ls}_x}{\mathrm{D}^s_x}\cdot(1+\tfrac{3}{8}i)\right\}$$

so that the complete total value is

$$\left\{(\text{present salary})\times\frac{\mathbb{N}^s_x-\frac{1}{2}({}^{d}\mathrm{M}^{ls}_x+{}^{w}\mathrm{M}^{ls}_x+{}^{r}\mathrm{M}^{ls}_x)}{\mathrm{D}^s_x}\cdot(1+\tfrac{3}{8}i)\right\}+\frac{\pi}{100}\left\{(\text{total past salary})\times\frac{(\mathrm{N}_{65}+\frac{1}{2}\mathrm{D}_{65})\cdot v^{\frac{5}{8}}+{}^{ra}\mathrm{M}_x}{\mathrm{D}_x}\right\}+\frac{\pi}{100}\left[(\text{present salary})\left\{\frac{\Sigma s_x}{s_x}\cdot\frac{(\mathrm{N}_{65}+\frac{1}{2}\mathrm{D}_{65})}{\mathrm{D}_x}\cdot v^{\frac{5}{8}}+\left(\frac{{}^{ra}\mathrm{R}^s_x}{\mathrm{D}^s_x}-\frac{1}{2}\frac{{}^{ra}\mathrm{M}^{ls}_x}{\mathrm{D}^s_x}\right)(1+\tfrac{3}{8}i)\right\}\right]$$

which looks more formidable than it really is.

In the second place, let us suppose that the retiring allowance is a fixed percentage of the last salary for every year of service, says π per-cent: then the value of (ii) will be

$$\frac{\pi}{100}\cdot\frac{(65-x)s_{64}\cdot(\mathrm{N}_{65}-\frac{1}{2}\mathrm{D}_{65})}{\mathrm{D}^s_x}\cdot v^{\frac{5}{8}},$$

and the value of (iii) will be

$$\frac{\pi}{100}\cdot\frac{({}^{ra}\mathrm{R}^{ls}_x-\frac{1}{2}{}^{ra}\mathrm{M}^{ls}_x)(1+\frac{3}{8}i)}{\mathrm{D}^s_x},$$

and the total value of the future salary and pension to the end of life, equated to 1 of salary at age x, will be

$$\left[\left\{\mathbb{N}^s_x-\tfrac{1}{2}({}^{d}\mathrm{M}^{ls}_x+{}^{w}\mathrm{M}^{ls}_x+{}^{r}\mathrm{M}^{ls}_x)\right\}(1+\tfrac{3}{8}i)+\frac{\pi}{100}(65-x)s_{64}\cdot(\mathrm{N}_{65}-\tfrac{1}{2}\mathrm{D}_{65})\cdot v^{\frac{5}{8}}+\frac{\pi}{100}\cdot({}^{ra}\mathrm{R}^{ls}_x-\tfrac{1}{2}{}^{ra}\mathrm{M}^{ls}_x)(1+\tfrac{3}{8}i)\right]\div\mathrm{D}^s_x.$$

If past service has to be taken into account in calculating the pension on retirement, we have to consider the number of years already served. Say the number is n, then the complete value of (ii), based on past and future service, will be

$$(\text{present salary})\times\frac{\pi}{100}\cdot\frac{(65+n-x)s_{64}\cdot(\mathrm{N}_{65}-\frac{1}{2}\mathrm{D}_{65})}{\mathrm{D}^s_x}\cdot v^{\frac{5}{8}},$$

and the value of (iii) will be

$$(\text{present salary})\times\frac{\pi}{100}\cdot\frac{n.{}^{ra}\mathrm{M}^{ls}_x+{}^{ra}\mathrm{R}^{ls}_x-\frac{1}{2}{}^{ra}\mathrm{M}^{ls}_x}{\mathrm{D}^s_x}\cdot(1+\tfrac{3}{8}i),$$

and the complete total value of the future salary and pension to the end of life will be

$$\text{(present salary)} \times \Big[\Big\{ {}^{\mathbb{N}}s_x - \tfrac{1}{2}({}^{d}\mathrm{M}^{ls}_x + {}^{w}\mathrm{M}^{ls}_x + {}^{r}\mathrm{M}^{ls}_x) \Big\}(1+\tfrac{3}{8}i) + \frac{\pi}{100}(65+n-x)s_{64}(\mathrm{N}_{65} - \tfrac{1}{2}\mathrm{D}_{65}).v^{\frac{5}{8}} + \frac{\pi}{100}(n^{ra}\mathrm{M}^{ls}_x + {}^{ra}\mathrm{R}^{ls}_x - \tfrac{1}{2}{}^{ra}\mathrm{M}^{ls}_x)(1+\tfrac{3}{8}i)\Big] \div \mathrm{D}^{s}_x.$$

The usual scales of retiring allowances can be introduced by appropriately applying the solutions given in Problems XIIIB, XIVB, and XVB.

Problem IXC.—To find the present value, in respect of a person of the age x at entrance, of the return of total salary and pension, without interest, on the death of a bachelor, equated to 1 of salary at age x.

This would be a very simple matter if it were not for the early retirements, because we could assume that all the bachelors in our Table No. 45 received full salaries up to the pension age, and then retired on full pension.

If the pension is π per-cent of average salary for the total number of years' service, then s for age 65 and over would be, taking 17 as the average age at entrance, $\frac{\pi}{100} \cdot \Sigma^{64}_{17} s$; or, if the pension at the pension age is $\frac{2}{3}$rds of last salary, then s for 65 and over would be $\frac{2}{3}s_{64}$.

We should then proceed in the same way as in Problem VB, and construct a Table of $v^{x+\frac{1}{2}}{}^{\mathrm{B}d}d_x$ for the whole of life, which we have called ${}^{\mathrm{B}d}\mathrm{C}_x$; sum these values like the M column, and call the new values ${}^{\mathrm{B}d}\mathrm{M}_x$, a column which we have already obtained in Problem VIC; multiply each value of ${}^{\mathrm{B}d}\mathrm{M}_x$ by s, calling the result ${}^{\mathrm{B}d}\mathrm{M}^{s}_x$, and then sum these values again, thus producing a column which we will call ${}^{\mathrm{B}d}\mathrm{R}^{s}_x$, and the value of the return of total salary and pension on the death of a bachelor, out of l_x persons (including pensioners) living at the age x would be represented by

$$\frac{{}^{\mathrm{B}d}\mathrm{R}^{s}_x}{\mathrm{D}^{s}_x}$$

where D^{s}_x is calculated on the l_x's in Table 3.

The work could be shortened by adding all the ${}^{\mathrm{B}}d_x$'s from age 65 inclusive to the end of life, thus producing ${}^{\mathrm{B}}l_{65}$, and multiplying ${}^{\mathrm{B}}l_{65}$ by $\frac{v^{65+\frac{1}{2}}\mathrm{R}^{(3)}_{65}}{\mathrm{D}^{(3)}_x} \times$ (average pension at age 65),

because no marriages are allowed after the pension age, and, consequently, the bachelors dying after that age are the numbers dying in the ordinary mortality Tables.

Even assuming that no one retires before the pension age, the values we have obtained would be in excess by half the last year's salary, namely, $\frac{1}{2}{}^{Bd}M_x^{ls}$, a value which has not been calculated.

If past service is to be taken into consideration, then the value will be—

$$\frac{(\text{total past salary})\ {}^{Bd}M_x + {}^{Bd}R_x^s}{D_x^s}.$$

The rule that no marriage after retirement on pension is recognized would enable us to make an exact valuation of the benefit, if anyone thought it worth while to take the trouble. We could find out how many bachelors retire at each age, and how many remain in the fund until the pension age. We could then deal with those remaining on the fund in the same way as above; and for those who retire early we should have to find an average pension at each age of retirement, and make the liability from the age of retirement the value of an increasing assurance of the retiring allowance according to Table 8; that is,

$$v^{x+\frac{1}{2}}\left(r_x \times \frac{R_x^{(r)}}{D_x^{(r)}} \times \text{average pension age at } x\right).$$

Another and equally good way would be to find an average age at retirement, by multiplying the number in the column "Retiring" in Table 4 by the age, including those who retire at the pension age, and dividing the sum of the results by the total number retiring.

The last plan would be quite safe, and as near accuracy as we might ever wish to get.

Problem Xc.—To find the present values, in respect of a person aged x at entrance, of the return of total salary and pension, without interest, on the death of a widower leaving no children under the age of (n), equated to 1 of salary at age x.

In the first place we have to find the number of widowers dying at each age leaving no children under the age of (n), as described in Problem VIIc; and then proceed as in Problem IXc, forming columns of $v^{x+\frac{1}{2}}(K-C)d_x$, ${}^{(K-c)d}M_x$, ${}^{(K-c)d}M_x^s$, and ${}^{(K-c)d}R_x^s$. The required value would then be represented by

$$\frac{{}^{(K-c)d}R_x^s}{D_x^s},$$

the D_x^s being calculated on Table 3.

There is no way of shortening this work as in the previous Problem, because widowers are constantly emerging from the husbands as they lose their wives. We, however, need not trouble about the early retirements, as the number of widowers dying do not become large until the pension age is reached.

Problem XIc.—A firm whose staff is represented by the particulars given on page 242, vol. xxxvi, is anxious to start a widows' pension fund, which all those now on the staff and of the age of 30 and under, and all future entrants, must join. The benefits are to be £20 a year to the widow, commencing from the moment of death of her husband, and, in the event of the early death of the widow, a continuation of the annuity until the youngest child reaches the age of 16. In the event of a widower dying leaving children, an annuity of £20 a year is to be paid until the youngest reaches the age of 16. On the death of a bachelor, the sum of £25 is to be paid to his relations. Each member of the fund is to contribute a percentage of his salary and pension until his death. The scale of pension is the Government scale, namely, $\frac{1}{60}$th of last salary for every year of service not exceeding 40, and retirement at 65 is compulsory. The firm will pay a sum down to start the fund, and will guarantee 4 per-cent interest on the investments, free of income tax. There are to be no expenses connected with the management of the fund. No one over the age of 30 is to be allowed to join except on payment of a fine commensurate with the risk. No second marriage is to be allowed except on payment of a fine commensurate with the risk, and no marriage after 65 is to be recognized. A widow on re-marriage is to forfeit her annuity.

What percentage of salary and pension should each member contribute to the fund, and what sum should the firm pay down to start the fund?

As our Tables do not provide for second marriages, nor for the forfeiture of the widows' pension on re-marriage, we must leave them out of our present calculations.

To find the proper contribution, it is necessary to determine the average age on entrance to the staff. This will vary according to the nature of the business, but in our imaginary staff I think 18 would be the proper age to take. It is not desirable to take too young an age, because you may be fixing the scale too low and endangering the Fund, while if you take an age higher than the average, you will have a margin to keep the Fund in safety.

On the benefit side we have

Present value of widow's annuity of £20 . . . . . $= \frac{{}^{wa}\mathrm{M}_{18}}{\mathrm{D}^{(3)}_{18}} \times 20$

Present value of continuation of annuity till youngest child is 16 $= \frac{{}^{\mathrm{E}(16)}\mathrm{M}_{18}}{\mathrm{D}^{(3)}_{18}} \times 20$

Present value of annuity of £20 to children of widower until youngest child is 16 . . $= \frac{{}^{\mathrm{K.YC}a(16)}\mathrm{M}_{18}}{\mathrm{D}^{(3)}_{18}} \times 20$

Present value of £25 on death of bachelor . . . . . $= \frac{{}^{\mathrm{B}d}\mathrm{M}_{18}}{\mathrm{D}^{(3)}_{18}} \times 25$

On the payment side we have

Present value of contribution during whole period of active service

$$= \frac{\mathrm{P}\left\{\mathbb{N}^{s}_{18} - \frac{1}{2}({}^{d}\mathrm{M}^{ls}_{18} + {}^{w}\mathrm{M}^{ls}_{18} + {}^{r}\mathrm{M}^{ls}_{18})\right\}(1+\frac{3}{8}i)}{\mathrm{D}^{(4)}_{18}}$$

Present value of contribution from those who retire on pension at age 65 (*see* Problem XIV B, *J.I.A.*, vol. xxxvii, p. 204)

$$= \mathrm{P} \cdot \frac{\frac{2}{3}s_{64}(\mathrm{N}_{65} + \frac{1}{2}\mathrm{D}_{65})v^{\frac{5}{8}}}{\mathrm{D}^{(4)}_{18}}.$$

Present value of contributions from those who retire on pension before the age of 65

$$= \mathrm{P} \cdot \frac{\frac{1}{60}({}^{ra}\mathrm{R}^{ls}_{18} - {}^{ra}\mathrm{R}^{ls}_{58} - \frac{1}{2}{}^{ra}\mathrm{M}^{ls}_{18})(1+\frac{3}{8}i)}{\mathrm{D}^{(4)}_{18}}.$$

As $\mathrm{D}^{(3)}_{18}$ is the same as $\mathrm{D}^{(4)}_{18}$, we shall find, by collecting all the benefits together, and all the payments together, that

$$\mathrm{P} = \frac{({}^{wa}\mathrm{M}_{18} \times 20) + ({}^{\mathrm{E}(16)}\mathrm{M}_{18} \times 20) + ({}^{\mathrm{K.YC}a(16)}\mathrm{M}_{18} \times 20) + ({}^{\mathrm{B}d}\mathrm{M}_{18} \times 25)}{\left\{\mathbb{N}^{s}_{18} - \frac{1}{2}({}^{d}\mathrm{M}^{ls}_{18} + {}^{w}\mathrm{M}^{ls}_{18} + {}^{r}\mathrm{M}^{ls}_{18})\right\}(1+\frac{3}{8}i) + \frac{2}{3}s_{64}(\mathrm{N}_{65} + \frac{1}{2}\mathrm{D}_{65})v^{\frac{5}{8}} + \frac{1}{60}({}^{ra}\mathrm{R}^{ls}_{18} - {}^{ra}\mathrm{R}^{ls}_{58} - \frac{1}{2}{}^{ra}\mathrm{M}^{ls}_{18})(1+\frac{3}{8}i)},$$

which produces ·0150, or 1½ per-cent.

To make a valuation in order to find what the reserve should be, or in other words to find the amount which the firm should pay down to start the fund, we must have a valuation schedule.

The most difficult part of this Problem is to find the present value of the future salaries and pensions on which we have to calculate the value of the contributions, so we will consider the contributions first.

From Problem VIIIc we find the value of future salary and pension (allowing for no more than 40 years' service to count for pension) to be:

Present salary multiplied by

$$\left\{\mathbb{N}^s_x - \tfrac{1}{2}({}^d\mathrm{M}^{ls}_x + {}^w\mathrm{M}^{ls}_x + {}^r\mathrm{M}^{ls}_x)\right\}(1+\tfrac{3}{8}i)$$
$$+\tfrac{1}{60}\binom{65+n-x}{\text{not exceeding } 40}s_{64}(\mathrm{N}_{65}+\tfrac{1}{2}\mathrm{D}_{65})v^{\frac{5}{8}}$$
$$+\tfrac{1}{60}(n\,{}^{ra}\mathrm{M}^{ls}_x + {}^{ra}\mathrm{R}^{ls}_x - {}^{ra}\mathrm{R}^{ls}_{x+40-n} - \tfrac{1}{2}\,{}^{ra}\mathrm{M}^{ls}_x)(1+\tfrac{3}{8}i)$$

and all divided by D^s_x.

The last two terms contain values based on past service; but the second term will not trouble us now, because in all cases in our supposed Fund, $65+n-x$ will exceed 40.

We will proceed then, first, to obtain the details for valuing the pensions on early retirement.

Particulars of Fund			Part of Valuation Schedule		
Number of Members	Number of years service $=n$	Annual salary receivable by all the members in col. (1)	Salary × past years of service (2)×(3)	$\frac{{}^{ra}\mathrm{R}^{ls}_{x+40-n}}{\mathrm{D}^s_x}$	Col. (3) ×Col. (5)
(1)	(2)	(3)	(4)	(5)	(6)
Present Age 20					
10	5	450	2,250	15·974	7,188·3
20	4	900	3,600	14·305	12,874·5
10	3	450	1,350	12·549	5,647·1
5	2	200	400	10·776	2,155·2
5	1	250	250	8·984	2,246·0
50	...	2,250	7,850	...	30,111·1
Present Age 30					
5	15	375	5,625	15·974	5,990·3
5	14	375	5,250	14·305	5,364·4
15	13	1,310	17,030	12·549	16,439·2
15	12	1,275	15,300	10·776	13,739·4
5	11	460	5,060	8·984	4,132·6
4	10	600	6,000	7·178	4,306·8
1	8	105	840	3·677	386·1
50	...	4,500	55,105	...	50,358·8

All the rest of the work, except when $(65+n-x)$ is less than 40, is based upon totals for each present age.

VALUATION SCHEDULE—(Continued)

Present age	Total annual salary	Total salary × past years of service	$\{\mathbb{N}^s_x - \frac{1}{2}({}^d\mathrm{M}^{ls}_x + {}^w\mathrm{M}^{ls}_x + {}^r\mathrm{M}^{ls}_x)\} \div \mathrm{D}_s$	(8) × (10)	$\frac{\mathrm{S}_{64}}{\mathrm{S}_x}$	$\frac{\mathrm{N}_{65}+\frac{1}{2}\mathrm{D}_{65}}{\mathrm{D}_x}$	(8) × (12) × (13)
(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
20	2,250	7,850	24·035	54,078·8	5·02	·193	2179·9
30	4,500	55,105	19·427	87,421·5	2·51	·487	5500·7
Totals	6,750	...		141,500·3	...	...	7680·6

VALUATION SCHEDULE—(Continued)

Present age	$\frac{{}^{ra}\mathrm{M}^{ls}_x}{\mathrm{D}^s_x}$	(9) × (15)	$\frac{{}^{ra}\mathrm{R}^{ls}_x}{\mathrm{D}^s_x} - \frac{1}{2}\frac{{}^{ra}\mathrm{M}^{ls}_x}{\mathrm{D}^s_x}$	(8) × (17)	Number of Members on Fund	$\frac{{}^{wa}\mathrm{M}_x}{\mathrm{D}_x}$
	(15)	(16)	(17)	(18)	(19)	(20)
20	1·136	8,917·6	42·039	94,587·8	50	·809
30	1·430	78,800·1	38·711	174,199·5	50	1·940
Totals	.	87,717·7	...	268,787·3	100	. .

VALUATION SCHEDULE—(Continued)

(19) × (20)	$\frac{{}^{E(16)}\mathrm{M}_x}{\mathrm{D}_x}$	(19) × (22)	$\frac{{}^{K\ YC a(16)}\mathrm{M}_x}{\mathrm{D}_x}$	(19) × (24)	$\frac{{}^{Bd}\mathrm{M}_x}{\mathrm{D}_x}$	(19) × (26)
(21)	(22)	(23)	(24)	(25)	(26)	(27)
40·45	·021	1·05	·027	1·350	·053	2·65
97·00	·048	2·40	·067	3·350	·066	3·30
137·45	...	3·45	...	4·700	...	5·95

Valuation Balance Sheet.

LIABILITIES.

By present value of annuity to widows of £20, col. (21)×20 . .	=	2,749·0
„ „ „ continuation of annuity after death of widow until youngest child is 16, col. (23)×20 .	=	69·0
„ „ „ £20 to children of widowers until youngest is 16, col. (25)×20	=	94·0
„ „ „ £25 on death of bachelors, col. (27)×25 . .	=	148·8
		3,060·8

ASSETS.

To present value of salary during active service, col. (11)		141,500·3	
Increase by $\frac{3}{8}i = 1\frac{1}{2}\%$ . . .		2,122·5	
			143,622·8
To present value of pension after 65, col. (14)×$\frac{2}{3}$=		5120·4	
multiplied by $v^{\frac{5}{8}}$ = decreased by $2\frac{1}{2}\%$ nearly . . .		128·0	
			4,992·4
To present value of pension on early retirement: col. (16)	=	87,717·7	
col. (18)	=	268,787·3	
		356,505·0	
deduct, col. (6) =	30,111·1		
	50,358·8	80,469·9	
		276,035·1	
increased by $1\frac{1}{2}\%$ . . .		4,140·5	
		280,175·6÷60 =	4,669·6
			153,284·8
$1\frac{1}{2}\%$ of £153,284·8 = ...			2,299·3
Deficit			761·5
			£3,060·8

The sum which the firm should pay down to start the Fund is £761. 10*s*.

Problem XIIc.—What is the present value, in respect of a person aged x at entrance, of an annuity to a widow based on the husband's pension, or the pension he would have been entitled to at the date of his death? And what would be the value of a continuation of the same annuity to the children of a widow or widower until the youngest reaches the age of (n)?

Here the question of retirement, and the mortality amongst those who retire, form important elements in the Problem; consequently we require to use Table 4, where those who retire early are separated from the others.

It will be necessary to divide the Problem into three parts, and investigate separately the values of the benefits in respect of

(1) Those who die while in the service.
(2) Those who retire before the pension age.
(3) Those who retire at the pension age.

For 1, the d's in Table 4 must be divided into $\mathrm{B}d_x$, $\mathrm{H}d_x$ and $\mathrm{K}d_x$; and for 2, the d's in Table 8 must be divided into $\mathrm{B}d_x^{(r)}$, $\mathrm{H}d_x^{(r)}$ and $\mathrm{K}d_x^{(r)}$, and extended to the end of life. This can stop at age 64 if the values of ${}^{w\alpha}\mathrm{M}_x$, &c., by Table 3 have already been calculated.

Now we know the value of the widow's annuity on the death of her husband, namely, $a_y + \frac{1}{2}$, but we want also to know the value of the widow's annuity per retired person *at the date of retirement*. It is necessary, therefore, to ascertain the values of $\frac{{}^{w\alpha}\mathrm{C}_x}{\mathrm{D}_x}$ and $\frac{{}^{w\alpha}\mathrm{M}_x}{\mathrm{D}_x}$ according to Table 8; and, for the children's annuities, to find the values of $\frac{{}^{\mathrm{E}(n)}\mathrm{M}_x}{\mathrm{D}_x}$ and $\frac{{}^{\mathrm{K.YC}a(n)}\mathrm{M}_x}{\mathrm{D}_x}$ also by the same Table.

Using the index (r) to distinguish the invalid pensioners Mortality Table, we may call

$\frac{{}^{w\alpha}\mathrm{M}_x^{(r)}}{\mathrm{D}_x^{(r)}} = \mathrm{W}a_x^{(r)}$ = the value, per retired person aged x, of a widow's annuity of 1.

$\frac{{}^{\mathrm{E}(n)}\mathrm{M}_x^{(r)}}{\mathrm{D}_x^{(r)}} = \mathrm{E}(n)_x^{(r)}$ = the value, per retired person aged x, of the addition to provide for continuation of the widow's annuity until the youngest child is (n) years of age.

$\frac{{}^{\mathrm{K.YC}a(n)}\mathrm{M}_x^{(r)}}{\mathrm{D}_x^{(r)}} = \mathrm{K.YC}a(n)_x^{(r)}$ = the value, per retired person aged x, of an annuity of 1 to the children of a widower until the youngest reaches the age of (n).

When these Tables are constructed we shall be in a position to proceed with the investigation of the Problem; but we must first know the principle on which the pensions are calculated. Let us, in the first place, take the case where the pension is based on the number of years' service of the husband, and the average salary he received.

Following the same reasoning as in my Problem XB (*J.I.A.*, vol. xxxvi, p. 238) we proceed thus:

Out of l_x persons living at the age x, in receipt of an average

salary of s each, the liability for widows' annuities, equal to the total amount of the salaries received by their husbands, would be—

In respect of $\mathrm{H}d_x$ persons dying—	And in respect of r_x persons retiring—
$\mathrm{H}d_x \times s_x \times (a_y + \frac{1}{2})$	$r_x \times s_x \times \mathrm{W}a_x^{(r)}$
In respect of $\mathrm{H}d_{x+1}$ persons dying—	And in respect of r_{x+1} persons retiring—
$\mathrm{H}d_{x+1}(s_x + s_{x+1})(a_{y+1} + \frac{1}{2})$	$r_{x+1}(s_x + s_{x+1})\mathrm{W}a_{x+1}^{(r)}$
In respect of $\mathrm{H}d_{x+2}$ persons dying—	And in respect of r_{x+2} persons retiring—
$\mathrm{H}d_{x+2}(s_x + s_{x+1} + s_{x+2})(a_{y+2} + \frac{1}{2})$	$r_{x+2}(s_x + s_{x+1} + s_{x+2})\mathrm{W}a_{x+2}^{(r)}$
⋮ ⋮	⋮ ⋮
In respect of $\mathrm{H}d_{64}$ persons dying—	And in respect of r_{64} persons retiring—
$\mathrm{H}d_{64} . \Sigma_x^{64} s . (a_{y+64-x} + \frac{1}{2})$	$r_{64} . \Sigma_x^{64} s . \mathrm{W}a_{64}^{(r)}$

In respect of l_{65} persons retiring at the pension age $l_{65}\Sigma_x^{64} s . \mathrm{W}a_{65}$.

The present value of all the annuities equated to 1 of salary at age x will be

$$\Big[\Big\{\mathrm{H}d_x \times (a_y + \tfrac{1}{2}) + r_x . \mathrm{W}a_x^{(r)}\Big\} s_x . v^{x+1} + \Big\{\mathrm{H}d_{x+1} \times (a_{y+1} + \tfrac{1}{2}) + r_{x+1} . \mathrm{W}a_{x+1}^{(r)}\Big\}(s_x + s_{x+1})v^{x+2} + \ldots + \Big\{\mathrm{H}d_{64} \times (a_{y+64-x} + \tfrac{1}{2}) + r_{64} a \mathrm{W}_{64}^{(r)}\Big\} \Sigma_x^{64} s . v^{65}\Big] + l_{65} . \Sigma_x^{64} s . \mathrm{W}a_{65}^{(r)} . v^{65}$$

divided by $l_x s_x v^x$.

Separating the symbols inside the large bracket, we have

$$(\mathrm{H}d_x(a_y + \tfrac{1}{2}) + r_x . \mathrm{W}a_x^{(r)}) s_x . v^{x+1} = (\mathrm{H}d_x . (a_y + \tfrac{1}{2}) + r_x \mathrm{W}a_x^{(r)}) s_x . v^{x+1}$$

$$(\mathrm{H}d_{x+1} . (a_{y+1} + \tfrac{1}{2}) + r_{x+1} . \mathrm{W}a_{x+1}^{(r)})(s_x + s_{x+1}) v^{x+2} = (\mathrm{H}d_{x+1} . (a_{y+1} + \tfrac{1}{2}) + r_{x+1} . \mathrm{W}a_{x+1}^{(r)}) s_x . v^{x+2} + (\mathrm{H}d_{x+1} . (a_{y+1} + \tfrac{1}{2}) + r_{x+1} . \mathrm{W}a_{x+1}^{(r)}) s_{x+1} v^{x+2}$$

⋮ ⋮ ⋮

If now we first form Tables of $\mathrm{H}d_x(a_y + \frac{1}{2})$ and $r_x . \mathrm{W}a_x^{(r)}$, then add the two values together, forming a Table of $\Big\{\mathrm{H}d_x(a_y + \frac{1}{2}) + r_x . \mathrm{W}a_x^{(r)}\Big\}$, and multiply the value by v^{x+1}, thus forming a Table of $\Big\{\mathrm{H}d_x(a_y + \frac{1}{2}) + r_x \mathrm{W}a_x^{(r)}\Big\} v^{x+1}$, which we will call ${}^{\mathrm{w.}ra}\mathrm{C}_x$; and sum these values, so that

$${}^{\mathrm{w.}ra}\mathrm{C}_x + {}^{\mathrm{w.}ra}\mathrm{C}_{x+1} + {}^{\mathrm{w.}ra}\mathrm{C}_{x+2} + \ldots {}^{\mathrm{w.}ra}\mathrm{C}_{64} = {}^{\mathrm{w.}ra}\mathrm{M}_x;$$

then multiply ${}^{\mathrm{w.}ra}\mathrm{M}_x$ by s_x making ${}^{\mathrm{w.}ra}\mathrm{M}_x^s$, and then sum these values so that

$${}^{\mathrm{w.}ra}\mathrm{R}_x^s = {}^{\mathrm{w.}ra}\mathrm{M}_x^s + {}^{\mathrm{w.}ra}\mathrm{M}_{x+1}^s + \ldots$$

we can represent the present value of an annuity to the

widow, of her husband's pension based on average salary and number of years' service equated to 1 of salary, in respect of a person aged x at entrance, and dying or retiring before the pension age, by the convenient symbol

$$\frac{{}^{\text{w.}ra}\text{R}^{s}_{x}}{\text{D}^{s}_{x}}.$$

Adding to this the value of the widow's pension in respect of the l_{65} persons who retire at the pension age,

$$=\frac{l_{65}v^{65}.\Sigma^{64}_{x}s.\text{W}a_{65}}{l_x s_x v^x}=\frac{\text{D}_{65}}{\text{D}^{s}_{x}}.\Sigma^{64}_{x}s.\text{W}a_{65}$$

we have for the complete value of the widow's annuity, per person aged x at entrance,

$$\frac{{}^{\text{w.}ra}\text{R}^{s}_{x}+\text{D}_{65}.\Sigma^{64}_{x}s.\text{W}a_{65}}{\text{D}^{s}_{x}}.$$

The value ${}^{\text{w.}ra}\text{R}^{s}_{x}$, however, requires adjustment, because it has been assumed that the full year's salary in the year of retirement was paid, and the whole has been over-discounted by half a year. The proper adjustment is to deduct $\frac{1}{2}{}^{\text{w.}ra}\text{M}^{ls}_{x}$ (which will be found later), and increase the difference by half a year's interest. The corrected value will therefore be

$$\frac{(1+\frac{1}{2}i)({}^{\text{w.}ra}\text{R}^{s}_{x}-\frac{1}{2}{}^{\text{w.}ra}\text{M}^{ls}_{x})+\text{D}_{65}.\Sigma^{64}_{x}s.\text{W}a_{65}}{\text{D}^{s}_{x}}.$$

For the continuation of the annuity for the benefit of the children of a widow or widower until the youngest child attains the age of n; we shall have (equating our values to a denominator of $l_x s_x v^x$)

In respect of $\text{H}d_x$ persons dying as husbands at age x,

$$\text{H}d_x\times\text{E}(n)\times s_x\times v^{x+1}$$

In respect of $\text{K}d_x$ persons dying as widowers at age x,

$$\text{K}d_x\times\text{YC}a(n)\times s_x\times v^{x+1}$$

In respect of r_x persons who retire at age x

$$(r_x\times\text{E}(n)^{(r)}_{x}+r_x\times\text{K.YC}a(n)^{(r)}_{x})(s_x\times v^{x+1}).$$

By separating the symbols as before, it will be found that a Table would have first to be constructed of

$$\{(\text{H}d_x\times\text{E}(n)+(\text{K}d\times\text{YC}a(n))+(r_x\times\text{E}(n)^{(r)}_{x})+(r_x\times\text{K.YC}a(n)^{(r)}_{x})\}v^{x+1}$$

which could be called ${}^{\text{YC}(n).ra}\text{C}_{x}$.

Then these values of ${}^{\text{VC}(n).ra}\text{C}_x$ would have to be summed to form a Table which might be called ${}^{\text{VC}(n).ra}\text{M}_x$. The values in this Table would have to be multiplied by s_x, forming a Table of ${}^{\text{VC}(n).ra}\text{M}^s_x$, and these new values summed again producing a Table of ${}^{\text{VC}(n).ra}\text{R}^s_x$.

In respect of those who retire at the pension age 65, the value equated to a denominator of $l_x s_x v^x$ will be $l_{65} \times \Sigma^{64}_x s\,(\text{E}(n)_{65} + \text{K}.\text{YC}a(n)_{65}).v^{65}$, and the complete value for the continuation of the annuity will be

$$\frac{{}^{\text{VC}(n).ra}\text{R}^s_x + \text{D}_{65}.\Sigma^{64}_x s(\text{E}(n)_{65} + \text{K}.\text{YC}a(n)_{65})}{\text{D}^s_x}.$$

The value ${}^{\text{VC}(n).ra}\text{R}^s_x$ is, however, too large by reason of our assuming that the last year's salary is paid in full, and by the whole being over-discounted by half a year. The correct value would therefore be

$$\frac{(1+\tfrac{1}{2}i)\left({}^{\text{VC}(n).ra}\text{R}^s_x - \tfrac{1}{2}.{}^{\text{VC}(n).ra}\text{M}^{ls}_x\right) + \text{D}_{65}.\Sigma^{64}_x s(\text{E}(n)_{65} + \text{K}.\text{YC}a(n)_{65})}{\text{D}^s_x}$$

Secondly, let us assume that the pension is based on last salary and number of years' service. The same process as before will have to be followed, up to the construction of the Table of values of ${}^{\text{W}.ra}\text{C}_x$, when each value would have to be multiplied by s_x, and the resulting values then summed, so that

$${}^{\text{W}.ra}\text{C}_x s_x + {}^{\text{W}.ra}\text{C}_{x+1} s_{x+1} + {}^{\text{W}.ra}\text{C}_{x+2} s_{x+2} + \ldots = {}^{\text{W}.ra}\text{M}^{ls}_x.$$

These values would have to be summed again, so that

$${}^{\text{W}.ra}\text{M}^{ls}_x + {}^{\text{W}.ra}\text{M}^{ls}_{x+1} + {}^{\text{W}.ra}\text{M}^{ls}_{x+2} + \ldots = {}^{\text{W}.ra}\text{R}^{ls}_x.$$

The value of the widow's pension, in respect of the l_{65} persons who retire at the pension age 65, will be

$$\frac{1}{l_x s_x v^x}.\, l_{65} v^{65}.(65-x) s_{64} \times \text{W}a_{65}\},$$

and the value of the widow's annuity per person aged x at entrance will be

$$\frac{{}^{\text{W}.ra}\text{R}^{ls}_x + \text{D}_{65}(65-x) s_{64} \times \text{W}a_{65}}{\text{D}^s_x},$$

or, using the proper adjustments,

$$\frac{(1+\tfrac{1}{2}i)\left({}^{\text{W}.ra}\text{R}^{ls}_x - \tfrac{1}{2}{}^{\text{W}.ra}\text{M}^{ls}_x\right) + \text{D}_{65}(65-x) s_{64} \times \text{W}a_{65}}{\text{D}^s_x}.$$

For the continuation of the annuity for the benefit of the children of a widow or widower until the youngest arrived at the age of (n) we follow the same process as above up to the construction of the Table of ${}^{\text{VC}(n).ra}\text{C}_x$. These values would have to be multiplied by s_x, and then summed, so that

$${}^{\text{VC}(n).ra}\text{C}_x s_x + {}^{\text{VC}(n).ra}\text{C}_{x+1}\, s_{x+1} + \ldots = {}^{\text{VC}(n).ra}\text{M}_x^{ls}.$$

Summing these values in like manner, we shall have

$${}^{\text{VC}(n).ra}\text{M}_x^{ls} + {}^{\text{VC}(n).ra}\text{M}_{x+1}^{ls} + \ldots = {}^{\text{VC}(n).ra}\text{R}_x^{ls},$$

and the value, in respect of the l_{65} persons who retire at the pension age 65, will be

$$\frac{1}{l_x s_x v^x}\{l_{65}v^{65}.(65-x)s_{64}.(\text{E}(n)_{65} + \text{K.YC}a(n)_{65})\},$$

so that the value of the addition to the value of the widow's annuity (making the usual correction) will be

$$\frac{(1+\frac{1}{2}i)({}^{\text{VC}(n).ra}\text{R}_x^{ls} - \frac{1}{2}{}^{\text{VC}(n).ra}\text{M}_x^{ls}) + \text{D}_{65}(65-x)s_{64}(\text{E}(n)_{65} + \text{K.YC}a(n)_{65})}{\text{D}_x^s}$$

By means of Problems XIII B, XIV B and XV B, it would be a simple matter to apply the above formulas to the usual modes of calculating pensions.

I have not attempted to make any Tables for this kind of pension to widows and children; because I have not found the plan to be in practical use with either of these modes of calculating pensions; but, from signs which I have observed, I should not be surprised to hear of an attempt to put this plan into operation. The younger members might do worse than exercise their powers on the construction of Tables for this kind of benefit.

Problem XIII C.—To ascertain the fine payable on second marriage.

One of the difficult problems connected with these Funds is to determine what fine should be paid on re-marriage. All the Tables are based on first marriage only; and if the contributions are based on such Tables, without any addition for the contingencies of second and third and subsequent marriages, then the widower who marries again is bringing on to the Fund a liability which he is not paying for. He undertook to make a contribution for the whole of life, and the Fund undertook the risk of making a payment if he died a bachelor, the risk of paying

an annuity to his widow by the first marriage, the risk of paying an annuity to his children, and the risk of paying a sum on his death as a widower. The risk of his dying a bachelor ceased when he married, the risk of paying an annuity to his widow by the first marriage ceased when his wife died, the risk of paying an annuity to his children may or may not have ceased, and the risk of paying a sum at his death is still in existence. He now wants to bring on to the Fund the risks attaching to a second marriage, that is, the risk of paying an annuity to his widow by the second marriage, and the risk of paying an annuity to his children by the second marriage; but by doing so he is reducing the risks left by his original contract, for the value of the annuity to his children by the first wife will be smaller, and the risk of his dying a widower without children under a certain age will be less.

If we assume that the value of the reduction in the old risks will cover the value of the annuity to the children of the new marriage, there remains the additional risk of the annuity to the widow, which is $a_y - a_{xy}$, and that is clearly the fine which should be paid.

The cost, however, is prohibitive; and you will never persuade a man that it is right and just that he should go on paying all his life for other people's widows, and not be allowed to leave his own widow on the Fund, even if it is by a second marriage, without paying an additional sum equal to the full value of the risk.

In all the Funds I have examined fine on second marriage is quite nominal compared with the correct amount; but then the contributions and the values of the risks have generally been calculated with an excess of caution which has justified the imposition of only a small fine.

If, however, we are going to calculate the values of the contributions more closely, we shall have to calculate the fines more closely, or find some way of making them merely nominal, or doing without them altogether. We can only do this by including the risk of second and subsequent marriages in the original calculations. But how?

The difficulty is considerably increased when the contribution is a fixed percentage of salary and pension.

Hypothetical Experience of Staff Pension Fund for Widows and Orphans.

TABLE 44.

Simple Commutation Columns (according to Tables 3 and 45).

4 PER-CENT				3 PER-CENT			
Age x	$D_x^{(3)}$	Age x	$D_x^{(3)}$	Age x	$D_x^{(3)}$	Age x	$D_x^{(3)}$
15	11105·	60	343·93	15	12837·	60	614·10
16	9839·0	61	319·28	16	11484·	61	575·61
17	8581·4	62	295·57	17	10113·	62	538·05
18	7469·1	63	272·79	18	8887·9	63	501·40
19	6552·1	64	251·01	19	7872·3	64	465·84
20	5793·8	65	230·41	20	7028·9	65	431·77
21	5158·9	66	211·11	21	6319·5	66	399·44
22	4627·2	67	192·95	22	5723·1	67	368·62
23	4176·1	68	175·63	23	5215·3	68	338·79
24	3789·7	69	159·14	24	4778·7	69	309·97
25	3456·0	70	143·49	25	4400·2	70	282·20
26	3164·7	71	128·69	26	4068·5	71	255·55
27	2908·0	72	114·74	27	3774·8	72	230·06
28	2680·8	73	101·66	28	3513·7	73	205·81
29	2478·6	74	89·462	29	3280·2	74	182·88
30	2297·3	75	78·141	30	3069·7	75	161·28
31	2134·2	76	67·716	31	2879·5	76	141·12
32	1986·3	77	58·186	32	2706·0	77	122·44
33	1852·3	78	49·538	33	2547·9	78	105·25
34	1730·0	79	41·768	34	2402·7	79	89·605
35	1618·0	80	34·851	35	2269·1	80	75·492
36	1515·1	81	28·755	36	2145·4	81	62·891
37	1420·6	82	23·449	37	2031·0	82	51·786
38	1333·2	83	18·883	38	1924·7	83	42·107
39	1252·5	84	15·005	39	1825·7	84	33·783
40	1177·5	85	11·757	40	1733·0	85	26·728
41	1107·5	86	9·076	41	1645·9	86	20·833
42	1042·6	87	6·897	42	1564·4	87	15·985
43	981·59	88	5·155	43	1487·2	88	12·063
44	924·76	89	3·789	44	1414·7	89	8·953
45	871·40	90	2·732	45	1346·0	90	6·517
46	821·26	91	1·933	46	1280·9	91	4·657
47	774·16	92	1·341	47	1219·1	92	3·263
48	729·93	93	·909	48	1160·6	93	2·233
49	688·10	94	·604	49	1104·7	94	1·497
50	648·54	95	·393	50	1051·4	95	·983
51	611·29	96	·248	51	1000·6	96	·627
52	576·21	97	·154	52	952·29	97	·392
53	542·90	98	·092	53	905·98	98	·237
54	511·32	99	·054	54	861·55	99	·139
55	481·02	100	·030	55	818·35	100	·078
56	451·72	...	...	56	776·00	...	...
57	423·02	...	...	57	733·74	...	...
58	395·95	...	...	58	693·44	...	...
59	369·45	...	...	59	653·31	...	...

Hypothetical Experience of Staff Pension Fund for Widows and Orphans.

TABLE 45.

Showing the number living and remaining on the staff at each age according to Table 3, and the numbers who die at each age as Bachelors, Husbands, and Widowers; no second marriage being allowed for.

Age	Living and remaining on Staff	Dying	Dying Bachelors	Dying Married	Dying Husbands	Dying Widowers	Age
x	$l_x^{(3)}$	$d_x^{(3)}$	$\mathrm{B}d_x$	$\mathrm{M}d_x$	$\mathrm{H}d_x$	$\mathrm{K}d_x$	x
15	20,000	72	72	...	...	...	15
16	18,428	68	68	...	...	...	16
17	16,716	64	64	...	...	...	17
18	15,131	59	59	...	...	...	18
19	13,804	55	55	...	...	...	19
20	12,695	52	51·9	·1	·1	...	20
21	11,756	49	48·7	·3	·3	...	21
22	10,966	47	46·2	·8	·8	...	22
23	10,293	45	43·4	1·6	1·6	...	23
24	9,714	43	40·5	2·5	2·5	...	24
25	9,213	42	38·3	3·7	3·7	...	25
26	8,774	42	37·0	5·0	4·9	·1	26
27	8,385	41	34·8	6·2	6·1	·1	27
28	8,039	40	32·5	7·5	7·4	·1	28
29	7,730	40	30·0	10·0	9·8	·2	29
30	7,451	40	28·0	12·0	11·8	·2	30
31	7,199	41	26·6	14·4	14·1	·3	31
32	6,968	41	24·6	16·4	16·0	·4	32
33	6,758	41	22·7	18·3	17·8	·5	33
34	6,564	42	21·2	20·8	20·2	·6	34
35	6,385	43	19·7	23·3	22·6	·7	35
36	6,218	43	18·0	25·0	24·2	·8	36
37	6,063	45	17·4	27·6	26·7	·9	37
38	5,918	46	16·5	29·5	28·4	1·1	38
39	5,782	47	16·1	30·9	29·7	1·2	39
40	5,653	49	16·1	32·9	31·5	1·4	40
41	5,530	50	16·0	34·0	32·4	1·6	41
42	5,414	53	16·4	36·6	34·8	1·8	42
43	5,301	54	16·2	37·8	35·9	1·9	43
44	5,194	57	16·5	40·5	38·3	2·2	44
45	5,090	60	16·9	43·1	40·6	2·5	45
46	4,989	63	17·4	45·6	42·8	2·8	46
47	4,891	65	17·7	47·3	44·2	3·1	47
48	4,796	69	18·6	50·4	46·8	3·6	48
49	4,702	73	19·4	53·6	49·5	4·1	49
50	4,609	76	19·8	56·2	51·6	4·6	50
51	4,518	79	20·2	58·8	53·7	5·1	51
52	4,429	83	20·8	62·2	56·5	5·7	52
53	4,340	87	21·4	65·6	59·2	6·4	53
54	4,251	92	22·4	69·6	62·4	7·2	54
55	4,159	97	23·2	73·8	65·6	8·2	55
56	4,062	103	24·3	78·7	69·3	9.4	56
57	3,956	108	25·1	82·9	72·4	10·5	57
58	3,851	114	25·8	88·2	76·4	11·8	58
59	3,737	119	26·4	92·6	79·5	13·1	59

Hypothetical Experience of Staff Pension Fund for Widows and Orphans.

TABLE 45—(*continued*).

Showing the number living and remaining on the staff at each age according to Table 3, and the numbers who die at each age as Bachelors, Husbands, and Widowers; no second marriage being allowed for.

Age	Living and remaining on Staff	Dying	Dying Bachelors	Dying Married	Dying Husbands	Dying Widowers	Age
x	$l_x^{(3)}$	$d_x^{(3)}$	$\mathrm{B}d_x$	$\mathrm{M}d_x$	$\mathrm{H}d_x$	$\mathrm{K}d_x$	x
60	3618·0	125·0	26·8	98·2	83·3	14·9	60
61	3493·0	130·0	26·9	103·1	86·4	16·7	61
62	3363·0	135·0	26·9	108·1	89·3	18·8	62
63	3228·0	139·0	26·8	112·2	91·2	21·0	63
64	3089·0	140·0	26·2	113·8	91·0	22·8	64
65	2949·0	139·0	25·5	113·5	89·0	24·5	65
66	2810·0	139·0	25·1	113·9	87·4	26·5	66
67	2671·0	142·5	25·4	117·1	87·8	29·3	67
68	2528·5	145·7	25·9	119·8	87·6	32·2	68
69	2382·8	148·4	26·4	122·0	86·9	35·1	69
70	2234·4	150·3	26·8	123·5	85·5	38·0	70
71	2084·1	151·6	27·1	124·5	83·6	40·9	71
72	1932·5	151·8	27·2	124·6	81·0	43·6	72
73	1780·7	151·0	27·1	123·9	77·8	46·1	73
74	1629·7	149·3	26·8	122·5	74·1	48·4	74
75	1480·4	146·2	26·4	119·8	69·6	50·2	75
76	1334·2	141·9	25·9	116·0	64·5	51·5	76
77	1192·3	136·6	25·4	111·2	59·0	52·2	77
78	1055·7	130·0	24·8	105·2	53·0	52·2	78
79	925·7	122·4	23·7	98·7	47·1	51·6	79
80	803·3	114·0	22·2	91·8	41·2	50·6	80
81	689·3	104·7	21·4	83·3	35·0	48·3	81
82	584·6	95·0	18·8	76·2	29·9	46·3	82
83	489·6	85·0	16·4	68·6	24·9	43·7	83
84	404·6	74·9	14·2	60·7	20·3	40·4	84
85	329·7	65·0	12·0	53·0	16·2	36·8	85
86	264·7	55·5	10·1	45·4	12·6	32·8	86
87	209·2	46·6	7·8	38·8	9·7	29·1	87
88	162·6	38·3	6·4	31·9	7·1	24·8	88
89	124·3	31·1	5·2	25·9	5·1	20·8	89
90	93·2	24·6	4·1	20·5	3·5	17·0	90
91	68·6	19·1	3·2	15·9	2·4	13·5	91
92	49·5	14·6	2·4	12·2	1·5	10·7	92
93	34·9	10·8	1·8	9·0	1·0	8·0	93
94	24·1	7·8	1·3	6·5	·6	5·9	94
95	16·3	5·6	·9	4·7	·4	4·3	95
96	10·7	3·8	·6	3·2	·2	3·0	96
97	6·9	2·6	·4	2·2	·1	2·1	97
98	4·3	1·7	·3	1·4	...	1·4	98
99	2·6	1·1	·2	·9	...	·9	99
100	1·5	·6	·1	·5	...	·5	100
101	·9	·4	·1	·3	...	·3	101
102	·5	·2	...	·2	...	·2	102
103	·3	·2	...	·2	...	·2	103
104	·1	·1	...	·1	...	·1	104
105	...	...	...	...	...	...	105

Hypothetical Experience of Staff Pension Fund for Widows and Orphans.

Table 46.

Value, on the Death of a Husband, of a continuous Annuity of 1 *to the Widow for the remainder of her life.*

Age of Husband at Death	Age of Widow at Death of Husband	Value of $a_y + \frac{1}{2}$		Age of Husband at Death
x	y	4 per-cent	3 per-cent	x
20	22·0	18·311	21·547	20
21	22·4	18·278	21·496	21
22	22·8	18·245	21·445	22
23	23·2	18·211	21·394	23
24	23·6	18·177	21·341	24
25	24·0	18·143	21·288	25
26	24·8	18·072	21·178	26
27	25·6	17·998	21·065	27
28	26·4	17·922	20·950	28
29	27·2	17·844	20·832	29
30	28·0	17·764	20·712	30
31	28·7	17·690	20·602	31
32	29·4	17·614	20·490	32
33	30·1	17·537	20·376	33
34	30·8	17·457	20·259	34
35	31·5	17·374	20·139	35
36	32·1	17·302	20·034	36
37	32·7	17·227	19·925	37
38	33·3	17·151	19·815	38
39	33·9	17·073	19·704	39
40	34·5	16·992	19·589	40
41	35·2	16·897	19·453	41
42	35·9	16·799	19·315	42
43	36·6	16·697	19·173	43
44	37·3	16·593	19·028	44
45	38·0	16·487	18·880	45
46	38·8	16·361	18·706	46
47	39·6	16·231	18·527	47
48	40·4	16·097	18·344	48
49	41·2	15·959	18·157	49
50	42·0	15·818	17·967	50
51	42·9	15·655	17·747	51
52	43·8	15·486	17·522	52
53	44·7	15·311	17·291	53
54	45·6	15·131	17·053	54
55	46·5	14·944	16·810	55
56	47·4	14·751	16·559	56
57	48·3	14·552	16·301	57
58	49·2	14·345	16·036	58
59	50·1	14·131	15·765	59

Hypothetical Experience of Staff Pension Fund for Widows and Orphans.

TABLE 46—(*continued*).

Value, on the Death of a Husband, of a continuous Annuity of 1 *to the Widow for the remainder of her life.*

Age of Husband at Death	Age of Widow at Death of Husband	VALUE OF $a_y + \frac{1}{2}$		Age of Husband at Death
x	y	4 per-cent	3 per-cent	x
60	51·0	13·911	15·488	60
61	51·9	13·687	15·205	61
62	52·8	13·457	14·919	62
63	53·7	13·224	14·628	63
64	54·6	12·986	14·334	64
65	55·5	12·745	14·037	65
66	56·5	12·471	13·702	66
67	57·5	12·191	13·362	67
68	58·5	11·904	13·015	68
69	59 5	11·609	12·662	69
70	60·5	11·305	12·300	70
71	61·5	10·993	11·929	71
72	62·5	10·671	11·551	72
73	63·5	10·339	11·164	73
74	64·5	9·998	10·769	74
75	65·5	9·648	10·366	75
76	66·5	9·290	9·957	76
77	67·5	8·928	9·545	77
78	68·5	8·572	9·143	78
79	69·5	8·233	8·760	79
80	70·5	7·908	8·395	80
81	71·5	7·594	8·044	81
82	72·5	7·291	7·705	82
83	73·5	6·996	7·376	83
84	74·5	6·705	7·054	84
85	75·5	6·418	6·736	85
86	76·5	6·130	6·419	86
87	77·5	5·843	6·104	87
88	78·5	5·556	5·791	88
89	79·5	5·267	5·478	89
90	80·5	4·977	5·163	90
91	81·5	4·679	4·842	91
92	82·5	4·370	4·513	92
93	83·5	4·049	4·171	93
94	84·5	3·718	3·821	94
95	85·5	3·388	3·473	95
96	86·5	3·060	3·129	96
97	87·5	2·734	2·790	97

Hypothetical Experience of Staff Pension Fund for Widows and Orphans.

Table 47.

Present value, on the death of a Married Man, of a continuous Annuity of 1 *until the youngest surviving child attains the age of* 21, 16, *or* 14.

Age of Widower at Death	4 per-cent			3 per-cent			Age of Widower at Death
	Annuity till Age of			Annuity till Age of			
	21	16	14	21	16	14	
x	$\text{YC}a(21)$	$\text{YC}a(16)$	$\text{YC}a(14)$	$\text{YC}a(21)$	$\text{YC}a(16)$	$\text{YC}a(14)$	x
25	11·74	9·59	8·61	12·79	10·25	9·12	25
26	11·71	9·56	8·57	12·75	10·21	9·08	26
27	11·68	9·52	8·53	12·71	10·17	9·03	27
28	11·65	9·48	8·48	12·67	10·12	8·98	28
29	11·62	9·43	8·43	12·63	10·07	8·93	29
30	11·58	9·38	8·38	12·59	10·02	8·87	30
31	11·54	9·33	8·33	12·54	9·96	8·81	31
32	11·50	9·28	8·28	12·49	9·90	8·75	32
33	11·46	9·23	8·22	12·44	9·84	8·69	33
34	11·41	9·18	8·16	12·38	9·78	8·63	34
35	11·36	9·13	8·10	12·32	9·71	8·56	35
36	11·31	9·07	8·04	12·26	9·64	8·48	36
37	11·26	9·00	7·97	12·20	9·56	8·39	37
38	11·20	8·93	7·89	12·13	9·48	8·30	38
39	11·13	8·85	7·80	12·05	9·39	8·21	39
40	11·06	8·76	7·71	11·97	9·29	8·11	40
41	10·98	8·67	7·60	11·88	9·19	8·00	41
42	10·90	8·57	7·50	11·78	9·08	7·88	42
43	10·82	8·46	7·39	11·68	8·96	7·76	43
44	10·74	8·35	7·27	11·57	8·84	7·63	44
45	10·64	8·24	7·15	11·46	8·71	7·49	45
46	10·54	8·12	7·02	11·35	8·58	7·35	46
47	10·44	8·00	6·89	11·23	8·44	7·20	47
48	10·33	7·88	6·75	11·11	8·30	7·05	48
49	10·22	7·75	6·60	10·98	8·15	6·89	49
50	10·10	7·60	6·45	10·84	7·98	6·72	50
51	9·97	7·43	6·27	10·69	7·80	6·52	51
52	9·82	7·24	6·08	10·52	7·60	6·30	52
53	9·65	7·04	5·86	10·32	7·37	6·06	53
54	9·57	6·82	5·61	10·10	7·12	5·80	54
55	9·26	6·56	5·33	9·86	6·85	5·51	55
56	9·02	6·28	5·00	9·60	6·54	5·17	56
57	8·75	5·90	4·62	9·29	6·19	4·77	57
58	8·44	5·58	4·21	8·94	5·78	4·34	58
59	8·08	5·13	3·76	8·54	5·30	3·86	59

Hypothetical Experience of Staff Pension Fund for Widows and Orphans.

TABLE 47—(*continued*).

Present value, on the death of a Married Man, of a continuous Annuity of 1 *until the youngest surviving child attains the age of* 21, 16, *or* 14.

Age of Widower at Death	4 PER-CENT			3 PER-CENT			Age of Widower at Death
	Annuity till Age of			Annuity till Age of			
	21	16	14	21	16	14	
x	$Y\bar{C}a(21)$	$Y\bar{C}a(16)$	$Y\bar{C}a(14)$	$Y\bar{C}a(21)$	$Y\bar{C}a(16)$	$Y\bar{C}a(14)$	x
60	7·65	4·61	3·27	8·04	4·75	3·34	60
61	6·97	4·08	2·76	7·34	4·20	2·82	61
62	6·32	3·48	2·26	6·64	3·58	2·32	62
63	5·68	2·87	1·80	5·94	2·95	1·86	63
64	5·04	2·26	1·37	5·23	2·32	1·42	64
65	4·41	1·67	1·06	4·53	1·70	1·09	65
66	3·78	1·29	·86	3·88	1·32	·88	66
67	3·22	1·03	·70	3·30	1·05	·72	67
68	2·74	·85	·58	2·81	·87	·59	68
69	2·32	·72	·48	2·38	·73	·49	69
70	1·94	·60	·39	1·99	·62	·41	70
71	1·60	·51	·32	1·64	·52	·33	71
72	1·30	·43	·26	1·33	·44	·27	72
73	1·10	·37	·21	1·13	·38	·22	73
74	·94	·31	·17	·96	·32	·17	74
75	·79	·26	·14	·81	·27	·14	75
76	·66	·22	·11	·68	·22	·11	76
77	·56	·18	·09	·57	·18	·09	77
78	·46	·15	·08	·48	·15	·08	78
79	·39	·12	·06	·40	·12	·06	79
80	·32	·10	·05	·33	·11	·05	80
81	·27	·09	·04	·28	·09	·04	81
82	·23	·07	·03	·23	·07	·03	82
83	·19	·06	·02	·20	·06	·02	83
84	·16	·04	·01	·16	·05	·01	84
85	·13	·03	...	·13	·03	...	85
86	·11	·02	...	·11	·02	...	86
87	·09	·01	...	·09	·01	...	87
88	·07	...	...	·07	...	...	88
89	·05	...	...	·05	...	...	89
90	·04	...	...	·04	...	...	90
91	·02	...	...	·02	...	...	91
92	·01	...	...	·01	...	...	92

Hypothetical Experience of Staff Pension Fund for Widows and Orphans.

TABLE 48.

Value, on the death of a Husband, of a continuous Reversionary Annuity of 1 *to commence on the death of the Widow, and continue until the youngest surviving child reaches the age of* 21, 16, *or* 14.

Age of Husband at Death	4 per-cent			3 per-cent			Age of Husband at Death
	Annuity till age of			Annuity till age of			
	21	16	14	21	16	14	
x	E(21)	E(16)	E(14)	E(21)	E(16)	E(14)	x
25	1·096	·698	·551	1·235	·765	·597	25
26	1·096	·696	·548	1·234	·762	·593	26
27	1·095	·694	·545	1·232	·758	·589	27
28	1·094	·691	·541	1·230	·754	·585	28
29	1·092	·687	·537	1·228	·750	·580	29
30	1·090	·682	·533	1·225	·745	·575	30
31	1·087	·677	·528	1·222	·740	·569	31
32	1·084	·672	·522	1·218	·735	·563	32
33	1·080	·667	·516	1·213	·729	·556	33
34	1·075	·661	·509	1·207	·722	·549	34
35	1·070	·655	·502	1·201	·715	·542	35
36	1·064	·648	·494	1·194	·706	·534	36
37	1·058	·640	·486	1·187	·697	·526	37
38	1·052	·632	·478	1·179	·688	·517	38
39	1·046	·623	·470	1·171	·678	·507	39
40	1·040	·614	·462	1·162	·668	·497	40
41	1·033	·605	·454	1·153	·658	·487	41
42	1·025	·596	·445	1·144	·648	·477	42
43	1·017	·587	·436	1·134	·637	·467	43
44	1·009	·578	·427	1·123	·625	·456	44
45	1·000	·568	·417	1·112	·613	·445	45
46	·991	·557	·406	1·101	·600	·433	46
47	·982	·546	·395	1·090	·587	·420	47
48	·972	·533	·383	1·078	·573	·406	48
49	·962	·519	·369	1·066	·557	·391	49
50	·951	·504	·354	1·051	·539	·375	50
51	·938	·486	·336	1·035	·519	·356	51
52	·922	·464	·316	1·016	·495	·334	52
53	·903	·440	·295	·993	·469	·311	53
54	·880	·415	·272	·966	·442	·286	54
55	·853	·388	·246	·934	·413	·258	55
56	·822	·356	·217	·898	·377	·228	56
57	·787	·320	·188	·857	·337	·196	57
58	·748	·284	·159	·811	·297	·164	58
59	·703	·248	·131	·759	·258	·134	59

Hypothetical Experience of Staff Pension Fund for Widows and Orphans.

TABLE 48—(*continued*).

Value, on the death of a Husband, of a continuous Reversionary Annuity of 1 *to commence on the death of the Widow, and continue until the youngest surviving child reaches the age of* 21, 16, *or* 14.

Age of Husband at Death	4 PER-CENT			3 PER-CENT			Age of Husband at Death
	Annuity till age of			Annuity till age of			
	21	16	14	21	16	14	
x	E(21)	E(16)	E(14)	E(21)	E(16)	E(14)	x
60	·650	·212	·109	·699	·220	·111	60
61	·574	·182	·091	·615	·189	·093	61
62	·498	·154	·077	·531	·160	·079	62
63	·422	·128	·066	·447	·133	·068	63
64	·352	·106	·057	·370	·110	·058	64
65	·282	·088	·049	·296	·092	·050	65
66	·230	·076	·042	·241	·080	·043	66
67	·192	·065	·036	·200	·068	·037	67
68	·161	·054	·031	·168	·057	·032	68
69	·139	·046	·026	·145	·048	·027	69
70	·123	·040	·021	·128	·042	·022	70
71	·111	·034	·017	·115	·036	·018	71
72	·101	·029	·014	·105	·030	·014	72
73	·092	·024	·011	·096	·025	·011	73
74	·083	·020	·009	·086	·021	·009	74
75	·075	·016	·007	·078	·017	·007	75
76	·068	·013	·005	·071	·014	·005	76
77	·061	·011	·004	·064	·011	·004	77
78	·055	·009	·003	·058	·009	·003	78
79	·050	·007	·003	·052	·007	·003	79
80	·045	·006	·002	·047	·006	·002	80
81	·040	·005	·002	·042	·005	·002	81
82	·035	·004	·001	·036	·004	·001	82
83	·030	·003	·001	·031	·003	·001	83
84	·025	·002	...	·026	·002	...	84
85	·020	·001	...	·021	·001	...	85
86	·016	...	...	·017	...	...	86
87	·012	...	...	·013	...	...	87
88	·009	...	...	·009	...	...	88
89	·006	...	...	·006	...	...	89
90	·004	...	...	·004	...	...	90
91	·002	...	...	·002	...	...	91
92	...	...	...	...	...	...	92

Hypothetical Experience of Staff Pension Fund for Widows and Orphans.

TABLE 49.

Value, on the death of a Married man, of continuous Annuities of 1 *to each of the children until they respectively attain the age of* 21, 16, *or* 14.

Age of Married man at Death	4 PER-CENT			3 PER-CENT			Age of Married man at Death
	Orphans' Annuities till age of			Orphans' Annuities till age of			
	21	16	14	21	16	14	
x	$O\alpha(21)$	$O\alpha(16)$	$O\alpha(14)$	$O\alpha(21)$	$O\alpha(16)$	$O\alpha(14)$	x
20	0·	0·	0·	0·	0·	0·	20
21	5·20	4·33	3·92	5·67	4·63	4·16	21
22	8·10	6·70	6·05	8·80	7·15	6·40	22
23	10·22	8·64	7·60	11·10	9·21	8·04	23
24	12·08	10·07	8·82	13·10	10·71	9·30	24
25	13·74	11·12	9·87	14·88	11·81	10·40	25
26	15·37	12·17	10·83	16·58	12·91	11·42	26
27	16·85	13·22	11·67	18·20	14·01	12·30	27
28	18·36	14·27	12·50	19·80	15·11	13·12	28
29	19·86	15·33	13·31	21·40	16·21	13·96	29
30	21·37	16·38	14·17	23·00	17·31	14·84	30
31	22·89	17·44	14·94	24·60	18·41	15·64	31
32	24·41	18·42	15·65	26·20	19·41	16·38	32
33	25·92	19·32	16·43	27·80	20·37	17·16	33
34	27·45	20·12	17·01	29·38	21·16	17·74	34
35	28·64	20·85	17·35	30·62	21·87	18·10	35
36	29·73	21·21	17·14	31·70	22·24	17·88	36
37	30·53	21·17	16·80	32·50	22·20	17·52	37
38	31·15	20·86	16·27	33·10	21·80	16·96	38
39	31·20	20·31	15·67	33·15	21·25	16·36	39
40	30·87	19·66	15·06	32·75	20·54	15·73	40
41	30·37	18·96	14·45	32·22	19·80	15·09	41
42	29·77	18·25	13·84	31·60	19·05	14·45	42
43	29·15	17·53	13·23	30·90	18·29	13·80	43
44	28·52	16·81	12·62	30·18	17·54	13·15	44
45	27·89	16·09	12·01	29·45	16·78	12·50	45
46	27·18	15·37	11·39	28·70	16·03	11·86	46
47	26·46	14·65	10·77	27·90	15·27	11·21	47
48	25·73	13·93	10·15	27·10	14·52	10·56	48
49	24·95	13·21	9·53	26·26	13·76	9·92	49
50	24·06	12·49	8·92	25·30	13·01	9·27	50
51	22·93	11·77	8·31	24·10	12·25	8·62	51
52	21·60	11·04	7·70	22·70	11·50	7·98	52
53	20·27	10·32	7·09	21·30	10·74	7·33	53
54	18·94	9·60	6·47	19·90	9·99	6·68	54
55	17·61	8·88	5·85	18·50	9·23	6·04	55
56	16·29	8·16	5·23	17·10	8·48	5·40	56
57	14·97	7·44	4·70	15·70	7·72	4·85	57
58	13·65	6·72	4·21	14·30	6·97	4·34	58
59	12·33	6·00	3·76	12·90	6·21	3·86	59

Hypothetical Experience of Staff Pension Fund for Widows and Orphans.

TABLE 49—(*continued*).

Value, on the death of a Married man, of continuous Annuities of 1 *to each of the children until they respectively attain the age of* 21, 16, *or* 14.

Age of Married man at Death	4 PER-CENT			3 PER-CENT			Age of Married man at Death
	Orphans' Annuities till age of			Orphans' Annuities till age of			
	21	16	14	21	16	14	
x	$O\alpha(21)$	$O\alpha(16)$	$O\alpha(14)$	$O\alpha(21)$	$O\alpha(16)$	$O\alpha(14)$	x
60	11·01	5·28	3·27	11·50	5·46	3·34	60
61	9·69	4·56	2·76	10·10	4·70	2·82	61
62	8·37	3·84	2·26	8·70	3·95	2·32	62
63	7·05	3·11	1·80	7·30	3·20	1·86	63
64	5·76	2·39	1·37	5·90	2·45	1·42	64
65	4·52	1·71	1·06	4·65	1·75	1·09	65
66	3·78	1·29	·86	3·88	1·32	·88	66
67	3·22	1·03	·70	3·30	1·05	·72	67
68	2·74	·85	·58	2·81	·87	·59	68
69	2·32	·72	·48	2·38	·73	·49	69
70	1·94	·60	·39	1·99	·62	·41	70
71	1·60	·51	·32	1·64	·52	·33	71
72	1·30	·43	·26	1·33	·44	·27	72
73	1·10	·37	·21	1·13	·38	·22	73
74	·94	·31	·17	·96	·32	·17	74
75	·79	·26	·14	·81	·27	·14	75
76	·66	·22	·11	·68	·22	·11	76
77	·56	·18	·09	·57	·18	·09	77
78	·46	·15	·08	·48	·15	·08	78
79	·39	·12	·06	·40	·12	·06	79
80	·32	·10	·05	·33	·11	·05	80
81	·27	·09	·04	·28	·09	·04	81
82	·23	·07	·03	·23	·07	·03	82
83	·19	·06	·02	·20	·06	·02	83
84	·16	·04	·01	·16	·05	·01	84
85	·13	·03	...	·13	·03	...	85
86	·11	·02	...	·11	·02	...	86
87	·09	·01	...	·09	·01	...	87
88	·07	...	...	·07	...	...	88
89	·05	...	...	·05	...	...	89
90	·04	...	...	·04	...	...	90
91	·02	...	...	·02	...	...	91
92	·01	...	...	·01	...	...	92

Hypothetical Experience of Staff Pension Fund for Widows and Orphans.

TABLE 50.

Commutation Columns for finding the Present Value per Member (whether Bachelor, Husband, or Widower) of an Annuity of 1 *to a Widow to commence at the moment of death of a Husband.*

Age	4 PER-CENT		3 PER-CENT		Age
x	${}^{wa}C_x$*	${}^{wa}M_x$†	${}^{wa}C_x$	${}^{wa}M_x$	x
20	·82	4686·13	1·18	8731·16	20
21	2·36	4685·31	3·42	8729·98	21
22	6·04	4682·95	8·82	8726·56	22
23	11·59	4676·91	17·09	8717·74	23
24	17·38	4665·32	25·86	8700·65	24
25	24·69	4647·94	37·07	8674·79	25
26	31·32	4623·25	47·41	8637·72	26
27	37·34	4591·93	57·00	8590·31	27
28	43·37	4554·59	66·77	8533·31	28
29	54·98	4511·22	85·36	8466·54	29
30	63·37	4456·24	99·21	8381·18	30
31	72·51	4392·87	114·49	8281·97	31
32	78·78	4320·36	125·44	8167·48	32
33	83·90	4241·58	134·74	8042·04	33
34	91·13	4157·68	147·60	7907·30	34
35	97·57	4066·55	159·38	7759·70	35
36	100·05	3968·98	164·83	7600·32	36
37	105·67	3868·93	175·59	7435·49	37
38	107·60	3763·26	180·34	7259·90	38
39	107·71	3655·66	182·07	7079·56	39
40	109·32	3547·95	186·38	6897·49	40
41	107·52	3438·63	184·84	6711·11	41
42	110·39	3331·11	191·38	6526·27	42
43	108·84	3220·72	190·27	6334·89	43
44	110·96	3111·88	195·58	6144·62	44
45	112·37	3000·92	199·73	5949·04	45
46	113·03	2888·55	202·53	5749·31	46
47	111·35	2775·52	201·12	5546·78	47
48	112·43	2664·17	204·71	5345·66	48
49	113·36	2551·74	208·07	5140·95	49
50	112·62	2438·38	208·38	4932·88	50
51	111·53	2325·76	207·96	4724·50	51
52	111·62	2214·23	209·74	4516·54	52
53	111·18	2102·61	210·55	4306·80	53
54	111·36	1991·43	212·50	4096·25	54
55	111·18	1880·07	213·80	3883·75	55
56	111·47	1768·89	216·00	3669·95	56
57	110·47	1657·42	215·68	3453·95	57
58	110·49	1546·95	217·38	3238·27	58
59	108·91	1436·46	215·90	3020·89	59

* ${}^{wa}C_x = v^{x+\frac{1}{2}} H d_x (a_y + \frac{1}{2})$.

† ${}^{wa}M_x = \Sigma_x^{\omega\, wa}C_x$.

Hypothetical Experience of Staff Pension Fund for Widows and Orphans.

TABLE 50—(*continued*).

Commutation Columns for finding the Present Value per Member (whether Bachelor, Husband, or Widower) of an Annuity of 1 *to a Widow to commence at the moment of death of a Husband.*

Age	4 PER-CENT		3 PER-CENT		Age
x	${}^{wa}C_x$*	${}^{wa}M_x$†	${}^{wa}C_x$	${}^{wa}M_x$	x
60	108·02	1327·55	215·77	2804·99	60
61	105·99	1219·53	213·31	2589·22	61
62	103·57	1113·54	210·02	2375·91	62
63	99·94	1009·97	204·18	2165·89	63
64	94·16	910·03	193·83	1961·71	64
65	86·91	815·87	180·23	1767·88	65
66	80·30	728·96	167·73	1587·65	66
67	75·82	648·66	159·53	1419·92	67
68	71·03	572·84	150·52	1260·39	68
69	66·07	501·81	141·04	1109·87	69
70	60·87	435·74	130·88	968·83	70
71	55·65	374·87	120·49	837·95	71
72	50·33	319·22	109·75	717·46	72
73	45·03	268·89	98·92	607·71	73
74	39·88	223·86	88·23	508·79	74
75	34·76	183·98	77·45	420·56	75
76	29·82	149·22	66·93	343·11	76
77	25·21	119·40	56·98	276·18	77
78	20·91	94·19	47·61	219·20	78
79	17·16	73·28	39·35	171·59	79
80	13·86	56·12	32·03	132·24	80
81	10·87	42·26	25·31	100·21	81
82	8·57	31·39	20·11	74·90	82
83	6·59	22·82	15·56	54·79	83
84	4·95	16·23	11·78	39·23	84
85	3·64	11·28	8·72	27·45	85
86	2·60	7·64	6·27	18·73	86
87	1·83	5·04	4·46	12·46	87
88	1·23	3·21	3·01	8·00	88
89	·80	1·98	1·98	4·99	89
90	·50	1·18	1·25	3·01	90
91	·31	·68	·78	1·76	91
92	·17	·37	·44	·98	92
93	·10	·20	·26	·54	93
94	·05	·10	·14	·28	94
95	·03	·05	·08	·14	95
96	·01	·02	·04	·06	96
97	·01	·01	·02	·02	97

* ${}^{wa}C_x = v^{x+\frac{1}{2}} \mathrm{H} d_x (a_y + \frac{1}{2})$.

† ${}^{wa}M_x = \Sigma_x^{\omega} {}^{wa}C_x$.

Hypothetical Experience of Staff Pension Fund for Widows and Orphans.

Table 51.

Commutation Columns for finding the present value per Member (whether Bachelor, Husband, or Widower) of an Annuity of 1 *to commence at the death of a Widow and continue until the youngest surviving child reaches the age of* 21, 16, *or* 14.

Age	4 per-cent			3 per-cent			Age
	Annuity till Age of			Annuity till Age of			
	21	16	14	21	16	14	
x	${}^{E(21)}M_x$*	${}^{E(16)}M_x$	${}^{E(14)}M_x$	${}^{E(21)}M_x$	${}^{E(16)}M_x$	${}^{E(14)}M_x$	x
20	229·04	118·96	84·32	398·95	197·91	137·50	20
21	228·99	118·93	84·30	398·88	197·87	137·47	21
22	228·85	118·84	84·23	398·68	197·75	137·38	22
23	228·49	118·61	84·05	398·17	197·44	137·13	23
24	227·79	118·17	83·70	397·18	196·83	136·65	24
25	226·74	117·50	83·17	395·68	195·90	135·93	25
26	225·25	116·55	82·42	393·53	194·57	134·89	26
27	223·35	115·34	81·47	390·77	192·86	133·56	27
28	221·08	113·90	80·34	387·44	190·81	131·97	28
29	218·43	112·23	79·03	383·52	188·41	130·11	29
30	215·07	110·11	77·38	378·49	185·34	127·73	30
31	211·18	107·68	75·48	372·62	181·77	124·98	31
32	206·72	104·90	73·32	365·83	177·66	121·82	32
33	201·87	101·89	70·99	358·37	173·16	118·37	33
34	196·70	98·70	68·52	350·35	168·34	114·69	34
35	191·09	95·25	65·86	341·56	163·08	110·69	35
36	185·08	91·57	63·04	332·06	157·42	106·40	36
37	178·93	87·82	60·18	322·24	151·61	102·01	37
38	172·44	83·89	57·20	311·78	145·47	97·37	38
39	165·84	79·92	54·20	301·05	139·21	92·66	39
40	159·24	75·99	51·23	290·23	132·95	87·98	40
41	152·55	72·04	48·26	279·17	126·59	83·25	41
42	145·98	68·19	45·37	268·21	120·34	78·62	42
43	139·24	64·27	42·45	256·87	113·92	73·89	43
44	132·61	60·44	39·61	245·62	107·60	69·26	44
45	125·86	56·58	36·75	234·08	101·18	64·57	45
46	119·04	52·71	33·91	222·32	94·70	59·86	46
47	112·19	48·86	31·11	210·40	88·20	55·17	47
48	105·45	45·11	28·40	198·57	81·83	50·61	48
49	98·66	41·39	25·72	186·54	75·44	46·08	49
50	91·83	37·70	23·10	174·32	69·06	41·60	50
51	85·06	34·11	20·58	162·13	62·81	37·25	51
52	78·38	30·65	18·19	150·00	56·73	33·08	52
53	71·73	27·31	15·91	137·84	50·80	29·08	53
54	65·17	24·11	13·77	125·75	45·09	25·29	54
55	58·69	21·06	11·77	113·71	39·58	21·73	55
56	52·34	18·17	9·94	101·83	34·33	18·45	56
57	46·13	15·48	8·30	90·12	29·41	15·48	57
58	40·16	13·05	6·87	78·78	24·95	12·89	58
59	34·40	10·86	5·65	67·79	20·92	10·67	59

* ${}^{E(n)}M_x = \Sigma v^{x+\frac{1}{2}} H d_x . E(n)_x$.

Hypothetical Experience of Staff Pension Fund for Widows and Orphans.

TABLE 51—(*continued*).

Commutation Columns for finding the present value per Member (whether Bachelor, Husband, or Widower) of an Annuity of 1 *to commence at the death of a Widow and continue until the youngest surviving child reaches the age of* 21, 16, *or* 14.

Age	4 PER-CENT			3 PER-CENT			Age
	Annuity till Age of			Annuity till Age of			
	21	16	14	21	16	14	
x	$^{E(21)}M_x$*	$^{E(16)}M_x$	$^{E(14)}M_x$	$^{E(21)}M_x$	$^{E(16)}M_x$	$^{E(14)}M_x$	x
60	28·98	8·95	4·64	57·40	17·39	8·83	60
61	23·93	7·30	3·79	47·66	14·33	7·28	61
62	19·48	5·89	3·09	39·03	11·68	5·98	62
63	15·65	4·70	2·50	31·55	9·43	4·87	63
64	12·46	3·73	2·00	25·31	7·57	3·92	64
65	9·91	2·96	1·59	20·31	6·08	3·14	65
66	7·99	2·36	1·26	16·51	4·90	2·50	66
67	6·51	1·87	·99	13·56	3·92	1·97	67
68	5·32	1·47	·77	11·17	3·11	1·53	68
69	4·36	1·15	·59	9·23	2·45	1·16	69
70	3·57	·89	·44	7·61	1·92	·86	70
71	2·91	·67	·33	6·25	1·47	·63	71
72	2·35	·50	·24	5·09	1·11	·45	72
73	1·87	·36	·17	4·09	·82	·32	73
74	1·47	·26	·12	3·24	·60	·22	74
75	1·14	·18	·08	2·54	·43	·15	75
76	·87	·12	·05	1·96	·30	·10	76
77	·65	·08	·03	1·48	·21	·07	77
78	·48	·05	·02	1·10	·14	·05	78
79	·35	·03	·01	·80	·09	·03	79
80	·25	·02	·00	·57	·06	·02	80
81	·17	·01	...	·39	·04	·01	81
82	·11	·00	...	·26	·02	·00	82
83	·07	...	...	·17	·01	...	83
84	·04	...	...	·10	·00	...	84
85	·02	...	...	·06	...	...	85
86	·01	...	...	·03	..	..	86
87	·00	...	...	·01	. .	...	87

* $^{E(n)}M_x = \Sigma v^{x+\frac{1}{2}} H d_x . E(n)_x .$

Hypothetical Experience of Staff Pension Fund for Widows and Orphans.

TABLE 52.

Commutation Columns for finding the present value per Member (whether Bachelor, Husband, or Widower) of an Annuity of 1 to commence on the death of a Widower and continue until the youngest surviving child reaches the age of 21, 16 *or* 14.

$$^{\text{K.YC}a(n)}\text{M}_x = \Sigma v^{x+\frac{1}{2}} \text{K} d_x . \text{YC}a(n).$$

Age	4 per-cent			3 per-cent			Age
	Annuity till age of			Annuity till age of			
	21	16	14	21	16	14	
x	$^{\text{K.YC}a(21)}\text{M}_x$	$^{\text{K.YC}a(16)}\text{M}_x$	$^{\text{K.YC}a(14)}\text{M}_x$	$^{\text{K.YC}a(21)}\text{M}_x$	$^{\text{K.YC}a(16)}\text{M}_x$	$^{\text{K.YC}a(14)}\text{M}_x$	x
25	259·00	156·10	121·33	476·46	276·03	210·96	25
26	259·00	156·10	121·33	476·46	276·03	210·96	26
27	258·59	155·76	121·03	475·88	275·56	210·55	27
28	258·19	155·44	120·74	475·32	275·11	210·15	28
29	257·81	155·13	120·46	474·77	274·67	209·76	29
30	257·08	154·54	119·93	473·71	273·83	209·01	30
31	256·38	153·97	119·42	472·69	273·02	208·29	31
32	255·37	153·16	118·69	471·21	271·84	207·25	32
33	254·08	152·12	117·76	469·30	270·32	205·91	33
34	252·54	150·88	116·66	466·99	268·49	204·30	34
35	250·77	149·46	115·39	464·31	266·37	202·43	35
36	248·79	147·87	113·98	461·29	263·99	200·33	36
37	246·63	146·14	112·44	457·95	261·37	198·02	37
38	244·30	144·28	110·79	454·33	258·53	195·53	38
39	241·58	142·11	108·87	450·05	255·19	192·60	39
40	238·74	139·85	106·88	445·55	251·68	189·54	40
41	235·58	137·34	104·67	440·49	247·75	186·11	41
42	232·13	134·62	102·28	434·92	243·44	182·36	42
43	228·42	131·71	99·73	428·88	238·79	178·32	43
44	224·69	128·79	97·18	422·75	234·08	174·24	44
45	220·57	125·58	94·39	415·92	228·86	169·73	45
46	216·11	122·12	91·39	408·46	223·19	164·85	46
47	211·35	118·45	88·22	400·42	217·11	159·64	47
48	206·33	114·60	84·90	391·87	210·68	154·16	48
49	200·78	110·37	81·27	382·33	203·55	148·11	49
50	194·77	105·81	77·39	371·91	195·81	141·57	50
51	188·36	100·99	73·30	360·70	187·56	134·62	51
52	181·61	95·96	69·06	348·80	178·88	127·36	52
53	174·47	90·69	64·64	336·09	169·70	119·75	53
54	166·89	85·16	60·04	322·50	160·00	111·77	54
55	158·76	79·37	55·28	307·98	149·76	103·43	55
56	150·15	73·27	50·32	292·30	138·87	94·67	56
57	140·91	66·83	45·19	275·31	127·30	85·52	57
58	131·28	60·33	40·10	257·48	115·42	76·37	58
59	121·24	53·69	35·09	238·76	103·32	67·28	59

Hypothetical Experience of Staff Pension Fund for Widows and Orphans.

TABLE 52—(*continued*).

Commutation Columns for finding the present value per Member (whether Bachelor, Husband, or Widower) of an Annuity of 1 to commence on the death of a Widower and continue until the youngest surviving child reaches the age of 21, 16 *or* 14.

$$^{K.YCa(n)}M_x = \Sigma v^{x+\frac{1}{2}} K d_x . YCa(n).$$

Age	4 per-cent			3 per-cent			Age
	Annuity till age of			Annuity till age of			
	21	16	14	21	16	14	
x	$^{K.YCa(21)}M_x$	$^{K.YCa(16)}M_x$	$^{K.YCa(14)}M_x$	$^{K.YCa(21)}M_x$	$^{K.YCa(16)}M_x$	$^{K.YCa(14)}M_x$	x
60	110·98	47·18	30·31	219·48	91·36	58·57	60
61	100·35	40·78	25·77	199·44	79·52	50·24	61
62	89·92	34·67	21·64	179·54	68·13	42·59	62
63	79·68	29·03	17·98	159·86	57·52	35·71	63
64	69·80	24·04	14·85	140·77	48·04	29·73	64
65	60·64	19·93	12·36	123·05	40·18	24·92	65
66	52·36	16·80	10·37	107·04	34·17	21·07	66
67	44·98	14·28	8·69	92·64	29·27	17·80	67
68	38·29	12·14	7·24	79·49	25·09	14·94	68
69	32·28	10·27	5·98	67·54	21·38	12·43	69
70	26·95	8·62	4·88	56·83	18·08	10·22	70
71	22·31	7·18	3·94	47·42	15·17	8·30	71
72	18·35	5·92	3·14	39·32	12·59	6·66	72
73	15·05	4·82	2·47	32·52	10·32	5·28	73
74	12·21	3·87	1·92	26·59	8·34	4·14	74
75	9·77	3·05	1·47	21·45	6·63	3·21	75
76	7·72	2·37	1·12	17·07	5·20	2·47	76
77	6·03	1·81	·84	13·44	4·01	1·88	77
78	4·64	1·37	·62	10·43	3·05	1·40	78
79	3·53	1·02	·44	7·98	2·29	1·00	79
80	2·65	·74	·30	6·04	1·68	·68	80
81	1·96	·52	·19	4·51	1·19	·44	81
82	1·43	·35	·11	3·32	·81	·26	82
83	1·02	·22	·06	2·39	·52	·13	83
84	·71	·13	·03	1·67	·31	·05	84
85	·48	·07	·01	1·13	·16	·01	85
86	·31	·03	...	·74	·07	...	86
87	·19	·01	.	·46	·02	...	87
88	·11	...	...	·27	...	...	88
89	·06	..	. .	·15	...	...	89
90	·03	...	...	·07	...	...	90
91	·01	.	...	·03	.	...	91
				·01	. .	...	92

Hypothetical Experience of Staff Pension Fund for Widows and Orphans.

TABLE 53.

Commutation Columns for finding the present value per Member (whether Bachelor, Husband, or Widower) of an Annuity of 1 *to each of the children until age* 16 *on the death of a married man (whether Husband or Widower).*

$$^{Oa(16)}M_x = \Sigma . v^{x+\frac{1}{2}} M d_x . Oa(16)_x .$$

4 PER-CENT				3 PER-CENT			
Age x	$^{Oa(16)}M_x$	Age x	$^{Oa(16)}M_x$	Age x	$^{Oa(16)}M_x$	Age x	$^{Oa(16)}M_x$
20	3411·5	60	246·42	20	5506·0	60	470·89
21	3411·5	61	198·09	21	5506·0	61	381·21
22	3410·9	62	155·94	22	5505·3	62	302·52
23	3408·7	63	120·17	23	5502·4	63	235·21
24	3403·2	64	91·25	24	5495·0	64	180·24
25	3393·6	65	69·58	25	5482·0	65	138·81
26	3378·5	66	54·71	26	5461·4	66	110·15
27	3357·0	67	43·89	27	5431·9	67	89·09
28	3329·1	68	35·35	28	5393·4	68	72·37
29	3294·1	69	28·38	29	5344·6	69	58·58
30	3245·9	70	22·63	30	5276·8	70	47·12
31	3186·5	71	17·94	31	5192·5	71	37·65
32	3113·5	72	14·09	32	5088·0	72	29·80
33	3029·1	73	10·94	33	4966·2	73	23·31
34	2934·1	74	8·37	34	4827·7	74	17·98
35	2825·9	75	6·31	35	4668·9	75	13·66
36	2705·2	76	4·69	36	4490·4	76	10·24
37	2578·5	77	3·44	37	4301·4	77	7·57
38	2444·2	78	2·49	38	4099·1	78	5·52
39	2308·3	79	1·78	39	3893·0	79	3·98
40	2175·0	80	1·25	40	3688·7	80	2·81
41	2042·9	81	·85	41	3484·6	81	1·92
42	1916·3	82	·56	42	3287·1	82	1·27
43	1790·1	83	·35	43	3088·6	83	·80
44	1669·8	84	·20	44	2897·5	84	·46
45	1550·9	85	·10	45	2706·8	85	·24
46	1434·5	86	·04	46	2518·3	86	·10
47	1321·3	87	·01	47	2333·4	87	·03
48	1213·7			48	2156·0		
49	1108·9			49	1981·5		
50	1007·3			50	1810·7		
51	910·40			51	1646·4		
52	818·57			52	1489·2		
53	730·95			53	1337·6		
54	647·91			54	1192·7		
55	569·10			55	1053·8		
56	494·76			56	921·7		
57	424·73			57	796·1		
58	360·05			58	679·1		
59	300·28			59	569·95		

Hypothetical Experience of Staff Pension Fund for Widows and Orphans.

Table 54.

Commutation Columns for finding the present value per Member (whether Bachelor, Husband, or Widower) of 1 *payable immediately on the death of a Bachelor.*

$$^{Bd}M_x = \Sigma . v^{x+\frac{1}{2}} . B d_x .$$

4 per-cent				3 per-cent			
Age x	$^{Bd}M_x$	Age x	$^{Bd}M_x$	Age x	$^{Bd}M_x$	Age x	$^{Bd}M_x$
15	467·00	60	40·04	15	632·62	60	79·51
16	427·80	61	37·54	16	587·09	61	75·03
17	392·20	62	35·13	17	545·34	62	70·66
18	359·98	63	32·81	18	507·18	63	66·42
19	331·42	64	30·59	19	473·03	64	62·32
20	305·82	65	28·50	20	442·13	65	58·43
21	282·59	66	26·55	21	413·81	66	54·75
22	261·63	67	24·70	22	388·02	67	51·23
23	242·52	68	22·90	23	364·26	68	47·78
24	225·25	69	21·14	24	342·59	69	44·36
25	209·75	70	19·41	25	322·96	70	40·98
26	195·66	71	17·72	26	304·93	71	37·64
27	182·57	72	16·08	27	288·03	72	34·37
28	170·73	73	14·50	28	272·59	73	31·18
29	160·10	74	12·98	29	258·59	74	28·09
30	150·67	75	11·54	30	246·05	75	25·13
31	142·20	76	10·17	31	234·68	76	22·30
32	134·47	77	8·88	32	224·20	77	19·60
33	127·59	78	7·66	33	214·79	78	17·03
34	121·49	79	6·52	34	206·36	79	14·59
35	116·01	80	5·47	35	198·71	80	12·33
36	111·11	81	4·53	36	191·81	81	10·27
37	106·81	82	3·65	37	185·69	82	8·35
38	102·81	83	2·91	38	179·95	83	6·71
39	99·16	84	2·29	39	174·66	84	5·32
40	95·74	85	1·77	40	169·65	85	4·15
41	92·45	86	1·35	41	164·79	86	3·19
42	89·31	87	1·01	42	160·10	87	2·41
43	86·21	88	·76	43	155·43	88	1·82
44	83·27	89	·56	44	150·95	89	1·35
45	80·39	90	·40	45	146·52	90	·98
46	77·55	91	·28	46	142·12	91	·70
47	74·74	92	·19	47	137·72	92	·49
48	71·99	93	·13	48	133·37	93	·33
49	69·21	94	·08	49	128·93	94	·22
50	66·43	95	·05	50	124·44	95	·14
51	63·70	96	·03	51	119·99	96	·09
52	61·02	97	·02	52	115·58	97	·06
53	58·37	98	·01	53	111·17	98	·04
54	55·75			54	106·77	99	·02
55	53·11			55	102·30	100	·01
56	50·48			56	97·80		
57	47·83			57	93·23		
58	45·20			58	88·64		
59	42·60			59	84·06		

Hypothetical Experience of Staff Pension Fund for Widows and Orphans.

TABLE 55.

Multipliers for use in a Valuation.

INTEREST 4 PER-CENT.

x	$\frac{{}^{wa}M_x}{D_x^{(3)}}$	$\frac{{}^{E(21)}M_x}{D_x^{(3)}}$	$\frac{{}^{E(16)}M_x}{D_x^{(3)}}$	$\frac{{}^{E(14)}M_x}{D_x^{(3)}}$	$\frac{{}^{K.YCa(21)}M_x}{D_x^{(3)}}$	$\frac{{}^{K.YCa(16)}M_x}{D_x^{(3)}}$	$\frac{{}^{K.YCa(14)}M_x}{D_x^{(3)}}$	$\frac{{}^{Oa(16)}M_x}{D_x^{(3)}}$	$\frac{{}^{Bd}M_x}{D_x^{(3)}}$	x
20	·809	·040	·021	·015	·045	·027	·021	·589	·053	20
21	·908	·044	·023	·016	·050	·030	·024	·661	·055	21
22	1·012	·049	·026	·018	·056	·034	·026	·737	·057	22
23	1·120	·055	·028	·020	·062	·037	·029	·816	·058	23
24	1·231	·060	·031	·022	·068	·041	·032	·898	·059	24
25	1·345	·066	·034	·024	·075	·045	·035	·982	·061	25
26	1·461	·071	·037	·026	·082	·049	·038	1·068	·062	26
27	1·579	·077	·040	·028	·089	·054	·042	1·154	·063	27
28	1·699	·082	·042	·030	·096	·058	·045	1·242	·064	28
29	1·820	·088	·045	·032	·104	·063	·049	1·329	·065	29
30	1·940	·094	·048	·034	·112	·067	·052	1·413	·066	30
31	2·058	·099	·050	·035	·120	·072	·056	1·493	·067	31
32	2·175	·104	·053	·037	·129	·077	·060	1·568	·068	32
33	2·290	·109	·055	·038	·137	·082	·064	1·635	·069	33
34	2·403	·114	·057	·040	·146	·087	·067	1·696	·070	34
35	2·513	·118	·059	·041	·155	·092	·071	1·747	·072	35
36	2·620	·122	·060	·042	·164	·098	·075	1·785	·073	36
37	2·724	·126	·062	·042	·174	·103	·079	1·815	·075	37
38	2·823	·129	·063	·043	·183	·108	·083	1·833	·077	38
39	2·919	·132	·064	·043	·193	·113	·087	1·843	·079	39
40	3·013	·135	·065	·044	·203	·119	·091	1·847	·081	40
41	3·105	·138	·065	·044	·213	·124	·094	1·845	·083	41
42	3·195	·140	·065	·044	·223	·129	·098	1·838	·086	42
43	3·281	·142	·065	·043	·233	·134	·102	1·824	·088	43
44	3·365	·143	·065	·043	·243	·139	·105	1·806	·090	44
45	3·444	·144	·065	·042	·253	·144	·108	1·780	·092	45
46	3·517	·145	·064	·041	·263	·149	·111	1·747	·094	46
47	3·585	·145	·063	·040	·273	·153	·114	1·707	·097	47
48	3·650	·144	·062	·039	·283	·157	·116	1·663	·099	48
49	3·708	·143	·060	·037	·292	·160	·118	1·612	·101	49
50	3·760	·142	·058	·036	·300	·163	·119	1·553	·102	50
51	3·805	·139	·056	·034	·308	·165	·120	1·489	·104	51
52	3·843	·136	·053	·032	·315	·167	·120	1·421	·106	52
53	3·873	·132	·050	·030	·321	·167	·119	1·346	·108	53
54	3·895	·127	·047	·027	·326	·167	·117	1·267	·109	54
55	3·909	·122	·044	·024	·330	·165	·115	1·183	·110	55
56	3·916	·116	·040	·022	·332	·162	·111	1·095	·112	56
57	3·918	·109	·037	·020	·333	·158	·107	1·004	·113	57
58	3·907	·101	·033	·017	·332	·152	·101	·909	·114	58
59	3·888	·093	·029	·015	·328	·145	·095	·813	·115	59

Hypothetical Experience of Staff Pension Fund for Widows and Orphans.

TABLE 55—(*continued*).

Multipliers for use in a Valuation.

INTEREST 4 PER-CENT.

x	$\frac{^{wa}M_x}{D_x^{(3)}}$	$\frac{^{E(21)}M_x}{D_x^{(3)}}$	$\frac{^{E(16)}M_x}{D_x^{(3)}}$	$\frac{^{E(14)}M_x}{D_x^{(3)}}$	$\frac{^{K.YCa(21)}M_x}{D_x^{(3)}}$	$\frac{^{K.YCa(16)}M_x}{D_x^{(3)}}$	$\frac{^{K.YCa(14)}M_x}{D_x^{(3)}}$	$\frac{^{Oa(16)}M_x}{D_x^{(3)}}$	$\frac{^{Ed}M_x}{D_x^{(3)}}$	x
60	3·860	·084	·026	·013	·323	·137	·088	·716	·116	60
61	3·820	·075	·023	·012	·314	·128	·081	·620	·118	61
62	3·767	·066	·020	·010	·304	·117	·073	·528	·119	62
63	3·703	·057	·017	·009	·292	·106	·066	·441	·120	63
64	3·626	·050	·015	·008	·278	·096	·059	·364	·122	64
65	3·541	·043	·013	·007	·263	·086	·054	·302	·124	65
66	3·453	·038	·011	·006	·248	·080	·049	·259	·126	66
67	3·362	·034	·010	·005	·233	·074	·045	·228	·128	67
68	3·262	·030	·008	·004	·218	·069	·041	·201	·130	68
69	3·153	·027	·007	·004	·203	·065	·038	·178	·133	69
70	3·037	·025	·006	·003	·188	·060	·034	·158	·135	70
71	2·913	·023	·005	·003	·173	·056	·031	·139	·138	71
72	2·782	·020	·004	·002	·160	·052	·027	·123	·140	72
73	2·645	·018	·004	·002	·148	·047	·024	·108	·143	73
74	2·502	·016	·003	·001	·136	·043	·021	·094	·145	74
75	2·355	·015	·002	·001	·125	·039	·019	·081	·148	75
76	2·204	·013	·002	·001	·114	·035	·017	·069	·150	76
77	2·052	·011	·001	·001	·104	·031	·014	·059	·153	77
78	1·901	·010	·001	...	·094	·028	·013	·050	·155	78
79	1·754	·008	·001	...	·085	·024	·011	·043	·156	79
80	1·610	·007	·001	...	·076	·021	·009	·036	·157	80
81	1·470	·006	...	.	·068	·018	·007	·030	·158	81
82	1·339	·005	...		·061	·015	·005	·024	·156	82
83	1·209	·004	...		·054	·012	·003	·019	·154	83
84	1·082	·003	. .		·047	·009	·002	·013	·153	84
85	·957	·002	. .	..	·041	·006	·001	·009	·151	85
86	·842	·001	..	. .	·034	·003	...	·004	·149	86
87	·731	...	. .	. .	·028	·001	..	·001	·146	87
88	·623	...	...	...	·021	...	...	...	·147	88
89	·523	...	...	...	·016	...	...	...	·148	89
90	·432	...	...	...	·011	. .	...	...	·146	90
91	·352	...	.	...	·005	..	. .	...	·145	91
92	·281	...	...	...	...	...	...	..	·142	92

Hypothetical Experience of Staff Pension Fund for Widows and Orphans.

TABLE 56.

Multipliers for use in a Valuation.

INTEREST 3 PER-CENT.

x	$\frac{^{wa}M_x}{D_x^{(3)}}$	$\frac{^{E(21)}M_x}{D_x^{(3)}}$	$\frac{^{E(16)}M_x}{D_x^{(3)}}$	$\frac{^{E(14)}M_x}{D_x^{(3)}}$	$\frac{^{K.YCa(21)}M_x}{D_x^{(3)}}$	$\frac{^{K.YCa(16)}M_x}{D_x^{(3)}}$	$\frac{^{K.YCa(14)}M_x}{D_x^{(3)}}$	$\frac{^{Oa(16)}M_x}{D_x^{(3)}}$	$\frac{^{Bd}M_x}{D_x^{(3)}}$	x
20	1·242	·057	·028	·020	·068	·039	·030	·783	·063	20
21	1·381	·063	·031	·022	·075	·044	·033	·871	·065	21
22	1·525	·070	·035	·024	·083	·048	·037	·962	·068	22
23	1·672	·076	·038	·026	·091	·053	·040	1·055	·070	23
24	1·821	·083	·041	·029	·100	·058	·044	1·150	·072	24
25	1·972	·090	·045	·031	·108	·063	·048	1·246	·073	25
26	2·123	·097	·048	·033	·117	·068	·052	1·342	·075	26
27	2·276	·104	·051	·035	·126	·073	·056	1·439	·076	27
28	2·429	·110	·054	·038	·135	·078	·060	1·535	·078	28
29	2·581	·117	·057	·040	·145	·084	·064	1·629	·079	29
30	2·730	·123	·060	·042	·154	·089	·068	1·719	·080	30
31	2·876	·129	·063	·043	·164	·095	·072	1·803	·082	31
32	3·018	·135	·066	·045	·174	·100	·077	1·880	·083	32
33	3·156	·141	·068	·046	·184	·106	·081	1·949	·084	33
34	3·291	·146	·070	·048	·194	·112	·085	2·009	·086	34
35	3·420	·151	·072	·049	·205	·117	·089	2·058	·088	35
36	3·543	·155	·073	·050	·215	·123	·093	2·093	·089	36
37	3·661	·159	·075	·050	·226	·129	·097	2·118	·091	37
38	3·772	·162	·076	·051	·236	·134	·102	2·130	·093	38
39	3·878	·165	·076	·051	·247	·140	·106	2·132	·096	39
40	3·980	·168	·077	·051	·257	·145	·109	2·129	·098	40
41	4·078	·170	·077	·051	·268	·151	·113	2·117	·100	41
42	4·172	·171	·077	·050	·278	·156	·117	2·101	·102	42
43	4·260	·173	·077	·050	·288	·161	·120	2·077	·105	43
44	4·343	·174	·076	·049	·299	·165	·123	2·048	·107	44
45	4·420	·174	·075	·048	·309	·170	·126	2·011	·109	45
46	4·489	·174	·074	·047	·319	·174	·129	1·967	·111	46
47	4·550	·173	·072	·045	·328	·178	·131	1·914	·113	47
48	4·606	·171	·071	·044	·338	·182	·133	1·858	·115	48
49	4·654	·169	·068	·042	·346	·184	·134	1·794	·117	49
50	4·692	·166	·066	·040	·354	·186	·135	1·722	·118	50
51	4·722	·162	·063	·037	·361	·187	·135	1·646	·120	51
52	4·743	·158	·060	·035	·366	·188	·134	1·564	·121	52
53	4·754	·152	·056	·032	·371	·187	·132	1·476	·123	53
54	4·755	·146	·052	·029	·374	·186	·130	1·384	·124	54
55	4·746	·139	·048	·027	·376	·183	·126	1·288	·125	55
56	4·729	·131	·044	·024	·377	·179	·122	1·188	·126	56
57	4·707	·123	·040	·021	·375	·174	·117	1·085	·127	57
58	4·670	·114	·036	·019	·371	·166	·110	·979	·128	58
59	4·624	·104	·032	·016	·366	·158	·103	·872	·129	59

Hypothetical Experience of Staff Pension Fund for Widows and Orphans.

TABLE 56—(*continued*).

Multipliers for use in a Valuation.

INTEREST 3 PER-CENT.

x	$\frac{{}^{wa}M_x}{D_x^{(3)}}$	$\frac{{}^{E(21)}M_x}{D_x^{(3)}}$	$\frac{{}^{E(16)}M_x}{D_x^{(3)}}$	$\frac{{}^{E(14)}M_x}{D_x^{(3)}}$	$\frac{{}^{K.YCa(21)}M_x}{D_x^{(3)}}$	$\frac{{}^{K.YCa(16)}M_x}{D_x^{(3)}}$	$\frac{{}^{K.YCa(14)}M_x}{D_x^{(3)}}$	$\frac{{}^{Oa(16)}M_x}{D_x^{(3)}}$	$\frac{{}^{Bd}M_x}{D_x^{(3)}}$	x
60	4·568	·094	·028	·014	·357	·149	·095	·767	·129	60
61	4·498	·083	·025	·013	·346	·138	·087	·662	·130	61
62	4·416	·073	·022	·011	·334	·127	·079	·562	·131	62
63	4·320	·063	·019	·010	·319	·115	·071	·469	·132	63
64	4·211	·054	·016	·008	·302	·103	·064	·387	·134	64
65	4·095	·047	·014	·007	·285	·093	·058	·321	·135	65
66	3·975	·041	·012	·006	·268	·086	·053	·276	·137	66
67	3·852	·037	·011	·005	·251	·079	·048	·242	·139	67
68	3·720	·033	·010	·005	·235	·074	·044	·214	·141	68
69	3·581	·030	·009	·004	·218	·069	·040	·189	·143	69
70	3·433	·027	·007	·003	·201	·064	·036	·167	·145	70
71	3·279	·024	·006	·002	·186	·059	·032	·147	·147	71
72	3·119	·022	·005	·002	·171	·055	·029	·130	·149	72
73	2·953	·020	·004	·002	·158	·050	·026	·113	·152	73
74	2·782	·018	·003	·001	·145	·046	·023	·098	·154	74
75	2·608	·016	·003	·001	·133	·041	·020	·085	·156	75
76	2·431	·014	·002	·001	·121	·037	·018	·073	·158	76
77	2·256	·012	·002	·001	·110	·033	·015	·062	·160	77
78	2·083	·010	·001	...	·099	·029	·013	·052	·162	78
79	1·915	·009	·001	...	·089	·026	·011	·044	·163	79
80	1·752	·008	·001	...	·080	·022	·009	·037	·163	80
81	1·593	·006	·001	...	·072	·019	·007	·031	·163	81
82	1·446	·005	...	..	·064	·016	·005	·025	·161	82
83	1·301	·004	...	..	·057	·012	·003	·019	·159	83
84	1·161	·003	...	...	·049	·009	·001	·014	·157	84
85	1·027	·002	...	...	·042	·006	...	·009	·155	85
86	·899	·001	...	...	·035	·003	...	·005	·153	86
87	·779	·001	...	...	·029	·001	.	·002	·151	87
88	·663	...	...	...	·022	...	...	...	·151	88
89	·557	...	...	...	·017	...	. .	...	·151	89
90	·462	...	...	...	·011	. .	...	...	·150	90
91	·378	...	...	...	·006	...	...	...	·150	91
92	·300	...	. .	...	·003	...	...	...	·150	92

Discussion.

Mr. Manly said that the Paper was not yet finished, and he hoped to be able to show how to ascertain the different benefits, not for individuals—bachelors, husbands, and widowers all put together—but for bachelors, husbands, and widowers separately; and to include the risks of second marriages. Such figures had not been produced before; and he never saw a way of arriving at them himself until he had made the very interesting table, which was marked No. 57*, tracing the husbands to their death or widowerhood. Table 58*, which had cost an enormous amount of labour, would also, he hoped, be considered an extremely interesting one, showing out of a given number of bachelors who started at the age of 15, namely, 200,000, how many died as bachelors, how many as husbands after the first marriage, and how many as widowers of the first marriage, husbands of the second marriage, widowers of the second marriage, and husbands of the third marriage, and widowers of the fourth marriage, and so on until, as would be noticed, they had really disappeared after the fourth marriage, for very few of them entered into that state. For this table he was indebted to Mr. Workman, who had shown the most exemplary patience in working the tables out for him. He thought that some very excellent results would be produced afterwards.

Mr. E. C. Thomas said that Mr. Manly had set himself many distinct problems, the number of which seemed to be only limited by the possible combinations of widows, children, and other relations on the one hand, with husbands, widowers, and bachelors on the other. Some of these problems had been already investigated by previous writers, but Mr. Manly had introduced two fresh complications, namely, the elements of withdrawal and average salary. One of the greatest difficulties with which Mr. Manly, in common with every other investigator, had had to contend, was to find suitable material on which to base his necessary tables, and one was at once struck with the great variety of the sources to which he had been obliged to resort for this purpose. Those sources comprised his own tables, based on the experience of certain railway pension Funds, Mr. Hewat's tables based on the experience of Scottish banks, the census returns of New Zealand, and the H^F Mortality Table. There might, therefore, be a possible objection raised to Mr. Manly's tables on account of the composite nature of their bases. Possibly a consideration of the various items involved might help to remove that objection. He thought that although the present paper was a continuation of Mr. Manly's previous paper, the kind of experience now in Mr. Manly's mind was more that of a bank than of a railway. He imagined that there would be no objection raised against the use of Mr. Hewat's tables drawn from the experience of banks. With regard to Mr. Manly's own tables, previously published, they had been criticized as comprising a very low mortality rate and a very high lapse rate. They might naturally expect that a low mortality rate would usually be experienced in these Funds, for not only were members generally selected at the outset, but selection operated at

* Tables 57 and 58 will be published later.

the other end by the retirement, compulsory or otherwise, of the weakly members. With regard to the withdrawal rate, experience had accumulated to show that for a hypothetical table, unaffected by any exceptional circumstances, Mr. Manly's table would probably be as suitable a basis as any other that could be devised. In one case—that of a bank's Fund—it had been possible to use Mr. Manly's withdrawal rate in its entirety. With regard to the figures based on the New Zealand census returns, it might be objected that New Zealand was a very much younger country than England, and subject to very different economic conditions; but this could be met by the unanswerable argument that the figures were apparently the only ones obtainable. One would naturally expect the number of children per marriage to be greater in New Zealand than in England, but in the discussion on Mr. King's paper on "Family Annuities", Mr. Morris Fox had stated that he considered that both the marriage rate and the birth rate were lower in New Zealand than in England. He took this to mean the birth rate per thousand of the population living, and this might follow upon a lower marriage rate, though not necessarily so. But what was required in the present instance was the number of children left per married man dying, and the New Zealand figures would probably err in this respect, if at all, on the side of safety. It was in accordance with reasonable expectation that the later figures obtainable by Mr. Manly showed a falling-off in this respect compared with the earlier statistics used by Mr. King. On the whole, he thought that they might conclude that Mr. Manly had been able to keep on the safe side in these figures as in the others. For lack of reliable information Mr. Manly had been forced in many cases to make assumptions. Mr. Manly seemed to have a remarkable gift for placing his hand upon the key of the situation, and this was exemplified in the short table on page 111. By calculating the extra liability involved in the continuance of the annuity to the youngest child after the death of the widow, the succeeding calculations for this benefit were greatly simplified. There were two assumptions made by Mr. Manly, both on the safe side, which would probably command general approval. The first was that the n youngest children tabulated for each age of the father at death, would represent the youngest children of the n fathers who died. This would certainly over-estimate the liability, particularly in respect of the fathers dying in middle life. The second assumption was that the annuity to the youngest child would be equivalent to the value of the annuity-certain for the same term, and would commend itself as an easy, effective, and sound solution of the difficulty. It was far less stringent than the assumption made by Mr. Huie that each husband dying would leave a child who would certainly live twenty-one years. It might be considered severe in the case of very young children, and it might be possible to obtain a little relief from the following consideration. The children that would take the place in the benefit of those who died, must necessarily be older than the originals, and the term would consequently be shortened. They might take it that on the average the difference would be about two years. They might, therefore, use the annuity-certain for two years less than the full

term, and add the value of the last two payments discounted with mortality as well as interest. Where the rate of child mortality was as low as in the present case, the difference would not be very large; but as they had so many assumptions on the safe side, he thought it would be desirable that they should obtain what relief was available. One of the greatest services which Mr. Manly had done was to provide them with a comprehensive system of notation, and it was a remarkable instance of his skill that he had been able to keep so successfully each distinct element apart, and provide them with a well-defined symbol for each, at the same time maintaining consistency with the scheme already propounded in his earlier paper. Although it might look complicated, it would be found to be in reality simple and straightforward. The paper was a further illustration, if such were needed, of the immense power of the commutation column. Mr. Manly extended into a much wider sphere than before the principle he developed so ably in his previous paper of providing a special commutation column for each kind of benefit, from which the value could be at once obtained. With reference to the benefit investigated in Problem VII c, a payment on the death of a widower without children under a certain age, it was a little difficult to follow Mr. Manly's assumptions as to the number of widowers which would come on to the Fund for that particular benefit. From ages 25 to 40 it appeared to be merely a question of the number dying childless; and the proportion of these among married men was stated as 33 per-cent. at 25, decreasing to 15 per-cent. after age 30. Unless, therefore, there were fewer widowers than husbands dying childless, they might expect the series to begin at 33 per-cent. at 25, and run down to 15 per-cent. at, say, age 40, and thence steadily rise to 100 per-cent. The series actually adopted by Mr. Manly started with 20 per-cent. and increased steadily up to 100 per-cent. at age 70. Perhaps the most interesting part of the paper was the short valuation schedule given on pages 128–130. It would be seen that all unnecessary labour had been avoided, but the essential features were set out with such clearness that they could be readily grasped and understood.

Mr. G. J. Lidstone said that after a diligent study of Mr. Manly's formulas, he could not help thinking that if Mr. Manly had worked throughout on the numbers existing in the middle of the year of age, his formulas would have been considerably simplified, and he would have got rid of the complicated adjustments referred to, more especially in the earlier paper. Then Mr. Manly's notation was distinctly awe-inspiring, and one was tempted to ask whether a pictorial notation could not be purchased at almost too great a price—one, in fact, almost sighed for a more arbitrary notation, which would be a little more easily written. Mr. Manly had given, with extraordinary completeness, theoretically exact solutions of a number of problems which arose in connection with these Funds. It was probable that in practice they would, if they could, use slightly rougher methods with a view of minimizing labour; but it was quite evident that a full study of these theoretically exact solutions must be the necessary preliminary to the consideration of what those rougher methods might be. The kernel of the paper was Mr. Manly's

adoption of what had been called the "Collective" method, suggested first, he believed, by Mr. Ralph P. Hardy to Mr. Meikle, and used by the latter with great success in 1875 in connection with the Ministers' Widows' Fund. The same method and principle was also used very successfully by Mr. King in his classic paper on "Family Annuities." According to that method, instead of investigating and seeking to combine various component forces, the statistics of which were quite insufficient for proper treatment, they dealt with the resultant effect of those causes as focussed and exhibited in those cases, whether of members living or of members dying, which they were able to bring under observation. As Mr. G. F. Hardy had said (*J.I.A.* xxxi., 469): "Given certain rates of mortality, marriage, &c., operating fairly constantly, it should follow that among the numbers surviving to successive ages the proportion of married, the average age of their wives"—one might add, also, the number and average age of their children—"and the average liability that would be created by the death of each member would be practically a function of the age." Having brought the calculations into that form, the benefits were treated as varying assurances payable at the death of the members, the amount of the assurance being the average value of the benefits at the moment of death of the member. It was important to observe that that average value could be determined equally well by examining the cases of members living, as it could by examining the cases of members who had actually died, for there was no reason to think that the members dying were anything else than an average sample of those existing. For example, Mr. Manly had applied the proportions of husbands, widowers, and married men existing to obtain the relative numbers of each dying in the year. When he came to deal with the values of orphans' annuities at the date of the death of the parents, he found it more convenient to derive his information from statistics of persons dying, because those statistics happened to be ready at his hand. But it was obvious that equally good results could have been obtained by examining the same number of persons living and taken at random from the population. This was a very important point when they had to consider how far the statistics could be derived from the records of the Funds with which they were dealing. The numbers dying in these Funds during a reasonable period of observation would doubtless generally be too small for statistical treatment. But if the figures were taken from the numbers living over a period of years, they might frequently get quite sufficient material for their purpose. For example, Mr. King based his valuation of orphan annuities on about 1,500 observations; the same number, he thought, might be easily obtained by an examination of the numbers living in an average widows' Fund. This point was of great importance, because it was obviously desirable to get as much information as they could from the Fund itself, if only for purposes of test. However much they might admire the way in which Mr. Manly had pieced together statistics drawn from various sources, they could not but have a slight feeling of anxiety as to how far such a delicate mosaic formed a solid and substantial basis for the erection of tables of valuation. It was desirable to get back to bed-rock whenever they could, and to examine

the statistics of the Funds with which they were dealing. There were one or two practical problems with which he had met, and to which it might be worth while to refer. As Mr. Thomas had pointed out, Mr. Manly had assumed that the widows' Fund was to be worked in conjunction with a pension Fund, and therefore he naturally treated the contribution as being a percentage of the salary. Where, however, the widows' Fund existed independently, they sometimes found that instead of the contribution being a percentage of the salary, the contribution varied, not continuously, but by steps according as the salary passed a certain limit. For instance: In a Fund with which he (Mr. Lidstone) had had to deal, the annual contribution was £1 for salaries under £100, £2 for salaries between £100 and £200, and £3 where the salary was £200 or upwards. There were two ways in which they could deal with a case of that kind. The first, and perhaps the best way, was to form a table of the average contributions payable at each age in the same way as Mr. Manly had formed a table of average salary. Another way was to determine the average age at which the increase of salary, and the consequent increase of contribution, took place, and to value the contribution as being unchanged between those points, treating the increment of contribution as taking place at those average ages. Another point which had to be borne in mind when a widows' Fund existed independently of a pension Fund, was that there was usually a certain option exercisable against the Fund at the pension age. He believed that it was usual to give pensioners the option of continuing in the Fund (subject to their paying the maximum contribution) or of retiring—indeed, it was difficult to prevent their having the option of retiring if there were no pension from which the contributions could be deducted. The probability was that those who were bachelors or widowers with no likelihood of marrying would drop their contribution to the Fund, because the benefit was worth practically nothing, whereas married men and men with families would probably maintain their membership in force. This, of course, had to be allowed for, as the benefits which were dropped at the same time as the contributions were practically worth very little, if anything, at the date of the pension age, so that there was practically a loss, in such cases, of the value of the future contributions. Of course, the easy way to get over this was to assume that a certain proportion of members would cease membership at the pension age, and to reduce the value of the contribution after that age by the same proportion.

Mr. George King said that the paper was an exceedingly heavy one, and he felt that he must repeat Mr. Manly's expression, and say that he rose to speak with an unprejudiced mind. The first thing that struck him was the immense magnitude and completeness of Mr. Manly's labours. When he (Mr. King) wrote the paper on "Family Annuities" he was quite unaware that anyone had done anything of the kind before, although he afterwards heard of what had been done by Mr. R. P. Hardy and Mr. Meikle, which had been mentioned by Mr. Lidstone. However, the work which he did on the subject had, he thought, put things in an entirely different light, and had shown how the benefits could best be valued.

When he (Mr. King) wrote the paper, he had specially in his mind what he called Family Annuities, and what Mr. Manly had now called Orphans' Annuities; but at the close of the paper he gave a formula for similar benefits for widows put in the same form, and just as he had transformed the integral for orphans' annuities into commutation columns, so he had had in his mind to transform the integral for widows' annuities also into commutation columns. But this Mr. Manly had now done. Ever since he (Mr. King) wrote the paper, he had been thinking of continuing it whenever opportunity offered, and whenever the right statistics came along, but so far his intention had not been fulfilled. However, only about six months ago he was collaborating in a Fund with Mr. H. J. Nahmer, a Fellow of the Faculty of Actuaries who resided in Dublin, and Mr. Nahmer and he made up their minds, from the statistics which they then had, to continue the paper. Mr. Manly had now relieved them of that labour, and he was sure that they were very much obliged to him. He thought, however, that some of the figures that were derived from their work might be useful, and he should refer to that subject later on. The difficulty which Mr. Manly had had, and the difficulty which stopped him (Mr. King) from continuing his paper, was the absence of suitable statistics. Mr. Manly had to a large extent overcome that difficulty by a most skilful use of the statistics of the Scottish banks, obtaining, however, facts from elsewhere to supplement them. He was not sure, however, that in preparing the tables the ideas had been fully carried out, although in using the tables they had been carried out. Those tables enabled them theoretically, as Mr. Lidstone had said, to solve exactly the most intricate problems which came before the actuary in this very special branch of consulting practice. In his paper on "Family Annuities", he said that they must not take the statistics from those living, but that, if they wanted to be exact, they must take them at the date of death. He did not remember exactly what was in his mind when he wrote that, but he would not now put the statement in the same form, because he entirely agreed that they might safely take the condition as to marriage and as to family, either at any point in life, or at the date of death, and if they used those statistics properly they got correct results. In fact, in the Fund which he had already referred to, he had, because of the very meagre number of deaths, made use of the statistics at a particular date of those living, and he believed that he got quite as good results that way as he would have had if he had had a large body of deaths. However, in the paper on "Family Annuities", in the same paragraph that he had been commenting upon, he found another statement: "To take the average difference between the ages of bridegrooms and brides would understate the values of annuities to widows, because the average difference between the ages of husbands who died and the ages of their wives is greater than in the case of bridegrooms and brides." He now found Mr. Manly, after having given the average difference of age of bridegrooms and brides, saying that there would not, of course, be so much difference of age between the wife and the

husband at the death of the husband, especially as the majority of marriages took place between the ages of 27 and 40. So here Mr. Manly apparently made a statement the exact opposite to his (Mr. King's), and one might think at the first blush that one or other statement or both of them must be wrong. But notwithstanding the extraordinary contradiction between their statements, he ventured to submit that they were both right. They could well imagine that of two people who had looked through a telescope, one might maintain that the telescope enlarged distant objects and brought them nearer, and that the other person might maintain that the telescope made distant objects smaller and sent them farther away. These statements would seem contradictory, but they were reconciled when it was found that the two persons had looked through different ends of the telescope. That was exactly what Mr. Manly and he had done. He took the average difference of age between bridegrooms and brides, and then he looked ahead, and he remembered that those who had wives older than themselves, or not much younger, were more likely to be left widowers than those who married young wives, and that, speaking of the death of husbands later on, they would be the survivors therefore of those who had comparatively young wives, because the others would be in the category of widowers and would not come under observation as husbands; so that from that point of view his statement was absolutely correct. Mr. Manly, however, went to the other end of the telescope, and looked at the ages of bridegrooms and brides, and then he took the deaths of husbands at that particular moment of age, and he remembered that those who were husbands then and dying, were the survivors of those who married earlier in life when the difference between the ages of bridegrooms and brides was not so great; and therefore Mr. Manly maintained, and perfectly correctly from that point of view, that the difference of age between husband and wife at the moment of the husband's death was not so great as the difference of age between brides and bridegrooms, the bridegroom being of the age of the husband of whom they were speaking. This reconciled the apparently absolutely contradictory statements; and it showed how statistics might bear various meanings, and give various lessons, according to the way in which they were regarded. There was one point in which he did not quite agree with Mr. Manly. Having got the average difference of age between husband and wife, he proceeded, in making his calculations, to take an annuity on the wife at that average age. A little investigation would show that, when the wives were young, that overstated the value of the annuity, and that when the wives were old it understated it. And they could not be sure which way the average error would go. In the case of the table that Mr. Manly had used, the turn he thought began somewhere about 70, but he believed that in the British Offices Annuity Tables it would be found to begin younger, somewhere about 60. He was not prepared to say exactly how much error there would be in this. In fact, the error would depend on the amount of the differences in the ages of the wives. There was a better plan, and a plan which he had himself followed. He multiplied the value of the annuity at each one of the

individual ages of the wives by the number of wives at that particular age, and then took the sum of all the annuities so found and divided by the total number of wives. That gave the average value of the annuity, as distinguished from the annuity at the average age. He thought that that gave more correct results, and it occasioned very little more trouble. He therefore would suggest that it was the better course to follow. He had spoken of the Fund which Mr. Nahmer and he had been dealing with. It was a Fund relating to the Irish clergy, and it was of rather a special kind. It was compulsory on all bachelor clergymen who entered the ministry of the church of Ireland, and it was voluntary on the part of all married clergymen who entered that ministry, so that every bachelor was compelled on entering the ministry to join the Fund, and if he remained a bachelor he never got any benefit from it at all. The Fund provided annuities to widows, and it provided annuities to children at the death of the father, whether the mother was alive or not, until those children reached the age of twenty-one. They did not value the Fund by means of its own experience by the new methods, but by means of adjustment of Mr. Hewat's tables on the old methods of Huie and Hewat. He had, however, worked out the statistics roughly, not with the same exactness as if he had had a larger body of facts, on the new method. He prepared the values of what Mr. Manly called the Orphans' Annuities, and he also got the value of Widows' Annuities. Unfortunately, on account of the smallness of the Fund, and it having been established barely more than thirty years, he could not carry the annuities for widows up to the necessary point. He could not carry them beyond age sixty-five, and, of course, they were wanted right on to the end of life. For the children he was able to get tables which were sufficiently complete for practical purposes. The fewness of the children per marriage was very remarkable. His table did not include the whole members of the Fund, whether husbands, or widowers, or bachelors, but only the husbands. That table he had the pleasure to submit. It showed the number of husbands, and the number of children of each age up to age twenty-one for each age of the husbands, and from that he had prepared a table of family annuities, and he might give a comparison with Mr. Manly's table. His were $3\frac{1}{2}$ per-cent., and he had taken the mean between Mr. Manly's 3 per-cent. and 4 per-cent. values. His figures did not extend below age thirty. He might remind the meeting that a man could not become a clergyman until he was twenty-four, and he probably would not enter the ministry of the church of Ireland until later than that, and in almost every case when he entered he was a bachelor.

As to the average ages of the wives compared with the husbands, he found that Mr. Manly gave a much greater difference in his tables than the Irish clergy showed. At age 30 the average difference according to the Irish clergy was 1·8, the wife being that much younger. Mr. Manly gave 2·5. So, at age 50, the Irish clergy showed 5·7, while Mr. Manly gave 8·5. Mr. Manly's tables would certainly err on the safe side, both as regarded the values of the children's benefits and the values of the widows' benefits; but he thought that that would be so seriously on the safe side as to upset

IRISH CLERGY.

Number of Children living at each Age, grouped according to Central Ages of Husbands, also value of Family Annuity and Average Age of Wife.

HUSBANDS		NUMBER OF CHILDREN AGED																				VALUE OF ORPHANS' ANNUITY TO AGE 21 3½ PER-CENT		Average Age of Wife
Central Age	Number	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	Irish Clergy	Manly	
30	39	12	5	...	4	...	...	1	...	...	...	...	...	...	...	...	...	...	...	...	...	14·9	22·1	28·2
35	91	37	20	13	12	8	12	9	5	4	4	1	1	1	1	1	...	...	...	...	...	17·0	29·6	31·9
40	113	41	22	25	17	24	14	19	15	15	14	8	4	9	4	4	1	...	...	...	...	24·2	31·8	35·7
45	94	15	14	12	14	13	10	17	13	20	16	16	9	18	8	12	10	5	7	4	5	21·8	28·7	39·8
50	64	7	4	5	8	6	4	4	9	7	10	5	13	8	9	8	9	14	6	6	4	18·7	24·7	44·3
55	32	4	...	2	1	1	1	1	1	1	4	1	4	6	3	4	4	5	4	4	3	11·5	18·1	49·3
60	16		...	...	...	...	1	1	1	2	1	1	3	1	3	3	1	3	5	3	3	7·1	11·3	54·3

altogether such a Fund as that of the Irish clergy. This led him to say that he thought that Mr. Manly's industry and thoroughness might perhaps be a source of danger, because they would tempt those persons who were not so industrious, and who had Funds of this kind to value, to use Mr. Manly's tables. His own experience was most emphatically that, although for mortality purposes it might be safe to use one experience for another Fund, yet for pension purposes, and for widows' and orphans' annuities purposes, the Funds were so very different that he did not think that they could find two sufficiently near for them to apply the same experience to both. He had submitted one example in the case of the Irish clergy. They could not possibly use Mr. Manly's tables for them. To do so would give results which would be out of the question. He had had another case not long ago. He had had a great deal to do with certain large railway superannuation Funds, and a very large and a very old-established industrial corporation came to him not long ago for him to advise them as to the establishment of a Fund for themselves. They thought that he could take the railway statistics and apply them. He said, "No; I must have the statistics of your own Service before I can do anything at all." The result was that he found them so totally different that the railway statistics could not possibly be applied to the commercial undertaking. The rate of withdrawal in the commercial undertaking was very much lower than in the railway. The rate of early superannuation was very much higher. The rate of mortality was not so different as to be impossible of adjustment. But the rate of mortality after pension age in the industrial undertaking was lower, and the rate of salary in the industrial undertaking was much higher and rose much more rapidly from the minimum to the maximum, so that the two sets of statistics were not in any way comparable, and to establish a Fund for an industrial undertaking on railway experience would be to launch out into

inevitable ruin. He had a case in which an industrial undertaking some ten or eleven years ago adopted the exact ratios and rates of pension of a railway Fund without any actuarial advice and applied it to their staff, and they had come to him to value it. He was sure it would prove to be in a bad way.

The PRESIDENT (Mr. Hughes) said that he was very sorry that the paper had not attracted a larger number of speakers, but it was not altogether surprising that it had not done so. The subject had been described by Mr. King as a very heavy one, and he could quite understand that members would feel hesitation in speaking on a paper like this without giving it a great deal more consideration than they had had the opportunity of doing. They must thank Mr. Manly, not only for the paper and for the information that it contained, but for having put into the hands of the profession an exceedingly useful tool. He had supplied them from time to time with useful tools, and this last was not the least useful of them. The paper might, moreover, be put to a use which was not contemplated by the author. The mere exhibition of it, with its formidable looking formulas, might serve to impress with some sense of the seriousness of the subject those amateur philanthropists and benevolent employers who were apt to call upon actuaries in an airy sort of way to devise some little scheme for their work-people, comprising impossible benefits at merely nominal contributions, and, of course, at an equally nominal fee for settling the scheme.

Mr. MANLY, in reply, said that he had put in one or two kinds of benefits which had not come into actual use as yet, so far as he knew; but from observations which he had heard dropped from various sources he was inclined to think that they were soon coming to the front. It was marvellous how benefits of that sort were introduced from time to time, and how old Funds which had been conducted upon quite cautious and conservative principles, and which simply provided for widows' annuities, were asked to give more benefits. They wanted to know whether they could not give some benefit for the children, and then it was said, "Well, you know, bachelors are very discontented, they do not see why they should go on subscribing for other people's widows and families, when they get nothing out of it themselves." So an additional benefit providing some consolation to the persistent bachelors was now being introduced in the form of the return of their subscriptions at death. But widowers with no children also wanted some consolation; and so it was coming now to be quite fashionable to give something on the death of a widower leaving no children; and all these new benefits were supposed to be so small that the Funds could easily bear them. If this paper had no other object it would do a great service in showing really what the costs of these benefits would be to such a Fund; and they totalled up pretty costly. They meant, of course, that the widow must go with less annuity, or that the contributions must be increased, neither of which changes would be received with any great amount of delight. He had listened with very great interest to Mr. King's experience of the different Funds he had valued. He (Mr. Manly) was not putting forward these tables as standard tables for use. They were intended more to show the way in which the

experience must be worked up in order to arrive at the values of the different benefits. He quite agreed with Mr. King that the experience of one Fund with regard to its withdrawals, its mortality, its rate of retirement, and its marriage rate, would not necessarily apply to another Fund. One would never expect that the rate of withdrawal in a service like, for instance, the Government Civil Service of the higher class, would be the same as in a poorly paid staff like that of a railway company. Then with regard to the question of the effect of these pension Funds on the rate of withdrawal. There was a general idea—though, curiously, he had not found the idea really confirmed by statistics—that the establishment of such Funds induced the men to remain in the services. It might do so when they received such a scale of salary and pension as they got in the Government service, but he thought that present income had much more to do with the movement of the members of a staff than future benefits. There was a very common feeling on the part of young men of from twenty to thirty years of age that they would never reach the far-off pension age of sixty or sixty-five, and they would consequently very much rather have an increase of their salary at the present time, when, they considered, the money would be much more valuable to them, than take the deferred pension into account. Moreover, the rule that the contributions shall be returned on withdrawal removes any feeling of loss when they leave. As to taking the average age of widows: If they had the material—that is to say, if they could ascertain from all the husbands living at each age the ages of their wives, and if they could make their average pension depend on the sum of the pensions to each of the wives, it would be perfection; but he did not think that they often got it. At any rate, he had no material of the kind on which to base his annuities, so he had to be content to assume an average age, as he had explained. He understood that Mr. Lidstone rather objected to his notation. He was very sorry for that. The notation which had been used before seemed to be very confusing, and he wanted something that would be very distinct, and that could not be mistaken for any other values or functions. He tried several methods before he could reduce the notation to anything like a system. If Mr. Lidstone, or anybody else, could make an improvement upon that notation, he should be quite glad to adopt it. He did not think that Old English type would have helped him. He had taken the clearest and most distinctive type which the printer had. As to the option to withdraw, to which Mr. Lidstone had referred, he took it that it was one of those Funds in which the contribution was a fixed contribution to the end of life, or something of that kind.

Mr. Lidstone said that his case was a widows' Fund without a pension Fund at all.

Mr. Manly said that the option allowed to the bachelor or widower to withdraw introduced another element which would have to be taken into consideration. It had been suggested by Mr. Lidstone that he (Mr. Manly) could have made his tables simpler and his work very much lighter by assuming that the payments took place in the middle of the year, and so avoiding the introduction of adjustments. In this paper it was assumed that they all took place in the middle

of the year, but he did not think that Mr. Lidstone had tried the problem of ascertaining the return of contributions based on salary, on withdrawal, or retirement. If they tried to value those benefits in the middle of a year they got mixed up with half-year's salaries, and it became a very awkward problem indeed. He found it work very much better to assume that the members went out exactly at the end of the year, and to make the adjustment afterwards. When they had simple benefits, such as in this paper, he agreed that it was easier to assume the payments to be made in the middle of the year. He would like to refer to his table number 45, that which he had called his working table, because he was not altogether satisfied with it. It did not altogether represent what it was intended to represent. He had said that it had been assumed that only first marriages had been allowed for; that no second marriages were allowed for, and that there were no marriages after sixty-five. But, having distributed the married men into husbands and widowers, according to the census returns, he felt that there certainly must be second marriages in them, but how many he could not tell. Further, he was unable to tell exactly whether the method he had adopted of extending the table from age sixty-five to the end of life had given him exactly what he wanted, namely, the numbers dying out of first marriages only. The supplementary tables had settled those points. The husbands dying in table 45 were more numerous than the husbands dying from first marriages in table 58, and took in, roughly, half the second marriages, and that applied up to the age of seventy-five. After seventy-five the husbands dying in table 45 were less than the deaths of husbands of the first marriage in table 58, which was calculated scientifically. He fancied that the increase in the husbands dying in the first part of the table and the decrease in the latter part arose entirely from their having taken the census returns. Taking the whole population, the marriages were earlier than in the class with which they were likely to deal, and were less in the later part of life. They knew that the working classes really attained to their maximum wage-earning power when they had reached maturity. From 22 to 24 or 25 was the time at which they earned best wages and the time when they married; but after 55 or 60 they were not able to earn wages enough to keep a wife. If they had children, the children generally kept them, but the children would not keep another mother if they could help it. He thought this explanation accounted for the difference in the two tables, coupled with the fact that the census returns included second and subsequent marriages.