The Actuarial Profession
making financial sense of the future

Mortality and longevity
Tim Gordon, Aon Hewitt

Weighted mortality experience analysis

Should weighted statistics be used in modern mortality analysis?

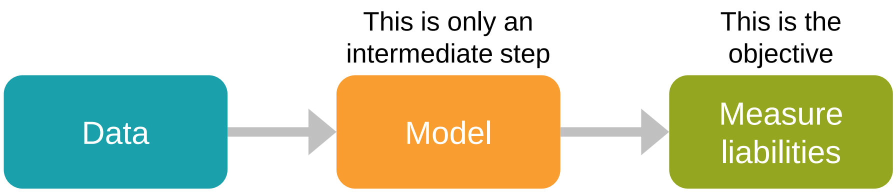
- Traditional mortality analysis means actual v expected deaths
  - unweighted (‘weighted by lives’), and
  - weighted, typically by revalued pension amount
- Actuaries have adopted survival modelling techniques
  - arose in fields such as medicine and biology
  - each life is equally significant
- Current actuarial mortality advice can be schizophrenic
  - amounts-weighted traditional analysis, but
  - unweighted survival modelling

  within the same firm, often within the same report

This (semi) session

- You can simultaneously, easily and naturally have
  - weighted mortality analysis, and
  - modern techniques
- Seen in this light, traditional weighted A/E analysis
  - is a subset; it's *not* inconsistent
  - has a sound justification; it's *not* ad hoc
- Weighted mortality modelling is
  - straightforward
  - best expressed in terms of A/E
  - best practice

What is our modelling objective?



- Our objective is *not* a model that works well for *individuals*
- We want a model that works well for *liabilities*
- Liability values are weighted
  - inaccuracy in mortality of individuals with higher liabilities has greater impact compared with individuals with lower liabilities
  - pension amount is a good proxy for liability magnitude

The science bit: Kullback-Leibler divergence

- Mortality is biology not physics: we want the best approximating model rather than the most likely simple truth
- Kullback-Leibler divergence
  - measures how far a model is from the truth
  - defined as:

$$\sum_{k \in \text{Data}} p_k^{\text{Truth}} \log\left(\frac{p_k^{\text{Truth}}}{p_k^{\text{Model}}}\right) = -\mathbf{E}LL + constant$$

- We want the best model for liabilities, so minimise KL divergence *weighted by (proxy) liabilities* or, equivalently,
- **Maximise the expected *weighted* log likelihood**

Why not just add pension as a rating factor?

- This is a standard suggestion, e.g. Pitacco et al (2009):

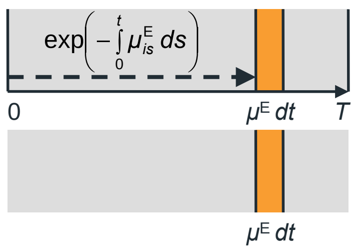
  'Actuaries sometimes weight their calculations by policy size to account for socio-economic differentials amongst policyholders. ... The pension size is thus used as a proxy of socio-economic group. However, this approach is somewhat ad hoc, and the amount of pension should better be included explicitly as a covariate in the regression models used for mortality projections.'
- This misses the point—we want a model that
  - is tuned in terms of financial impact
  - automatically uses the all the data most parsimoniously
- (We can still have pension as a rating factor if we want)

Three standard problems

1. Confidence intervals for A/E
2. Model fitting
3. Model selection

Preliminaries I
Calculations using exposed to risk

- Mortality statistical analysis requires that we ‘forget something’
  - Approach (a)
    Use *expected* survival
    (and forget time of death)
  - Approach (b)
    Just use *actual* survival



- Actuaries
  - worry about independence—approach (a) ‘feels right’
  - not always clear on distinction—sometimes mix (a) and (b)
- **Tractability and practicality strongly favour approach (b)**

Preliminaries II
A and E revisited

- For an individual *i* at time *t*
  - μ_{it} is instantaneous mortality rate (‘force of mortality’)
  - ω_{it} is weighting factor
- E2R for *i* starts at v_i, ends at τ_i, if died then θ_i = 1 else 0
- Work with A and E *operators*—map a factor ω to a number
  - Expected deaths: $$\mathrm{E}\,\omega = \sum_{i \in \mathrm{All}} \int_{t=v_i}^{\tau_i} \mu_{it}\,\omega_{it}\,dt,$$
  - Actual deaths: $$\mathrm{A}\,\omega = \sum_{i \in \mathrm{All}} \theta_i\,\omega_{i\tau_i} = \sum_{i \in \mathrm{Deaths}} \omega_{i\tau_i}$$

1. Confidence intervals

- Treat Aω, i.e. deaths weighted by ω, as a random variable
  - Expected value of Aω is Eω
  - Variance of Aω is Eω^2
- Confidence intervals are *wider* (e.g. 2 or 3×) for weighted statistics—*unweighted are misleading for model performance*
- Assuming Aω ~ N(Eω, Eω^2) tends to be reasonable—if the approximation breaks down the variance is large anyway
- **90% two-tailed confidence interval:** $$\frac{\mathbf{A}\boldsymbol{\omega}}{\mathbf{E}\boldsymbol{\omega}} = 1 \pm 1.645 \frac{\sqrt{\mathbf{E}\boldsymbol{\omega}^2}}{\mathbf{E}\boldsymbol{\omega}}$$

2. Model fitting

- Weighted log likelihood
  - $LL = -\mathrm{E}\omega + \mathrm{A}\omega \log \mu$
  - Reasonably uncontroversial—see e.g. Richards (2008)
- Let's be specific and use the proportional hazards model
  - $\mu_{it} = \mu_{it}^{\mathrm{Ref}} \exp(\beta^{\mathrm{T}} \varphi_{it})$, where φ_{it} and β are vectors
  - Ubiquitous because of its power and tractability
- Straightforward to obtain numerical solution using Newton-Raphson (including confidence intervals for β)
- More interestingly, we can show $\mathrm{E}\omega\varphi = \mathrm{A}\omega\varphi$

2. Model fitting—implications of E*ωφ* = A*ωφ*

- **If we fit an unweighted model without due care, we should not be surprised to find that it performs poorly**
  - Pensions actuaries do not expect lives and amounts-weighted mortality experience analyses to tie up
  - More inclined to use the amounts-weighted result
- **Variance of deaths is *not* fitted by maximum likelihood**
  - Dispersion (frailty) matters because it affects liability value
  - Variance *is* accounted for when comparing different models
  - But there's no remedy if all candidate models are poor

3. Model selection

- The Kullback-Leibler relative information is expected log likelihood under truth: $KL = \mathbf{E}\,LL$
- In the unweighted case, if our model is reasonable, we can estimate *KL* as $LL_{\text{Max}} - \dim(\varphi)$,

  commonly known as the Akaike Information Criterion (AIC)
- For the proportional hazards model, we can generalise to the weighted *KL* as

$$LL_{\text{Max}} - \text{Tr}\left(\text{E}\,\omega^2\varphi\varphi^{\text{T}} / \text{E}\,\omega\varphi\varphi^{\text{T}}\right)$$

3. Model selection—implications

- **We can select the most parsimonious model *in terms of financial impact***
- Usual modelling/AIC caveats apply
  - Don't data mine—hypothesise and test
  - Small number of parameters relative to data
- AIC tends to cross validation with increased data, but this is not good enough for a generic model applied to schemes
  - schemes are not random cross sections
  - need additional checks/steps in fitting process

Wrapping up

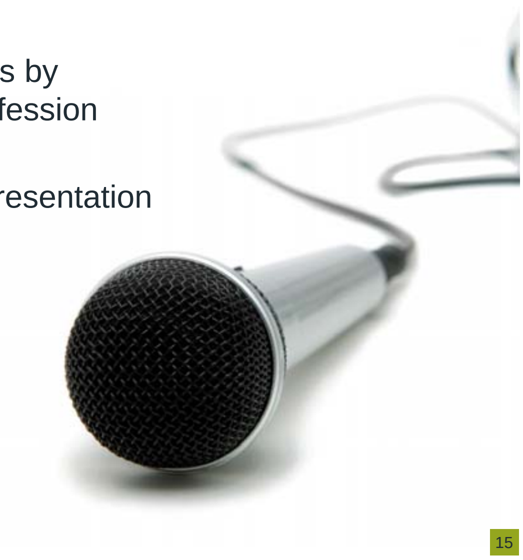
- There are drawbacks to weighted mortality modelling
  - Pension revaluation for deaths often tricky
  - Implicit assumption that weight distribution of the experience data matches the valuation data—weights have noise too

  But these are not as significant as the drawbacks of using only unweighted models
- Weighted mortality modelling
  - historic actuarial practice is justified, it's not ad hoc
  - is best practice when applying survival models to liabilities

Questions or comments?

Expressions of individual views by members of The Actuarial Profession and its staff are encouraged.

The views expressed in this presentation are those of the presenter.



15

References

- Richards, S.J. (2008). Applying survival models to pensioner mortality data. *British Actuarial Journal* **14**, 257-326
- Pitacco, E. et al (2009). *Modelling longevity dynamics for pensions and annuity business*, Oxford University Press ISBN 978-0-19-954727-2