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THE Z-METHOD FOR ASSURANCES

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ABSTRACT

The use of the Z-method for finding the probability of survival of at least r lives, and the corresponding pure endowment and annuity functions, is well known in acturial circles. (See, for example, Neill [1; section 7.4].) The purpose of this note is to show that similar formulae apply to assurances (including varying, temporary and deferred assurances).

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1. *An extension of the Z-method*

Consider m independent lives aged $x_1, x_2, \ldots x_m$ respectively, and let T be the time (in appropriate units, for example years) until the $(m+1-r)^{\text{th}}$ death. (When $r = 1$, T is the time to the death of the m^{th} life, i.e. the last survivor.)

We have

$$\Pr\{T \le t\} = \text{the distribution function of T}$$
$$= 1 - {}_t p_{\overline{x_1 x_2 \ldots x_m}}^{\;r} \tag{1.1}$$

since

$$\Pr\{\text{at least } m - r + 1 \text{ have died by time } t\} = 1 - \Pr\{\text{at least } r \text{ survive}\}.$$

Let is define

$${}_t q_{\overline{x_1 \ldots x_m}}^{\;r} = 1 - {}_t p_{\overline{x_1 \ldots x_m}}^{\;r} \tag{1.2}$$

where the r may be omitted if it equals 1. The Z-method ([1; formula 7.4.3]) shows that ${}_t q_{\overline{x_1 \ldots x_m}}^{\;r}$ may be evaluated by the formula $1 - \dfrac{Z^r}{(1+Z)^r}$, which is understood to represent the expansion

$$1 - \left(z_r + \binom{-r}{1} Z_{r+1} + \ldots + \binom{-r}{m-r} Z_m \right) \tag{1.3}$$

where

$$Z_k = \sum_{i_1 < \ldots < i_k} {}_t p_{x_{i_1} \ldots x_{i_k}}$$

and the symbol $\binom{-r}{t}$ denotes the binomial coefficient

$$\frac{(-r)(-r-1) \ldots (-r-t+1)}{t!}$$

We observe that, on considering the case when all m lives are certain to survive (i.e. when ${}_t p_{x_i} = 1$ for each $i = 1, 2, \ldots, m$),

$$1 = \Pr\{\text{at least } r \text{ lives survive}\} = Z'_r + \binom{-r}{1} Z'_{r+1} + \ldots + \binom{-r}{m-r} Z'_m$$

where $Z'_k = \sum_{i_1 < \ldots < i_k} 1$. On substituting in formula 1.3, we obtain

$$
\begin{aligned}
{}_t q_{\overline{x_1 \ldots x_m}}^{\;r} &= (Z'_r - Z_r) + \binom{-r}{1}(Z'_{r+1} - Z_{r+1}) + \ldots + \binom{-r}{m-r}(Z'_m - Z_m) \\
&= Z^*_r + \binom{-r}{1} Z^*_{r+1} + \ldots + \binom{-r}{m-r} Z^*_m \qquad (1.4) \\
&= \frac{(Z^*)^r}{(1+Z^*)^r} \qquad (1.5)
\end{aligned}
$$

where $Z^*_k = \sum_{i_1 < \ldots < i_k} {}_t q_{x_{i_1} \ldots x_{i_k}}$ (1.6)

Hence a Z-formula applies to ${}_t q_{\overline{x_1 x_2 \ldots x_m}}^{\;r}$, and this leads to its use in valuing certain assurances, as shown below.

2. *Applications to assurances*

Consider the present value of an assurance of $b(t)$ payable immediately on the death at time t of the $(m+1-r)^{\text{th}}$ life. Since the distribution function of T, the time to death of $(m+1-r)^{\text{th}}$ life, is ${}_t q_{\overline{x_1 \ldots x_m}}^{\;r}$, we have

$$\text{mean present value of benefit} = \int_0^\infty b(t) v^t \, d\left({}_t q_{\overline{x_1 \ldots x_m}}^{\;r}\right) \qquad (2.1)$$

Now apply the Z-formula 1.4, which leads to the formula

$$\text{mean present value of benefit} = \frac{Z^r}{(1+Z)^r} \qquad (2.2)$$

which is understood to represent the expansion $Z_r + \binom{-r}{1} Z_{r+1} + \ldots + \binom{-r}{m-r} Z_m$ with

$$Z_k = \sum_{i_1 < \ldots < i_k} \int_0^\infty b(t) v^t \, d\left({}_t q_{x_{i_1} \ldots x_{i_k}}\right) \qquad (2.3)$$

The terms Z_k may be evaluated by the usual methods for joint-life assurances.

Similar formulae hold for assurances in which the benefits are payable at the end of the year of death. The mean present value of the benefits may be written as

$$\sum_{t=0}^{\infty} v^{t+1} b(t+1) \left[{}_{t+1} q_{\overline{x_1 \ldots x_m}}^{\;r} - {}_t q_{\overline{x_1 \ldots x_m}}^{\;r} \right] \qquad (2.4)$$

where $b(t+1)$ is the death benefit in year $t+1$. The Z-formula 1.4 may then be applied to ${}_t q_{\overline{x_1 \ldots x_m}}^{\;r}$ and ${}_{t+1} q_{\overline{x_1 \ldots x_m}}^{\;r}$, leading to formula 2.2 with

$$Z_k = \sum_{i_1<\dots<i_k} \left[\sum_{t=0}^{\infty} v^{t+1} b(t+1)\, {}_{t|}q_{x_{i_1}\dots x_{i_k}} \right] \tag{2.5}$$

Formulae for the value of assurances payable at the end of the $1/p$ year of death may be derived in a similar way.

3. *Last-survivor assurance*

In the special case when $r = 1$, we have ${}_tq_{\overline{x_1 \dots x_m}}$ = the distribution function of the time until the death of the last survivor of the m lives.

By formula 2.2, the mean present value of a benefit of $b(t)$ payable immediately on death at time t is represented by $\frac{Z}{1+Z}$, which is to be interpreted as

$$Z - Z_2 + Z_3 - \dots + (-1)^{m+1} Z_m \tag{3.1}$$

where

$$Z_k = \sum_{i_1<\dots<i_k} \int_0^{\infty} b(t) v^t \, d\left({}_tq_{x_1 \dots x_m}\right) \tag{3.2}$$

In the particular case when the death benefit is 1 at all future times, the mean present value of the assurance is

$$\bar{A}_{\overline{x_1 \dots x_m}} = \int_0^{\infty} v^t \, d\left({}_tq_{\overline{x_1 \dots x_m}}\right)$$

This may be evaluated by the formula $\frac{Z}{1+Z}$, which is understood to represent the expansion 3.1 with

$$Z_k = \sum_{i_1<\dots<i_k} \bar{A}_{x_{i_1} \dots x_{i_k}} \tag{3.3}$$

Consider, for example, the case when $m = 3$, and let the three lives be (x), (y) and (z). Formula 3.1 and 3.3 show that

$$\bar{A}_{\overline{xyz}} = \left(\bar{A}_x + \bar{A}_y + \bar{A}_z\right) - \left(\bar{A}_{xy} + \bar{A}_{xz} + \bar{A}_{yz}\right) + \bar{A}_{xyz}$$

This formula may also be obtained from the conversion relationship

$$\bar{A}_{\overline{xyz}} = 1 - \delta\bar{a}_{\overline{xyz}}$$

but our method may be used in more general circumstances.

When the death and survivorship benefits are the same, last-survivor endowment assurances may be dealt with by conversion relationships such as

$$\bar{A}_{\overline{xyz}:\overline{n}|} = 1 - \delta\bar{a}_{\overline{xyz}:\overline{n}|}$$

(see Neill), or by regarding the benefits as the sum of those for a term assurance and a pure endowment: the former benefit may be evaluated by formula 3.1, and the pure endowment by the "traditional" Z-method (as given in Neill). On combining the two expansions (which for last-survivor assurances and pure endowments are both given by formula 3.1 with appropriate values of Z_k), the mean present value of a last-survivor endowment assurance (with death and survival benefit of 1 unit) may be written as in formula 3.1 with

$$Z_k = \sum_{i_1<\ldots<i_k} \bar{A}_{x_{i_1} \ldots x_{i_k} : \overline{n}|}$$

This "term assurance plus pure endowment" approach is more general than that using conversion relationships: it may be extended to cover policies with varying death benefits, and policies with survival benefits whose size may be independent of that of the death benefits, and which may be payable at more than one time.

A similar argument applies to Z-formulae for policies providing benefits on the death of the $(m + 1 - r)^{\text{th}}$ life of a group of m lives, plus benefits on the survival of at least r lives to one or more specified times. (When $r = 1$ we obtain the last-survivor case just discussed.)

4. *An Example*

Find a formula, in terms of temporary joint-life assurance functions, for the mean present value, at a given rate of interest per annum, of an assurance of £100,000 payable immediately on the fifth death of seven independent lives aged x_1, x_2, ..., x_7 respectively, all subject to the same mortality table, if this event occurs within 30 years.

Solution

We have $m = 7$, so we set $r = 3$. By formulae 2.1 to 2.3, the mean present value is

$$100{,}000\int_0^{30} v^t \, d\left({}_t q_{\overline{x_1 x_2 \ldots x_7}}^{\;3} \right)$$

or

$$100{,}000\frac{Z^3}{(1+Z)^3}$$

or

$$100{,}000\left(Z_3 + \binom{-3}{1} Z_4 + \ldots + \binom{-3}{4} Z_7 \right)$$

where $Z_k = \sum_{i_1<\ldots<i_k} \bar{A}_{\overbrace{x_{i_1} \ldots x_{i_k}}^{1} : \overline{30}|}$

References

1. Neill, A. (1977). **Life Contingencies. Heinemann, London.**