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ONE-YEAR RETURNS AND THE DEGREE OF RISK

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INTRODUCTION

In the early days of actuarial science investments were selected with the emphasis on capital security, with maximum interest as a secondary consideration. In such conditions pure interest represented almost the whole of the investment return and it was natural that interest should be regarded as a stable element in actuarial calculations, while risk arose almost entirely under the mortality heading.

2. In subsequent years investment managers progressively adopted policies widened to embrace higher risk investments in the expectation of obtaining higher returns (these returns including capital profits as well as interest receipts) so that at the present time only a small proportion of a typical investment portfolio can be described as capable of providing complete capital security from year to year. Increasing attention is therefore being given to the assessment of the degree of risk attaching to particular investments and combinations of investments, as well as to the estimation of expected long-term returns, and in Parts I and II of this paper it is shown how the concept of one-year investment returns can be of value for this purpose.

3. Free reserves are necessary to provide against both investment and mortality risks and in Part III some of the implications for financial management decisions are discussed, including optimization of the size of the reserve, valuation methods, bonus decisions and new business regulation. It is suggested that by treating liabilities as negative investments, with their own one-year returns, immunization problems can be studied more closely than before. Although reference is made to life office planning, the principles can be applied to any investment fund held to meet future liabilities.

4. The development of ideas on the treatment of investment risk has been relatively slow. In 1948 Pegler[1] proposed that the main principle of investment should be the maximization of expected yield, treating capital and interest alike, and arguing that capital security, which hitherto had first priority, depended upon security of income. H. G. Clarke[2] pointed out that the higher investment returns were normally associated with the higher risks and, further, the nature of the liabilities should be taken into account. In spite of the inherent conflict between maximum return and minimum risk, he incorporated these points into his first aim, which was

'to maximize the expected yield with the minimum of error, having regard to the nature and incidence of the liabilities'.

5. Nearly ten years later, Pepper[3] stated the fundamental principle of investment in similar terms to Clarke, although in the meantime Day[4] had pointed out that availability of reserves put a limit on the degree of risk which could be afforded, and had suggested that the aim should be to maximize yields subject to the restrictions on one's policy. The degree of risk which can be afforded is in practice still largely a matter of subjective judgment, but there is in train a movement towards quantification of both the risks and the restraints.

6. Pepper advocated a dynamic approach to gilt-edged investment based on short-term assessments over a period of, say, twelve months and he calculated one-year yields made up of interest plus capital appreciation, and dealt with the risk of capital loss from unexpected short-term price movements. In the more general investment field Benjamin[5] used one-year returns as a 'measure of gain' while, outside actuarial circles, Markowitz [6] used them in developing a theory for optimum portfolio selection, given any restraints, to maximize the expected return at any specified level of risk. In his paper on Operational Research in Business, Moore[7] described Markowitz's approach to the problem.

7. The link between investment risks and general financial management lies in the availability of free reserves, which may limit acceptance of new risks on the liability as well as on the asset side. In Part III of this paper an attempt is made to extend the one-year return concept to the overall business risk, and it is suggested that these returns, which are in the nature of 'returns on capital employed', not only provide some measure of overall efficiency in the use of limited resources but they also provide a means of assessing the degree of overall risk to the life office.

PART I. THE NATURE OF ONE-YEAR INVESTMENT RETURNS

Definition

8. The one-year return from an investment is the algebraic sum of all interest or dividend income from that investment received during any one-year period under review, all capital distributions or repayments in the year and the increase (or decrease) in mid-market value during the same one-year period. The rate of one-year return is the ratio which the return bears to the mid-market value at the beginning of the year, and is usually expressed as a percentage. Fluctuations in returns are large enough to make unnecessary any adjustment for the timing of interest receipts.

9. Where investments are unquoted it is necessary for the purpose of one-year returns, to estimate a market value consistent with market conditions at the time. No adjustment is needed for large holdings because

we are concerned with investments which are being held, as distinct from those being bought or sold.

Net one-year returns

10. One-year returns may be gross, or net of tax. This paper will deal mainly with gross returns but the discussion applies equally to net returns made up of interest or dividends net of any tax on investment income, plus increase in capital value net of any tax on capital gains. There are two alternative approaches to net capital gains. The first is to treat the tax as part of the expense of selling (together with dealing margins, broker's commission, etc.). This approach recognizes that incidence of the tax depends upon the decision to sell, but the gross market value is an overstatement of the value to the holder (where the contingent tax liability is positive).

11. The more satisfactory alternative is to define a net market value for each investment at any time as equal to the gross market value less any tax liability contingent upon selling at that price, and the net appreciation during the year is then the increase in net value during the year. Each holder will have his own net price which will vary with the gross price and he can, if he wishes, plot both on the same price chart. As the net market value gives no credit for postponement of tax (where positive) due to retention of the investment it is normally an understatement of value, but there is the advantage that the effect of tax emerges smoothly in the returns, year by year, as it should, with no distortion in the final year.

Relationship between one-year returns and redemption yields

12. Actuaries are accustomed to thinking of investment returns in terms of a flow of interest or dividends and a final capital repayment producing an equivalent level long-term compound interest return in the form of a redemption yield. It is not suggested that this traditional view be abandoned, but rather that it be broken down into a series of one-year steps which combine together to give the same long-term yield, while in the process giving additional information about shorter-term returns over any selected period, and also about the degree of risk. As will be seen later, the benefits of 'spreading the risk' can then readily be demonstrated and, to some extent, quantified, making possible more scientific treatment not only of investment portfolios but also of premium calculations and valuation methods, in so far as they contain an interest assumption.

13. The relationship between one-year returns (in retrospect) and the redemption yield can be examined algebraically as follows:

Let $M_1, M_2, \ldots M_n$ be the recorded *xd* market values at the ends of years 1 to n, and let $D_1, D_2, \ldots D_n$ be dividend payments assumed to have been received at the end of each year. If M_0 is the initial value and $V_1, V_2, \ldots$

V_{n-1}, M_n are the end-year values to give the same redemption yield i, then the following equations hold, where $r_1, r_2, .. r_n$ are the rates of one-year return in the respective years.

$$M_0(1+i)-D_1 = V_1 \qquad M_0(1+r_1)-D_1 = M_1$$
$$V_1(1+i)-D_2 = V_2 \qquad M_1(1+r_2)-D_2 = M_2$$
$$\cdots\cdots\cdots\cdots\cdots \qquad \cdots\cdots\cdots\cdots\cdots$$
$$V_{n-1}(1+i)-D_n = M_n \qquad M_{n-1}(1+r_n)-D_n = M_n$$

Eliminating $D_1, D_2, .. D_n$ we have,

$$M_0(1+i)-M_0(1+r_1) = V_1-M_1$$
$$V_1(1+i)-M_1(1+r_2) = V_2-M_2$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$V_{n-1}(1+i)-M_{n-1}(1+r_n) = 0.$$

Progressively eliminating $V_1, V_2, .. V_{n-1}$ we end with

$$M_0(i-r_1)(1+i)^{n-1}+M_1(i-r_2)(1+i)^{n-2}+\ldots+M_{n-1}(i-r_n) = 0$$

Whence $$i = \frac{M_0 r_1 + vM_1 r_2 + v^2 M_2 r_3 + \ldots\ldots + v^{n-1} M_{n-1} r_n}{M_0 + vM_1 + v^2 M_2 + \ldots\ldots + v^{n-1} M_{n-1}}$$

14. This equation may be solved for i, but its significance is more in its form than in its solution. It shows that i, the redemption yield, is the weighted mean of the one-year return rates, the weights being the market values discounted to the starting point at the rate i. In normal circumstances the weights nearer to the starting point are greater than the more distant ones, so that r_1 is usually the most important of the one-year return rates. It is of interest to note that if the investment is of an accumulating, non-interest-paying type, the increasing M values counterbalance the discounting effect and the weights lose their declining trend. As a rule, the greater the interest element in the return, the more important is the contribution of the early values to the redemption yield.

15. When looking to the future, an investment can be regarded as providing a sequence of one-year returns and its suitability judged by weighting and averaging the rates of return over any relevant period. It will, however, be normal practice to select investments giving good prospects in the early years. Fluctuations in market value will introduce uncertainty into future returns, and this will be discussed later in this paper.

When reviewing the past record of an investment, any weighting of the realized rates usually seems unnecessary, and the more recent values have the most significance for current and future trends.

16. Using the same notation it can be shown that $r_1, r_2, \ldots r_n$ make up a variable rate of interest, of which i is the equivalent single rate. Progressively eliminating $M_1, M_2, \ldots M_{n-1}$ from the equations used above, we have

$$M_0 = \frac{D_1+M_1}{1+r_1} = \frac{D_1}{(1+r_1)} + \frac{D_2+M_2}{(1+r_1)(1+r_2)}$$

$$= \ldots\ldots$$

$$= \frac{D_1}{(1+r_1)} + \frac{D_2}{(1+r_1)(1+r_2)} + \ldots + \frac{D_n+M_n}{(1+r_1)(1+r_2)\ldots(1+r_n)}$$

17. Thus the present value of an investment is the sum of the future income receipts and the final capital repayment, all discounted at the variable interest rate defined by the rates of the one-year returns. This gives a rather more complete view of the present value than does the use of the equivalent single rate i, being derived from information about intervening market values.

Measurement of degrees of risk using one-year returns

18. Some writers appear to regard 'risk' as having much the same meaning as 'chance' where the probabilities are known, and distinguish it from 'uncertainty' where the probabilities are not known. Others regard risk as related directly to the variance of the returns, or to the variance of price movements about a trend line, but without bringing in any question of capital loss. Moody[8] provided estimates of the extent of price falls to be expected in normal, unusual and exceptional circumstances, but without attempting to estimate their respective probabilities. Other measures are no doubt available—for example the average ratio of high to low price limits, taken annually, would give relative risk but would not provide an absolute measure which could be applied at any price level.

19. In this paper the word 'risk' is used in relation to events which may have a variety of possible results, some of which are associated with financial loss, i.e. involving loss of risk capital. The probabilities are not usually known precisely and must then be estimated from past experience. Each possible result has its own 'expectation', being the product of the probability and the corresponding gain or loss. The total expectation will usually be positive, otherwise the risk would not be accepted. The sum of the probabilities associated with the various levels of financial loss is the

'chance of loss' which, it is suggested, provides a satisfactory absolute measure of the 'degree of risk'. When there is a multiplicity of events and the probabilities can be combined, there is a chance of loss, or degree of risk, applicable to the whole. Further, if risk capital is limited as it must be, a higher degree of risk, viz. the chance of total loss exceeding the available risk capital, could be calculated from the same probability distribution. This higher degree of risk would be the chance of insolvency. In theory, with the same chance of loss and the same amount of risk capital it would be possible to have different degrees of insolvency risk, depending upon the shape of the probability distribution.

20. There are many business risks which may contribute to the chance of loss but at this stage the discussion will be confined to investment risks and it is suggested that the return over a one-year period provides a convenient 'event' in their treatment. It is convenient because (a) interest, dividends and rents are usually paid at regular intervals of a year or fractions of a year, (b) the accounts of the investor are usually made up annually and (c) when the return is negative it measures the loss sustained. There is, however, no reason why returns should not be calculated more frequently, if desired, and a move towards shorter periods appears likely with increasing mechanization of accounting and closer management control.

21. Certainly, the use of shorter periods would result in the provision of a larger volume of data for analysis but, outside the gilt-edged market, periods of less than one year are probably too short for investment decisions and financial planning. Data based on calendar years is used later in this paper and it will be seen that information derived from alternative investments can be used to supplement limited data available from a single stock.

22. Another way of increasing the data would be to use a series of one-year returns starting at intervals of less than one year—for example they could be monthly. These would pick up many price fluctuations which would otherwise pass unnoticed. Against this suggestion, however, it can be argued that unless the term of years covered by the original observations was long enough to be completely representative of a varied economic background, there would be a good deal of overlapping of the experience. One-year return rates for a particular stock can be plotted against time as a continuous line which will pass through the annual values. It is clear that the intermediate values will, while showing fluctuations, tend to lie between the corresponding end-year values which themselves could be well away from average, reflecting, perhaps, a particular stage of a cycle. The value of additional data from this source is therefore questionable.

23. It may be asked whether it is correct, in the assessment of investment risk, to include the relatively stable income element instead of operating only upon price fluctuations. In favour of its inclusion, the main

argument is that to an investor who owns both capital and income the two are, after tax has been paid, indistinguishable in his hands, where surplus income becomes capital. High income is often associated with a high degree of risk and it is right to take it into account when specifying the chance of loss.

24. A possible objection, which can be dealt with at this stage, is that one-year returns should not be used in the assessment of investment risk when the profits and losses have not in fact been realized by sale in the market, and there may yet be a completely secure ultimate redemption value. This objection is somewhat illogical as it would imply that an addition to an existing holding had, after excluding dealing margins and expenses, a different value to a gross fund from an equal nominal amount of the original holding. If the objection is based only upon the existence of somewhat arbitrary day-to-day price fluctuations, the answer is that these may not affect the final outcome but they do imply the existence of a degree of risk, and they should not be ignored. If it is held that profits and losses are not real until converted into cash then it may be pointed out that certificates of value are always based on a market assessment. (An investment loss evidenced by a price change might, of course, be made good, overall, by an equal fall in value of matched liabilities but this is an insufficient argument for disregarding fluctuations in market value.)

25. The definition of the one-year return specified mid-market value and the effect is to separate dealing expenses, including price margins, from the basic profit or loss, and thus from the degree of risk. Should activity be contemplated then dealing expenses would be taken into account. It may be remarked that since market value is made by the interaction of the views of a large number of stockholders it automatically takes account of new risks, not yet evident in the past record, but still liable to affect the future.

The shape of the probability distribution

26. The range of possible returns from a single investment in a particular year and their respective probabilities are dependent upon human activities, largely directed towards the one objective of maximizing investment yields, and it could be expected that extreme results would be unlikely and that the highest density in the probability distribution would occur near the mean of the range. It therefore seems reasonable to postulate that this distribution may be approximately Normal. If it can be shown that this is a workable hypothesis then the measurement of risk in absolute terms will be greatly simplified, since the mean value and the standard deviation together will then define the distribution, and if these two factors can be estimated, then their ratio can be used to derive not only the degree of risk (chance of loss), but also the chance of loss exceeding any given figure.

27. Assuming a Normal distribution, the chance of loss is the proportion of the area under the curve which lies to the left of the line $x = 0$, the range of possible returns having values x, mean value $\bar{x}$, and standard deviation of the distribution σ. A selection of values for the chance of loss, i.e. degree of risk, for corresponding values of $\frac{\bar{x}}{\sigma}$ is given in Table 1.

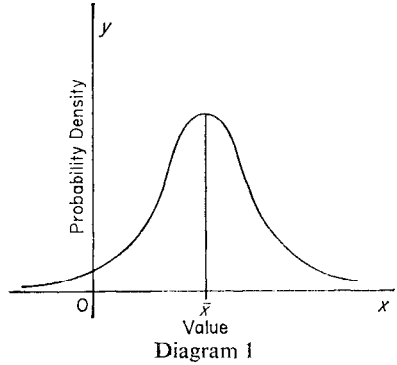

Diagram 1

Table 1

$\frac{\bar{x}}{\sigma}$	0·0	0·5	1·0	1·5	2·0	2·5	3·0
Degree of risk	·500	·309	·159	·067	·023	·006	·001

28. The ratio $\frac{\bar{x}}{\sigma}$ is the inverse of the coefficient of variation and is a more convenient measure where, as in investment work, the mean may on occasion be zero. This ratio itself provides a measure of relative risk but the values are less easy to comprehend than are those giving 'chance of loss' directly, i.e. an absolute measure of the degree of risk.

29. In practice each event is unique since the economic background is continuously changing and even against the same background in a particular year investments differ in character and performance. A judgment of the probability distribution in advance of a unique investment event can only be formed from a study of performance of the investment in earlier years and from performance of other investments in similar conditions. It will be shown that quite small samples of shares taken in a single year display features resembling those of a Normal distribution, giving weight to the assumption that a similar distribution applies to the probabilities surrounding a single future event, and also that there is some evidence to show that if the mean of the distribution can be estimated then so can the standard deviation and the degree of risk.

The pattern of one-year returns of shares

30. In Table 2 there is displayed the record of one-year return rates recorded by thirty leading ordinary shares over ten consecutive calendar years. The shares are divided into three groups of ten shares each, representing capital goods, consumer goods and miscellaneous industries. The observations may be regarded as a sample drawn from a large popula-

tion which extends back in time, and across many other companies, but the sample is far from being a random one. Sequences of observations from individual shares, for example, are not independent, being linked by market values and by such continuing factors as company dividend policy and profitability of its capital employed. Nor are observations in a single year independent since the same economic and political background affects them all to a more or less extent. However, at the beginning of any one year the thirty shares represented alternative choices to the investor and were all quoted on a market which should, in theory, have adjusted the price so as to make them equally attractive, judged on the basis of investment returns over a limited period, and they may therefore be regarded, to this extent, as a homogeneous group.

31. Examination of the data shows that taken year by year, actual losses conform reasonably closely to Normal for the same mean and variance, with 85 recorded compared with a total of 87·9 from a Normal distribution. Statistical tests show a highly significant divergence from Normality, which is, perhaps, only to be expected, but it is nevertheless suggested that the assumption is still of practical value.

32. In Table 3 the actual distribution about the 30-share means is compared with the Normal.

33. The actual distribution is positively skewed, and this must be inevitable in view of the occurrence of values exceeding $+100$ at one end while at the other end the limit occurs with total loss at -100. This difficulty could be avoided by using $\log_{10} x'$ where $x' = x+100$, and the resulting distribution is more symmetrical as shown in Table 3. The degree of risk can be obtained from the distributions of $\log x'$ since the chance of loss is then the area to the left of the line $\log_{10} x' = 2$. Comparing expected losses, however, the result shows little difference, with 89·5 compared with 87·9 before, and 85 actual. The explanation lies in the way the experience curve crosses the Normal curve around the value $x = \bar{x} - \sigma$. Since for shares σ usually exceeds $\bar{x}$, the area to the left of the line $x = 0$ may well be much the same for the two cases in spite of the skewness of the 'actual' curve, and for estimations of the degree of risk, the Normal assumption applied to a natural one-year return expectation would probably provide a reasonable approximation. The objection to the use of $\log_{10} x'$ is that it is unsuitable for the other purposes for which one-year returns are used, in particular, as an alternative to interest in financial calculations.

34. When estimating the chance of insolvency, i.e. losses in excess of available risk capital, the Normal assumption applied to the distribution of one-year return probabilities, would most likely result in the risk being overstated, although in this connexion the data may be criticized on the grounds that shares selected in retrospect automatically exclude companies which have gone bankrupt during the period under review, thus introducing a bias towards the positive skewness which has been observed.

Table 2. *Recorded one-year investment returns %*

	1958	1959	1960	1961	1962	1963	1964	1965	1966	1967	10 year Mean	10 year σ
Capital goods group												
A_1	9·3	—3·0	5·2	—15·0	17·0	41·0	2·5	26·0	13·0	19·7	11·6	15·7
A_2	21·8	—5·0	—3·6	—13·0	—17·4	20·5	26·5	20·0	29·5	—0·9	7·8	17·5
A_3	65·6	79·6	—1·0	18·3	13·8	35·2	—3·0	—3·9	4·3	14·1	22·3	29·2
A_4	81·0	69·5	7·0	13·0	23·0	31·0	—5·4	1·6	—10·4	—2·0	20·8	31·5
A_5	44·5	50·0	3·9	7·0	—4·2	9·8	3·5	—5·8	—9·8	22·2	12·1	20·6
A_6	51·0	52·5	—8·9	5·8	—·5	·8	5·6	2·4	7·5	33·0	14·9	22·2
A_7	45·9	52·6	—7·5	—10·4	—14·5	35·5	—4·5	·4	—7·4	29·4	12·0	25·9
A_8	26·0	15·6	—22·3	3·3	6·2	18·0	—18·3	7·9	30·6	20·9	8·8	17·6
A_9	50·2	51·0	—10·0	29·4	4·8	20·0	—8·6	41·0	—2·2	·4	17·6	24·0
A_{10}	60·0	66·7	46·5	—10·4	—16·5	46·1	—27·5	14·3	—23·7	55·5	21·1	37·8
Mean	45·5	43·0	·9	2·8	1·2	25·8	—2·9	10·4	3·1	19·2	14·9	17·9
σ	21·5	29·9	18·2	14·9	14·4	14·4	14·5	14·9	17·7	17·8		
Consumer goods group												
B_1	50·0	128·1	—9·0	47·8	—8·3	12·5	·0	—5·0	2·0	23·1	24·1	42·5
B_2	74·0	59·4	—1·5	17·8	7·8	50·3	7·0	—12·0	—5·0	24·3	22·2	29·4
B_3	45·1	35·5	—16·8	18·5	2·2	22·5	—12·0	1·3	—10·6	43·2	12·9	23·3
B_4	75·5	43·5	12·8	28·1	—17·4	10·2	—18·4	—7·4	—7·5	12·5	13·2	29·5
B_5	19·5	48·5	55·0	6·3	46·3	—5·0	—12·2	14·9	—14·3	2·5	16·2	23·6
B_6	34·5	35·0	21·5	—10·8	—11·5	50·6	—27·5	·8	—16·5	10·8	8·7	26·0
B_7	68·8	97·5	—11·2	—3·5	37·0	47·0	14·9	5·0	—8·5	43·9	29·1	36·1
B_8	57·0	38·0	6·0	—1·0	—5·1	33·5	17·0	32·0	4·5	26·0	20·8	19·9
B_9	94·9	91·8	—11·0	26·0	—22·6	23·4	—8·0	48·2	6·9	73·3	32·3	31·4
B_{10}	68·5	49·5	5·2	18·0	—12·5	33·5	—13·7	8·0	—7·8	32·6	18·1	27·6
Mean	58·8	62·7	5·1	14·7	1·6	27·8	—5·3	8·6	—5·7	29·2	19·8	24·7
σ	22·1	31·9	21·3	17·5	23·0	18·6	14·6	18·7	7·8	20·5		

Misc. industries group												
C_1	106·0	51·0	—16·5	6·0	4·0	—1·8	29·4	11·2	3·0	54·7	24·7	36·6
C_2	13·1	13·4	—32·2	3·4	47·6	112·2	24·4	17·1	—2·1	99·3	29·6	45·0
C_3	10·1	49·6	4·5	28·2	2·0	98·0	—17·8	4·1	—22·7	52·4	20·8	36·9
C_4	108·5	131·0	—29·7	4·5	5·2	28·0	3·0	2·0	—10·5	6·4	24·8	52·3
C_5	14·0	86·0	1·0	18·9	—9·8	29·0	38·5	8·0	—10·6	59·2	23·4	30·8
C_6	62·5	42·5	—15·0	41·8	3·2	40·5	2·9	9·5	·3	34·8	22·3	25·1
C_7	151·0	11·5	30·0	—11·0	1·5	—6·2	8·0	19·8	14·2	125·3	34·4	56·3
C_8	49·3	62·0	9·1	—7·0	7·0	30·0	—5·5	5·2	—12·0	43·2	18·1	26·1
C_9	57·8	47·0	—13·6	—24·9	·0	59·3	2·8	4·7	—1·5	34·2	16·6	30·4
C_{10}	66·0	55·0	—7·8	—11·8	—44·5	33·0	8·0	17·5	8·5	5·1	12·9	32·4
Mean	63·8	54·9	—7·0	4·8	1·6	42·2	9·4	9·9	—3·3	51·5	22·8	24·9
σ	46·6	34·4	18·8	20·2	22·2	38·3	16·9	6·3	10·8	34·3		
30-Share Mean	56·0	53·5	—0·3	7·4	1·5	31·9	·4	9·6	—2·0	33·3	19·1	22·7
σ	32·1	32·1	19·5	17·8	19·5	26·1	16·2	13·8	12·9	29·2		
$\frac{\bar{x}}{\sigma}$	1·74	1·67	—·02	·42	·08	1·23	·02	0·70	—·16	1·14		
Degree of Risk	·04	·05	·51	·34	·46	·11	·49	·24	·56	·13		
											Total	
Normal losses	1·2	1·4	15·3	10·2	13·8	3·4	14·7	7·2	16·8	3·9	87·9	
Actual losses	—	2	17	11	13	3	14	5	18	2	85	
											10 year	
											Mean	σ
$3\frac{1}{2}\%$ War Loan	10·5	3·9	—3·8	—6·2	24·8	2·0	—1·9	4·4	3·4	2·2	3·9	8·7

Table 3

Range of values	Less than $\bar{x}-3\sigma$	$\bar{x}-3\sigma$ to $\bar{x}-2\sigma$	$\bar{x}-2\sigma$ to $\bar{x}-\sigma$	$\bar{x}-\sigma$ to $\bar{x}$	$\bar{x}$ to $\bar{x}+\sigma$	$\bar{x}+\sigma$ to $\bar{x}+2\sigma$	$\bar{x}+2\sigma$ to $\bar{x}+3\sigma$	Over $\bar{x}+3\sigma$
Actual %	—	·3	14·0	41·7	30·0	8·3	5·0	·7
Normal	·1	2·2	13·6	34·1	34·1	13·6	2·2	·1
$\log_{10}(x+100)$	·3	1·4	14·0	34·3	36·0	9·0	5·0	·0

The relationship between the mean and the standard deviation

35. In general, random samples from a large population can be expected to show a range of mean values, distributed around the population mean, with the variance of the observations in the samples unrelated to their mean values. It has already been pointed out that one-year return rates, taken from a group of shares in a particular year, are not independent and although each year may be regarded as providing a sample with its own mean, it can be shown that a relationship with the standard deviation appears to exist, thus facilitating estimates of the degree of risk.

36. In Diagram 2, means are plotted against the standard deviations as a scatter, the observations being for the groups of ten shares taken a year at a time, and for the whole, also in separate years. It is fairly clear that, far from being unrelated, high variance goes with a high mean, and vice versa. The straight line given by the equation $\sigma = 0{\cdot}3\bar{x}+16{\cdot}0$, appears to fit the 30-share values fairly well and the fact that this line fits high points in three cycles as well as the low values of 1961 and 1966 leads one to think the relationship may persist in the future. Applying it to the probability distribution surrounding a single share in the future, it is suggested that if the mean of the distribution can be estimated, then an estimate of the degree of risk is determined.

Performance of separate shares over time

37. The sample of observations which represents the experience of a single share is much less homogeneous than that of different shares in the same year. A random price movement at the end of one year would affect two observations in opposite directions, while economic cycles and profitability trends could introduce predictable features into future results. Nevertheless, over a very long period, a company's returns, taken separately, could be expected to produce a distribution not dissimilar to the combined single year experience curve for the 30-share group (Table 3). The distribution in relation to each share's own mean and standard deviation, combined into the main groups, is shown in Table 4.

38. The marked positive skewness can be attributed to the exceptional years 1958–59 which were responsible for the high mean values and, to some extent, the cluster of low values in subsequent years. Thus the shape

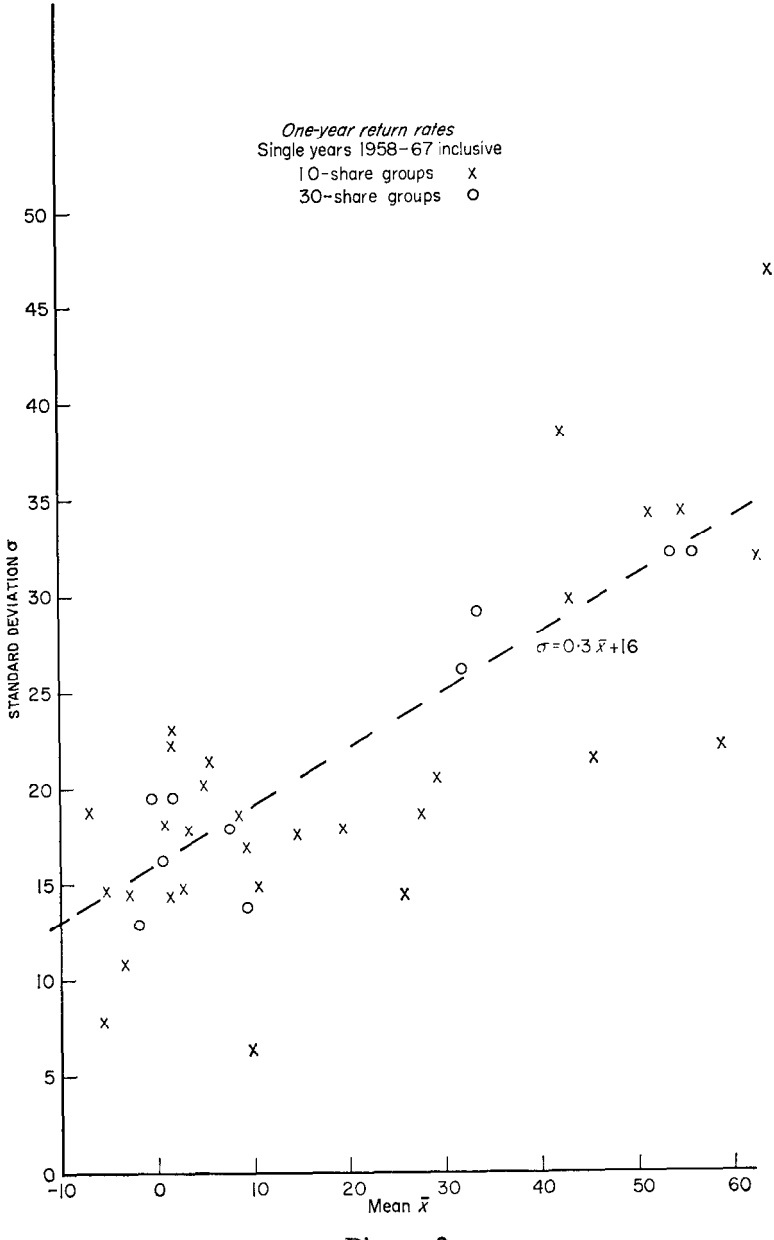
One-year return rates
Single years 1958–67 inclusive
10-share groups X
30-share groups O
STANDARD DEVIATION σ
Mean $\bar{x}$
$\sigma = 0{\cdot}3\,\bar{x} + 16$
0
5
10
15
20
25
30
35
40
45
50
-10
0
10
20
30
40
50
60

Diagram 2

C

of the distribution is related more to economic conditions in the past decade than to risk in the future.

39. The performance of individual shares over a period is, however, always interesting and it must be used in the process of share selection. It can be seen from Table 2 how varied the performance has been, with ten-year means ranging, among the 30 shares, from 7·8% to 34·4%. Plotted as a scatter in Diagram 3 it is again fairly clear that a high mean value is associated with a high variance, in general, the standard deviation in this particular period being on average about 1·6 times the mean. This is

Table 4

Range of values	Less than $\bar{x}-3\sigma$	$\bar{x}-3\sigma$ to $\bar{x}-2\sigma$	$\bar{x}-2\sigma$ to $\bar{x}-\sigma$	$\bar{x}-\sigma$ to $\bar{x}$	$\bar{x}$ to $\bar{x}+\sigma$	$\bar{x}+\sigma$ to $\bar{x}+2\sigma$	$\bar{x}+2\sigma$ to $\bar{x}+3\sigma$	Over $\bar{x}+3\sigma$
Group A	—	—	12	46	24	18	—	—
Group B	—	—	18	40	22	18	2	—
Group C	—	—	11	52	19	13	5	—
Totals %	—	—	13·7	46·0	21·7	16·3	2·3	—
Normal	·1	2·2	13·6	34·1	34·1	13·6	2·2	·1

a different formula from that suggested earlier as applicable to a single event, and the ten-year sequence has different uses from the single-year range of returns. Single share returns represent emerging experience which can be used as a platform for projecting future returns, as will be discussed in Part II. Another use is connected with portfolio management.

Spreading the risk

40. The principle of 'spreading the risk' is fundamental in insurance and its value has for long been recognized in the investment field. The effects can readily be studied in the records of one-year returns—for example in Table 2 it can be seen that the ten-share means formed a more stable sequence than did most of the individual shares. In Diagram 3 the position of the 'A' share group mean and the average of the single share standard deviations has been marked at the point (14·9, 24·2) and this can be regarded as a typical result from investing in a single share in this group. If, however, all ten 'A' shares had been held throughout (with weights equalized to the same value at the start of each year), the portfolio result would have shown the same mean of 14·9% but with the standard deviation reduced by 6·3 to 17·9, while the number of years showing a loss would have been reduced from probably three for the single share to one only, and that quite small in amount, for the portfolio.

41. Similarly, the standard deviation of the 'B' portfolio came out at 4·2 less than the single share average, and for the 'C' group the difference was as high as 12·3. But it is of considerable interest to find that upon further consolidation to a 30-share portfolio the change was only marginal,

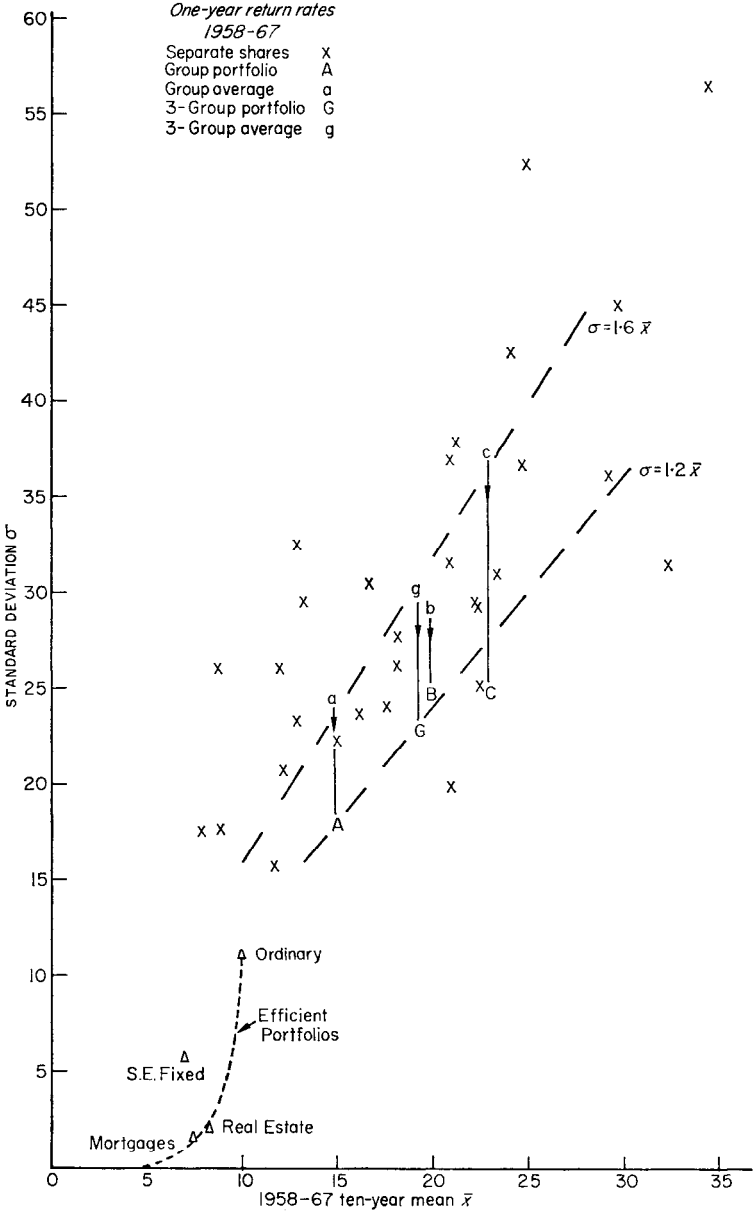
One-year return rates
1958-67
Separate shares X
Group portfolio A
Group average a
3-Group portfolio G
3-Group average g
σ=1·6 x̄
σ=1·2 x̄
STANDARD DEVIATION σ
1958-67 ten-year mean x̄
Ordinary
Efficient Portfolios
S.E. Fixed
Real Estate
Mortgages
0
5
10
15
20
25
30
35
40
45
50
55
60

Diagram 3

from 22·5 average to 22·7 consolidated. This suggests that even within a main group the variance from year to year can be considerably reduced by holding as few as ten shares, but further additions, including shares in other main groups, may produce no further improvement. Thus there is probably a limit, imposed by economic swings, to the degree of stability of a portfolio consisting only of ordinary shares, although it should be possible to combine above-average returns with below-average risk by judicious selection and weighting.

Variance and co-variance

42. Clearly if all share prices moved in sympathy there would be no benefit to be derived from a spread of share-holdings. The benefit arises when they behave independently, over time. With complete independence, but with each showing random movements, the sum of expected means and the sum of expected variances give an expected portfolio result—for example, had the 'A' shares been independent and their recorded means and variance (σ^2) been a representative sample of their individual characteristics, the standard deviation of the group returns could have been expected to be around 7·9 (instead of 17·9 as recorded).

43. Co-variance determines the correlation, or dependence of two series and, clearly, negative correlation has even more stabilizing power in a portfolio than mere independence. The records of one-year returns provide a simple means of observing the degree of correlation displayed in the past, enabling preference to be given (other things being equal) to shares with negative correlation to the portfolio. For example, in Table 2 it can be seen that shares A_1 and A_2 produced losses in 1959 and above average figures in 1964–66, thus reducing the variance of the 'A' portfolio. However, the ten-year mean values for these shares were both low and if the higher variance had been acceptable a better overall result would have been produced without them.

Fixed-interest classes

44. In this paper ordinary shares have been chosen for study thus far because the wide price fluctuations create a more interesting body of data. One-year returns can, however, equally well be recorded for other classes of investment and the emerging values studied in the same way as for shares. As an example, the rates of return for 3½% War Loan have been added at the foot of Table 2 for comparison. Expressed in this way the complete returns, year by year, from investments of different classes can be directly compared, and there is no difficulty in checking, from the records, to see how, for example, the fixed-interest classes could best have been used, not only in maximizing the return, but also in reducing the risk to an acceptable level.

45. Since expected yields to redemption from fixed-interest stocks can

be estimated with a relatively high degree of confidence, prices move closely in line under the influence of the market yield basis and it is to be expected that the distribution of the rates of return in a particular year from a number of redeemable securities with the same outstanding term would be fairly tight about the mean, confirming that the need to hold a spread of investments in this class, rather than a single stock, is less marked. The more important aspect of the recorded returns from fixed-interest classes is therefore the progression from year to year, and the resulting mean and variance over time.

Mixed portfolios

46. In Table 5 there is given an estimate of the rates of return which might have been obtained from a portfolio of holdings in each of the main classes of investment. Those for mortgages assume part to be in loans fixed for varying terms and therefore liable to fluctuations in market value. The returns from a mixed portfolio will depend upon the weights given to each class, and an estimate is given of the overall return from one weighted in accordance with the average U.K. life office distribution (assumed to be based on market value) at the given dates.

47. Over the period 1958–67 the quoted fixed-interest class showed a degree of negative correlation with the ordinary shares, thereby improving the stability of the portfolio, although, as things turned out, the mean return from the fixed-interest class was low and held back the overall yield.

48. The above figures are given purely for illustrative purposes. Individual offices would experience quite different results, depending upon their own distribution by class, and upon selection of individual investments.

Adjustment for new investments

49. Investment portfolios are usually subject to purchases and sales during any one-year period under review and if the overall performance is being examined a fairly simple adjustment for new investments can be made when calculating the one-year return rate.

Let M_0 be the initial and M_1 the closing total market values, I the total investment income, N the new investment cost and P the sale proceeds. Then if r is the overall rate of return and if purchases and sales are assumed to be spread evenly over the year,

$$M_0(1+r)+(N-P)(1+\tfrac{1}{2}r) = M_1+I$$

whence

$$r = \frac{(M_1-M_0)-(N-P)+I}{M_0+\tfrac{1}{2}(N-P)}.$$

50. The cost of new investments will include dealing margins and expenses which may well absorb the first half-year's return. If returns excluding new investment costs are being compared, and sales are a minor feature, an approximation which simplifies the formula is to omit $\frac{1}{2}(N-P)$ from the denominator.

Degree of risk as applied to a mixed portfolio

51. A large well-spread portfolio is almost certain to produce results (from an inactive policy) fairly representative of the whole market so there is probably no need for a life office to concern itself with the chance of overall loss arising purely from too heavy a concentration in any one investment. The principal investment risk remains that associated with swings in the whole market, over time, as displayed by price indices and by records of portfolio one-year returns. An attempt should be made to quantify this risk as it has a bearing upon management decisions concerning such diversified matters as premium rates, bonus distributions, reinsurance and new business policy, by virtue of its impact upon the free reserves.

52. The one-year returns from broad classes of investment, recorded over a period of years provide valuable basic data. The method is to estimate a mean and a variance for each class, suitable for application in the future, and then to weight and combine them to provide corresponding estimates for various distributions of classes in the whole portfolio. Results can only emerge singly, year by year, but looking more than the next few years into the future any one year is very like another and the estimates can be regarded as referring to a probability distribution of possible returns for any given year. If the shape of this probability distribution can also be estimated then a degree of risk for the portfolio can be calculated, and, after allowing for liability risks, there can also be calculated the amount of free reserves (risk capital) necessary to reduce the overall business risk to an acceptable level.

53. Estimates of future mean returns will be discussed in Part II. The present problem is to extract from the available data some indication of a probability distribution about the mean for any year in the not-too-distant future. It is suggested that as a first approximation the group ordinary share relationship derived from Diagram 3 be used giving, for this class, a standard deviation of 1·2 times the expected mean. Thus for an expected mean of 10% the associated standard deviation would be 12 and variance 144. Similarly, for fixed-interest stocks a standard deviation about 0·8 times the mean looks reasonable, while for mortgages the ratio could be 0·2 and for real estate 0·25. For example, if a portfolio consisted of 25% in each of these four main classes, and there was, for simplicity, assumed to be no correlation between the class returns, the overall mean, assuming the other class means to be fixed-interest 7%, mortgages 7½% and real

Table 5. *U.K. life office investment portfolios, 1958–67*

Calendar year	1958	1959	1960	1961	1962	1963	1964	1965	1966	1967	Mean	σ
Returns %												
Ordinary shares	56·0	66·5	2·0	16·5	3·0	21·0	—4·5	11·5	—4·5	36·0	20·4	25·0
Real estate	8·0	10·5	14·0	6·0	8·5	10·0	9·0	9·5	6·5	8·5	9·0	2·2
S.E. fixed-interest	10·0	7·0	1·0	2·0	15·0	5·5	2·0	3·5	3·0	5·5	5·4	4·3
Mortgages and loans	7·0	6·5	4·5	5·0	7·0	7·5	4·5	4·5	7·0	6·5	6·0	1·2
Weights												
Ordinary shares	17·7	19·1	20·8	22·1	23·0	22·0	22·4	22·0	22·6	22·1		
Real estate	9·1	9·5	9·8	10·1	10·3	10·6	10·4	9·9	10·1	11·3		
S.E. fixed-interest	50·8	49·1	47·4	45·2	43·6	44·0	44·0	44·0	42·6	42·1		
Mortages and loans	18·0	18·1	17·9	18·5	19·1	19·3	19·1	19·4	20·4	21·1		
Weighted return %	17·9	19·1	3·2	6·3	9·8	10·0	1·7	6·2	2·4	33·0	9·0	6·2

estate 8%, would come out at 8·1% with standard deviation 3·4. Correlation between returns for main classes can be allowed for by the introduction of additional terms involving co-variance.

54. As it is the estimated probability distribution of possible returns from a unique event in the future which is being considered, there is no need to attempt any very sophisticated calculations, and it is suggested that once again the Normal distribution be used. Admittedly the assumption of Normality may be thought even more dubious than before, since it is applied to a mixture of distinctly different classes, but on the other hand the means are fairly close, and the variances high in comparison. In the example given above, the chance of loss in any one year would then be estimated, from Table 1, at 0·01 and the chance of losses in excess of 5% of the total value at the start of the year would be about ·0005. For comparison, the chance of a 5% loss on an all-equity portfolio would be as high as ·1, which might be an unacceptable risk.

55. Points for other-class portfolios, added to Diagram 3, can all be enclosed by a curve of increasing gradient, drawn from the origin (which represents a completely secure portfolio of cash, with no return) up to the 100% equity portfolio. Portfolios on this line would conform to the accepted rule that higher returns can be expected from higher risk investments, for not only does the variance increase with the increasing mean return, but the increasing gradient has the effect of determining an increasing degree of risk, assuming Normal distribution. The assumptions for mortgages and real estate look out of line with the quoted classes, reflecting the fact that higher returns can be obtained by acceptance of a lower degree of marketability as well as from a higher degree of risk.

Efficient portfolios

56. The different classes can be weighted and combined in an infinite variety of ways, producing a wide scatter of points on the diagram. Clearly there will be a curve drawn from the origin to the ordinary share portfolio which can be made to pass through the points representing portfolios with the highest mean return at each level of variance, being also the lowest variance at each level of return. These special combinations were called 'efficient portfolios' by Markowitz, and computer programmes have been written to facilitate their selection, given the necessary input of expected returns and their variance, together with any restraints imposed by management.

57. The example given above was of a single year in the future, there being no exact knowledge of opening market values. The same method can be applied in the special case of the next twelve-month period, for which the opening values are known and more precise estimates of the returns and variances can be made. Similarly, calculations of efficient distributions can be made for the coming year in much the same way as for the general future.

PART II. FROM THE PAST INTO THE FUTURE

Expected returns

58. Actuaries have the continuing problem of estimating future trends in the light of what has happened in the past, and then making appropriate provision in their valuations and in premium rates. The rate of interest usually presents some difficulty and it is suggested that when dealing with this aspect, attention should be focused, not on interest only, but on one-year returns.

59. When looking from the past into the future we look from fact to uncertainty. Since each value represents a complete investment return for the year, and since the records can be presented in the form of a frequency distribution of returns secured in the past, they are very suitable for use when attempting to grapple with the investment uncertainties of the future. Asset-income can be pictured in terms of a linked sequence of one-year returns extending from the past record forward into the future, the link from one to the next being the market value, but whereas annual returns in the past may be looked upon as single random selections from a distribution of possible values, the future must be thought of in terms of expectations.

60. The concept of a probability distribution of all possible returns applicable to a single event has been discussed earlier. The sum of the products of the returns and their respective probabilities gives a value for the total expectation and this can be equated with an equivalent certain return, which can be called the expected one-year return. If the distribution is symmetrical, and it is assumed to be for this purpose, the expected value is the central one. It is in order, therefore, to replace the range of possible values, each year in the future, by single central values, and to build forward in this way. In practice the expectations contain a large subjective element because their quantification involves, at one or more stages, the selection of a value which is thought as likely to prove too high, in the event, as too low.

61. Since central values are used there is no indication of the associated variance, or degree of risk, and when dealing with expectations this absence must be borne in mind otherwise there might be a tendency to take higher risks than are justified by the size of the reserves. Earlier in this paper an attempt was made to relate the variance of the distribution to the level of the mean return, thus enabling a future variance estimate to be put alongside the expected return. The past record of one-year returns provides almost the only source of data for risk measurement and little more remains to be said on this aspect of financial planning. On the other hand there is a great deal of information available to assist in producing a sequence of expected returns, leading off from the past record into the future. Thus, although past returns were analysed fairly closely for signs

of a persistent trend in variance, it would be very unwise to rely only on them for estimates of future returns.

62. As each year is completed, the previous expectation of that year is replaced in the record by the single achieved value taken from the range of possible values, and a new train of expectations conceived. If the divergence of the actual from expected is of a minor nature and if no new factors have appeared, the following year's new expectation may well bring the projected sequence of returns back on to the same line as before but, more likely, new information would make necessary a revision of long-term expectations. In view of the wide range of possible results it is clear that few expectations will prove, after a year has elapsed, to be an accurate forecast, although, when a large number of results are compared with expectations, trends in the relationship between actual and expected mean values become apparent and the judgment of the operator can be appraised.

63. The practice of estimating future performance and then comparing actual results has recognized value in financial planning, and the system described enables investments to be brought into this field on a basis which includes the whole investment return in the year in which it is earned, and enables the return from this part of the business to be compared with returns from other areas. As the investment return is derived from securities held during the year it emphasizes the point that the decision to continue to hold an investment is an important one and is the responsibility of the managers in that year. Historic costs tend to confuse the issue, and apart from tax implications, are best ignored.

Investment selection

64. Assuming that the degree of risk is acceptable and that account has been taken of the nature of the liabilities, the investment problem is one of selecting and holding investments to give the portfolio the maximum expected yield at market price. Used in this sense the expected yield is the long-term compound interest rate which equates present value with the discounted value of all possible future receipts, allowing for their respective probabilities. The concept is assumed to apply to inactive holdings. The problem, put this way, appears formidable but it becomes more manageable if the equivalent certain one-year return is substituted for the range of possible receipts in the year, and the level compound interest expected yield is replaced by the variable stream of expected returns, on the lines demonstrated in the formulae in § 16. Maximizing the expected yield is then seen to be a matter of maximizing expected one-year returns, step by step, subject, of course, to restrictions upon marketability. As shown in § 13, the return in the first year usually carries most weight in a redemption yield, and the same applies to the first year's expected return in an expected yield.

65. This view, reminiscent of Pepper's 'dynamic approach' to gilt-edged investments, may be criticized on the grounds that it is dangerous to take a short-term view of long-term business. There is this danger, certainly, but in practice it is necessary to take a long view in arriving at short-term expectations, since the market itself looks ahead, and to forecast a price twelve months on necessitates taking a long-term view from that point.

66. Characteristics of the different classes have been fully described elsewhere, and in this paper discussion of investment selection will be confined to techniques for breaking down long-term yields into one-year expected returns, with the object of facilitating the building up of portfolios suited to the manager's requirements.

67. For dated fixed-interest securities the obvious method is to make a prediction of the market yield basis for as far ahead as is thought necessary —say 10 years—and use it to calculate market values allowing for the reducing outstanding term. Lack of accuracy in estimating the more distant values will not matter greatly as the first year is the important one and the other estimates will be revised annually. For long-dated and irredeemable securities the pattern of the returns may give a quite different impression from that given by the redemption yields, depending upon the long-term view taken of the course of interest rates. Market yields have in the past formed patterns which can be explained in economic terms and these could form a basis for forecasting. If thought necessary, some allowance can be made for the possibility of default on the one hand, and, on the other, the chance of improved terms following capital reorganisation. At the foot of Table 6 expected one-year returns for $3\frac{1}{2}\%$ War Loan have been reproduced, based on the subjective estimate that yields will continue to rise by about 1% every five years and that the five-year economic cycle will continue.

68. Real estate long-term returns involve prediction of both rents and property values at review dates, but once this has been done a flow of one-year expected returns can be brought out in the same way as for fixed interest, making allowance for expected changes in the market yield basis. In the case of short leasehold properties, the rent would probably be high in relation to market value, making up for the annual depreciation.

69. Ordinary shares, as a class, are the most difficult to handle. Projection of past trends is especially dangerous as a negative correlation between returns over successive periods in the past has often been pointed out, and the high average values recorded over the last decade are unlikely to be repeated. Techniques for selecting shares using a system of ranking by 'cheapness' are not well suited to the prediction of either long- or short-term returns, although it may be found that a portfolio operating on this basis will, over time, produce results sufficiently consistent for prediction purposes.

70. The most promising approach to the treatment of shares is, it is

suggested, by way of projecting the company's equity 'profit on capital employed' ratio in order to arrive at estimates of future earnings and dividends. Marshall[9] discussed the overall stability of this rate and the author[10] used long-term profitability trends to arrive at estimates of expected yield. The formula used for this purpose, viz. expected yield = dividend yield + growth rate, where the growth rate equals the retained proportion of the equivalent level future net equity profitability, is quite convenient for use in producing a flow of expected returns: a projected market expected yield basis can be applied, through the formula, to each year-end in the future (in much the same way as the projected market fixed-interest yield was applied), in order to bring out, via the growth rate and dividend yield, an expected market price. This technique was applied in the early part of 1968 to the shares used in Table 2 to produce the expected returns shown, purely as an example, in Table 6. The same subjective estimate of a rising fixed-interest yield pattern was used as in the case of 3½% War Loan, in order to maintain consistency between ordinary share market expected yield levels and fixed-interest yield levels.

71. Economic cycles are expected to continue and, although precise advance timing of peaks and troughs is impossible, a cyclical effect was introduced into the estimates of company results and market price levels in order to observe the effect on central expectations. The overall picture is one of quite wide variations from share to share both in particular years and in their ten-year means. The 30-share means taken year by year show the cyclical effect, with three years out of the ten giving negative mean returns. However, the overall mean of 8·0% shows a useful margin over the expected fixed-interest performance. Being central expectations, the values do not indicate the range of possible results or the degree of risk, but an estimate can be obtained by use of the formula $\sigma = 0{\cdot}3\bar{x} + 16$. For example, if the overall mean of the 'A' shares at 8·6% is taken as a typical value, the associated standard deviation would be 18·6 and the degree of risk 0·32.

72. The need to take a long view in order to arrive at short-term expectations has already been mentioned, and this may be regarded as producing a stabilizing effect in the market, as a result of investors whose actions are directed towards maximizing long-term expectations being attracted towards stocks which appear temporarily depressed and likely to benefit from a subsequent recovery. In the process of drawing attention towards under-valued stocks, the long view also tends to point out the stocks inversely correlated with the market, i.e. possessing the greatest stabilizing power. This tendency for the market to try to put shares on the same expected yield basis has been used in producing the expected returns in Table 6. It may be likened to the other economic force (allowed for in making the company earnings projections), which directs risk capital towards the areas of highest profitability thus tending to stabilize swings in the profit/capital employed ratio.

Table 6. *Subjective estimates of one-year expected returns*

	1968	1969	1970	1971	1972	1973	1974	1975	1976	1977	10 year mean
Capital goods group											
A_1	43	21	—1	—1	15	20	21	3	—1	16	13·6
A_2	13	4	—4	—1	13	19	13	—1	0	4	6·0
A_3	29	15	—7	—8	16	23	21	—1	—7	2	8·3
A_4	29	8	—6	—4	11	16	12	—3	—4	12	7·1
A_5	24	7	—5	—4	9	22	17	0	—2	2	7·0
A_6	42	19	—5	—5	15	18	16	—1	—3	5	10·1
A_7	37	28	0	—2	10	23	23	3	—2	4	12·4
A_8	23	11	—5	—3	16	17	13	0	—2	12	8·2
A_9	12	—5	—5	1	20	20	7	—4	—2	17	6·1
A_{10}	12	12	—4	—4	7	20	20	2	—2	4	6·7
Mean	26·4	12·0	—4·2	—3.1	13·2	19·8	16·3	—0·2	—2·5	7·8	8·6
Consumer goods group											
B_1	29	26	2	3	11	26	24	3	3	10	13·7
B_2	39	21	—2	—5	12	31	7	0	0	9	11·2
B_3	18	13	0	—2	9	20	18	2	—5	18	9·1
B_4	16	6	—3	—2	9	18	12	2	—2	8	6·4
B_5	31	8	—5	—3	10	19	17	0	—4	12	8·5
B_6	19	4	—6	—5	5	19	15	—2	—2	13	6·0
B_7	16	17	—4	1	17	22	22	1	—4	9	9·7
B_8	27	19	—2	—1	14	24	23	2	—3	17	12·0
B_9	—1	7	—9	—8	8	22	—4	—2	0	9	2·2
B_{10}	3	10	—4	—4	6	17	15	6	—3	7	5·3
Mean	19·7	13·1	—3·3	—2·6	10·1	21·8	14·9	1·2	—2·0	11·2	8·4
Misc. industries group											
C_1	30	—7	—10	—1	12	11	4	—5	—1	11	4·4
C_2	12	—8	—9	—4	10	15	13	—4	—5	2	2·2
C_3	25	—3	—4	—1	14	16	12	0	—5	10	6·4
C_4	33	4	—5	—5	10	21	22	—3	—4	4	7·7
C_5	5	—2	—4	—3	12	18	17	0	—3	6	4·6
C_6	45	14	—5	—6	9	20	22	0	—6	9	10·2
C_7	1	—4	—11	—7	8	17	19	7	—5	3	2·8
C_8	30	18	—2	—3	8	21	19	—3	—4	9	9·3
C_9	3	22	3	—1	18	27	26	8	—2	16	12·0
C_{10}	16	16	14	—1	2	17	19	3	1	15	10·2
Mean	20·0	5·0	—3·3	—3·2	10·3	18·3	17·3	·3	—3·4	8·5	7·0
30-share mean	22·0	10·0	—3·6	—3·0	11·2	20·0	16·2	·4	—2·6	9·2	8·0
3½% War Loan	1·4	7·3	—1·5	5·6	12·0	9·2	·6	1·7	6·8	12·6	5·6

Random walks in stock prices

73. There is a body of statistical evidence[11] which appears to show that in the short term (less than one year) stock prices do not display any predictable pattern, and past prices are therefore no help in forecasting prices in the near future. According to the random walk theory the most likely short-term movement is always a sideways one, being independent of the actual price level. In terms of probability this theory would mean that the probability distribution of possible future price changes is always symmetrical about the zero line. It is necessary to see how this theory fits into the proposition that past records of one-year returns can be used in making estimates of future experience.

74. In the first place, the theory is applied in the form stated above only to short-term movements and it is admitted that in the long term (i.e. periods of one year or more) the share price increments do have an upward bias. However, investment analysts must attempt to predict short-term price movements, which brings one to the second important point; this is the fact that the statistics referred to above related only to the past record of price movements and did not attempt to take account of additional background knowledge regarding the company and the economic scene, which is available to the investor. To calculate a probability distribution from restricted data is, of course, to court failure. The theorists explain actual price movements in terms of an almost instantaneous adjustment to a new intrinsic value following upon new information available to the market, thus giving to operators credit for being able to recognize cheap and dear shares (and hence to predict short-term movements given new information). This is difficult to reconcile with the conclusion that cheapness and dearness cannot be recognized at any point of time.

75. An alternative explanation is that the market is a long way from being as sophisticated as the random walk theory suggests, and may in fact be quite illogical within a fairly wide band in which prices move as a result of buying and selling decisions made for various reasons, many themselves of a random nature. In other words, the market does not instantaneously take full account of all new information in the share price but, instead, the price takes some time to adjust itself and in the process may move from one side of its intrinsic value to the other, because nobody knows precisely what the right price should be. There would, however, always be outer areas in which shares came to look distinctly cheap or dear by the various, often crude, yardsticks employed, soon leading to a reaction in price. There is so much randomness both in the decisions of investors and in the decisions which affect company results that it will never be possible consistently to forecast correctly the next price movement, but it is nevertheless still reasonable to suggest that there is always in the background a real probability distribution of price movements to any chosen future date, and the shape and mean value of this distribution, far from remaining symmetrical about zero, changes with

the price level. According to this proposition, once prices have moved into the outer area, the mean of the distribution will shift progressively making a reaction more and more likely with each further movement.

76. The mean and standard deviation of these 'real' probability distributions can, of course, only be estimated. Some skilled operators may make consistently better estimates than the average. Their ability, depending as it does upon the complex process of share evaluation, can only be tested by publication of such estimates. The current emphasis on performance will no doubt lead to production of evidence of this nature, and the effort given to share selection will be seen to be justified.

Portfolio returns

77. Consolidation of expected returns for individual investments into portfolio returns is a simple matter of weighting and adding. Production of separate expectations for all investments would probably be quite uneconomic, but there seems to be a good case for applying long-term expectations, based on sample findings, to large classes of investments for the purpose of financial planning over long periods. In doing this, the expected returns and their associated variance, are treated as derived from an inactive portfolio, thus enabling account to be taken of the nature of the liabilities which, for the purpose of valuations, are also treated as inactive. A theoretical investment policy, as applied to a mixed portfolio held to secure long-term liabilities, first requires a ruling upon the permitted degree of overall risk, and then subject to this and to any other restraints imposed by management, the portfolio is continuously adjusted by purchases of investments calculated to raise the portfolio long-term expected yield (and therefore likely to perform well in the short term) while disposing of holdings with low long-term expected yield and poor short-term prospects. Regular valuations and adjustments of income expectations will be necessary in order to keep a watch on the overall position which will, of course, also be affected by changes taking place in long-term liabilities.

PART III.

THE OVERALL DEGREE OF RISK INCLUDING LIABILITIES

78. In Parts I and II attention was concentrated upon investments. Results were analysed using one-year returns in order to derive a measure of risk and a means of applying it to future expected returns. It was accepted that investment risks are not the only ones which have to be covered by overall resources of risk capital and that the nature of the liabilities must be borne in mind when formulating policy and making investment selections. In this Part an attempt will be made to extend the one-year return concept to liability valuations and to the measure of liability returns and risks. Some possible implications for those manage-

ment decisions which affect the adequacy of the reserves will be discussed in a general way against a life office background, although the same principles can be applied to other funds holding assets to meet future liabilities.

Liabilities as 'negative investments'

79. The present value of an actuarial liability is usually represented as a compound interest discount of expected future net outgo at a constant interest rate. In this respect it is not very different from the market value of an asset, which can be equated with expectations of asset-income, discounted at a constant rate, viz. the long-term expected yield.

80. If the market value of an asset can usefully be regarded also as a discount, at the variable one-year return rates, of expected future asset-income it may be worth considering whether the present value of a liability can with advantage be similarly regarded. Certainly a liability to a borrower is an asset to the lender and has a market value. Insurance policies can be bought and sold and, although the market is small, life policies can be valued in relation to quoted investments, using yields based on the market level. Thus, against the background of a stock market subject to changes in interest rates, the market value of a policy can be thought of as fluctuating in much the same way as does a long-term investment, producing a sequence of one-year returns. To a policyholder, the returns, made up of increase in market value less the gross annual premium, would often be negative, but ending with a large positive value in the event of a death claim. That which is positive to the policyholder is negative to the life office and policies can therefore be regarded as negative assets with their own expected returns and degrees of risk.

81. Clearly, for a single policy still in force, the past record will not give a proper measure of mean return or degree of risk, but when a large number of policies are grouped, so that the mortality risk is more effectively spread, one-year returns, being the premiums received less claims, expenses and increase in liabilities, take on more meaning and there is no difficulty, using standard valuation methods, in producing one-year return rates by class (assuming a break-down of expenses by class to be available) enabling the corresponding mean rates to be calculated and, in addition, the variance, degree of risk and the co-variance with the overall investment returns.

82. A display of one-year returns for different classes of policy will permit correlation between returns of separate pairs of classes to be measured and, as with investment classes, the benefits of negative correlation will be seen when two or more classes are combined. When all classes are consolidated the degree of stability of the liability returns will be seen. A great part of their fluctuation will be attributable to changes in the market yield basis, so that (allowing for the change in sign) there will be negative correlation with the one-year return rates derived from the assets, with its well-known immunizing effect. It is suggested that this presenta-

tion will give a more complete picture of the need for, and the success of, matching policy than would a mere calculation and comparison of the mean terms of assets and liabilities.

83. The final step in the consolidation is the deduction of end-year liabilities from end-year assets and one-year liability returns from one-year asset returns to produce a statement showing the net values from year to year, the overall annual returns and the overall rates of one-year return, calculated in relation to the net value, i.e. the free reserves. This net value is the portion of the assets not needed to meet expected liabilities, and it has also been called 'surplus funds' and 'the estate'. To some, the terms 'equity' or 'risk capital' might be preferred as being more descriptive of the main function of this net value, which is to absorb residual profits and act as a buffer against adverse fluctuations in experience, thus protecting against insolvency.

84. Skerman [12] described this main function and discussed the determination of net value. He came down in favour of a bonus reserve valuation of liabilities at a market interest rate, varying from year to year, combined with a market valuation of assets, thus, in effect, isolating overall one-year returns. In his list of functions he emphasized the protection afforded against adverse contingencies, but felt that it was 'not possible to assess closely what reserve should be held'. It is hoped that the use of one-year returns to estimate degrees of risk will enable progress to be made on this point.

85. If the net value is the equity (risk capital employed by the fund), the increase in the equity in one year, as before any bonus increase or other distribution, is the true equity earnings in the year and the rate of overall one-year return is the equity profitability. When bonus rates are increased beyond those assumed in the valuation the effect is to appropriate part of the risk capital, making the fund more highly geared and therefore more exposed to fluctuations in experience.

Market value of liabilities

86. It appears, therefore, that there is scope for using one-year returns on the liability side with the same advantages to management as are evident on the asset side, viz. performance can be recorded from year to year for separate classes and for whole portfolios in a way which makes comparison easy and which enables the complete return each year to be examined in isolation; the degree of risk can be quantified and the beneficial effect of combining unlike classes can be seen and measured.

87. In practice there are difficulties, not least of which being the selection of the rate of interest to be used in the liability valuation. It is the intention to establish a market value, but the liabilities can be likened to a large block of unquoted negative investments which should be valued, it may be thought, at rates appropriate to their respective risks. Further, the concept of a flow of variable expected one-year rates must not be

forgotten. The simple answer is to use the overall long-term expected yield appropriate to the whole asset portfolio at its market price, which yield is, of course, the single compound rate equivalent to the flow of expected variable rates of one-year return. This choice may be justified on the grounds that although the liabilities are exposed to an investment risk, the interest element in the valuation is secured by a surplus of assets and, unless the assets and liabilities are grossly mismatched, the investment returns required by the liability valuation are already secured in the asset-income expectation.

88. The one-year return approach, as far as the valuation of the liabilities is concerned, is similar to that adopted by Skerman, and his analysis of the return according to source needs no adjustment. In particular, it will be normal practice to value liabilities each year at two rates of interest, viz. the present and the previous long-term expected yield rates, in order to isolate the effect of the change in the interest basis.

89. Springbett[13] also valued assets and liabilities at the same rate of interest but preferred a notional rate, unchanged over a period of years, chosen to represent the rate at which future surplus income could be invested. This makes for convenience on the liability side but complicates the asset valuation. Further, not only might it result in concealed losses, but it eliminates any possibility of estimating a degree of risk from variations in overall experience.

90. The emerging overall one-year return rates will show considerable variations from year to year, accentuated by the gearing effect of using the net value, but in the same way that investment portfolio rates of return can be averaged and their variance used to estimate an overall degree of risk, so can the overall rates for the equity return be used to estimate an overall degree of risk to the equity. These overall returns will, of course, embrace profit margins in the premiums and will show the effect of fluctuations in mortality experience as well as investment experience.

Investment of risk capital

91. Skerman suggested that the portion of the risk capital held to protect against other than investment losses should be kept in risk-free investments, but adoption of that policy would seem to reduce the return unnecessarily when there is probably already sufficient risk capital to ensure that the risk of total loss lies well below the maximum acceptable level. Springbett, on the other hand, favoured the investment of the estate in fixed-interest perpetuities. These solutions both appear to be in conflict with the general investment objective of maximizing the expected yield, subject to an acceptable degree of risk, and it is suggested that it is better to treat the assets as a whole for investment purposes, with the risk capital invested in a cross-section of the portfolio. The overall investment expectations then need no adjustment before use as a basis for valuation and premium rates.

Components of the overall risk

92. When discussing the functions of the risk capital, Skerman distinguished between temporary fluctuations in experience as regards interest, mortality, expenses and taxation, and permanent changes, the latter requiring a change in valuation assumptions. A market value of assets used to determine an expected yield for the portfolio as described earlier, does have the effect of combining the temporary with the permanent interest changes each year, and if realistic assumptions are made on other counts the permanent and temporary will be combined also, and both will contribute to the annual return and to the measure of overall risk.

93. The record of one-year liability returns was assumed to have been built up from returns recorded for separate classes of policy and, after combining with investment returns, an overall degree of risk calculated for the equity of the fund. This would apply to the more distant years, assuming inactive portfolios, but more importantly to the next year, for which the risks will receive special examination. For this purpose the expected return for the next year will be broken down into its components of interest, mortality and expenses (treating taxation separately, if desired). A further main source of profit will be net new business, after deducting withdrawals. Although it has been suggested that the past overall record could be used for future estimates it is likely that in practice more accuracy could be attained from separate estimates of the components of the expected return, and their separate variances. Investment expectations have already been discussed and the following notes deal with some of the implications of the one-year approach to the other components.

Mortality risks

94. Beard[14] has discussed the distribution of possible claims on one-year temporary insurances, the reduction of the risk by reinsurance, the need for risk capital and for a profit margin in the premium rate. The one-year approach has the effect of reducing all life insurance to a sequence of one-year insurances and Beard's work is therefore relevant. If expected mortality rates can be estimated fairly accurately, as they can, then for a group of lives each of the same age and insured for the same amount the probability distribution (binomial) of possible total claims in a given year is known and can be defined in terms of the expected mean of the distribution and its variance. Building up to include all ages and all sums insured, the variance of the total expected claims in the next year can be estimated. The liability valuation will provide for average experience so the contribution to the return will be derived from the divergence from the expected. If there is a margin in the mortality assumption the expected contribution will be positive and the chance of mortality loss will decrease with increasing volume of business. In any case the variance of the expected contribution to the return should not be greatly affected by a

margin in the assumption and if this variance is calculated from assumed rates it will be adequate for present purposes.

Expenses risk

95. It should be possible to make a fairly close estimate of the next year's expenses so that, having made due allowance in the bonus reserve valuation, the contribution from expenses on existing business to the expected return, and to the variance will both be relatively small.

New business contribution

96. The investments and the liability valuations both having been made on the basis of inactive portfolios, one would expect to see very little by way of contribution to future equity returns from existing policies, since any profit margins in the premiums would already be capitalized in the value of the equity (although this does not mean that they are available for distribution), leaving the investment earnings on the equity itself to provide the major contribution. Compared with past years the expected rate would look low because new business profits would not have been included, and in estimating the next overall return allowance must be made for these. This item is subject to chance variations to some extent, although it is more under control than is mortality or, in a large unmarketable portfolio, investment returns. There is no compulsion to write unprofitable business, so the possible variations in the return from new business should leave most of the distribution on the positive side, and generally this source should have the effect of reducing the risk overall.

97. It may appear, from this point of view, that no limit need be put on the intake of profitable new business, but this would only be the case if its contribution to equity earnings was sufficient to build up the equity to the level needed to cover extra mortality and investment risks.

98. The idea of taking credit for profit margins in premiums not yet received (but after the contract has been signed) may be thought at first to be wildly incautious, but in justification the point must be made that, so long as they are not actually distributed, the valuation does no more than recognize their existence. The risk analysis approach may, in fact, inspire as much caution as did traditional methods, while imparting more knowledge of the true position. In particular, the importance of profitable new business in maintaining the return on the equity and in building up risk capital will be recognized.

Bonus decisions

99. The liability valuation provides for bonus continuing at current rates and an increase in the rate will not only reduce the available risk capital, but also the margin for profit on new participating business.

Meanwhile the variance in the expected overall return will remain, arising as it does mainly on the mortality and investment components (and since these two risks can be regarded as independent the overall variance will be the sum of the separate variances). Thus, at the point of bonus decision, various possible combinations of new business profit and reduced risk capital can be tested against whatever overall degree of risk is regarded as the maximum which can be accepted. In converting overall expected returns and variance into a degree of risk it is suggested once again that the Normal assumption and the use of Table 1 will give sufficiently accurate results. In justifying this, it can be held that the distribution of possible divergences of actual claims from expected will approximate to Normal, being based on binomial distributions for large numbers, each age-group having a small positive mean in its distribution. Thus the mortality and investment risks can be combined for this purpose. The contribution of new business profit to the return, being under control, can conveniently be added direct to the risk capital, when using Table 1, as an alternative to inclusion in the expected mean return.

Expanding Funds

100. Redington[15] drew attention to the impossibility, due to the absence of suitable investments, of completely immunizing many rapidly growing funds, and to the risk that a period of falling interest rates could conceivably render such funds insolvent. This would happen if the investments, having a shorter mean term than the liabilities, appreciated less in value and in consequence the risk capital, or estate, came to be eroded. This risk can be dealt with on a year to year basis, as already described, but as a long period of falling interest rates might occur again, as it has in the past, it is desirable to consider whether the risk capital might be used as an automatic safeguard.

101. The risk capital is assumed to be a proportion of each and every investment and will therefore appreciate, when interest rates fall, in the same way as the portion of assets held against liabilities. Thus for every set of liabilities and every mean asset term there will be an optimum amount of risk capital such as to immunize even the most rapidly expanding fund against the effect of changing interest rates. For this purpose the interest rate is the expected yield of the fund, which will probably include equity classes of investment, and the changes in interest rate with which we are concerned are regarded as taking place independently of any changes in expectation regarding future asset income.

102. Suppose, for example, that the fund has mean liability term of sixty years, so that 1% fall in the market rate of interest would increase the value by about 77·5%. Assets with a mean term of, say, fifty years, including equity type investments, would only increase by about 61·5%, but risk capital amounting to 10% of the liabilities would provide immunization against this risk. Clearly, as an expanding fund matures, its mean

liability term is likely to become shorter and its risk capital requirement, as a proportion of total assets, is likely to decline. Ultimately, when mean asset and liability terms are matched, risk capital will no longer be needed for this particular purpose, although the requirement to provide against other risks will remain.

Profit margins in premium rates

103. In conditions of expanding business, both mortality and investment risks increase in pace with the expanding fund and to provide comparable security to the policyholder, reserves of risk capital must grow at nearly the same rate. They can expand by rolling up their own investment earnings, but net of tax this source may produce only about 5% per annum and funds subject to more rapid expansion must find new sources in the form of retained profit from the insurance business. In these circumstances it becomes necessary to have a profit margin in the premium rates and it is recognized that these are, traditionally, concealed in the mortality and interest assumptions.

104. If the valuation interest rate, based on the expected investment return, is above that assumed in the premium rate, the margin will be released and capitalized, as already mentioned, to supplement the equity of the fund. Equally, if a rise in security prices leads to a reduction in the valuation rate, some or all of the margin may be returned. Thus, the one-year approach leads on to the concept of a variable profit margin in the premiums.

105. This concept can be regarded as a means of reconciling Springbett's desire to value at an interest rate which represented a future new investment rate (and therefore a likely future premium interest assumption) with the advantages of a variable market rate. On the one hand, it can be argued that the current expected yield rate on the assets is in fact the best estimate of the future new investment rate since it represents both the current investment policy, i.e. distribution of investments by class, and the balance of views in the market of the right return for the various investment risks, i.e. yields which are, in the future, as likely to go up as down. On the other hand, the current expected yield can also be regarded as the rate assumed in the 'in force' premiums, the profit margin having adjusted itself to the new conditions.

Conclusion

106. It was suggested earlier that progress towards measurement of the degree of risk has in the past been relatively slow. It is hoped that this paper will stimulate new thought on this important subject and perhaps point the way to further advances. This could lead, in turn, to improved management efficiency in the use of limited resources to provide a service to policyholders (the efficiency being measured in terms of the return on risk capital employed at whatever level of risk is deemed acceptable)

resulting in better benefits to some and, to others, less risk of their company becoming insolvent.

107. The results of the risk analysis and the year by year study of overall performance would be intended as an aid to management and not normally for publication. It is not suggested that, if adopted, these methods would necessarily lead to any marked change in policy, although it is to be hoped that they would enable future decisions in areas of uncertainty to be taken with a higher degree of confidence than otherwise.

108. The views expressed in this paper are personal ones, but I am indebted to many colleagues for help and advice.

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ABSTRACT OF THE DISCUSSION

The author, in presenting his paper, said that, although he did not know of any developments in Great Britain since he had finished the paper, there was a steady flow of work on investment risk in America. He could not hope to read all the references on the subject but, in the latest edition of the American *Financial Analysts Journal*, there was an article entitled 'Yield-risk performance measurements' by Prof. Soldofsky, Professor of Finance at an American university, which gave some idea of developments in America. The article defined a concept called 'Annual yield' or 'Annual rate of return' which was in fact what the author had called 'one-year returns'. However, on the question of risk, Prof. Soldofsky had used the standard deviation as a measure of risk and the author was slightly critical of that. He felt it was an unsatisfactory definition compared with his 'chance of loss' because it was using risk in a way which was rather foreign to insurance ideas. Using the standard deviation, as they seemed to be doing in America, it was possible to have a risky investment where a big profit was almost certain, and what might be called a safe investment, where a loss was most likely. That seemed unsatisfactory.

The paper was on management accounting and, although he had nothing against historic cost accounting, for it was obviously necessary to record what had happened in the past, his own feeling was that really two sets of accounts were needed, one in historic cost and another in market values which could be used for management purposes and looking into the future.

He had put a few examples of expectations in the paper but there was always the fear that some non-actuarial readers might think they represented forecasts. To him, an expectation was something different from a forecast in that, when the event happened and it was different from the expectation, it could still be said that the expectation had been right! It was like telling somebody he had an expectation of life of twenty years; if that person walked out and fell under a bus, it could still be said that the statement was quite correct: the person was just unlucky.

He felt slightly guilty that, having described 'insolvency' as a case of losing risk capital, he had talked about life offices and, having described certain surpluses as risk capital, had said that the chance of losing the risk capital could be calculated. A life office normally would not be insolvent if it lost its estate (or free reserves) because it still had the ability to reduce its bonus rate.

Mr C. Brill, in opening the discussion, said that the author had very properly taken as his opening text the aims of investment policy of a managed fund. Those aims had been defined in various ways in the past but could be crystallized as being the maximization of expected yield having regard to the constraints imposed by the nature and incidence of the liabilities of the fund. The term 'maximization of expected yield' implied the taking of risks which, in the context of the paper, was related to (but not equated to) the variability of investment returns. The constraints imposed by the liabilities acted to dampen down the risks taken in order to comply with statutory and internal requirements of solvency and, in the case of a life office, gradualism in connexion with the emergence of surplus. Thus there were two conflicting forces influencing the financial management of the fund and it was the size and flexibility of the estate or free reserves which governed the reconciliation of the two forces.

The science of the measurement of liability risks had been fairly fully developed.

The too s were possessed in the form of mortality and other decremental tables and those probabilities had been combined with the liability payments to provide the expected liabilities, which could be easily costed. A parallel treatment for investment returns had received scant attention and that ground had lain almost barren up to ten years previously when Markowitz and other investment statisticians (mainly American) had begun sowing seeds which seemed likely to take root rapidly.

The author had related risk to the variability of investment returns but had chosen as his variable neither the market value nor the expected redemption yield, but the one-year yield combining cash income and capital appreciation. Furthermore, he had introduced a much more precise definition of risk by considering not the one-year return itself but that return multiplied by its probability density (i.e. the expectation) and had taken as his measurement of risk, the expectation less than zero divided by the total expectation (i.e. the risk of loss). However, he much preferred the author's subsequent variation of that definition in which it was not the chance of loss that was measured but the chance of the return falling short of a control yardstick. Two interesting possibilities flowed from that adaptation of the risk measurement definition. One, which the author had mentioned, provided a quantitative way of relating the aggregate investment risk to the amount of the estate and thence enabled a figure to be put on the risk of insolvency. The other placed a performance yardstick on the control return, so that what was measured was the chance that the one-year return fell short of X where X, the control yardstick, could be, for example, the return available on twelve-month deposits or similar virtually riskless securities. In that way, the term 'risk premium' of equities over gilts, or of debentures over gilts, acquired a more precise numerical meaning. Alternatively, the yardstick X could be the one-year return on a given equity market index: it was then possible to isolate the risk taken by investing in the equity market, and the return gained thereby, from the additional risk taken by individual selection and active timing within the equity market, and the return gained from that additional risk activity. It was the relationship between the additional return produced by active management and the additional risk taken, over and above the equity market risk that was the true test of management performance.

The author had neatly put up arguments against the use of one-year returns only to knock them down again very convincingly. The opener agreed that one-year returns gave a much clearer picture of the anatomy of an investment return than did the overall compound interest yield. The one-year return concept in effect replaced an investment held over n years by n separate yearly investments and, in doing so, had the great merit of focusing attention on the fact that the negative act of continuing to hold an investment itself constituted as much a dynamic management decision as selling or buying it, since an investment held from one year to the next was notionally being sold and re-purchased instantaneously at each accounting date. If, therefore, investment management continually concentrated on maximizing the return over the ensuing year, it was obviating the need to worry about the long-term, except in so far as it was concerned about the market price in one year's time which, as the author had said, would reflect the market's view of the long-term.

By operating on the one-year return, the author had attempted to quantify in a single measure a concept which had formerly been assessed only subjectively or intuitively. He had, however, used a global or aggregate approach and it was that that was likely to engender some criticism. The one-year return was made up of price change and cash income and each of those components was subject to a combination of a variety of risks. For example, the price change of an individual security could be affected by a change of status of the individual security *vis-à-vis* the market or (among other things)

by a change of status of the equity market as a whole. What was required ideally was an investigation of one-year return changes broken down by class or source of risk. It might be, for example, that the risk attributable to loss of status of the equity market was the risk most feared and therefore the risk that it was desired to minimize most of all. It was appreciated, however, that such an analysis of the change in the one-year return would reintroduce many of the subjective elements of assessment that the author had wished to minimize and, on balance, the use of the global one-year return was probably unavoidable in practice.

The author's postulation of a theoretical statistical model was logical and, indeed, vital to the practical development of his whole argument. The one-year return was admitted to be far from random and also far from independent, both in regard to time sequence and the observations in a single year. However, the actual distribution about the mean of one-year returns on the thirty shares described did show an impressive closeness to the Normal distribution having the same mean and standard deviation albeit with a positively skewed shape. What was important, however, was whether the shape of the actual distribution over the period of years chosen was influenced by events over that period which were in any way exceptional, to the extent that the results were biased in a particular direction and therefore rendered dangerous for use as a basis for projecting future returns. More precisely, could the sort of linear relationship between the mean and standard deviation deduced by the author from Diagrams 2 and 3 be used with any degree of confidence in order to deduce the standard error, and so the degree of risk, or in projecting future returns? That encroached deeply into the broader question of the extent to which the past was any guide to the future in the field of investment data, a question which was crucial to the practical relevance of Parts I and II of the paper. The author's exercise in curve fitting was impressive but the opener preferred to see data based on a broader period. There had not been time, for example, to assess the full impact of the combination of capital gains tax and corporation tax on the volatility of either ordinary share or fixed interest returns. It was possible that a few more years' experience of the current tax system might alter the volatility of investment returns to the extent of completely nullifying the linear relationship between the mean and standard deviation or, at any rate, of necessitating a change in the parameters of the regression line.

The majority of institutions possessed equity funds numbering several hundred different holdings. If their investment committees had been warming themselves in the knowledge that spread by means of a large number of holdings provided protection and stability, §§ 41 and 42 of the paper would come as a well-timed shock. The author's findings on the extent to which a spread of equity holdings provided increased stability had, of course, a certain support from a general consideration of the nature of the different industries underlying the economy. There was a great deal of interdependence between the prosperity of the industrial and service sectors underlying the equity market and all sectors were increasingly influenced by Government action, although to varying degrees. However, quite apart from the general argument, a considerable amount of analytical work had been done in recent years on the subject, notably by Markowitz, who had shown that as the number of securities in a portfolio was increased the decline in the aggregate variance (reflecting the declining variability of the portfolio's return) became less dependent upon the variances of the individual securities and more dependent on their co-variances (reflecting the interdependence or correlation of the individual securities). In other words, the more correlated the securities of the portfolio were, the more minimal was the reduction of risk beyond a relatively small number of securities. Both in theory and in practice, therefore, unless the constituents of a portfolio

had a small or negative degree of correlation, the added stability afforded by a large spread of holdings was largely illusory.

The author had touched on that aspect only obliquely since he was more interested in maximizing the expected return for a given level of risk (provided the level of risk was quantified) rather than concentrating on reducing risk for its own sake. Ultimately, to maximize the return given the maximum permitted risk or, alternatively, to minimize the risk given the required return (i.e. to achieve an efficient portfolio) was a matter of weighting and selection both within and between classes of security. That, of course, they may have known intuitively but the author had provided a set of tools and a formal approach which could be used to quantify what they were thinking and talking about instead of, as in the past, relying entirely upon an intuitive estimate the basis of which was generally unspoken and unpublished and often lay in the mind of the investment manager only.

In Part III of the paper, the one-year return concept was interwoven through assets and liabilities in a very impressive way, bringing into the weave the problems of bonus policy, free reserves and new business. Such a consistent approach to assets and liabilities was surely right since they were opposite, if unequal, sides of the same coin. It was for this reason that he regarded that part as the most important in the paper. By investigating the relationship between one-year liability returns and one-year asset returns the author basically had refined the Haynes and Kirton matching technique by considering not the emerging cash flows of the assets and liabilities but what could be termed as the true earnings flows, which took into account the change in value of both the assets and liabilities between the beginning and end of the year. That approach threw a fascinating light on the dynamics of a long-term fund and of a life office in particular. The use of the difference between the one-year asset return and the one-year liability return to define profitability had a striking parallel with the accounting of an ordinary commercial company, which struck its profit by deducting from its net cash income an adjustment to allow for the change in the value of the company's assets and stocks over the period. The author had demonstrated that precisely the same technique was applicable to a life office and, indeed, one of the underlying themes of the paper could be sensed to be that the financial problems of a life office, whilst certainly specialized, were not so fundamentally or sacrosanctly different from those of any company which was in business to make a profit as many had conditioned themselves into believing.

He was less happy regarding the practical value of the independent application of one-year liability returns. Their practical use, for example, in comparing the performance of separate classes of business seemed rather doubtful since no action could be taken by the management concerning existing poor performers in the liabilities in the same way as was possible in respect of investments. Any conclusions drawn from such an investigation would in any case be obscured by the existence of policies within the same class having different scales of premiums.

In § 96, the new business adjustment which was made in estimating the next overall return had been assumed to be a positive adjustment on the grounds that almost by definition new business would be profitable, and that, provided its profit contribution was sufficient to cover the additional mortality and investment risk, the net effect of new business was to reduce the overall risk. That was clearly true provided, of course, that realistic valuation bases were being used which capitalized the expected profit from the new business but he wondered whether it was not more valuable in practice to apply that technique in considering the impact of new business not upon the overall risk (referring presumably to the risk of insolvency) but to the trend of profitability of the whole business. If, for example, the bonus earning power of new business were less

than that of the business in force, even excluding the contribution from interest on the estate, each new policy written, although profitable in itself, would gradually dilute the total equity or the total profitability of the whole business. The tracing of overall rates of return by class of business, as suggested by the author, seemed to provide a natural and valuable way in which the potential watering down of the profitability of existing business by new business could be watched for, detected and, where possible, remedied.

The paper had provided a considerable stimulus to the great deal of thought that had been given in actuarial circles over the previous fifteen years on the link between assets and liabilities of a long-term fund and of a life office in particular. It was by now generally accepted that investments and liabilities were not separate compartments, one being run independently by an investment manager who saw his task solely in terms of maximizing the size of the available cake, while the other compartment was concerned purely with safeguarding the liabilities and matching the premiums and bonuses of competitors. The paper had demonstrated unmistakably the totality and unity of a life office and, indeed, of any long-term fund, and had pointed to a consistent approach leading to a measurement of overall profitability and a measurement of overall risk relative to the size of the estate or risk capital. In short, the way was open in principle for the testing of performance, not just of investment, but of the life office management as a whole, a test that would encompass the year-by-year unfolding of both assets and liabilities.

A question which would inevitably be asked related to the extent to which the implementation of the author's techniques was practicable. The array of practical problems was clearly formidable. Among the more obvious were the forecasting technique, the practical relevance of middle market prices for large holdings, the capital gains tax complications and, in the final analysis, someone, or some committee, had to put a figure on the maximum degree of overall risk acceptable. However, the best way to find out whether the whole technique was practicable was to try it out. It was a practical exercise which introduced a discipline, a discipline which insisted on an annual calculation of the estate and a more or less regular ruling upon the permitted degree of overall risk. It was an exercise and a discipline which, if it did nothing else, would concentrate the mind of a life office management wonderfully on the interplay of those elements which made up its business.

Mr K. J. Goodare mentioned that the paper had developed ideas with which he had been toying for some time further than he had ever dreamed possible but that there were a few points on which he had some difference with the author. On the liabilities side, the values of the one-year returns must be largely a function of the valuation basis and it seemed to him that they could tell little which was not implicit in the idea of the analysis of surplus. On the assets side, however, one-year returns could tell them a good deal and he would concentrate on that.

With regard to §§ 10 and 11, he felt sure that any investor would want to calculate net one-year returns on his own portfolio or parts of it. Dividends would be net of income tax and market values would be net of the capital gains tax which would arise in the case of sale at that price. It was not really true that that overstated the tax because the tax was not payable until the investment was sold; after all, if the investor did not sell he did not receive the market price either. Hopefully, he would sell later, receive a higher price and pay more tax but, until then, current market price less tax was surely a most convenient yardstick. Whether he took middle market prices or net proceeds of sale would make little difference after the first year. One result of bringing in capital gains tax was that if two investors, A and B, held the same security but A had unrealized

capital gains while B had not, the security would give higher one-year returns to A, who would thus have less incentive to sell, which seemed very reasonable.

In § 33, the author had suggested taking logarithms of the one-year rates but had dismissed it as unsuitable. For some purposes, he felt it was quite suitable. In considering the performance of a single share over a period of years, taking the arithmetic mean of the one-year returns overstated the true yield. In Table 2, the true mean yield on each security (assuming all dividends to be reinvested) was the geometric mean of the ten one-year returns for the ten years, which was always less than the arithmetic mean. For 3½% War Loan, the least variable, the geometric mean was 3·6% against the arithmetic mean of 3·9%. For Security C_7, the most variable, the geometric mean was 26·2% against 34·4%. It might be reasonable to calculate, for each security, the mean and standard deviation of the natural logarithms of the $(1+r)$s for various years, and that mean logarithm would give the required geometric mean yield.

With regard to the criteria used for selecting investments, the first aim of H. G. Clarke as quoted in § 4, was 'to maximize the expected yield with the minimum of error, having regard to the nature and incidence of the liabilities', though it was pointed out that maximum return and minimum risk were conflicting requirements. Strictly, of course, it was impossible to optimize two functions at once and that was a fundamental dilemma. The same dilemma was encountered by Kennedy and himself in a joint paper to the Eighteenth International Congress on Reassurance Strategies; in effect, how could the 'best' of a number of probability distributions be chosen when the distributions with the more desirable mean values tended to have large standard deviations? Their solution had been to find the one with the optimum mean for a given standard deviation although there was another possibility they had contemplated, namely, to multiply each distribution by a 'utility function' which suitably expressed the fact that the more money there was, the less important was acquiring, or losing, a marginal pound. That was an old idea in economics but strangely seemed to have been disregarded by actuaries. The principle was simple. A suitable utility function in the form of a system of weights which gave increasing weight to losses and less weight to gains had to be selected and multiplied by each of the probability distributions under consideration, and the product with the greatest expected utility found. They had gone no further than that; the mathematics seem to be intractable and they saw no hope of finding the 'best' distribution out of an infinity of possibilities. On the other hand, Kennedy and Howroyd had shown in principle how to find the distribution with the greatest mean for a given standard deviation in *J.S.S.* 1956, **13**, 260. However, the concept of the maximum expected utility expressed more accurately what was really wanted and he hoped that a mathematician could tell him a reasonable utility function which could be multiplied by, say, a Normal distribution without becoming unmanageable. Meanwhile, a simple way of ranking investments, or groups of investments, in order of desirability might be to use as a criterion the mean yield in the recent past less some constant multiple of the standard deviation. That was roughly equivalent to judging investments or groups of investments on the basis of that yield which had an $x\%$ chance of being exceeded in future. A diversified portfolio would appear more desirable than a single security, as it obviously should, because of its lower standard deviation. It was a rough criterion and he did not propose to suggest suitable values for the constants, but at least it was one solution of the perpetual dilemma of trying to optimize simultaneously two characteristics which were fundamentally irreconcilable.

Mr M. McIvor had little doubt that there would be further occasions on which discussions of risk would take place as it was a subject about which a great volume had

been spoken but little written in the United Kingdom. Risk was used in the paper in terms of financial loss but that covered a variety of sins. Many times in the past cash had been referred to as a riskless investment but there were many people who no longer shared that view, at least according to the newspapers, and so he thought it was important that, in any definition of risk for investment purposes, it should be emphasized that cash would be an acceptable form of payment at the termination of the investment. It was perhaps fortunate that life offices had been able to express both assets and liabilities in money terms.

Even though the past returns from equity investment could not be disputed, he had grave doubts about using those returns to estimate the future. The period covered in the investigation of the paper was one in which investors had come to accept a continuing pattern of inflation. What would the attitude of actuaries have been if during that period price stability had become accepted? It was with that factor in mind that the risk of investment in equities should have been divided into two parts, the first involving the company and the second involving the stock market. The first depended on the economy as a whole, the desire and possibility of manufacturing a product or offering a service, and the size of the operation possible at the time and in the future. Those factors made it necessary to consider the supply and demand situation, political pressures, management ability and the rest. The risks involving the stock market were those of liquidity preference by equity investors, taxation factors, and how well-informed and interested the investing public was in the particular equity being considered. The fashion factor, the display of public interest, had increased in the United States and might well do so in the United Kingdom.

Markowitz in his study of efficient portfolios allowed more than one definition of risk and the author's idea was set out in the paper. However, they might well in their pursuit of risk quantification be adopting what would in the future be regarded as a quasi-Chartist's approach. Any investment technique was acceptable provided both better results were forthcoming and its limitations, including risk, were fully recognized but, in so many cases, the limitations overrode the advantages.

He hoped the author would amplify his remarks in § 51 concerning too heavy a concentration in any one investment and asked whether the word heavy was used in absolute terms or relative to other holdings.

Mr W. Perks referred to the suggestion by Mr Goodare that utilities should be used in the probability set-up. He suggested that the appropriate theory would be that of 'decision functions' in which utilities and Bayesian prior distributions were used.

He was surprised that nobody so far had questioned the validity of talking in terms of probability merely because a set of observed numbers could be put into a cocked-hat shape that looked rather like a Normal distribution. Probability did not enter into such a situation until the process of selection was defined. Was it a random selection or by judgment? If it were by judgment, the whole concept of an expected value and a standard deviation fell flat on its face. That was not to say that he did not find it very interesting to see the one-year term rates set out in Table 2 for a period of ten years for a number of securities and averages taken of them both over the same year and over time. Nevertheless, he had to comment on the averaging over the period of ten years, 1958 to 1967, in a situation in which the author claimed to see two five-year cycles, so that there were three peaks in ten years: one peak at the beginning, another in the middle and a third at the end of the period. The plain fact was that averaging over the ten years in question was weighting the average pretty heavily on the high side in favour of equities.

He had said on a number of occasions that he was thoroughly unhappy about the

use of stock exchange prices at particular dates in relation to a large portfolio of investments: he did not believe it was sound. Stock exchange prices represented the marginal deals made between willing buyers and willing sellers and nobody could possibly either buy or sell a large portfolio at those prices. For that reason he did not get much guidance in relation to a life office fund by looking at the market prices of one day in each successive year. The author himself had ignored every price between the beginning and end of his years and the speaker asked why he had done so if market prices were important. He thought the author had done it because he instinctively knew that the only market prices that really mattered were the prices paid for the securities and the prices at which they were sold. The other things that mattered were the dividends received in the meantime. In the case of all the formulae at the beginning of the paper, if they were manipulated algebraically the intermediate prices all dropped out.

The definition of 'expected yield' in probability terms in § 64 seemed logically sound and it was the interpretation that he had always held.

Table 6 represented the projections and he noted that the author pointed out in § 60 that the expectations contained a large subjective element. He asked whether it was correct to suppose that the figures in Table 6 could not be repeated by any other operator: did they depend on the personal judgment of the author? If so, it was difficult to see what they could mean to anybody else unless he had had a long experience of the author and either respected his judgment or not!

He questioned the whole validity of thinking in terms of past experience to form a judgment of the expected yield of an investment in the future.

He referred to the period of time in which they were living because he did not believe that the previous ten years had anything whatever to do with what would happen in the investment field in the following ten years. The 1958–59 peak was a special non-repeatable post-war operation, related to the holding back of dividends and to the controls that went on long after the end of the war. Then there was the passage from the positive yield gap between gilts and equities to a negative yield gap reflecting ideas of growth, followed by the move from prices based on dividend yield to prices based on earnings yield. Then there developed a negative gap between the gilts rate and earnings yields and the effect of corporation tax and capital gains tax. A point had been reached at which earnings yields by and large were not much more than half the gilt rate. In effect, present prices were discounting the continuance of the inflationary effect of recent years and, unless that kind of inflation went on, he could not see that present prices had any justification in comparison with an 8% yield for gilts. In other words, everything that had led from the prices of 1958 to the prices of 1968 was non-repeatable except in terms of the inflationary argument.

A lot was heard about growth. Particularly in the middle of the last ten-year period, there had been a great confusion between real growth and inflationary growth. With regard to real growth, it appeared that much of it came from ploughed-back profits but he was also led to believe that ploughed-back profits were often uneconomically invested. It was often said that because the gross national product was likely to go up by about 3% or 4% per annum, the prices of present holdings of equities could be expected to show a similar rate of growth. That was a logical *non sequitur*. The gross national product growth came in large measure from the injection of fresh capital into the economy which needed to earn its keep and, if that fresh capital were in the form of loan capital, its remuneration was a charge on the whole profits of the organizations concerned. However, the political situation was quite simply that the growth in the gross national product was being pre-empted by three or four different groups of people. It was first of all wanted by the workers, it was wanted by the Government to finance

Government expenditure and Government investment, and it was wanted by the loan stockholders; finally, if there was anything left that was not a large negative element, it would be available for the equity shareholders.

In § 70 there was a practical definition of 'expected yield', viz. expected yield = dividend yield + growth rate . . . , and the author had gone on to say that capital values were based on that formula. That seemed to involve the concept of growth in perpetuity and the speaker could not imagine a more nonsensical idea.

Mr J. M. Brew felt that apart from laying down the principles which should be followed in relating investment policy to the nature of the liabilities, it was surprising that the analysis of risk had not been carried further. In his opinion much of the American work on the subject had been to very little practical purpose but he was pleased to see that Markowitz had been mentioned several times. People might think that the analysis of risk was much more difficult than Markowitz or Hemsted made it appear but the ideas, even if only accepted in qualitative terms, seemed to be extremely important. The question of how many different shares, and how many different industries, should be in a portfolio could only be properly answered by the use of the theory of investment risk. He did not think it was generally realized how low the theoretical figure for the number of shares needed in a portfolio actually was. The Americans put it at about twenty: after that there was not much benefit to be obtained from having more.

His main point of difficulty was that he had not yet seen a really convincing method of measuring risk or of checking the results of such measurement in the light of past experience. It was all very well to use the variance of the periodic rates of return but the really important risks were left out of the analysis. The volatility of the historical price movements gave no indication that West Driefontein might be badly flooded or that Villiers Engineering would suddenly have to face sharply increased competition but those were exactly the risks of which a well-informed analyst might have been aware and would have included in his calculations.

Accepting for the moment the method of measurement, there was another point which worried him about the analysis of risk, which applied to Markowitz just as much as to Hemsted. Diagram 2 reassured the reader that on the basis of historical data high yield went with high risk, which, of course, seemed intuitively likely, but the argument seemed to be circular in an important respect. The shares which had shown a high average one-year return in the past would pretty certainly have risen in price by a more than average amount in the process and the capital element in the one-year return was likely to have been much more volatile than the income element. Using past volatility as a measure of risk, they must surely expect high yield to be strongly associated with high risk. It would have been more satisfactory if risk were measured using relative price movements, relative to a market increase, rather than absolute ones.

Mr Goodare had mentioned the methods of linking one-year returns to arrive at an effective return over the period. With regard to the formula for i, it might not be immediately obvious that the rs and the Ms were dependent on one another. In fact, it was slightly misleading to describe i, as the author did in § 14, as the weighted mean of the one-year returns. Using the relationship between the rs and the Ms to simplify the formula for i, all except M_0 and M_n cancelled out and what was left was a conventional equation equating M_0 with the present value of future dividends and the final capital value. Although the author would realize that, the text did not make it clear.

The equation in § 16 struck him as much more interesting because it could be used to shed light on the question of how a series of one-year returns should be linked together to arrive at the equivalent effective annual rate of return for the whole period.

In Table 2, the arithmetic mean was used but the author would agree that it would be theoretically right to be a little more sophisticated. In the absence of a better based forecast, the best indicator of the future effective return was the effective average rate of return in the past, which was not the same as the average of the past one-year returns. Returning to the equation in § 16, it would be of interest to consider what the position would be if all the Ds were zero: in other words, if the share never paid a dividend. It was clear that $(1+r_n)$ was the geometric mean of all the $(1+r)$s which must be the correct way of linking the rs together in that case. In the case of non-zero dividends, the same formula for linking would be found if it were assumed that all dividends were reinvested each year at the ruling one-year rate and the accumulated result were compared with the uniform growth of £1 at compound interest.

He considered another example in which M_0 and M_n were put equal to 1 and all the Ds equal to i, which represented a share which paid a fixed dividend and held its price exactly, having fluctuated during the period under review. That gave a completely different explicit formula for i which was just another way of expressing the familiar truth that the difference between the annuity which could be bought for £1 and the sinking fund needed to repay £1 was the rate of interest that £1 would fetch. The formula assumed that the effective return could be taken away each year and spent, and that it could be derived even when the Ds were not all equal. That standard of comparison, a stock standing at par, giving a fixed rate of return and no capital gain, was the one used in fixed interest markets when redemption yields were quoted.

He had made some calculations in relation to Table 2 to supplement those shown by Mr Goodare. He had looked not at C_7 but at B_4. The mean return was 13·2%. On reinvesting dividends, which was like taking the geometric mean, the return came down to 10·0%. Taking out the benefit each year, no less than 16·2% would have been realized. There were enormous discrepancies due to the volatility of those returns. Share C_2, which ostensibly gave a mean return of 29·6%, on reinvesting the dividends came down to 15·6% and, on taking out the yield each year, came down again to 14·7%. In those sums it mattered at which stage in holding the share the high returns came. C_2 looked at one way, was actually a less good share to hold than B_4 and the present value would give yet a fourth answer. The question of evaluating the historical benefit derived from an investment depended critically on what was to be measured, and that was a general rather than a particular point.

Mr G. Mills, F.F.A., (a visitor) said that after Mr Perks's contribution he had to confess to being an unrepentant believer in equities and, after Mr Brew's warnings on the way in which the figures in the paper could be interpreted, he felt sure that those who had used similar statistics as one of the general aids in selecting shares for investment would remember that they were always told by the Continuous Mortality Investigation Committee, in connexion with mortality reports, that the statistics were there but it had to be an actuarial judgment in the end which determined how the figures should be used. A similar warning in relation to statistics in connexion with investments was just as important.

Like most others, he had at various times in his career felt that the only real value of an ordinary share was its market value. That was the view he held when his office experience was concerned solely with the liabilities side of the business. Since moving to the other side of the balance sheet, he had become increasingly aware of the irrelevance of the market values of ordinary shares in many actuarial calculations. As a previous speaker had said, it only really meant the value at which one dealt and, when it came to valuing a portfolio, other considerations were involved.

It might be thought that he was against one-year returns, but he was not. A one-year return was a statistic which, if its imperfections were appreciated, would help some operators in exercising their investment judgment.

Turning from the purely investment aspects of the paper, it seemed to him that it was a particularly interesting field to be developed. The author had not pretended to have taken the subject as far as was possible and there was much to be done. In the management of assurance and annuity funds in relation to valuation problems for the purposes of distribution there was indeed a need to know the elements of risk especially with life offices having increasingly high proportions of their assets invested in equities, whether ordinary shares or real estate, and being subject to commercial pressure to distribute to the immediate generation of policyholders as much as possible of the profit earned in that generation. It was in that direction that he sincerely hoped future papers would go.

Mr J. Plymen suggested that, in producing papers on investment, it was very easy to talk vaguely about expected yields and the degree of risk. Consequently, he much appreciated the author's efforts on this subject for, instead of talking vaguely about an expected yield, he had tried to work it out. It was really a theoretical paper illustrated by practical figures and, clearly, the practical side bristled with difficulties and required an enormous amount of further development. In his courage in trying to explore the practical aspects of risk, the author could be likened to a sort of actuarial astronaut: in one or two respects he took the reader up to the moon and left him there, but that was due to the intense difficulty of the problem with which he was struggling.

He was very interested to see that the author advocated the principle of equity investment as maximizing the one-year return. That was the theme advocated in a paper about investment in gilt-edged securities some years previously and he could not see that there was any question that the same principle should surely apply to equity investment. There was a theme advocated in some quarters that there were certain rather mysterious investments which were to be bought now and which would be very good in the long run but might not be profitable over the next year. If that were the case, they should be bought a year later when they would be cheaper!

The author had given some most interesting figures, using past experience to assess the risk. Other speakers had commented on that point, and obviously there were lots of shares where something had happened to make past experience not necessarily a good guide to the future. All the same, the demonstration in the paper was most illuminating and drew attention to several very valuable principles of investment.

He had met in America many people who had been trying to develop the Markowitz principle. He had visited certain major American banks where hordes of people were working on it but he had yet to see them produce anything of practical value. His view was that the Markowitz principle, in requiring the analyst to estimate the co-variance between the expected yield performance of different shares, was just an impossible assignment. His impression was that the workers in America were turning away from the Markowitz principle and trying to select the best portfolio by computer or by other methods.

He was most interested in the application to life office finance, and he felt that the most important section of the paper. It was intriguing that the question had been raised of measuring earnings on the capital employed in a life business. In a paper he had submitted to the Chartered Insurance Institute, he had suggested that non-life insurance ought to control its finance by watching carefully the earnings on the capital employed and by regularly using the business sanctions of industry. He was very glad that the author had advocated the same theme in the life assurance industry. That was one

instance in which the author had taken the reader to the moon and left him there! The question of what was the capital base of a life assurance business bristled with difficulty and he asked whether life office business should be valued as it was at the time of the valuation or as a going concern. He felt that the usual method of valuation made provision for the expense involved in getting new business but did not take credit for the profit from the new business and there was a certain unbalance there.

He found Table 6 very intriguing and felt that the author had been extremely courageous in deriving it. He agreed with Mr Perks that it was obviously based on the author's judgment and he had a very high opinion of that judgment. The table apparently showed for the thirty shares a ten-year mean of 8% and presumably the standard deviation of that would be about 18·5%. That seemed to reflect Mr Perks's qualms over the equity market for, if the author's equities were going to yield 8% with a standard deviation of 18%, that did not seem an impressive performance, particularly when compared with, say, a ten-year redeemable Government stock with a redemption yield of 7¾%. He was not sure what the author would calculate as the expected yield but it seemed to be better than 8% with an 18% standard deviation.

Mr R. E. Beard commented mainly on the part of the paper that related to the theory of risk. The formal model of an insurance operation was a rather complex random process and there was a long history of attempts to devise models which had some relation to the real situation. He believed that it effectively dated from Filip Lundberg's 1903 paper but Prof. Buhlmann had suggested to him that Bachelier might have allowed for some random movements in interest in his model of a life office in his paper written in 1900. The basic premise in those models was to find the distribution function of the expected gain or loss in a year so that it was possible to make some probabilistic statements regarding the margins required to meet fluctuations in experience. In recent years, the limitations imposed on the early models had been modified quite considerably and de Finetti in particular had made a number of suggestions. More recently Karl Borch had developed some of the models still further. He had used utility functions combined with probability distributions and had discussed various operations of insurance companies on that basis. It was not necessary to go through the decision theory approach because Borch had done the analysis for them in his book *The Economics of Uncertainty*, which also included discussion of the Markowitz process, particularly from the point of view of an insurance operation.

For the reasons put forward by the author in § 1, the random variation in those early models had been that arising from the liabilities, and it was proper to consider what form of model should be used to represent the whole of the operations of an insurance activity, i.e. when variations in assets and liabilities were both brought into the picture.

There was a fundamental difference between the liabilities and the assets. As far as the liabilities were concerned, they were represented by formal contracts whose continuance to their natural completion was effectively under the control of the policyholder. From the office point of view, the future course of the contracts arose from a number of factors outside its control and it therefore made sense to endeavour to describe the operation as a probability model, thus linking up with the point made by Mr Perks, but on turning to the assets side there was a completely different picture. The whole of the assets side was subject to control by the office, since such factors as the type of investment and the timing of changes in the mix of the portfolio were all matters upon which decisions could be made in the light of opinions which were subject to continuous change. Variations in the assets side were thus not properly to be described by a random

process. Extension of the collective risk model to include asset variations did not seem appropriate and a different approach to the assets seemed to be called for. It was not clear that the author's ideas formed a justifiable approach since it seemed necessary to treat variations in assets and liabilities on different bases.

The classical form of valuation could be looked at in terms of emerging costs where the items of income and outgo were separated into their accounting years and a discount factor applied to the results to arrive at a present value. The discount factor had the useful effect that it reduced the weight attached to items more remote in time where it was more difficult to estimate the probability of their distribution. If an insurance operation were being examined, it was possible to look at the emerging cost on the liabilities side and build up a distribution of the emerging cost. It was then possible, in effect, to work backwards and build up a distribution of redeemable assets (assuming there were stocks of the right term available) in which the liabilities could always be met as determined. A bonus assumption could be built in and a completely matched position could be constructed. At that point, when bringing the stochastic variation into a notional matched distribution of assets to meet the liabilities, the emergence of the liabilities in the future could be randomized and the randomization of the liabilities would give a 'fluffy' distribution of the required asset distribution in terms of fixed maturity dates.

The problem of practical application had been reduced to the comparison of the actual portfolio at a point of time with the theoretical distribution discussed above. The whole question turned on those comparisons. In practice, there would be three situations developing. The first was that a switch could not immediately be made from an existing asset situation into a hypothetical one because the market practice would in general not permit it and so one could say only that that was the amount required to switch from one to the other. Then there were those liabilities which could not be matched because they were new and a theoretical matching distribution of assets could not be derived. Finally, there were cases where the actual distribution of assets differed so much from the theoretical that a major operation would be needed to match them. Criteria would then have to be devised to enable statements to be made about the company. Risk capital came into the calculation because, at the point of time considered, the quantity sought was the amount of margin required to get from the actual asset situation to the hypothetical so that any future movements in the stock markets could be disregarded. There were some obvious consistency points in the two sides of the equation to be matched.

His feeling was that there were two completely different kinds of variation involved, and thus the two sides of the account had to be treated separately and by a different calculus. The liabilities could be discussed in terms of a stochastic variation but the assets side would not lend themselves to it. On that basis, the valuation took on a completely different concept from the traditional one but, in that way, a meaningful model might be developed from which to talk sense.

Mr R. S. Skerman was very interested in Part III of the author's paper, in which liabilities and assets were linked together. It had always been his concept in looking at the financial position of a life office to consider emerging costs combining the liability outgo with the asset income. That would produce a series of positive and negative figures year by year which would be an assessment of the future net income and outgo of the office. The figures could then be discounted to the valuation date in order to evaluate what might be called the free assets or the estate. The income and outgo should be discounted using the best estimate which could be made of the future rate of interest. It was reasonable

if assets were valued at market values to use the current market rate of interest, but a more refined method would be to make the best allowance possible for future changes in the rate of interest. That brought him very close to the idea the author put forward of the one-year return.

He differed a little from the author's approach for what he had in mind would be to tabulate as best he could the liability income and outgo in the future and the asset income in the future, which would mean including in the asset income an assessment of the changes in the income from equity shares. Therefore the one-year return basis he would use would be the one-year return which reflected changes in the rate of interest only but did not reflect changes in the assessment of growth prospects in equity shares. That provided a clear basis on which to value the liabilities correspondingly.

In § 87, the author stated that the simple answer in regard to the liabilities was to value them at the yield appropriate to the asset portfolio of the current market price. That might provide a useful practical answer but he would rather use the one-year return rates year by year for two reasons. The first was that the yield on the asset portfolio at its market price was affected by the yield on equity investments. The author had quoted the speaker in § 84 as advocating the use of the yield on the asset portfolio at its market value, but the paper quoted was one written to the Fourteenth International Congress of Actuaries, which was long before growth prospects of equities were reflected in their market values. In a paper presented to the Institute in November 1967 (*J.I.A.* **94**, 53), the speaker had suggested that, in arriving at the value of the liabilities at a rate of interest corresponding to the market rate on the assets, it would be better to assume that equity shares were earning the gilt-edged yield. That would be his preference. The second shortcoming in using just the market rate on the portfolio was that, although that yield was equivalent, on the author's assumptions, to the one-year return year by year for the assets, it was not necessarily equivalent to that return for the liabilities because the incidence of the liability income and outgo might be considerably different from that of the asset income.

The question remained as to whether the one-year return would enable an advance practically and he could foresee very considerable difficulties. In order to use the approach, the rates of one-year returns in the future had to be estimated first and it could not be disputed that subjective judgments were being made. It was difficult enough to judge an average rate of interest for the future: to judge the trend and the variation from year to year was an extremely subjective operation. Then, in valuing the liabilities, it was proposed to use a bonus reserve valuation and the future rate of bonus assumed would presumably be the current rate. The result was to estimate the effect of the assumptions as to the future rate of interest on the free assets. If it were assumed that the future rate of interest would decline very considerably, the answer was probably unrealistic in practice. If that did happen, a change in the bonus rates must be expected and it was necessary to be careful in interpreting what was eventually produced as an answer.

In § 91, the author stated that in the speaker's paper to the Fourteenth International Congress of Actuaries he had advocated that the portion of the risk capital held as protection against other than investment losses should be kept in risk-free investments. That had not been his intention, which had been limited to suggesting what should be the length of term of that portion of the free assets if the matching position of the office were to be unaffected whatever rate of interest ruled. If the position of the office were such that a greater degree of risk was acceptable than was involved in the assets other than free assets, he quite agreed with the author that some risk would be acceptable in investing the free assets if that were considered to be justified by the prospect of a higher investment yield.

Mr D. G. R. Ferguson considered that the subject had two sides: firstly, the formation (and updating) of an investment policy as a framework within which daily investment decisions could be taken; and secondly, the actual taking of decisions to buy, hold or sell. Books and papers on investment tended to be theoretical when contributing to the investment policy side and practical when contributing to the investment decision-taking. Redington's outstanding 'matching' paper (*J.I.A.* 1952, **78,** 286) was an example of the former and the stimulating paper by Weaver and Hall (*J.I.A.* 1967, **93,** 165) an example of the latter. However, when reading the author's paper he had frequently felt that it failed to say anything which would be useful either to policy-making or to decision-taking in a practical sense and that was a pity because the paper had a good title and the concepts of the one-year return and the degree of risk both seemed to be useful. For smaller life offices they were particularly useful. Mr Perks's objection to using market prices was then of less significance since the sums invested were relatively small and, secondly, since, as far as the degree of risk was concerned the degree of insolvency was real whereas, for the very large offices, it was so small as to be negligible. When talking about net one-year returns the incidence of capital gains tax was very important as the author had pointed out in §§ 10 and 11.

Unlike Mr Brew, he found the formula derived in § 16 rather misleading. The author had offered a prospective formula for the market price which depended on rates r which had been assessed retrospectively, and for two reasons the speaker disagreed that that gave 'a rather more complete view of the present value than the use of the single rate i'. Firstly, there was no view at all unless the rates r could be estimated and any estimate was likely to assume that all the rs were equal. Secondly, he thought that the rate r should be that which it was estimated would apply to the market as a whole, not to the individual share. If those two alterations were made and, moreover, it was assumed that dividends would grow at a rate equivalent to a level annual growth rate and also that the share was a perpetuity, then the formula would simplify to the well-known relationship that

$$\text{expected yield} = \text{dividend yield} + \text{growth rate}$$

which the author used later in the paper. A corollary was that taking the expected yield as being that currently available on long-term gilts, then the growth rate which the market was discounting on dividends was the same as the reverse yield gap. It seemed that the growth rate on dividends which the market was discounting was a more useful starting point for assessing the cheapness or dearness of a share than either the price–earnings ratio or the dividend yield.

He disagreed with the opener on the pattern of one-year returns of shares and found it remarkable that the author could select a sample which was far from being a random one, postulate a Normal distribution, admit that statistical tests (on this unrandom sample) show a highly significant divergence from Normality, and then state that the assumption might still be of practical value. The author had not even stated that the thirty shares had been randomly selected. As a result, the following pages in Part I of the paper were of dubious value and it was a pity that such conclusions as, for example, that reached in § 41, where the author had suggested that ten shares were as good as thirty for spreading the risk, were unsupported by sound empirical argument. More serious was the reference in § 34 to estimating the chance of insolvency: to do that properly, the shape of the tail of the distribution had to be considered and not only did he not believe that the dubious Normal distribution in question had a Normal tail but doubted whether a reinsurer asked to reassure the insolvency risk would do so.

Mr L. G. Hall, in closing the discussion, referred to the Presidential Address (*J.I.A.*

1961, **87**, 1) delivered by Mr J. H. Gunlake, in which he had spoken somewhat critically of the assumption tacitly made by actuaries that they were dealing with a problem in statics and not dynamics and that they could adequately understand in terms of actuality a problem which was essentially one of futurity. He had wondered whether they could not set about their valuations in a straightforward way, producing a series of forecasts in revenue account form—in other words, said the closer, based on one-year returns. Mr Gunlake has continued that in that way actuaries would be able to measure with greater ease and speed the effect of changing their assumptions. They could take account of future inflation at various assumed rates, they could investigate the consequences of fluctuating rates of interest, and so on. And then perhaps actuaries might feel themselves equipped to amplify their help to those whose decisions they guided by presenting them, as other statisticians sometimes but actuaries seldom did, not with a single appraisal of the future but with a range of estimates, to which it might be possible to attach degrees of likelihood. Of course (said Mr Gunlake), the development of those relatively untried procedures would require much research and experiment and their operation much skill, and all that would be undeniably difficult.

The author had undertaken such work and developed the theory of one-year returns and the degree of risk, the value of which had been foreseen by Mr Gunlake eight years before, and the profession owed him its admiration and gratitude for what he had done. He had attempted to be precise where so many actuaries had spoken in generalities. He had himself said, in the discussion on Skerman's paper in 1967, that the life office investment manager could face with equanimity the risks he took because he knew he had a cushion of past success (in other words, an estate) on which he could always rely. They had spoken of facing risks with equanimity; how much better if they could know the degree of risk they were facing. The concept of the one-year return on an investment was the natural concomitant of modern investment thinking. The life offices were in strenuous competition and not only with each other; they were fighting for their share of the savings of the country. To win the fight they needed a dynamic investment policy. Simply buying a bond and holding it to maturity, having previously precisely calculated the yield, just would not do.

In Part I of the paper, the author had made some interesting practical points. For example, the greater the interest element in the return, the more important was the contribution of the early values to the redemption yield. The closer concluded that in high yielding gilt-edged and fixed-interest securities it was the return in the early years which was vital, and that depended very much on the time of the purchase. In buying assured growth stocks, with low dividend payments, the timing was of less consequence to the final yield. The author had also said that, outside the gilt-edged market, periods of less than one year were probably too short for investment decisions and financial planning. He did not altogether agree with him and maintained that it was not so in the case of investment decisions. Investment decisions must be moment by moment. If he bought a share for 45*s*. today because he thought it was worth 50*s*., and if it went up to 90*s*. tomorrow, he would sell it; he wished it happened more often. Financial planning was different. A period of less than a year was usually too short; indeed, investments were bought because they were thought to be good value for the long term but they needed to be watched over in the short term and sometimes swift action had to be taken. Table 2 had fascinated him and he would have liked to know the thirty companies included.

The opener had spoken about the need to compare, on the one hand, the performance of equities against gilt-edged and, on the other hand, the performance of equities against the market index. He had made the very valid points that the return obtained by active management of a fund against the extra risk taken in achieving that return was

the real test of performance, and also that a 'no action' decision was just as much a management decision as a decision to buy or sell.

Mr Perks had said that he was unhappy about the use of stock exchange prices in relation to a large portfolio but he agreed with some other speakers who had asked, in effect, what else could be used. It was not possible to sell a very large investment portfolio at the current market price but, nonetheless, current market prices must be used as the best estimate available of the current situation.

Mr Skerman had spoken of using a one-year return reflecting changes in the rate of interest only and not reflecting changes in the market value of the equities. The closer argued in favour of looking all the time at the changes in market value of the equities and at the appreciation which they showed since, in order to do justice at all to the current generation of policyholders, some of the appreciation must be considered as available for distribution.

He turned to Part II of the paper and to Table 6, which he found very much less fascinating than Table 2. It was based, in the case of War Loan, on the assumptions that yields would continue to rise by about 1% every five years and that the five-year economic cycle would continue. For the thirty ordinary shares, the same assumptions were superimposed on projections of the profit on capital-employed ratios. He could not begin to believe in Table 6 or find anything useful in it but acknowledged that several speakers had pointed out that it depended on the author's subjective ideas. It was no doubt very useful to the author but, of course, it was subjective. It was an attempt to look ten years ahead, not by considering all aspects of a company but by looking at profitability trends. Two weeks earlier, he and the author had both attended the Fifth International Congress of the European Federation of Financial Analysts' Societies in Germany. The degree of sophistication in investment analysis and the quality of the information available in company accounts varied enormously from one country to another and the various discussion groups involved the sophisticated and the unsophisticated. Nevertheless, he thought it useful to quote a brief press release about one of the groups which had had as its subject 'Estimating future earnings—methods in use':

> 'The discussions brought out the problems facing the analyst seeking to forecast profits, many of which seemed insuperable in the context of the information at present available. This raised the question whether it was worth while to the analyst to attempt to forecast profits. The question was answered clearly in the affirmative by the London member of the preparing commission.'

Of course it was!

> 'In relation to the problem of long-term forecasting, it was agreed that it would be useful but not entirely practicable to forecast net profits per share. However, one opinion was expressed that even the forecast of the trend in earnings was of no value because of the high degree of error. Long-term forecasting techniques would demand careful attention to the qualities of management but the necessary techniques to measure this quality had not yet been developed. It was thought that investigations should be made in this field. It was agreed that investment analysis techniques could best be developed with the active co-operation of company managements. This co-operation should be actively sought and in cases when it was not forthcoming the reasons should be investigated.'

The quotation showed just how difficult the whole problem of looking forward was. The United Kingdom was a great deal further forward than most of the Continental countries but, nonetheless, it was a very difficult matter.

The doubts expressed about the merit of even attempting to forecast profits were clearly no more than a reflection of how much more complete the information coming from companies was in the United Kingdom than on the continent of Europe. Forecasting profits was at the very heart of investment analysis as known in the United Kingdom. Nevertheless, there was no lack of unanimity in the discussion group about the risk of a high degree of error in long-term earnings forecasting and about the vital importance of the quality, and changes in the quality, of management. It was necessary also to study and forecast a company's markets, turnover and margins of profit. The basic task of investment analysis was to set out the longer-term background and superimpose on it a short-term estimate of earnings to the highest possible degree of accuracy. After achieving the earnings estimate, the next task was to bring in some thought about the likely price-earnings ratio, say, a year ahead and the related questions of how interest rates would move and how the rival forms of investment (such as gilt-edged) would perform. It was subjective, and so it would remain: highly subjective ideas developed against the best possible statistical and economic and political background. Mr McIvor had said very much the same thing about the approach to investment analysis and the evaluation of ordinary shares.

Mr Perks had given some very interesting comments about what had happened to equity prices in the last ten years and why it was not going to happen again. He believed that Mr Perks was too severe on equity prospects. Mr Mills, on the other hand, had proclaimed himself to be an unrepentant believer in equities. He would compromise, and say that the best investments of all were the right equities. He did not mean that an equity portfolio which was spread right across the index or the market was necessarily going to do very well and, with gilt-edged securities yielding 8%, he had reached the position that he would rather hold gilt-edged securities than an equity portfolio which was spread right across the market. Such equity portfolios were, however, out of fashion and attempts were made to find the right equities. He felt that the heavy demand for the right equities would continue, in spite of the fact—and he fully agreed with Mr Perks—that many of the factors bringing equities up in the previous ten years were not likely to be repeated.

The paper was one of more than theoretical elegance. It was a step along that road which Mr Gunlake saw stretching into the distance eight years before and it would have its honoured place among actuarial contributions to investment thought and, indeed, not only investment thought but thought about the way in which the whole corporate entity of a life office developed. It was a step on the road to precision in producing the annual earnings per share on the shares of a proprietary life office and the surplus on the policies of a mutual life office too, provided the life offices revealed the true value and performance of their assets. He believed that competition would oblige them to do so, and sooner rather than later.

A number of speakers had referred to Part III of the paper and Mr Brill had rightly suggested that it was the most important part. It was Mr Mills who had made the very important point that the real need to know the element of risk was because it would enable proper justice to be done to current generations of policyholders.

The author had expressed the hope in his conclusion that his paper would stimulate new thoughts on the important subject of the measurement of the degree of risk and perhaps point the way to further advances. He had hoped that it would lead to improved management efficiency in the use of limited resources and would enable future decisions in areas of uncertainty to be taken with a higher degree of confidence than otherwise, and that the ideas he had expounded would yield practical results. The closer also hoped so and thought that over the years they would. Nevertheless, he ended his remarks with

some comments which were in no way critical of the author but which came into his mind on an evening when they had been discussing a paper which, for all its possible practical consequences, would be seen by most of its readers as highly theoretical. The actuary's reputation in investment must be for more than theoretical elegance: it must be for practical handling of investment decisions, for a knowledge of the economic factors underlying the gilt-edged and equity markets, for an ability to sum up a management, a company and an industry, and, indeed, the stockbroker who might be giving him advice. It was just because they as a profession were capable of producing elegant theories that they must be on their guard. They must take care that the world did not think they were so concerned with the theory of risk and the requirements of immunization that they could not see the wood for the actuarial trees. He hoped that actuaries, and in particular the younger generation of actuaries concerned with investments, would always make the fullest contribution they could, not only to the proceedings of the Institute but also to the work, and especially the practical work, done in the investment field outside Staple Inn and outside their own professional journals. He hoped that the author would develop the work further, particularly in regard to the handling of the whole of a life office on the lines of Part III of the paper.

The President (Mr J. B. H. Pegler), in proposing a vote of thanks to the author, spoke of a personal interest in the paper because some years previously he had dipped a tentative toe in the expected yield pool. He had been seeking after a single criterion to describe why one security—and particularly one ordinary share—should be purchased rather than another. After the criticism his own ideas had received in the discussion, he realized that they rested on somewhat shaky theoretical foundations and he had always hoped that someone would clean them up. The author had gone a long way towards doing that. On the previous occasion, one distinguished critic had said that he had, significantly, given no arithmetical examples. Mr Hemsted, on the other hand, had given many. It had been altogether a most interesting paper and it was to be hoped that it would encourage others to proceed further in the same direction.

The author, in reply, noted that Mr Brill had said that more experience was necessary before coming to any judgment but said that a very important point in investment work was that the judgment had to be made immediately on whatever experience there was. Certainly judgment should be improved with added experience.

Mr Goodare had spoken of geometric means. When looking to the future, the author declared himself definitely a compound interest man and there was no question of taking arithmetic or geometric averages of one-year returns and saying 'This is what our return will be'. The whole purpose of the one-year return exercise was that the long-term compound interest return emerged a year at a time in the one-year returns. When looking at means, he had been trying to find some way of relating a degree of risk to the actual one-year return expectation.

Mr Perks had asked about his expectations. They were certainly subjective but it was purely an example to show that any person who wished to try could break down his future compound interest expectation into a pattern of expected one-year returns. Each person's expectation obviously would differ but, in the event, people who held the same investments would get the same returns. The thirty shares were not his subjective choice; they were a reasonably random selection.

Mr Brew had spoken about the author's formulae. Obviously, again, it was an attempt to break down the redemption yield into steps. It was true that market values could be eliminated and the redemption yield formula obtained but the formulae did not lose

their significance for that reason. It was surely useful to show how the formulae could be derived from returns emerging in the past and applied again to expectations for the future.

Mr Plymen had mentioned gilt-edged securities, and ten-year gilt-edged securities in particular. The point of his breaking down the yields on such securities into one-year returns was to show, just as an example, how a trend towards higher interest rates could produce continuing lower returns. A ten-year gilt-edged security over the period would end up with probably a better average one-year return than the long-term gilt-edged security.

Mr Beard had given a very learned contribution which would require some study later. With regard to the differentiation between the assets and the liabilities, the author's approach was to treat the assets as inactive in the first place, on an expected yield basis, and in that sense they were the same as liabilities which were very largely outside one's control. The fact that, having looked at them on an inactive basis, some action could then be taken on the assets did not invalidate the theoretical comparison of assets and liabilities nor the bringing of them together on an inactive basis.

Several speakers had mentioned overall risk. It seemed to him to be the one really important decision a management had to make: what degree of risk were they to run? He looked forward to the stage where, having made that decision, the rest of the decisions could be almost mechanical, including the selection of investments, the linking of them together and the selection of new business. The original risk decision could not be avoided and the fact that it was not really quantified at the moment was perhaps a reflection on the profession. He was looking forward to seeing more progress towards better quantification of that basic management decision.

Mr S. Benjamin later submitted the following written contribution:

I would like to ask eleven detailed questions on the paper:

1. The author states in § 15 that the more recent values of the realized rates of an investment have the most significance for current and future trends. Does he have any statistical evidence to support this assertion?

2. In § 37, he states that over a very long period, a company's returns could be expected to produce a distribution similar to that produced by a single year's experience for a large group of shares. If we think of the likely secular trend over the long term, we would expect this not to be true. Does he have any statistical evidence to support his assertion?

3. In § 64, he says that 'maximizing the expected yield is then seen to be a matter of maximizing expected one-year returns'. However, if a portfolio of shares S_1 would maximize the return over year 1 starting from a cash position, and a portfolio S_2 would maximize the return over year 2 starting from a cash position at the beginning of year 2, then unless S_2 is the same as S_1 there will be a loss on costs of reinvestment and capital gains tax. There may well be another portfolio S_3 which maximizes over the two-year period. Would he agree that this puts a serious limitation on the use of one-year returns?

4. He says in § 74 that a market operator uses new information and hence recognizes cheapness and dearness. Does he have any statistical evidence that this recognition is effective? (I am thinking of the fact that unit trust results seem to be random in the sense that one is no better than another.)

5. In § 80 where he says that the liability to the office is an asset to the policyholder, he seems to have in mind the market value to the policyholder. In what follows, however, he seems to be talking about the reserve in the office. Is he assuming these are the same?

6. I find §§ 81, 82 and 83 obscure. Is the author basically suggesting that it would be useful to look at the statistical distribution of one year's surplus or what our American friends would call the net operating gain? Would he also agree that this has little relevance to matching?

7. Is there a contradiction between his suggestion in § 87 of a future average rate of interest in the liability valuation and the main point of §§ 81, 82 and 83?

8. In § 87 he justifies his choice of a single compound rate on the grounds that, although the liabilities are exposed to an investment risk, the interest element in the valuation is secured by a surplus of assets. Surely the purpose of the exercise was to investigate the surplus of assets which he now assumes?

9. I agree with the author's remark in § 91 that it is better to treat the assets as a whole for investment purposes with the risk capital invested in a cross-section of the portfolio, but I believe this is unusual. Does this approach follow directly from the paper or is it a separate idea?

10. In § 100 is the author suggesting that the distribution he has produced might not be typical and therefore should not be used? Would he agree that the whole approach could be dangerous as a basis for valuation and solvency theory on exactly these grounds and that it is better to continue with the theory we have which is based on the idea of a historical cautious approach?

11. Would he agree that the paper *Long and Short-Term Rates of Interest* by H. B. Rose, (*J.S.S.* 1957, **14**, 22) would be a useful reference and pertinent to the subject?

The author subsequently supplemented his verbal reply to the discussion with the following written remarks:

Mr McIvor pointed out that a safe investment in money terms could be risky in real terms. I accept this, and suggest that real risks can only be tackled by first quantifying expectations and degrees of risk in money terms and then adjusting for an expected change, year by year, in the value of money. For this purpose, I would think it sufficient to adjust only the central expectations as the variance appears to be related to the mean in money terms rather than in real terms. With regard to his suggested subdivision of equity risks, I agree that, in forecasting performance, the company and the market must be considered separately. The expected yield formula effectively separates these components in the form of dividend yield and growth factor. Further, one can invest in company performance by deciding upon an inactive policy but, if the concept of one-year returns is accepted, one cannot escape the market risk. In reply to the question regarding § 51, by heavy concentration I meant a high proportion of the fund in one investment, i.e. in relation to other holdings.

Mr Perks pointed out that the ten-year period analysed probably produced exceptionally high mean returns. This is admitted, but the object of the analysis was to examine the distribution and its relation to the mean, and to derive a degree of risk. A range of mean values was required for Diagrams 2 and 3 but the overall mean has no relevance and does not invalidate the result. He expressed the opinion that growth in perpetuity is a nonsensical idea. To me, it is no more nonsensical than fixed-interest in perpetuity for, if it is a condition that a proportion of each year's interest is reinvested, then we have growth in perpetuity similar in some respects to an equity investment.

Mr Brew suggested that the really important risks were left out of the analysis but I do not believe that this is the case. The price movement of a share reflects the views of the well-informed analyst and, where there is a greater chance of catastrophic loss, even if past experience has been good, there is likely to be extra volatility in relation to the mean. He appeared to be under the mistaken impression that I was using past

volatility as a measure of risk. If one takes chance of loss, as I do, as a measure of risk, the higher expected return situations, even with high volatility, are often safer than the lower ones. Mr Brew complicated the issue by introducing reinvestment assumptions. I prefer to treat each investment on its own and to regard the reinvestment of income as a new problem to be dealt with, if it still exists, when the time comes.

I quite agree with Mr Skerman that the concept of an expected emerging net annual income or outgo statement for the office would be valuable, and the discount would give an estimate of the free reserves. My approach does, in fact, provide the same estimated free reserves, although without the intermediate information on net income or outgo. Using the one-year return approach, however, the net income position, which disregards changes in market value, loses some of its importance. Incidentally, the rate at which I discount is the market expected yield, i.e. the single rate equivalent to the expected flow of one-year return rates, and not the market income yield basis, as Mr Skerman appears to think. I agree that the incidence of liability income and outgo is likely to be considerably different from that on the asset side but, provided there are always surplus assets, the situation will look better on a one-year return basis. Either way, the discount may involve assumptions about the yield basis on which investments have to be realized in the future, and the extent of any mis-matching will contribute to the overall risk.

I was sorry Mr Ferguson thought the paper was failing to say anything useful on policy-making or decision-taking. In my opinion too many policies are made and decisions taken without understanding the nature of the expected returns or trying to measure the degree of risk. With regard to the formula in § 16, it is common practice to calculate a realized yield using a prospective formula. When actually assessing the future I said in § 64 that the formula had to be applied to expectations; but I consider that each investment has its own expected yield and it would not usually be correct to apply a market expected yield indiscriminately to select portfolios. Dealing with my suggested Normal assumption, it must not be forgotten that the object of the analysis in Table 2 was to try to estimate the probability distribution surrounding a unique event in the future, by studying different investments in the same period and the same investment in different periods. It is not good enough to decry practical approximations unless something better can be suggested.

Mr S. Benjamin has asked eleven detailed questions to which my answers are as follows:

1. The statement in § 15 concerning the significance of the more recent rates of returns was based on common sense rather than statistical evidence. I made it clear later (§§ 61 and 74) that it would be unwise to use only the record of past one-year return rates in projecting future returns—if anything, there is likely to be a negative correlation. But the investment return can be analysed into company performance and market assessment. In each of these components, the results depend upon human decisions and, the further back one goes into the record, the more one is aware that the political and economic background has changed, the companies have changed their nature and the decision-taking individuals have, in general, learnt from experience although many have passed entirely from the scene. Consequently, when considering current and future trends, the more distant the observation the less significant they become.

2. I was concerned in § 37 with the distribution of the possible rates of return on either side of the mean in terms of standard deviations. In thinking of a likely secular trend over the long term, I would think first of the trend in mean values, and I am not convinced that there is a likely secular trend—on the contrary, economic forces tend to keep the mean within limits. But the discussion concerned the distribution rather than the mean,

and its divergence from Normality. If it is hard to picture a likely secular trend in means, it is even harder to picture a trend in distributions, which I take to depend upon human error in attempts to maximize the investment returns. The divergence from Normality lies in the skewness and I would expect this to continue with occasional very high values coming up amid a marked cluster around the mean.

3. I certainly agree that if a change of investments is under consideration, costs of reinvestment and capital gains tax must be taken into account (see § 25), but surely this is an argument for looking at short-term estimates when activity is contemplated. The serious limitation in activity arising from the incidence of these costs is there already. One-year expectations help the operator to draw the line between changes where the expected gains adequately cover the cost and those where the reverse applies.

4. The random walk theory is based on statistical evidence which is interpreted as showing that, in the short term, the most likely next move is sideways, and this itself may be regarded as statistical evidence for the ability of operators to recognize cheapness or dearness. Even if the random walk theory is not accepted, there are plenty of operators who claim to beat the market consistently and I have records of one-year estimates over a period of eight years which provide satisfactory statistical evidence relating to the expected yield method. With regard to Unit Trust performance, I would say that, far from one being no better than another, they show a considerable spread which may include a random element but the mean is, I believe, significantly better than the index of market performance.

5. In § 80, I was making a case for treating liabilities as negative assets, with their own rates of return. Limitation of marketability means that prices realized by policyholders who sell their policies are not much of a guide, but my view is that the actuarial reserves can be equated with a block of assets sufficient, taken with future premium income and investment returns, to provide the assured benefits, and the actuarial reserves then have a negative market value matching that of the earmarked portion of assets.

6. In §§ 81 to 83, I developed the concept of treating liabilities as negative assets and calculating one-year rates of return for classes of liability as is possible for classes of assets. I suggested that there is some considerable relevance to matching because not only can the benefit of spreading the risk be observed among different liability classes (as with asset classes) but, when assets and liability returns are combined, the benefits of matching appear as a correlation in the returns, which has the effect of eliminating much of the overall risks. The one-year increase in risk capital, as described in § 85, may well be the same as 'net operating gain' in America. As the components of this one-year increase are to some extent under control, I do not see that there is much to be gained by looking at the statistical distribution, although the actual values and trend are important as an indication of the efficiency of the management.

7. I do not see any contradiction. Each year, a rate of interest must be applied in the valuation of future liability outgo in order to calculate one-year liability returns. It is quite logical to adopt the rate of interest (i.e. expected yield) determined by the market value of assets and the expected asset income.

8. I agree that one of the main purposes of the exercise is to calculate the surplus of assets. If there is a surplus, then the assumption of the same compound interest rate for liabilities as is determined by the assets is valid. If there is no surplus, then management is technically unable to maintain its bonus and must take suitable steps to create a surplus, at which stage the assumption is again valid.

9. The idea of having risk capital invested in a cross-section of the portfolio is almost essential for calculation purposes. In any case, given particular assets and liabilities, there can be only one degree of insolvency risk whether assets are segregated or not,

although the risk of overall loss would be higher if safe assets are earmarked for risk capital.

10. In § 99, I suggested that a Normal assumption would, in fact, give sufficiently accurate results to be of some value. I do not agree that we should continue to follow a historical cautious approach because, in competitive conditions and in a completely new environment of higher fixed interest rates, equity investment and higher bonuses, the risk may be much greater than is appreciated.

11. I agree that H. B. Rose (*J.S.S.* 1957, **14**, 22) should have been included in my references. His examination of the relationship between long-term and short-term (one-year) yields has some resemblance to my approach, especially as his assumptions make capital appreciation equivalent to income and thus make the bond coupon irrelevant. Rose also found that the relationship between long-term and short-term yields was useful in dealing with uncertainty in the bond market.