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AN OPTION PRICING APPROACH TO BONUS POLICY

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1. INTRODUCTION

1.1 In this paper I discuss a simple application of option pricing theory to bonuses on with-profits life assurance policies. The approach is a new one and I have been able to make only an introductory exploration of its possible applications. Nevertheless it seems to me to give actuaries some sort of a handle for gripping the rather intractable problem of the relationship between reversionary bonus, terminal bonus and the proceeds of comparable unit-linked policies. I look at the problem very much from the policyholder's side, and I am not concerned here with aspects of valuation or solvency. These require further investigation.

1.2 The approach relies on the use of options, particularly put options, so it is first necessary to explain what is meant by an option. A *call option* on an ordinary share gives the purchaser the *right* (but not the *obligation*) to *buy* one share on a specified date (the *exercise date*) at a specified price (the *exercise price*) from the *writer* of the option. In exchange for this option the purchaser usually pays the writer a price or premium.

1.3 A *put option* is similar to a call option except that the purchaser has the right to *sell* one share at the exercise price at the exercise date.

1.4 A so-called 'European' option gives the purchaser the right to exercise the right only *on* the exercise date. A so-called 'American' option gives the purchaser the right to exercise the option at any date *on or before* the exercise date. In fact both sorts of option are traded in different markets in both Europe and America, and the names applied to them are purely conventional, though well established.

1.5 Those who are already familiar with options and the theory of option pricing as exemplified by the Black-Scholes (or other) formulae should read on. So also should those who do not care about the derivation of the formulae. Those who wish to progress on the basis of a better knowledge of the theory of option pricing should study the Appendix now, returning to the main body of the paper afterwards.

1.6 Those who have not studied the Appendix need only to be aware that the Black-Scholes formula for the value of a European call option is given by

$$W(\text{call}) = P.\,\mathrm{N}(d_1) - Ee^{-\delta t}.\,\mathrm{N}(d_2),$$

where W(call) is the value of a call option on one share at a period t before the exercise date, if the price of the share at that time is P, the exercise price of the option is E, δ is the risk-free money force of interest per unit time, and σ is the standard deviation per unit time of the share price change. It is assumed that the share pays no dividends (or that they are accumulated within the share price).

N(d) is the normal distribution integral:

$$N(d) = \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx,$$

$$d_1 = \ln(P/Ee^{-\delta t})/\sigma\sqrt{t} + \sigma\sqrt{t}/2,$$

and

$$d_2 = \ln(P/Ee^{-\delta t})/\sigma\sqrt{t} - \sigma\sqrt{t}/2$$

1.7 The corresponding formula for a European put option is:

$$W(\text{put}) = Ee^{-\delta t}.N(-d_2) - P.N(-d_1).$$

1.8 Thus, given the values of P,E, t, δ and σ, the values of a call and of a put option can be derived. Various authors have elaborated the formulae to allow for more complicated descriptions of the share price process and the process describing the risk-free rate of interest, but I ignore these in what follows in order not to introduce too many complications.

2. THE INVESTOR

2.1 Let us now imagine that a man has £1,000 to invest. He has in mind to invest it in accumulation units of a unit trust, in which all income is reinvested rather than being distributed. Units have a current price of 100p, so he could buy 1,000 units. He intends to retain the units for a period of 20 years, and to sell them at that date. However, rather than just buying the units he would also like some sort of surety that he will get at least a minimum sum at the end of the 20 years. So he considers buying rather fewer than 1,000 units and spending the balance on buying (European) *put options* on the same number of units at some suitable exercise price.

2.2 Let us assume that he finds that 20-year put options on these units are available for any exercise price he chooses. For particular exercise prices he finds that the prices of put options are as follows:

Exercise price	Price of put option
80p	1.9125p
100p	3.5939p
120p	5.8202p
140p	8.5546p
200p	19.3567p
300p	43.6404p
400p	72.7860p
600p	138.5553p
1,000p	281.8077p
10,000p	3,668.8976p

2.3 If he chooses an exercise price of 100p, then one unit plus one option will cost him 100p + 3.5939p = 103.5939p, and he could afford to buy 965.31 units and corresponding options with his £1,000. The minimum return he will get in 20 years will be attained if he has to exercise the option and will be £965.31. Similar calculations for various exercise prices are shown below:

Exercise Price	Cost of Unit plus option	Number of Units	Minimum Value
p	p		£
80	101.9125	981.23	784.99
100	103.5939	965.31	965.31
120	105.8202	945.00	1,134.00
140	108.5546	921.20	1,289.67
200	119.3567	837.83	1,675.65
300	143.6404	696.18	2,088.55
400	172.7860	578.75	2,315.00
600	238.5553	419.19	2,515.14
1,000	381.8077	261.91	2,619.12
10,000	3,768.8976	26.53	2,653.30

2.4 As the exercise price increases the price of the put option also increases, so he can buy fewer units. His minimum return increases, but he will find that, however high the exercise price he chooses, he cannot achieve a minimum return of more than £2,653.30, and that is with a huge exercise price and very few units. In this case, since he has spent almost all his investment on the put option, he has almost no investment in units, and whatever the price of his units at the end of 20 years his return is fixed at £2,653.30.

2.5 At the other extreme, he could choose not to bother with the put option, simply buy 1,000 units, and hold them till the end of the 20 years; in this case his minimum possible return is nil, but he gets the advantage of the maximum investment in units.

2.6 Let us assume that he might first choose a minimum value of £1,500. He has to find what exercise price will give him this guarantee. He finds that a put option with an exercise price of 170.3649p will cost 13.5766p so he can buy 880.46 units to give him exactly a guaranteed amount of £1,500.

2.7 He can carry out the same calculations for various desired minimum values. Particular examples are shown below (the values being rounded):

Minimum Value	Exercise Price	Cost of Unit plus option	Number of Units
£1,000	104.00p	104.00p	961.57
£1,250	134.73p	107.79p	927.75
£1,500	170.36p	113.58p	880.46
£1,750	214.18p	122.39p	817.06
£2,000	272.95p	136.48p	732.72

2.8 At this point a little algebra may clarify matters. Let the price of units be P and the term to go be t; if P and t are fixed the value of a put option can be treated as a function of the exercise price, E, say $W(E)$. Let the sum he wishes to invest be A; let him buy n units and n put options; and let this provide a minimum value of G. Then the price of one share plus one put option is $P + W(E)$. The number of shares plus options he can buy is given by

$$n = A/(P + W(E))$$

The minumum value provided by these n options is given by:

$$G = n.E = AE/(P + W(E)).$$

This minimum value therefore is a function of E, say $G(E)$. If the value of E is chosen, then the value of $G(E)$ is fixed. If a particular minimum value is desired, say g, then the corresponding exercise price, e, must be chosen so that

$$g = G(e).$$

This is an equation in e that has only one root, provided $0 \leqslant g \leqslant M$, where M is the maximum value of G achievable.

2.9 It can be shown that as E increases, the value of $G(E)$ increases asymptotically to M. If W is derived from the Black-Scholes formula, then $M = A.e^{\delta t}$, and it is the amount that would be produced by investment of the sum A at the risk-free (constant) force of interest δ for the period t.

2.10 Let us assume that after reflection the investor decides that a minimum guaranteed amount of £1,250 will satisfy him. He purchases 927.75 units plus corresponding put options, and forgets about his investment for one year.

2.11 One year later he finds that the price of his units now happens to be 122.85p. The value of his put options has changed for two reasons: they are now 19-year options and the price of the share has changed. The put options are now worth 6.0040p each, and his total investment has a value of £1,195.48. If he had invested wholly in units, without buying any options, then his investment would have been worth £1,228.55, but if the price of the units had fallen, the investment he chose would have been more valuable than this at this time.

2.12 The investor now has a similar range of choice as he had initially. He can sell his put options, and use the proceeds, together with the value of his units, to purchase a new combination of units plus put options at a new exercise price to provide a new minimum guaranteed value. This could be either higher or lower than his current minimum value of £1,250.

2.13 Note that I assume, here and throughout, that the investor incurs no transaction costs on rearranging his investments. I ignore commission, bid/offer spread, expenses and taxes. At this stage I wish to concentrate on the principles involved in the option procedure.

2.14 After some calculation he decides to increase his minimum guaranteed value by 5%, to £1,312.50. He can do this by changing his investment to 920.86 units together with put options at an exercise price of 142.5298p, which will cost 6.9678p each. He then carries out these transactions and forgets about his investment for another year.

2.15 A year later he finds that the price of his units has fallen a little, to 121.39p. But his put options have increased in value to 7.5537p, so the value of his total investment has fallen a little less than his units have, to £1,187.43p. Again he can rearrange his investments, and again he chooses to increase his minimum guaranteed amount by 5% to £1,378.13. He continues on these lines for the rest of the 20 years. The prices of units, the corresponding prices of the put options he has purchased, and details of his investment are shown in Table 2.1. (The values

Table 2.1 *Specimen investment in units and put options*
Guarantee £1,250 increasing at 5% pa

Year	Price of units	Amount to invest	Exercise price	Price of put	Number of units	Guaranteed amount	Price of put at end
	p	£	p	p		£	p
0	100	1,000	134.73	7.79	927.75	1,250	6.00
1	122.85	1,195	142.53	6.97	920.86	1,313	7.55
2	121.39	1,187	151.10	8.80	912.06	1,378	4.57
3	180.22	1,685	159.33	5.36	908.19	1,447	6.32
4	169.97	1,601	168.34	7.43	902.54	1,519	9.32
5	155.02	1,483	178.56	10.99	893.46	1,595	6.23
6	212.11	1,951	188.49	7.39	888.72	1,675	4.52
7	266.15	2,405	198.57	5.42	885.78	1,759	6.29
8	253.40	2,300	209.52	7.56	881.46	1,847	24.39
9	145.68	1,499	227.53	30.21	852.28	1,939	21.43
10	184.79	1,758	244.67	26.40	832.30	2,036	17.49
11	232.60	2,081	261.08	21.56	818.87	2,138	12.44
12	295.67	2,523	276.82	15.45	810.94	2,245	10.58
13	338.64	2,832	292.94	13.33	804.61	2,357	6.63
14	412.23	3,370	309.00	8.55	800.95	2,475	4.65
15	466.90	3,777	325.50	6.17	798.37	2,599	1.65
16	575.73	4,610	342.17	2.33	797.43	2,729	1.05
17	602.60	4,814	359.60	1.58	796.73	2,865	.01
18	882.20	7,029	377.58	.02	796.72	3,008	.00
19	967.30	7,707	396.46	.00	796.72	3,159	nil
20	1,217.94	9,704	–	–	–	–	–

here and elsewhere have been accurately calculated to a high number of significant figures, and are quoted in the table rounded; they therefore may not be exactly reproduced by calculation with the rounded values.)

2.16 At the end of the period of 20 years the investor finds that he has 796.72 units plus the same number of options at an exercise price of 396.46p, arranged at the end of the 19th year. The price of the units turns out to be 1,217.94p, so it is not worth exercising the option, the put options are valueless, and his final proceeds resulting from the sale of his units are £9,703.65. If he had not purchased any put options he could have achieved £12,179.43 from investing in units alone, so the guarantee he bought has cost him £2,475.78, or 20.3% of his maximum proceeds.

2.17 We can express his proceeds alternatively by saying that he achieved his desired minimum value of £3,158.69, which equals his original £1,250 increased by 5% compound on 19 (not 20) occasions, and in addition received an extra sum of £6,544.96 making up £9,703.65 as before.

2.18 In this example the investor was actually able to achieve his desired increase in guarantee on each occasion. The unit price increased sufficiently to allow this. He could not have purchased this amount of guarantee initially. In §2.4 it was noted that the maximum guarantee he could have purchased initially was £2,653.30.

2.19 The effect of an insufficient increase in the unit price can be seen by assuming that the investor initially chose a guaranteed amount of £1,750, increasing it by 5% a year. Table 2.2 shows the results after some years. At the end of 10 years he has available £1,739.43 and he wishes to arrange a guarantee of £2,850.57. But the maximum guarantee that he can obtain is only £2,833.36. He cannot effect his desired transaction. He has a great many alternative choices; indeed he can choose any guaranteed amount he likes not exceeding £2,833.36. Two reasonable options are:

(a) increase his guarantee to the maximum possible; if he does this he will have no units left, and at the end of the period of 20 years he will certainly get £2,833.36 with no possibility of anything extra;

(b) keep his investment in units and put options unchanged at this time, foregoing his desired 5% increase; he could then reschedule his future desired increases in a variety of ways.

2.20 Let us assume that he chooses option (b) and reschedules his desired future guarantee amounts to forego this year's increase entirely. Thus he plans just to increase his guaranteed amount by 5% in each future year, if he can. He is able to implement his plan for one further year, but at the end of 12 years he again cannot achieve his desired 5% increase, and again maintains his investment unchanged. This happens repeatedly, and as Table 2.2 shows, he only achieves his desired increase on two further occasions, at the end of years 14 and 18.

2.21 At the end of year 20 he has only 197.27 units plus put options with an exercise price of 1,593.09p. Since the units only have a value of 1,217.94p each it is worth while exercising the option to sell his units, and he ends up with his guaranteed amount of £3,142.75 and nothing more. This guarantee has cost him £9,037, compared with investment wholly in units with no guarantee.

Table 2.2 *Specimen investment in units and put options*
Guarantee £1,750 increasing at 5% pa

Year	Price of units	Amount to invest	Exercise price	Price of put	Number of units	Guaranteed amount	Price of put at end
	p	£	p		£	p	
0	100	1,000	214.18	22.39	817.06	1,750	18.83
. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .
8	253.40	1,996	388.95	46.83	664.76	2,586	99.29
9	145.68	1,628	541.20	178.94	501.63	2,715	161.96
10	184.80	1,739			desired:	2,851	impossible
		maintain:	541.20	161.96	501.63	2,715	141.07
11	232.60	1,874	764.86	270.36	372.69	2,851	240.34
12	295.67	1,998			desired:	2,993	impossible
		maintain:	764.86	240.34	372.69	2,851	227.48
13	338.64	2,110			desired:	2,993	impossible
		maintain:	764.86	227.48	372.69	2,851	193.98
14	412.23	2,259	1,113.63	428.38	268.77	2,993	415.98
15	466.90	2,373			desired:	3,143	impossible
		maintain:	1,113.63	415.98	268.77	2,993	357.85
16	575.73	2,509			desired:	3,143	impossible
		maintain:	1,113.63	357.85	268.77	2,993	370.01
17	602.60	2,614			desired:	3,143	impossible
		maintain:	1,113.63	370.01	268.77	2,993	182.21
18	882.20	2,861	1,593.09	567.98	197.27	3,143	550.95
19	967.30	2,995			desired:	3,300	impossible
		maintain:	1,593.09	550.95	197.27	3,143	375.15
20	1,217.94	3,143	–	–	–	–	–

2.22 Although the price of units fell considerably during year 9 (from 253.39p to 145.68p), he was in fact able to maintain his original plans during this period (though only just). But to increase his guarantee by the planned 5% required him to sell too many units, and the subsequent increases in the unit price meant losses almost every year in his holding of options.

2.23 If he had chosen a yet higher initial guarantee, say £2,000, increasing that at 5%, he would indeed have run into trouble at the end of year 9, at the time when the unit price was at its lowest. He would have had to forego his planned increase both that year and in all but one of the following years, ending with a guaranteed amount of £3,102.66, and no more.

2.24 Table 2.3 summarises the results that our hypothetical investor would have achieved over this specimen period of 20 years, with the unit prices and put option prices as already discussed, and with various amounts of initial guarantee, from £1,000 to £2,000, and various desired rates of increase of guarantee from 2% a year to 6% a year. If the desired target could not be met, strategy (b) has been adopted, i.e. maintaining the investment until an increase could be achieved. The column headed 'Guarantee Achieved' shows the guarantee resulting from this strategy, and the next column, '% of Target', shows how far (if at all) the achievement fell short of the initial target. The final two columns in the table show the 'Cost of Guarantee', i.e. the amount by which the 'Total Proceeds' fall short of the maximum total proceeds of £12,179, which would have been obtained with no guarantee at all, and also the cost as a percentage of this maximum.

2.25 It is clear that the higher the initial guarantee the lower the total proceeds and the higher the cost, provided that the target is achieved. Similarly, the higher the rate of increase of guarantee the lower the total proceeds, again provided that the target is achieved. This regular pattern, however, may be broken when the investor is unable to meet his target and has to cut back his planned guarantee. This happens whenever the initial amount is high enough or the rate of increase is high enough, and the precise outcome depends then on which years the restrictions happened to apply. But in the examples in the table, whenever the target guarantee is restricted, which occurs in eight of the examples, the resulting proceeds are conspicuously low, and the 'Cost of Guarantee' is between 72.5% and 76.6% of the maximum proceeds. The investor, in this particular example of unit prices, would have run into trouble if he required and maintained too high a guarantee over the period of low unit prices in years 9 to 11.

2.26 This is only one example, though a pertinent one, of a possible sequence of unit prices and put option prices. Before I discuss alternative possible sequences, I wish to introduce the regular saver, and discuss his possible strategies, using the same sequence of prices.

Table 2.3 *Investment of £1,000 in specimen units and put options*

Initial guarantee	Increasing at %	Guarantee achieved	% of target	Extra proceeds	Total proceeds	Cost of guarantee	Cost as % of maximum proceeds
£		£		£	£	£	
none	–	–	–	12,179	12,179	nil	nil
1,000	2	1,457	100	10,118	11,575	604	5.0
1,250	2	1,821	100	9,177	10,998	1,181	9.7
1,500	2	2,185	100	7,948	10,133	2,046	16.8
1,750	2	2,549	100	6,284	8,833	3,346	27.5
2,000	2	2,914	100	3,655	6,569	5,611	46.1
1,000	3	1,754	100	9,707	11,461	718	5.9
1,250	3	2,192	100	8,546	10,738	1,441	11.8
1,500	3	2,630	100	6,955	9,585	2,594	21.3
1,750	3	3,069	100	4,509	7,577	4,602	37.8
2,000	3	2,852	81.3	0	2,852	9,328	76.6
1,000	4	2,107	100	9,188	11,295	884	7.3
1,250	4	2,634	100	7,709	10,342	1,837	15.1
1,500	4	3,160	100	5,489	8,649	3,530	29.0
1,750	4	3,278	88.9	69	3,347	8,832	72.5
2,000	4	3,079	73.1	0	3,079	9,100	74.7
1,000	5	2,527	100	8,520	11,047	1,132	9.3
1,250	5	3,159	100	6,545	9,704	2,476	20.3
1,500	5	3,790	100	2,705	6,495	5,684	46.7
1,750	5	3,143	71.1	0	3,143	9,037	74.2
2,000	5	3,103	61.4	0	3,103	9,077	74.5
1,000	6	3,026	100	7,638	10,664	1,516	12.4
1,250	6	3,782	100	4,741	8,522	3,657	30.0
1,500	6	3,199	70.5	0	3,199	8,980	73.7
1,750	6	3,134	59.2	79	3,213	8,967	73.6
2,000	6	3,188	52.7	0	3,188	8,992	73.8

Table 3.1 *Regular Investment of £50 a year*

Guarantee £1,000 increasing at 4% per annum

Year	Price of units	Amount to invest	Maximum present guarantee	Maximum future guarantee	Desired total guarantee	Guaranteed so far	Exercise price	Price of put	Number of units	Price of put at end
	p	£	£	£	£	£	p	p		p
0	100	50.00	133	1,603	1,000	76	174.85	14.40	43.71	11.68
1	122.85	108.80	275	1,477	1,040	163	211.86	18.37	77.04	20.00
2	121.39	158.93	382	1,357	1,082	238	211.53	19.93	112.46	11.91
3	180.22	266.06	610	1,242	1,125	370	286.59	25.63	129.25	29.76
4	169.97	308.16	673	1,133	1,170	436	280.37	28.26	155.45	34.29
5	155.02	344.27	716	1,029	1,217	499	270.50	31.57	184.52	21.11
6	212.11	480.33	951	930	1,265	640	329.11	35.00	194.39	25.94
7	266.15	617.78	1,165	836	1,316	766	376.17	37.13	203.70	43.00
8	253.40	653.75	1,174	746	1,369	837	381.42	44.55	219.42	95.60
9	145.68	579.40	991	660	1,423	854	301.27	56.68	283.52	45.96
10	184.80	704.23	1,147	579	1,480	984	350.84	66.35	280.41	50.92
11	232.60	845.04	1,311	501	1,539	1,114	401.07	71.74	277.66	51.74
12	295.67	1,014.61	1,499	427	1,601	1,246	451.44	72.00	275.96	60.25
13	338.64	1,150.77	1,619	357	1,665	1,364	494.69	78.65	275.77	55.93
14	412.23	1,341.06	1,797	290	1,732	1,491	541.83	75.11	275.18	59.43
15	466.90	1,498.35	1,912	226	1,801	1,610	585.54	77.90	275.03	44.86
16	575.73	1,756.80	2,135	166	1,873	1,738	629.90	60.89	275.96	52.95
17	602.66	1,859.21	2,152	108	1,948	1,855	671.86	70.67	276.12	9.22
18	882.20	2,511.38	2,769	53	2,026	1,988	709.55	14.10	280.19	2.34
19	967.30	2,766.87	2,905	0	2,107	2,107	739.56	3.95	284.88	nil
20	1,217.94	3,469.64	–	–	–	–	–	–	–	–

3. THE SAVER

3.1 Let us now imagine a different investor, who wishes to invest, not a single amount of £1,000, but a regular amount of £50 a year, at the beginning of each of 20 years. He invests in the same units as in the first example, and knowing the progress of unit prices in that example, we can at once calculate the proceeds of his investment if he were to put his entire amount into units, with no guarantee at all. It would come to £5,463.70.

3.2 But our saver also wishes some sort of guarantee of a minimum benefit. He has a considerable choice as to how to arrange this. I shall describe only one possible strategy; others would also be worth considering. At the beginning of the first year the saver has £50 to invest. He decides that he wants a minimum guarantee of £1,000 (which happens to equal 20 times £50). He calculates, knowing the current prices of put options, that the maximum present guarantee he could purchase with his first £50 is £132.66, and that (if the prices of put options remain on their present basis) his 19 subsequent investments could in total purchase a maximum future option of £1,603.30. He could thus obtain a maximum guarantee of £1,735.96. Since his desired guarantee is 57.60% of this he decides to purchase 57.60% of the maximum present guarantee, or £76.42. He arranges this by purchasing 43.71 units at 100p plus the same number of put options at an exercise price of 174.85p, which cost 14.40p each.

3.3 At the end of the first year the unit price is 122.85p (as before) and the price of his options is 11.68p. His investment so far is worth £58.80. Now he has a further £50 to invest, making £108.80 altogether. The maximum present guarantee he can obtain with this is £274.94, and the maximum future guarantee from his next 18 payments is £1,476.95, a total maximum of £1,751.89. He decides that he wishes to increase his planned guarantee by 4%, to £1,040, which is 59.36% of this maximum. Again apportioning the planned guarantee between present and future, he purchases a present guarantee of £163.22, which he arranges by buying 77.04 units at 122.85p plus the same number of put options at an exercise price of 211.86p, which cost 18.37p each.

3.4 He proceeds in this way for the whole of the 20 years. On no occasion is he unable to achieve his planned guaranteed amount. Details of his progress are shown in Table 3.1. At the end of the period he has 284.88 shares, together with put options at an exercise price of 739.56p, which are not worth exercising. His final proceeds are £3,469.64, some £1,362.79 higher than his planned (and achieved) target of £2,106.85. His guarantee has cost £1,994.06 compared with what he could have attained with investment wholly in units with no put options at all.

3.5 It is interesting to note that, in this example, the number of units purchased remains roughly constant from the end of year 9 to the end of the period. Almost all of each year's new investment of £50 has gone towards increasing the guaranteed amount by buying put options at a higher exercise price.

3.6 It is also useful to note that the desired guarantee, which at commencement was 57.60% of the maximum possible, is, from years 9 to 17 inclusive, over 80% of the maximum possible, reaching 86.19% at the end of each of years 9 and 17. The progress of this percentage may be a useful guide to the progress of his investment arrangements.

3.7 In this example it was always possible to maintain the desired guarantee. The percentage just referred to never reached or exceeded 100%. In other cases this would not be the case. Again, various strategies could be used to reduce the planned guarantee in these circumstances. I have used the following strategy, based on strategy (b) for the single payment case described above. In the first place the desired guarantee is maintained at last year's level, all future years being scaled down appropriately. In many cases this is a sufficient reduction for the guarantee to be maintainable this year, at this reduced level, on the usual proportionate basis. In some cases, however, the new desired guarantee cannot be maintained on a proportionate basis. I then keep last year's units and put options unchanged, and invest this year's £50 in new units and put options at the same exercise price. Because this throws a higher requirement on future payments it is likely that the same procedure will be necessary in future years. Indeed it is possible in these circumstances that the final achieved guarantee is less than had been planned at an earlier stage. This strategy, however, seems a realistic one for the individual saver.

3.8 Table 3.2 is similar to Table 2.3, in that it shows the results of investment, this time of £50 a year for 20 years, with the same prices of units and put options as previously, with various initial guarantees and rates of increase. The same conclusions can be drawn: higher guarantees mean lower final proceeds, provided the desired guarantee can be met. If the saver has to scale down his planned guarantee schedule the results are rather similar and rather irregular. In many cases, where he has had to scale down his planned guarantee he ends up with nothing extra; the exercise price he had reached was higher than the final unit price. But in some cases this is not the case; the schedule had to be abandoned at such a time that the then exercise price was less than the final unit price. Whether or not any extra proceeds are available, the total proceeds in the cases where the planned guarantee is not met range from £1,771 to £2,043, whereas the lowest final proceeds where the planned (and more modest) guarantee could be met exceed (in this table) £3,345. There is quite a sharp contrast between the cases which run through smoothly and those where the guarantee needs to be cut back. These observations apply only to this particular example. Whether they are generally true requires further investigation.

4. ANALOGIES WITH LIFE ASSURANCE POLICIES

4.1 The analogy between the investment strategies of my two investors and various types of life assurance contract must by now be apparent. If the investor chooses to have no minimum guarantee he has the equivalent of a unit-linked endowment assurance with no minimum guarantee – ignoring the possibility of death before maturity. At the other extreme, if he chooses the maximum possible guarantee at the outset, and hence thereafter, he has the equivalent of a without-profits endowment assurance. These are simply the extremes of a continuous spectrum which includes all with-profits assurances as intermediate steps.

Table 3.2 *Investment of £50 per annum in specimen units and put options*

Initial guarantee £	Increasing at %	Guarantee achieved £	% of target	Extra proceeds £	Total proceeds £	Cost of guarantee £	Cost as % of maximum proceeds
none	–	–	–	5,464	5,464	nil	nil
1,000	2	1,457	100	3,480	4,937	527	9.6
1,100	2	1,602	100	3,030	4,633	831	15.2
1,200	2	1,748	100	2,422	4,170	1,293	23.7
1,300	2	1,894	100	1,451	3,345	2,118	38.8
1,400	2	1,847	90.6	0	1,847	3,616	66.2
1,000	3	1,754	100	2,756	4,509	955	17.5
1,100	3	1,929	100	1,933	3,861	1,602	29.3
1,200	3	2,043	97.1	0	2,043	3,421	62.6
1,300	3	1,800	78.9	0	1,800	3,664	67.1
1,400	3	1,827	74.4	0	1,827	3,637	66.6
1,000	4	2,107	100	1,363	3,470	1,994	36.5
1,100	4	1,905	82.2	8	1,913	3,550	65.0
1,200	4	1,847	73.1	0	1,847	3,616	66.2
1,300	4	1,850	67.6	0	1,850	3,613	66.1
1,400	4	1,771	60.1	0	1,771	3,692	67.6
1,000	5	1,886	74.6	0	1,886	3,578	65.5
1,100	5	1,881	67.7	0	1,881	3,582	65.6
1,200	5	1,955	64.5	0	1,955	3,509	64.2
1,300	5	1,921	58.5	33.1	1,954	3,510	64.2
1,400	5	1,876	53.0	72.8	1,949	3,515	64.3

4.2 The investment with a single payment corresponds to a single premium policy. The initial minimum guarantee corresponds to the basic sum assured (possibly including the first year's reversionary bonus if the life office allows this in advance). The increases in the guaranteed amount from time to time correspond to declared reversionary bonus. The extra proceeds that result if the value of units exceeds the amount of the guarantee correspond to terminal bonus. If the guarantee cannot be increased as planned (and in my strategy (b) it is either increased as planned or not increased at all) this corresponds to an office declaring no reversionary bonus in that year.

4.3 The regular saving of a fixed amount each year corresponds to an annual premium policy. Again the initial planned guarantee corresponds to the initial sum assured (possibly including the first year's reversionary bonus), successive increases in the planned guarantee correspond to reversionary bonus declared, and the extra proceeds at the end correspond to terminal bonus. The only difference between my strategy and that of a real life policy occurs when bonus cannot be maintained. I distinguish between guarantee purchased so far, and that which is intended to be purchased in future. While my strategy ensures that the former is never reduced, it is possible that the latter cannot be achieved, so in effect it is possible for 'declared bonus' to be reduced. Alternative strategies may be able to avoid this problem or to reduce the chance of it.

4.4 The methodology I have described automatically produces 'surrender values' at any time. The surrender value is simply the value of the units plus the put options, and it is shown in the columns headed 'Amount' in Tables 2.1, 2.2 and 3.1. In the annual premium case the 'paid-up value' is also readily seen. It is the amount of guarantee already purchased, shown in Table 3.1 as 'Guaranteed so far'. If future premiums cease the saver (policyholder) has the units and put options purchased to date, and these can be treated in future as a single premium policy.

4.5 Further variations in a policy can easily be handled: variable premiums, alteration of policy term, partial surrenders, increase of guaranteed amount (within limits) or reduction of guaranteed amount (even to nil), all result in a variation in the number of units and put options held and the exercise price for the put options. All these possible changes simply result in a changed terminal bonus, for that particular policy.

4.6 The costs of treating with-profits policies on the lines I have described are twofold: first, a significant record-keeping and calculation problem, since the details for each policy would have to be held individually; the calculation of the appropriate number of units and exercise price could also be on an individual basis. And secondly, the loss of consistency, since the terminal bonus for each policy should be different, depending on the price of units on the day of maturity and on all previous relevant days. This would make such policies much more like unit-linked than the usual with-profits ones. At the extreme, the policyholder might be able to choose his own reversionary bonus (within limits) with a consequential effect on his terminal bonus. This would be a truly flexible with-profits unit-linked policy, and an appropriate extension of the 'universal life' policy, wherein the policyholder may (within limits) choose his own premium and sum assured on death.

5. VARYING THE FORCE OF INTEREST

5.1 It is now time to disclose the basis that has been used in the examples in Sections 2 and 3. The unit prices have been derived from a rolled-up index, based on the values of the Financial Times – Actuaries All Share Index on the last day of June in each year from 1965 to 1985 inclusive, with dividends after tax at 35% throughout reinvested, and no allowance at all for expenses. The put option prices have been calculated using the Black-Scholes formula, with σ, the standard deviation of the share price, as .2 per year, and δ, the risk-free force of interest, at a uniform .0488 throughout, which corresponds to a rate of interest of 5% per annum (the equivalent of 8% net of tax at 37.5%).

5.2 It can be seen that the rate of interest has an identifiable, and not unexpected, influence on the results. The maximum guarantee available on the initial payment of £1,000 for 20 years was £2,653.30 or £1,000 × 1.05^{20}. The maximum guarantee available on future annual payments of £50 for n years was £50 × $\ddot{s}_{\overline{n}|}$ at 5%. At the start of the 20-year annual payment plan the first £50 could ensure a maximum guarantee of £133 = £50 × 1.05^{20}, and the 19 subsequent payments could ensure a maximum guarantee of £1,603 = £50 × $\ddot{s}_{\overline{19}|}$ at 5%.

5.3 The first variation to be considered is therefore variation of the risk-free force of interest. Two possibilities are: a different fixed rate; and a rate which depends on current conditions from time to time. The table below shows the yield on 2½% Consols, later the Financial Times – Actuaries British Government Securities Irredeemables index, for the last day of June in each year from 1965 to 1985 inclusive. The force of interest has been taken as the net equivalent of these rates, subject to tax at 37.5%, and these rates are also shown in the table.

	Gross Rate %	Net Force		Gross Rate %	Net Force
1965	6.66	.0408	1976	13.87	.0831
1966	6.89	.0422	1977	12.97	.0779
1967	6.80	.0416	1978	12.41	.0747
1968	7.75	.0473	1979	11.42	.0689
1969	9.12	.0554	1980	11.96	.0721
1970	9.47	.0575	1981	12.97	.0779
1971	9.25	.0562	1982	12.56	.0756
1972	9.33	.0567	1983	10.01	.0607
1973	10.43	.0632	1984	10.63	.0643
1974	15.58	.0929	1985	10.07	.0610
1975	14.89	.0890			

5.4 A constant rate of .0488 is therefore rather low, assuming that the rate on irredeemable stock is a reasonable indicator of the general level of fixed interest rates at any time (which itself is an arguable proposition). Tables 5.1 and 5.2 show the results, corresponding to those in Table 2.3, for the single payment of £1,000 on the assumption that put options are priced, first at a uniform force of interest of .0677, equivalent to a rate of 7% per annum, and secondly at the forces of interest shown above. Tables 5.3 and 5.4 do the same for the annual payment of £50, corresponding to Table 3.2.

Table 5.1 *Investment of £1,000 in specimen units and put options*

Put options valued using 7% interest

Initial guarantee £	Increasing at %	Guarantee achieved £	% of target	Extra proceeds £	Total proceeds £	Cost of guarantee £	Cost as % of maximum proceeds
nil	–	–	–	12,179	12,179	–	–
1,000	2	1,457	100	10,519	11,976	204	1.7
1,250	2	1,821	100	9,940	11,761	418	3.4
1,500	2	2,185	100	9,255	11,441	739	6.1
1,750	2	2,549	100	8,443	10,992	1,187	9.7
2,000	2	2,914	100	7,469	10,383	1,797	14.8
1,000	3	1,754	100	10,175	11,929	251	2.1
1,250	3	2,192	100	9,461	11,653	526	4.3
1,500	3	2,630	100	8,599	11,230	950	7.8
1,750	3	3,069	100	7,542	10,611	1,568	12.9
2,000	3	3,507	100	6,202	9,709	2,470	20.3
1,000	4	2,107	100	9,751	11,858	322	2.6
1,250	4	2,634	100	8,856	11,489	690	5.7
1,500	4	3,160	100	7,738	10,898	1,281	10.5
1,750	4	3,687	100	6,285	9,972	2,207	18.1
2,000	4	4,214	100	4,189	8,402	3,777	31.0
1,000	5	2,527	100	9,223	11,750	430	3.5
1,250	5	3,159	100	8,074	11,233	947	7.8
1,500	5	3,790	100	6,558	10,348	1,831	15.0
1,750	5	4,422	100	4,317	8,739	3,440	28.2
2,000	5	4,366	86.4	0	4,366	7,814	64.2
1,000	6	3,026	100	8,556	11,582	597	4.9
1,250	6	3,782	100	7,032	10,814	1,365	11.2
1,500	6	4,538	100	4,781	9,319	2,860	23.5
1,750	6	5,295	89.0	407	5,119	7,061	58.0
2,000	6	3,797	62.7	0	3,797	8,323	68.8

Table 5.2 *Investment of £1,000 in specimen units and put options*

Put options valued using specimen interest rates

Initial guarantee	Increasing at %	Guarantee achieved	% of target	Extra proceeds	Total proceeds	Cost of guarantee	Cost as % of maximum proceeds
£		£		£	£	£	
nil	–	–	–	12,179	12,179	–	–
1,000	2	1,457	100	9,884	11,340	839	6.9
1,250	2	1,821	100	8,767	10,588	1,592	13.1
1,500	2	2,185	100	7,282	9,468	2,712	22.3
1,750	2	2,459	100	5,194	7,743	4,436	36.4
2,000	2	2,914	100	965	3,878	8,301	68.2
1,000	3	1,754	100	9,517	11,271	909	7.5
1,250	3	2,192	100	8,224	10,246	1,754	14.4
1,500	3	2,630	100	6,482	9,112	3,067	25.2
1,750	3	3,069	100	3,784	6,852	5,327	43.7
2,000	3	2,688	76.6	0	2,688	9,492	77.9
1,000	4	2,106	100	9,072	11,179	1,000	8.2
1,250	4	2,634	100	7,569	10,202	1,977	16.2
1,500	4	3,160	100	5,413	8,573	3,606	29.6
1,750	4	3,687	100	703	4,390	7,790	64.0
2,000	4	2,632	62.5	0	2,632	9,548	78.4
1,000	5	2,527	100	8,527	11,054	1,125	9.2
1,250	5	3,159	100	6,717	9,876	2,304	18.9
1,500	5	3,790	100	3,835	7,626	4,554	37.4
1,750	5	3,143	71.1	0	3,143	9,037	74.2
2,000	5	2,553	50.5	25	2,578	9,602	78.8
1,000	6	3,026	100	7,850	10,876	1,304	10.7
1,250	6	3,782	100	5,573	9,355	2,824	23.2
1,500	6	4,282	94.3	183	4,464	7,715	63.3
1,750	6	3,134	59.2	0	3,134	9,045	74.3
2,000	6	2,525	41.7	0	2,525	9,654	79.3

Table 5.3 *Investment of £50 per annum in specimen units and put options*

Put options valued using 7% interest

Initial guarantee	Increasing at %	Guarantee achieved	% of target	Extra proceeds	Total proceeds	Cost of guarantee	Cost as % of maximum proceeds
£		£		£	£	£	
nil	–	–	–	5,464	5,464	–	–
1,000	2	1,457	100	3,836	5,293	171	3.1
1,100	2	1,602	100	3,578	5,180	283	5.2
1,200	2	1,748	100	3,275	5,023	440	8.1
1,300	2	1,894	100	2,913	4,806	657	12.0
1,400	2	2,040	100	2,463	4,502	961	17.6
1,000	3	1,754	100	3,356	5,110	354	6.5
1,100	3	1,929	100	2,966	4,895	569	10.4
1,200	3	2,104	100	2,474	4,578	885	16.2
1,300	3	2,280	100	1,805	4,084	1,380	25.2
1,400	3	2,455	100	563	3,018	2,446	44.8
1,000	4	2,107	100	2,651	4,757	706	12.9
1,100	4	2,318	100	1,972	4,290	1,174	21.5
1,200	4	2,528	100	773	3,301	2,162	39.6
1,300	4	2,341	85.5	0	2,341	3,122	57.1
1,400	4	2,155	73.1	5	2,161	3,303	60.5
1,000	5	2,527	100	1,383	3,910	1,554	28.4
1,100	5	2,401	86.3	0	2,401	3,063	56.0
1,200	5	2,263	74.6	0	2,263	3,201	58.6
1,300	5	2,223	67.7	0	2,223	3,240	59.3
1,400	5	2,172	61.4	0	2,172	3,292	60.2

Table 5.4 *Investment of £50 per annum in specimen units and put options*

Put options valued using specimen interest rates

Initial guarantee £	Increasing at %	Guarantee achieved £	% of target	Extra proceeds £	Total proceeds £	Cost of guarantee £	Cost as % of maximum proceeds
nil	–	–	–	5,464	5,464	–	–
1,000	2	1,457	100	3,420	4,877	587	10.7
1,100	2	1,602	100	3,051	4,654	810	14.8
1,200	2	1,748	100	2,612	4,360	1,103	20.2
1,300	2	1,894	100	2,063	3,957	1,506	27.6
1,400	2	2,040	100	1,294	3,334	2,130	39.0
1,000	3	1,754	100	2,909	4,662	801	14.7
1,100	3	1,929	100	2,384	4,313	1,151	21.1
1,200	3	2,104	100	1,684	3,789	1,675	30.7
1,300	3	2,280	100	348	2,627	2,837	51.9
1,400	3	2,056	83.7	0	2,056	3,408	62.4
1,000	4	2,107	100	2,159	4,265	1,198	21.9
1,100	4	2,318	100	1,223	3,540	1,923	35.2
1,200	4	2,248	88.9	0	2,248	3,216	58.9
1,300	4	2,078	75.9	0	2,078	3,385	62.0
1,400	4	1,993	67.6	0	1,993	3,471	63.5
1,000	5	2,527	100	594	3,121	2,343	42.9
1,100	5	2,178	78.4	68	2,246	3,218	58.9
1,200	5	2,155	71.1	0	2,155	3,309	60.6
1,300	5	2,118	64.5	0	2,118	3,346	61.2
1,400	5	1,970	55.7	0	1,970	3,494	63.9

5.5 Note first that the maximum possible proceeds, with nil guarantee, is the same in these examples as previously; secondly that the planned guarantee is also the same in each example as previously, and therefore produces the same value if the target is met.

5.6 Since a higher risk-free force of interest reduces the price of put options in all cases, the cost of the guarantee is less when 7% interest is used (Table 5.1) than with 5% (Table 2.3) in all cases. Where the guarantee itself was low, the additional proceeds are small. But in only three of the examples in Table 5.1 is the planned guarantee not met in full, compared with eight in Table 2.3, and in some cases the total proceeds are very much larger, sometimes more than three times as much, in one case as in the other. Compare £2,000 initial, increasing at 3%. In Table 2.3 the target is not reached, and the total proceeds are only £2,852; in Table 5.1 the target is reached successfully and the total proceeds are £9,709. Yet when the target is not reached in Table 5.1 the results are strikingly low.

5.7 Table 5.2 is based on variable rates of interest, which are often, but not always, higher than the level rate of Table 2.3; the level rate of Table 5.1 is an intermediate one. The results are striking. At low levels of guarantee the cost turns out to be higher than in Table 2.3, and much higher than in Table 5.1. At high levels of guarantee the cost is sometimes lower than in Table 2.3, though always higher than in Table 5.1. In seven cases the target is not reached, and in the one case where it is reached here, but not in Table 2.3, the resulting proceeds are noticeably higher (initial £1,750 increasing at 4%: £4,390 against £3,347, and £9,972 in Table 5.1).

5.8 When the results in Tables 5.3 (7% interest) and 5.4 (variable interest rates) are compared with those in Table 3.2 (5% interest) similar features can be observed though the contrasts are less. Table 5.3 always shows higher total proceeds than Table 3.2, sometimes much higher (£1,400 at 2%: £4,502 against £1,847). Table 5.4 always shows lower total proceeds than Table 5.3, but higher than in Table 3.2 in all but one of the examples (£1,000 at 2%). Where the target is not reached Table 5.4 is usually closer to Table 3.2 than to Table 5.3. Whereas the target is not reached in thirteen cases in Table 3.2, this is reduced to eight cases in Table 5.4 and to six in Table 5.3. However, the final proceeds in these six cases are strikingly low in all the Tables.

6. VARYING THE BONUS RATE

6.1 In all the examples so far I have assumed that the investor planned to increase his guaranteed amount by a uniform compound percentage. This may not be a realistic strategy in all possible circumstances, though it worked out not too badly with the specimen unit prices I have used so far. An alternative strategy would be to base the increase in the guarantee in some way on the performance of the unit itself. Although I have quoted only the rolled-up price of the 'units', I actually know the amount of dividend added each year and the dividend yield on the underlying share price (since all has been based on the known values of the FTA All-Share Index and the dividend yield thereon). I have assumed that the investor also knows the amount of dividend and that he prefers to consider increases in the underlying dividend income, being more solidly based than increases in the share price due solely to changes in the dividend yield basis. The percentage increases in dividends in the successive years considered are:

Year ending in June	Dividend Increase %	Year ending in June	Dividend increase %
1966	3.34	1976	10.04
1967	-3.14	1977	14.81
1968	1.04	1978	12.87
1969	6.18	1979	10.79
1970	3.89	1980	30.81
1971	3.08	1981	5.18
1972	8.72	1982	8.34
1973	5.99	1983	6.45
1974	9.32	1984	13.51
1975	8.38	1985	20.35

6.2 I consider also the investor transforming the rate of dividend increase into a rate of increase in his guarantee in a variety of ways, which can be expressed as:

Change % in guarantee

= Maximum (a% + $k \times$ Change % in Dividend; b%).

For example, if $a\% = b\% = 0$, and $k = .75$, he increases his guarantee by three quarters of the percentage increase in dividends, but does not reduce his guarantee. If $a\% = -3\%$, $k = 1.0$ and $b\% = -25\%$, he changes his guarantee by 3% less than the full increase in dividends, reducing his guarantee if dividends fall or their increase is less than 3%, but never reducing his guarantee by more than one quarter.

6.3 Table 6.1 shows the results of a single investment of £1,000 and Table 6.2 the results of an annual payment of £50, for the same period of 20 years as before, using the same specimen unit prices and the varying rates of interest used for Tables 5.2 and 5.4, for various amounts of minimum guarantee and various values of k, with a and b both being taken as 0%. Tables 6.3 and 6.4 show similar results, respectively, with initial guarantees of £1,500 and £1,000 respectively, and various values of a and k, with b again taken as 0%.

6.4 Comparison of Tables 6.1 and 5.2 suggests the following rough equivalence, in terms of final proceeds, for this particular example:

Value of k	Uniform compound increase
1.0	6%
.75	4%
.5	3%
.25	2% (or less)

Comparison of Tables 6.2 and 5.4 yields the following very approximate equivalence:

Value of k	Uniform compound increase
1.0	5%
.75	4% - 5%
.5	3% - 4%
.25	2% (or less)

Table 6.1 *Investment of £1,000 in specimen units and put options*

Put options valued using varying interest rates

Guarantee increased by k × *increase in Dividends*

Initial guarantee £	Value of *k*	Guarantee achieved £	% of target	Extra proceeds £	Total proceeds £	Cost of guarantee £	Cost as % of maximum proceeds
nil	-	-	-	12,179	12,179	-	-
1,000	1.0	4,613	100	6,277	10,891	1,289	10.6
1,250	1.0	5,767	100	2,740	8,506	3,673	30.2
1,500	1.0	4,660	67.3	156	4,817	7,363	60.5
1,750	1.0	3,495	43.3	805	4,300	7,879	64.7
2,000	1.0	3,064	33.2	517	3,581	8,598	70.6
1,000	.75	3,202	100	8,033	11,234	945	7.8
1,250	.75	4,002	100	6,159	10,162	2,018	16.6
1,500	.75	4,803	100	2,986	7,788	4,391	36.1
1,750	.75	3,744	66.8	0	3,744	8,436	69.3
2,000	.75	2,953	46.1	0	2,953	9,226	75.8
1,000	.5	2,198	100	9,153	11,351	828	6.8
1,250	.5	2,748	100	7,814	10,562	1,617	13.3
1,500	.5	3,298	100	5,982	9,279	2,900	23.8
1,750	.5	3,847	100	2,941	6,789	5,391	44.3
2,000	.5	2,884	65.6	0	2,884	9,296	76.3
1,000	.25	1,492	100	9,915	11,407	773	6.3
1,250	.25	1,865	100	8,857	10,722	1,457	12.0
1,500	.25	2,238	100	7,475	9,713	2,467	20.3
1,750	.25	2,611	100	5,576	8,188	3,992	32.8
2,000	.25	2,984	100	2,310	5,294	6,885	56.5

Table 6.2 *Investment of £50 per annum in specimen units and put options*

Put options valued using varying interest rates

Guarantee increased by k × *increase in Dividends*

Initial guarantee £	Value of *k*	Guarantee achieved £	% of target	Extra proceeds £	Total proceeds £	Cost of guarantee £	Cost as % of maximum proceeds
nil	–	–	–	5,464	5,464	–	–
1,000	1.0	2,415	52.4	0	2,415	3,048	55.8
1,100	1.0	2,284	45.0	0	2,284	3,180	58.2
1,200	1.0	2,155	38.9	14	2,170	3,294	60.3
1,300	1.0	2,058	34.3	0	2,058	3,406	62.3
1,400	1.0	2,045	31.7	0	2,045	3,419	62.6
1,000	.75	2,601	81.2	1,048	3,648	1,815	33.2
1,100	.75	2,332	66.2	0	2,332	3,132	57.3
1,200	.75	2,250	58.6	434	2,684	2,780	50.9
1,300	.75	2,194	52.7	39	2,233	3,231	59.1
1,400	.75	2,017	45.0	103	2,120	3,344	61.2
1,000	.5	2,198	100	2,313	4,511	953	17.4
1,100	.5	2,418	100	1,530	3,948	1,515	27.7
1,200	.5	2,472	93.7	0	2,471	2,993	54.8
1,300	.5	2,247	78.6	0	2,247	3,216	58.9
1,400	.5	2,138	69.5	260	2,398	3,065	56.1
1,000	.25	1,492	100	3,457	4,949	514	9.4
1,100	.25	1,641	100	3,109	4,750	714	13.1
1,200	.25	1,791	100	2,696	4,486	977	17.9
1,300	.25	1,940	100	2,188	4,127	1,336	24.5
1,400	.25	2,089	100	1,502	3,591	1,873	34.3

Table 6.3 *Investment of £1,000 in specimen units and put options*

Put options valued using varying interest rates

Guarantee initially £1,500 increased using given parameters

Value of *a*	Value of *k*	Guarantee achieved	% of target	Extra proceeds	Total proceeds	Cost of guarantee	Cost as % of maximum proceeds
%		£		£	£	£	
0	1.0	4,660	67.3	156	4,817	7,363	60.5
−1	1.0	4,852	82.8	692	5,544	6,635	54.5
−2	1.0	5,001	100	2,817	7,819	4,361	35.8
−3	1.0	4,265	100	4,643	8,908	3,272	26.9
−4	1.0	3,693	100	5,654	9,347	2,832	23.3
0	.75	4,803	100	2,986	7,788	4,391	36.1
−1	.75	4,062	100	4,812	8,874	3,305	27.1
−2	.75	3,457	100	5,913	9,370	2,809	23.1
−3	.75	2,975	100	6,651	9,626	2,553	21.0
−4	.75	2,605	100	7,142	9,747	2,432	20.0
0	.5	3,298	100	5,982	9,279	2,900	23.8
−1	.5	2,787	100	6,809	9,596	2,584	21.2
−2	.5	2,383	100	7,389	9,773	2,407	19.7
−3	.5	2,089	100	7,759	9,848	2,332	19.1
−4	.5	1,889	100	7,997	9,886	2,293	18.8

Table 6.4 *Investment of £50 per annum in specimen units and put options*

Put options valued using varying interest rates

Guarantee initially £1,000 increased using given parameters

Value of *a*	Value of *k*	Guarantee achieved	% of target	Extra proceeds	Total proceeds	Cost of guarantee	Cost as % of maximum proceeds
%		£		£	£	£	
0	1.0	2,415	52.4	0	2,415	3,048	55.8
–1	1.0	2,492	63.8	0	2,492	2,972	54.4
–2	1.0	2,589	77.6	1,246	3,835	1,629	29.8
–3	1.0	2,843	100	550	3,394	2,070	37.9
–4	1.0	2,462	100	1,900	4,362	1,101	20.2
0	.75	2,601	81.2	1,048	3,648	1,815	33.2
–1	.75	2,708	100	1,024	3,732	1,731	31.7
–2	.75	2,305	100	2,187	4,492	972	17.8
–3	.75	1,984	100	2,790	4,774	690	12.6
–4	.75	1,736	100	3,171	4,907	556	10.2
0	.5	2,198	100	2,312	4,511	953	17.4
–1	.5	1,858	100	2,948	4,806	658	12.0
–2	.5	1,589	100	3,364	4,953	511	9.4
–3	.5	1,392	100	3,629	5,022	442	8.1
–4	.5	1,259	100	3,800	5,059	404	7.4

6.5 It is more interesting to note that, whatever the target guarantee in Tables 5.4 and 6.2, the guarantee achieved is hardly ever over £2,500. A more ambitious target is always frustrated *en route*. The total proceeds are also irregular when the target has not been met. As before, this depends on when the target had to be cut back, and the level of guarantee that had been reached by that time. Similar irregularities are seen in Table 6.4.

7. VARYING THE DATES

7.1 I have already explained that the specimen 20 years experience used so far is based on the FTA All-Share Index from June 1965 to June 1985. It is interesting to see how the results would compare when different time periods are used. However, because the All-Share Index commenced only in 1962 I have reduced the term to 15 years for the following comparisons.

7.2 I have calculated the results for a single payment of £1,000 with an initial guarantee of £1,250 (Table 7.1), an annual payment of £120 with an initial guarantee of £1,800 (Table 7.2), and a monthly payment of £10 with an initial guarantee of £1,800 (Table 7.3). In each case I show the results for the guarantee rising at (a) 4% per annum and (b) 5% per annum. In the monthly payment case the guarantee is assumed to rise every month, at the stated annual effective rate; but it rises for a further 11 months as compared with the annual payment case. As unit prices I used the All-Share Index, rolled up with tax on dividends at 35%, as previously, and to price the put options I use the appropriate yield on irredeemables, net of tax at 37.5%. I show results for investments commencing at the end of June for each year from 1962 to 1971 inclusive (hence maturing in June each year from 1977 to 1986 inclusive).

7.3 It is interesting to observe how often the target guarantee fails to be achieved particularly with a 5% increase; also that although the maximum possible proceeds increase almost uniformly with calendar year, the total proceeds are more irregular, not infrequently falling when the maximum proceeds rise in the single payment examples, and not rising *pari passu* in the annual and monthly payment examples. Indeed the total proceeds from the investment strategy with a guarantee seem to be more volatile than those where there is no guarantee. It is also notable that the annual payment cases do not always provide higher proceeds than the monthly payment ones.

7.4 A comparison over a shorter time period is provided by Tables 7.4, 7.5 and 7.6 which show results corresponding precisely to those in Tables 7.1(a), 7.2(a) and 7.3(a) respectively, for investments commencing at the end of each month from January 1969 to December 1970 inclusive, hence maturing at the end of each month from January 1984 to December 1985 inclusive.

7.5 In almost every month the target guarantee has been met, but even so it is interesting to see how variable the results are. As a percentage of the maximum possible proceeds the final proceeds from the monthly payment schemes are much more stable than those from the annual payment or single payment scheme. It is not surprising when the maximum proceeds rise or fall from one month to the next that the total proceeds should move in the same direction. It is however, surprising that this does not always happen, as in the single payment cases ending in September 1984, December 1984, January 1985 and March 1985.

Table 7.1 *Investment of £1,000 over various 15-year periods*

Unit prices as experience from All-Share Index

Put options priced at current interest rates

Starting date (June)	Finishing date (June)	Guarantee achieved £	% of target	Extra proceeds £	Total proceeds £	Maximum proceeds £	Cost of guarantee £	Cost as % of MP
(a) Guarantee £1,250 rising at 4% per annum								
1962	1977	2,001	92.5	0	2,001	3,396	1,395	41.1
1963	1978	1,924	88.9	0	1,924	3,442	1,517	44.1
1964	1979	1,924	88.9	0	1,924	3,780	1,856	49.1
1965	1980	2,165	100	1,256	3,420	4,669	1,249	26.7
1966	1981	2,165	100	1,097	3,261	4,686	1,425	30.4
1967	1982	2,165	100	1,661	3,825	4,964	1,139	22.9
1968	1983	2,165	100	858	3,023	4,895	1,872	38.2
1969	1984	2,165	100	2,526	4,690	5,691	1,000	17.6
1970	1985	2,165	100	4,823	6,987	7,857	869	11.1
1971	1986	2,165	100	4,628	6,792	8,062	1,269	15.7
(b) Guarantee £1,250 rising at 5% per annum								
1962	1977	2,138	86.4	0	2,138	3,396	1,259	37.1
1963	1978	2,036	82.3	0	2,036	3,442	1,406	40.8
1964	1979	2,036	82.3	0	2,036	3,780	1,744	46.1
1965	1980	2,475	100	413	2,888	4,669	1,781	38.1
1966	1981	2,357	95.2	0	2,357	4,686	2,329	49.7
1967	1982	2,475	100	1,057	3,532	4,964	1,432	28.8
1968	1983	2,036	82.3	215	2,251	4,895	2,644	54.0
1969	1984	2,475	100	1,844	5,691	5,691	1,371	24.1
1970	1985	2,475	100	4,348	6,823	7,857	1,034	13.2
1971	1986	2,475	100	3,941	6,416	8,062	1,646	20.4

Table 7.2 *Investment of £120 per annum over various 15-year periods*

Unit prices as experience from All-Share Index

Put options priced at current interest rates

Starting date (June)	Finishing date (June)	Guarantee achieved £	% of target	Extra proceeds £	Total proceeds £	Maximum proceeds £	Cost of guarantee £	Cost as % of MP
			(a) Guarantee £1,800 rising at 4% per annum					
1962	1977	2,771	88.9	123	2,894	3,693	799	21.6
1963	1978	2,771	88.9	0	2,771	3,900	1,129	28.9
1964	1979	2,882	92.5	0	2,882	4,391	1,509	34.4
1965	1980	3,117	100	0	3,117	4,595	1,478	32.2
1966	1981	3,117	100	596	3,713	5,123	1,410	27.5
1967	1982	3,117	100	738	3,855	4,900	1,045	21.3
1968	1983	3,117	100	2,198	5,315	6,476	1,161	17.9
1969	1984	3,117	100	2,501	5,618	6,588	970	14.7
1970	1985	3,117	100	3,482	6,599	7,587	988	13.0
1971	1986	3,117	100	5,251	8,368	9,497	1,129	11.9
			(b) Guarantee £1,800 rising at 5% per annum					
1962	1977	2,792	78.4	0	2,792	3,693	900	24.4
1963	1978	2,932	82.3	0	2,932	3,900	968	24.8
1964	1979	2,932	82.3	0	2,932	4,391	1,459	33.2
1965	1980	2,932	82.3	0	2,932	4,595	1,663	36.2
1966	1981	3,079	86.4	0	3,079	5,123	2,045	39.9
1967	1982	3,233	90.7	0	3,233	4,900	1,667	34.0
1968	1983	3,394	95.2	844	4,239	6,476	2,238	34.6
1969	1984	3,564	100	1,371	4,935	6,588	1,654	25.1
1970	1985	3,564	100	2,502	6,066	7,587	1,521	20.0
1971	1986	3,564	100	4,264	7,828	9,487	1,670	17.6

Table 7.3 *Investment of £10 per month over various 15-year periods*

Unit prices as experience from All-Share Index

Put options priced at current interest rates

Starting date (June)	Finishing date (June)	Guarantee achieved £	% of target	Extra proceeds £	Total proceeds £	Maximum proceeds £	Cost of guarantee £	Cost as % of MP
(a) Guarantee £1,800 rising at 4% per annum (monthly)								
1962	1977	2,863	88.6	0	2,863	3,607	744	20.6
1963	1978	2,807	86.9	0	2,807	3,811	1,004	26.3
1964	1979	2,765	85.6	0	2,765	4,301	1,536	35.7
1965	1980	2,888	89.4	0	2,888	4,485	1,596	35.6
1966	1981	3,231	100	0	3,231	5,040	1,809	35.9
1967	1982	3,231	100	431	3,663	4,721	1,058	22.4
1968	1983	3,231	100	2,061	5,292	6,337	1,045	16.5
1969	1984	3,231	100	2,367	5,598	6,458	860	13.3
1970	1985	3,231	100	3,265	6,496	7,367	870	11.8
1971	1986	3,231	100	5,067	8,298	9,304	1,006	10.8
(b) Guarantee £1,800 rising at 5% per annum (monthly)								
1962	1977	2,725	73.1	0	2,725	3,607	882	24.4
1963	1978	2,837	76.1	0	2,837	3,811	974	25.6
1964	1979	2,918	78.3	0	2,918	4,301	1,383	32.1
1965	1980	2,842	76.2	0	2,842	4,485	1,643	36.6
1966	1981	2,975	79.8	0	2,975	5,040	2,066	41.0
1967	1982	3,193	85.7	0	3,193	4,721	1,527	32.4
1968	1983	3,492	93.7	0	3,492	6,337	2,845	44.9
1969	1984	3,727	100	1,045	4,772	6,458	1,686	26.1
1970	1985	3,727	100	2,132	5,859	7,367	1,507	20.5
1971	1986	3,727	100	3,949	7,676	9,304	1,629	17.5

Table 7.4 *Investment of £1,000 over various 15-year periods*

Unit prices as experience from All-Share Index

Put options priced at current interest rates

Guarantee £1,250 rising at 4% per annum

Starting date	Finishing date (end of month)	Guarantee achieved	Extra proceeds	Total proceeds	Maximum proceeds	Cost of guarantee	Cost as % of MP
		£*	£	£	£	£	
1969	1984						
	January	1,924	0	1,924	4,540	2,615	57.6
	February	2,165	1,228	3,386	4,929	1,543	31.3
	March	2,165	1,400	3,565	5,235	1,670	31.9
	April	2,165	2,096	4,260	5,477	1,217	22.2
	May	2,165	2,087	4,252	5,227	975	18.7
	June	2,165	2,526	4,690	5,691	1,000	17.6
	July	2,165	2,924	5,089	6,041	952	15.8
	August	2,165	3,180	5,345	6,340	995	15.7
	September	2,165	3,064	5,229	6,487	1,259	19.4
	October	2,165	3,538	5,703	6,902	1,199	17.4
	November	2,165	3,330	5,494	6,823	1,329	19.5
	December	2,165	3,156	5,321	6,847	1,525	22.3
1970	1985						
	January	2,165	3,350	5,515	6,797	1,282	18.9
	February	2,165	3,619	5,783	6,990	1,206	17.3
	March	2,165	3,603	5,768	7,025	1,258	17.9
	April	2,165	4,729	6,894	7,787	893	11.5
	May	2,165	5,626	7,790	8,654	863	10.0
	June	2,165	4,823	6,988	7,857	869	11.1
	July	2,165	4,592	6,757	7,765	1,008	13.0
	August	2,165	5,084	7,248	8,243	994	12.1
	September	2,165	4,338	6,503	7,641	1,138	14.9
	October	2,165	5,009	7,174	8,238	1,064	12.9
	November	2,165	5,978	8,143	9,224	1,081	11.7
	December	2,165	5,266	7,430	8,524	1,093	12.8

*100% of target except for January 1969-84, 88.7% of target.

Table 7.5 *Investment of £120 per annum over various 15-year periods*

Unit prices as experience from All-Share Index

Put options priced at current interest rates

Guarantee £1,800 rising at 4% per annum

Starting date	Finishing date (end of month)	Guarantee achieved	Extra proceeds	Total proceeds	Maximum proceeds	Cost of guarantee	Cost as % of MP
		£*	£	£	£	£	
1969	1984						
January		3,117	2,338	5,456	6,669	1,213	18.2
February		3,117	1,971	5,088	6,437	1,349	21.0
March		3,117	2,555	5,672	6,940	1,268	18.3
April		3,117	2,430	5,547	6,770	1,223	18.1
May		3,117	2,198	5,315	6,258	943	15.1
June		3,117	2,501	5,618	6,588	970	14.7
July		3,117	2,466	5,583	6,509	925	14.2
August		3,117	3,245	6,362	7,061	699	9.9
September		3,117	4,036	7,154	7,475	321	4.3
October		3,117	4,148	7,265	7,675	410	5.3
November		3,117	4,949	8,066	8,113	47	.6
December		3,117	5,047	8,164	8,383	219	2.6
1970	1985						
January		3,117	3,553	6,670	7,856	1,186	15.1
February		3,117	3,134	6,251	7,566	1,314	17.4
March		3,117	3,466	6,583	7,780	1,197	15.4
April		3,117	3,192	6,309	7,463	1,154	15.5
May		3,117	3,693	6,810	7,864	1,055	13.4
June		3,117	3,482	6,599	7,587	988	13.0
July		3,117	3,671	6,788	7,771	984	12.7
August		3,117	4,317	7,434	8,202	768	9.4
September		3,117	4,700	7,817	8,205	388	4.7
October		3,117	5,250	8,368	8,840	473	5.3
November		3,117	6,196	9,313	9,471	158	1.7
December		3,117	5,665	8,782	9,097	315	3.5

*100% of target in all cases.

Table 7.6 *Investment of £10 per month over various 15-year periods*

Unit prices as experience from All-Share Index

Put options priced at current interest rates

Guarantee £1,800 rising at 4% per annum (monthly)

Starting date	Finishing date (end of month)	Guarantee achieved	Extra proceeds	Total proceeds	Maximum proceeds	Cost of guarantee	Cost as % of MP
		£*	£	£	£	£	
1969	1984						
January		3,231	2,566	5,797	6,763	966	14.3
February		3,231	2,470	5,701	6,632	931	14.0
March		3,231	2,821	6,053	7,024	971	13.8
April		3,231	2,935	6,166	7,139	973	13.6
May		3,231	2,260	5,491	6,345	854	13.5
June		3,231	2,367	5,598	6,458	860	13.3
July		3,231	2,303	5,434	6,258	824	13.2
August		3,231	2,701	5,932	6,821	889	13.0
September		3,231	2,852	6,083	6,985	902	12.9
October		3,231	2,915	6,146	7,047	901	12.8
November		3,231	3,078	6,309	7,222	913	12.7
December		3,231	3,418	6,649	7,602	952	12.5
1970	1985						
Janaury		3,231	3,625	6,856	7,826	970	12.4
February		3,231	3,537	6,769	7,715	946	12.3
March		3,231	3,600	6,831	7,773	943	12.1
April		3,231	3,637	6,868	7,805	936	12.0
May		3,231	3,731	6,962	7,903	940	11.9
June		3,231	3,265	6,496	7,367	870	11.8
July		3,231	3,344	6,576	7,450	874	11.7
August		3,231	3,737	6,968	7,885	917	11.6
September		3,231	3,482	6,713	7,589	876	11.5
October		3,231	3,920	7,151	8,076	925	11.4
November		3,231	4,154	7,385	8,332	948	11.4
December		3,231	3,961	7,192	8,110	917	11.3

*100% of target in all cases.

8. STOCHASTIC SIMULATION

8.1 So far I have used the actual experience of the last 24 years in various overlapping periods. But what of the more general case? In order to investigate this I have used my earlier stochastic model, described in Wilkie (1985, 1986a, 1986b) to simulate the possible progress of share prices and of interest rates. This stochastic model generates simulated values of share dividends and share yields, hence share prices, and also values of the rate of interest on irredeemable fixed-interest stocks, all on a consistent basis. The details of the simulation model are not important here; alternative models would have exemplified the method and the specimen results equally well. However, I believe the model to be a realistic one for use in a United Kingdom context.

8.2 I have used the parameters that I call the Standard Basis with neutral starting values. These values and their implications are described in the papers referred to above. I assume tax at 35% on share dividends and at 37.5% on interest, as I have done above. I have carried out a mere 101 simulations for each of the examples below; the same 101 sets of values have been used in each case.

8.3 I revert to a 20-year investment period, and I consider investment of £1,000 in a single payment or of £50 per year, payable annually.

8.4 Since the same set of 101 simulations is used for each example below, the distribution of final unit prices is the same for each example. The mean final unit price after 20 years is 744.37p, equivalent to a 10.56% compound return. The standard deviation is 565.44p, and the distribution is very positively skewed ($\sqrt{\beta_1} = 2.03$) and very fat-tailed ($\beta_2 = 8.75$). The 101 samples of the final unit price range from 118.67p (equivalent to a compound return of .86%) to 3,601.30p (equivalent to a compound return of 19.63%).

8.5 The maximum proceeds are the same, whatever the level of guarantee chosen, and are just 1,000 times the final unit prices given above for the single investment of £1,000. For the annual payment cases the mean of the maximum proceeds is £3,578.02 with a standard deviation of £2,164.20 ($\sqrt{\beta_1} = 1.41$ and $\beta_2 = 4.65$) and a range in the 101 simulations of £891.58 to £11,555.41. The former does not coincide with the lowest final unit price, though the latter does occur in the simulation with the highest final unit price.

8.6 For the single payment cases I try:

(a) initial guarantee £1,250, increasing at 4%, target £2,634;
(b) initial guarantee £1,500, increasing at 4%, target £3,160;
(c) initial guarantee £1,250, increasing at .75 times the rate of increase of dividends (but not decreasing);
(d) initial guarantee £1,500, increasing at .75 times the rate of increase of dividends (but not decreasing).

For cases (c) and (d) the target guarantee depends on the particular simulation. The mean targets are £3,306 and £3,968 respectively, so on average they are much more ambitious than (a) and (b).

8.7 Guarantee (a) reaches the target 69 times out of 101, and the average percentage achieved is 95.3%. The worst case provides 67.6% of the target, only £1,779. Guarantee (b) reaches the target just 50 times, and the average percentage achieved is 90.0%. The worst case provides 62.5% of the target, £1,974 (this is the same simulation as the worst case in (a)).

8.8 Guarantee (c) has a variable target, but it reaches it 87 times out of 101 and the worst percentage achieved is 98.2%. The worst case provides 71.1% of the target, £2,271 instead of £3,193. Guarantee (d) also has a variable target, and it reaches it 51 times out of 101. The average percentage achieved is 92.0% and the worst case is 61.5%, £2,442 instead of £3,968. Thus strategy (c) is rather better than (a) at meeting the target, and (d) is about the same as (b).

8.9 When guarantee (a) is used, the final unit price exceeds the exercise price of the option in 69 simulations, and falls below it in 32. In these 32 simulations the option is exercised, and there is no extra amount available beyond the guarantee. In most of these 69 simulations the target guarantee is also achieved, though in 4 simulations the target is achieved fully only by exercise of the option. By chance there are also 4 simulations where the target is not achieved, but the final unit price is high enough for there to be extra proceeds available. There are also 8 simulations where the cost to the investor is negative, i.e. his final proceeds are higher with the guarantee than they would have been with no guarantee; these include the simulations with a very low final unit price, but they are not just the 8 lowest.

8.10 Continuing with the results of guarantee (a), i.e. £1,250 increasing at 4%, we find that the mean proceeds are £6,237, some 83.8% of the mean maximum proceeds. The mean cost is £1,207, some 16.2% of the mean maximum proceeds. The mean value of cost as a percentage of the maximum is close to this, at 16.0%. The standard deviation of the cost is £1,005 and that of the cost as a percentage of the maximum is 18.7%. The range of this percentage in the 101 simulations is −69.9% (i.e. the final proceeds were 169.9% of the maximum, £2,081 compared with £1,225 with no guarantee) to +66.6% (proceeds £1,924 compared with a maximum of £5,760).

8.11 Comparable figures for the other guarantees for the single payment case are given in Table 8.1. Comparing (b) with (a) or (d) with (c) we see that the higher initial guarantee (£1,500 instead of £1,250) means that we are less likely to reach the target, and the cost of the guarantee is greater. The higher initial guarantee produces a wider spread of the cost of guarantee, but the final proceeds, which have a lower mean, also have a lower standard deviation, so there is somewhat greater certainty of money return. Obviously if the guarantee were taken to its extreme, with the maximum possible guarantee chosen initially, the money return would be certain and the cost of guarantee (which is measured relative to the return on an investment wholly in units) would be very variable.

8.12 Now comparing (c) with (a) or (d) with (b), i.e. increasing the guarantee at three quarters of the rate of increase of dividends instead of at 4% fixed, we see that the 'dynamic' strategy is more likely to reach its target, and has on average a higher target and a higher achieved guarantee. Because of the higher guarantee achieved the cost of the guarantee is greater and the resulting proceeds rather smaller. But the variability of final proceeds is very much reduced, as is the variability of the cost of guarantee. A dynamic strategy for adjusting the guarantee in the light of actual investment performance is more likely to be satisfactory than a rigid one. Guarantee (c) is better in this respect than (a) by more than (d) is better than (b); thus the rough equivalence between some fixed rate of increase of guarantee, such as 4%, and the fraction of dividend increase used, such as .75, is not the same for all amounts of initial guarantee.

Table 8.1 *Summary of 101 simulations, £1,000 single payment, 20 years*

Guarantee Basis	(a)	(b)	(c)	(d)
Initial Guarantee £	1,250	1,500	1,250	1,500
Increasing at	4%	4%	.75 × dividends	.75 × dividends
Mean target guarantee	2,634	3,160	3,306	3,968
(s.d.)	(0)	(0)	(1,189)	(1,427)
Mean guarantee achieved £	2,510	2,845	3,261	3,670
(s.d.)	(299)	(382)	(1,223)	(1,481)
Mean percentage of target	95.3	90.0	98.2	92.0
(s.d.)	(8.7)	(12.1)	(5.6)	(10.6)
Mean final proceeds £	6,237	5,479	6,114	4,932
(s.d.)	(5,153)	(4,678)	(4,734)	(3,643)
Mean maximum proceeds £	7,444	7,444	7,444	7,444
(s.d.)	(5,654)	(5,654)	(5,654)	(5,654)
Mean cost £	1,207	1,964	1,330	2,511
(s.d.)	(1,005)	(1,903)	(1,114)	(2,494)
Mean percentage cost	16.0	22.6	16.7	26.9
(s.d.)	(18.7)	(26.2)	(11.4)	(22.0)
(range)	(−69.9 to 66.6)	(−96.1 to 85.8)	(−34.2 to 64.5)	(−57.9 to 67.7)
Number of simulations in which:				
target achieved	69	50	87	51
target not achieved	32	51	41	50
option not exercised	69	54	82	53
option exercised	32	47	19	48
target achieved and option exercised	4	2	10	10
target not achieved but extra proceeds available	4	6	5	12
cost of guarantee negative	8	12	5	9

8.13 For the annual payment case I try in all cases an initial guarantee of £1,000 increasing:

(e) at 3%, target £1,754;
(f) at 4%, target £2,107;
(g) at .5 times the increase in dividends;
(h) at .6 times the increase in dividends.

The results are shown in Table 8.2. Similar conclusions can be drawn from these results as for the single payment case. I leave it to the reader to observe this from the table.

9. FURTHER CONSIDERATIONS

9.1 In section 4 I pointed out analogies between my hypothetical investment strategy using put options and with-profits policies with reversionary and terminal bonus. My single payment, annual payment and monthly payment results are equivalent to the results from single premium, annual premium and monthly premium endowment assurances, but only to a limited extent. The actual amount of proceeds is not strictly comparable with the proceeds from a real life insurance policy for a number of reasons.

9.2 In the first place I have made no allowance for expenses. A real life office has to incur selling and administrative expenses. It also incurs expenses in managing its investments and in purchasing and selling securities, as reflected in its bid/offer spread on unit-linked policies and annual management charges deducted from the income of units. Allowance for actual or hypothetical expenses would not be difficult. Expenses would also be incurred in the purchase or sale of real traded options, but my options are hypothetical ones, and I do not see any need to make an adjustment to the results for this reason.

9.3 Secondly, I have made no allowance for mortality and have assumed survival to the end of the term. For short term endowment assurances at young ages the effect of mortality, though present, is not large, and an approximate allowance for the cost of mortality could be made. But to convert my method to a whole of life policy requires more adjustment, and conventional methods may need some adaptation.

9.4 Thirdly, I have assumed investment in a unit whose price moved exactly in line with the F.T.A. All-Share Index, which represents U.K. ordinary shares. But life offices invest in a whole range of other assets, and it would be more appropriate to consider the actual portfolios in which the assets relating to with-profits policies may have been invested.

9.5 Fourthly, a life office has additional sources of profit (or loss), from mortality, expenses, surrenders, etc, quite apart from investment profits. With-profits policyholders expect to share in the fortunes of the office on these accounts too.

9.6 Fifthly, and most importantly, I have looked at the investment wholly from the policyholder's (or investor's) point of view. The life office is seen as a writer of put options which the policyholder buys, exchanges and sometimes exercises. The extra proceeds, or terminal bonus, that I have calculated gives the policyholder 'fair value' for his investment, taking account of the guaranteed amount, or

Table 8.2 *Summary of 101 simulations, £50 p.a. payment, 20 years*

Guarantee Basis	(e)	(f)	(g)	(h)
Initial Guarantee £	1,000	1,000	1,000	1,000
Increasing at	3%	4%	.5 × dividends	.6 × dividends
Mean target guarantee	1,754	2,107	1,911	2,176
(s.d.)	(0)	(0)	(456)	(625)
Mean guarantee achieved £	1,689	1,875	1,877	2,044
(s.d.)	(91)	(205)	(458)	(578)
Mean percentage of target	96.3	89.0	98.2	94.6
(s.d.)	(5.2)	(9.8)	(3.8)	(8.5)
Mean final proceeds £	2,904	2,667	2,821	2,601
(s.d.)	(1,807)	(1,624)	(1,569)	(1,304)
Mean maximum proceeds £	3,578	3,578	3,578	3,578
(s.d.)	(2,164)	(2,164)	(2,164)	(2,164)
Mean cost £	674	911	757	977
(s.d.)	(587)	(865)	(813)	(1,224)
Mean percentage cost	15.7	19.7	16.9	20.0
(s.d.)	(20.1)	(25.5)	(16.4)	(21.0)
(range)	(−74.7 to 56.1)	(−94.2 to 55.9)	(−58.9 to 52.8)	(−69.9 to 75.6)
Number of simulations in which:				
target achieved	59	32	77	55
target not achieved	42	69	24	46
option not exercised	64	45	72	51
option exercised	37	56	29	50
target achieved and option exercised	3	1	14	16
target achieved and extra proceeds available	8	14	9	12
cost of guarantee negative	13	15	12	13

reversionary bonus, declared *en route*. But since, in my examples, the guarantee on average costs the policyholder something (only in a minority of simulations was the cost negative), the life office correspondingly makes a profit. If we imagine a proprietary life office that issues contracts precisely in the form I have described, i.e. unit-linked policies with explicit guarantees, then we see that the shareholders of such a company need to put up sufficient capital to finance the writing of the options, and require to receive a suitable profit thereon. But in a mutual life office (or in a proprietary office that behaves like one) the with-profit policyholders themselves carry all (or nearly all) the risks, and just as the cost to the office of providing a guarantee for policyholders whose policies mature falls on the remaining with-profits policyholders, so the profit to the office if the guarantee is charged for on the basis I have described falls to the credit of the remaining policyholders, among whom could be numbered the holders of currently maturing policies. That is, the with-profits policyholders may share also in the profits and losses of writing options for each other, which can be effected by adjusting their unit price.

9.7 Nevertheless, I suggest that the calculations I have made would give guidance on the extent to which terminal bonus should be modified to allow for the previous amounts of guaranteed reversionary bonus. The 'asset share', or accumulation of premiums (net of expenses and a charge for mortality) as invested in the actual portfolio of the office (or a notional portfolio with similar return), corresponds to my maximum proceeds, and provides a starting point for consideration of the amount of terminal bonus. My cost of guarantee, perhaps taken as a percentage of the maximum proceeds, gives a starting point for consideration of how much to alter the asset share.

9.8 I said at the start that I was not going to consider questions of valuation or solvency. Nevertheless some comments are appropriate. The derivation of the option pricing formula assumes a 'hedge ratio', or proportion invested in the underlying security balanced by a proportion invested in the 'risk free asset', i.e. in fixed interest securities of an appropriate date. For the combination of share and put option that I have described the theoretical hedged or 'matched' position can be calculated. But just as with Redingtonian immunisation, in order to maintain the hedged position at all times it is necessary to change it continuously and costlessly, and for this action to have no effect on market prices. This is impractical. Indeed, those who organise markets in traded options recognise these aspects by requiring the writers of options to put up reserves in the form of 'cover' or 'margin' to ensure that they will be able to meet their obligations, and also by restricting the number of options written and still open at any one time to a small fraction of the total number of corresponding shares outstanding. Thus, while the theoretical hedged position is of interest, the requirement on a life office to maintain sufficient assets with sufficient solvency margins seems to me to be unchanged by my analysis.

9.9 A further observation may, however, be useful. In my various examples of investment strategies, a relevant figure to calculate was the ratio of the guarantee desired at each revision date to the maximum guarantee that could have been achieved at that date (taking into account the guarantee that it was expected could be obtained from future payments). In Table 2.2, for example, at the end of year 8 the investor had arranged a guarantee of £2,586, which was 72.1% of the maximum that he could have chosen at that time. At the end of year 9 he increased his guarantee by 5%, to £2,714, which was 97.5% of the maximum possible at that time. Thereafter he ran into trouble. If he had adopted a different strategy, of not increasing has guarantee if that were to take him to over 90% of the maximum possible, then he would have had to maintain his guarantee unchanged on only three occasions (instead of seven as in Table 2.2) and he would have ended up with a higher guarantee of £3,820 (compared with £3,143) and much higher final proceeds of £5,695 (compared with £3,143). One can suggest an *ad hoc* rule, that the guarantee should not be increased to more than 90% of the maximum possible (though if it is above 90% it can be maintained). Translated into life office terms, this suggests that, if an office values its assets at market value and its liabilities on a comparable realistic basis but with no allowance for future bonus, and finds that the value of its liabilities exceeds some percentage, such as 90%, of the value of its assets, then it should consider very carefully whether or not it should declare *any* reversionary bonus. Long before this point, of course, it may have had to reconsider whether it could maintain unchanged its current *rate* of reversionary bonus.

10. CONCLUSION

The analogy between an option and a guarantee on a unit-linked life policy is not new, as papers by Boyle and Schwartz (1977), Fagan (1977), Brennan and Schwartz (1979), Collins (1982) and Beenstock and Brasse (1986) demonstrate. The application to bonus policy on with-profits policies I believe to be new, though it was inspired by the work of the authors just named. My presentation of the idea in this paper, although lengthy enough, is still only an introduction to possible applications of the idea. In particular, more work could be done on (a) how to allow for mortality, (b) what is the matching investment policy for the writer of such options, how should he value them and what margins does he then need, (c) alternative bonus strategies, (d) the implications for bonus illustrations, (e) the respreading of profits and losses on the hypothetical options among other with-profits policyholders, (f) the effect of using different option pricing models (see Appendix A.6.3), and (g) the effect of using different asset models in the simulations. But these can wait till another day.

APPENDIX

OPTION PRICING THEORY

A.1 INTRODUCTION

A.1.1 The modern theory of option pricing has not received consideration at a sessional meeting of either the Institute or the Faculty, although a paper entitled 'Investment Options and Traded Trusts' by Muckart and Smith was presented to the Faculty of Actuaries Students' Society in 1984. This, however, discussed traded options from a speculative rather than an actuarial point of view. It is therefore necessary to introduce the mathematical-statistical approach to options to the profession, so that it can also become the actuarial one.

A.1.2 The statistical approach to options involves a number of ideas that are not familiar to all actuaries, and do not appear in any part of the present syllabus. I shall therefore have to derive my results from first principles. I hope those who have travelled the road, or part of it, before, will not mind going over some familiar ground, and that those who find this all new will find something of interest along the way as well as at the final destination.

A.1.3 None of what follows is new. The approach I use leans heavily on that of Collins (1977). Those who have studied stochastic processes may find my development somewhat too elementary, so if anyone wishes to present the ideas more rigorously, they are welcome to do so.

A.2 PRELIMINARIES

A.2.1 *The Discrete Random Walk*

A.2.1.1 Consider a variable X which takes integral values, and which changes between integral time intervals, $t = 0, 1, 2, \ldots$. Denote its value at time t by $X(t)$. Its changes are governed by the rule:

$$X(t+1) = X(t) + 1 \text{ with probability } p,$$
$$X(t) - 1 \text{ with probability } q = 1-p.$$

Successive changes in the value of X are independent.

A.2.1.2 Let us choose our origins in space and time so that $X(0) = 0$.

Then:

$$P(X(1) = +1) = p,$$
$$P(X(1) = -1) = q,$$

and it is easily verified that:

$$E(X(1)) = p-q,$$
$$\text{Var}(X(1)) = 4pq.$$

A.2.1.3 By time t, $X(t)$ can take alternate integral values in the range $(-t, +t)$, and, because successive steps are independent, its probability distribution over these values is binomial, with

$$E(X(t)) = t(p-q),$$
$$\text{Var}(X(t)) = 4tpq.$$

A.2.1.4 Since we choose arbitrary origins, and since p and q do not depend on time, we see that the distribution of $X(u)-X(t)$ is also binomial, depending only on $(u-t)$, with

$$E(X(u)-X(t)) = (u-t)(p-q),$$
$$\text{Var}(X(u)-X(t)) = 4(u-t)pq.$$

A.2.1.5 As $(u-t)$ increases, the distribution of $X(u) - X(t)$ approaches the normal distribution, with the mean and variance given above.

A.2.1.6 An example of such a discrete space, discrete time 'random walk' is given by two gamblers, who play a series of games, in each of which the probabilities of one of them, say A, winning or losing are p and q respectively $(p+q=1)$ and a stake of one unit is paid by the loser to the winner of each game. Then $X(t)$ represents A's accumulated gains (losses if negative) from the start of the series to the end of the tth game.

A.2.1.7 There are many interesting results to be obtained from a consideration of this discrete random walk, but we are not concerned with them here.

A.2.2 *The Continuous Random Walk*

A.2.2.1 Now let us change our unit of space (or currency) to ΔX and our unit of time to Δt. In time interval $(0,t)$ there are then $t/\Delta t$ steps, each of size ΔX. We then get

$$E(X(t)) = \frac{t}{\Delta t}(p-q)\,\Delta X,$$

$$\text{Var}\,(X(t)) = \frac{4t}{\Delta t}\,pq\,(\Delta X)^2.$$

A.2.2.2

$$\text{Now put } p = \frac{1}{2} + \frac{\mu}{2\sigma^2}\,\Delta X$$

$$q = \frac{1}{2} - \frac{\mu}{2\sigma^2}\,\Delta X$$

$$\text{so that } (p-q) = \frac{\mu}{\sigma^2}\,\Delta X,$$

$$pq = \frac{1}{4} - \frac{\mu^2}{4\sigma^4}\,(\Delta X)^2,$$

$$E(X(t)) = t\,\frac{\mu}{\sigma^2}\,\frac{(\Delta X)^2}{\Delta t},$$

$$\text{Var}\,(X(t)) = 4t\left(\frac{1}{4} - \frac{\mu^2}{4\sigma^4}(\Delta X)^2\right)\frac{(\Delta X)^2}{\Delta t}.$$

A.2.2.3 Now let $\Delta X \rightarrow 0$, and $\Delta T \rightarrow 0$ in such a way that $\frac{(\Delta X)^2}{\Delta t} \rightarrow \sigma^2$. The steps in space become infinitesimal, and their frequency becomes infinite; the probabilities p and q both tend to 1/2, but from opposite sides. But

$$E(X(t)) = t\,\frac{\mu}{\sigma^2}\,\frac{(\Delta X)^2}{\Delta t} \rightarrow t\,\frac{\mu}{\sigma^2}\,.\,\sigma^2 = \mu\,t.$$

$$\text{Var}\,(X(t)) = t\left(1-\frac{\mu^2}{\sigma^4}(\Delta X)^2\right)\frac{(\Delta X)^2}{\Delta t} \rightarrow \sigma^2\,t.$$

A.2.2.4 There are infinitely many steps between 0 and t so $X(t)$ is distributed normally with mean $\mu\,t$ and variance $\sigma^2 t$. Likewise $X(u) - X(t)$ is normally distributed with mean $\mu\,(u-t)$ and variance $\sigma^2(u-t)$. We could say that μ and σ^2 are the mean and variance of the distribution of $X(u)-X(t)$ per unit of time difference.

A.2.2.5 $X(t)$ is continuous, but nowhere differentiable at least according to the usual rules of differentiation. Instead we introduce ideas of stochastic differentiation.

$$X(t+\Delta t) - X(t) \sim \text{N}(\mu\,\Delta t,\, \sigma^2\,\Delta t)$$

or, if you will excuse the notation,

$$\sim \mu\,\Delta t + \sigma.\text{N}\,(0,\,\Delta t),$$

$$\frac{X(t+\Delta t) - X(t)}{\Delta t} \sim \mu + \frac{\sigma}{\Delta t}\,.\,\text{N}\,(0,\,\Delta t),$$

$$\frac{\text{d}X}{\text{d}t} = \underset{\Delta t \rightarrow 0}{\text{Lt}}\left(\frac{X(t+\Delta t) - X(t)}{\Delta t}\right) \sim \mu + \frac{\sigma}{\text{d}t}\,.\,\text{N}\,(0,\,\text{d}t).$$

It is convenient to write $\text{d}z \sim \text{N}(0,\text{d}t)$,

whence $\frac{\text{d}X}{\text{d}t} = \mu + \sigma\,\frac{\text{d}z}{\text{d}t}$

or $\text{d}X = \mu\,\text{d}t + \sigma\,\text{d}z$,

where $\text{d}z$ is understood as representing an infinitesimal random variable which has a normal distribution with mean zero and variance $\text{d}t$.

A.2.2.6 If $\mu = 0$ and $\sigma = 1$ we obtain

$$\text{d}X = \text{d}z.$$

The movement of X in this case is called the Wiener process.

A.2.2.7 It is possible to define continuous space, continuous time processes where

$$\text{d}X = \mu\,(t,X(t))\,\text{d}t + \sigma(t,X(t))\,\text{d}z,$$

i.e. μ and/or σ are not constant, but are some functions of t and/or the value of $X(t)$. Such processes are Gauss-Wiener processes, and have many applications that we shall not consider further here.

A.2.2.8 When a sunbeam shines into a dark room you can see particles of dust dancing in the air. The motion of such particles in space was first studied scientifically by Robert Brown in 1826. The particles of dust are knocked around by innumerable collisions with the very small molecules of air. The resulting motion of the particles is known as Brownian motion, and it is described statistically, in three dimensions, by just the sort of process we have described above.

A.3 OPTIONS

A.3.1 *Description*

A.3.1.1 A *call* option gives the *purchaser* of the option the right (if he wishes to exercise it) to *purchase* from the *writer* of the option a given number of units of some *underlying security* (eg, an ordinary share) at a fixed price (the *exercise price*), at a fixed date (the *exercise date*), in exchange for a fee (or *premium*) paid by the purchaser to the writer.

A.3.1.2 A *put* option has a similar definition, except that the purchaser has the right to *sell* a given number of units of the underlying security to the writer, instead of purchasing them.

A.3.1.3 A so-called *European* option restricts the exercise of the option to a fixed date, as in the definitions above. A so-called *American* option allows the owner of the option the right to exercise it *on or before* the exercise date.

A.3.1.4 With such definitions the range of possible options that could be written is unlimited. Traded options restrict the range to the shares of a small number of large companies, and define the permitted exercise dates (at three month intervals) and exercise prices (in tidy round numbers). Instead of each purchaser and writer having to find the other, both to create the option and to settle it, the exchange facilitates the creation of options and pools the final settlements. The options traded on the London exchange are 'American' ones.

A.3.1.5 Commission is payable on the purchase or sale of an option, as it is on ordinary shares, but in what follows I shall ignore all consideration of commissions, expenses, stamp duties and taxes. Would that this were possible in real life!

A.3.2 *Profit Profiles*

A.3.2.1 Let us consider first a call option for one share at an exercise price E. Let the price of the share on the exercise date be P. Then, if P is greater than E, it is worth exercising the option; the holder gets a share, value P, at a price of E, so his profit on exercising is $P - E$. If P is less than E, it is not worth exercising the option, so he gets nothing and pays nothing, value 0. We can put:

$$\text{Value of option at exercise date} = \begin{cases} P - E & \text{if } P > E, \\ 0 & \text{if } P \leqslant E. \end{cases}$$

If $P = E$ the two expressions give the same result, 0, so it does not matter where we put the equality sign. I assume that an option with zero value is not worth the trouble of exercising.

A.3.2.2 Since the purchaser paid a premium, say F, for the option, his (accounting) profit is given by:

$$\begin{array}{ll} P - E - F & \text{if } P > E, \\ 0 - F & \text{if } P \leqslant E. \end{array}$$

This can be shown in the diagram below, where the sloping line is at an angle of 45° to the horizontal.

Purchaser of Call Option

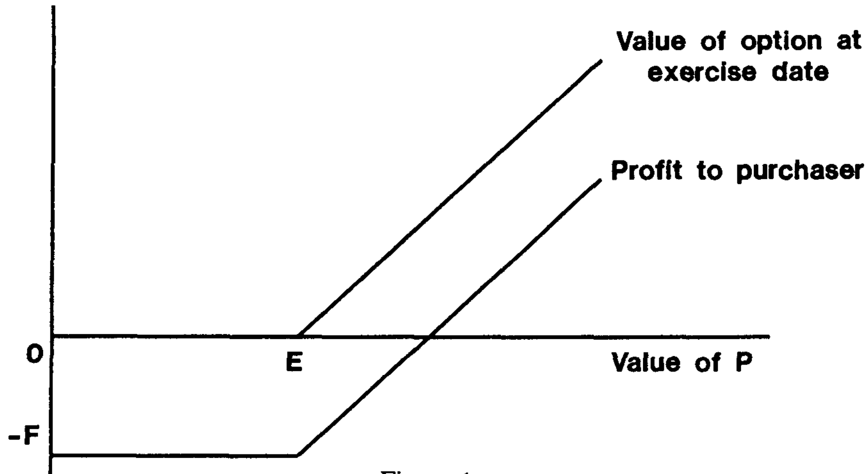

Figure 1

A.3.2.3 The profit (loss) to the purchaser provides an exactly opposite loss (profit) to the writer, whose profit is thus:

$$\begin{array}{ll} E + F - P & \text{if } P > E, \\ F & \text{if } P \leqslant E, \end{array}$$

as shown below.

Writer of Call Option

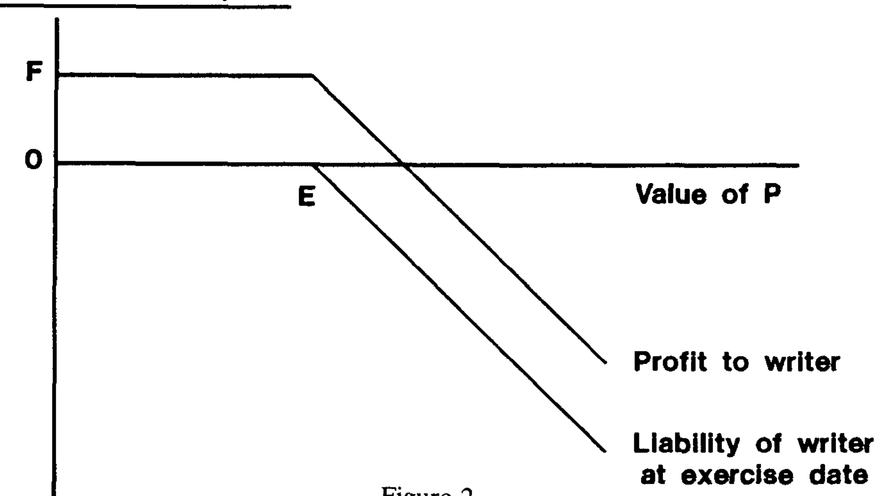

Figure 2

A.3.2.4 The holder of a put option will choose to exercise on the exercise date if P is less than E. The value to him at the exercise date is:

$$\begin{array}{ll} E - P & \text{if } P < E, \\ 0 & \text{if } P \geqslant E, \end{array}$$

and his profit if he paid a premium of F is

$$\begin{array}{ll} E - P - F & \text{if } P < E, \\ -F & \text{if } P \geqslant E, \end{array}$$

as shown below.

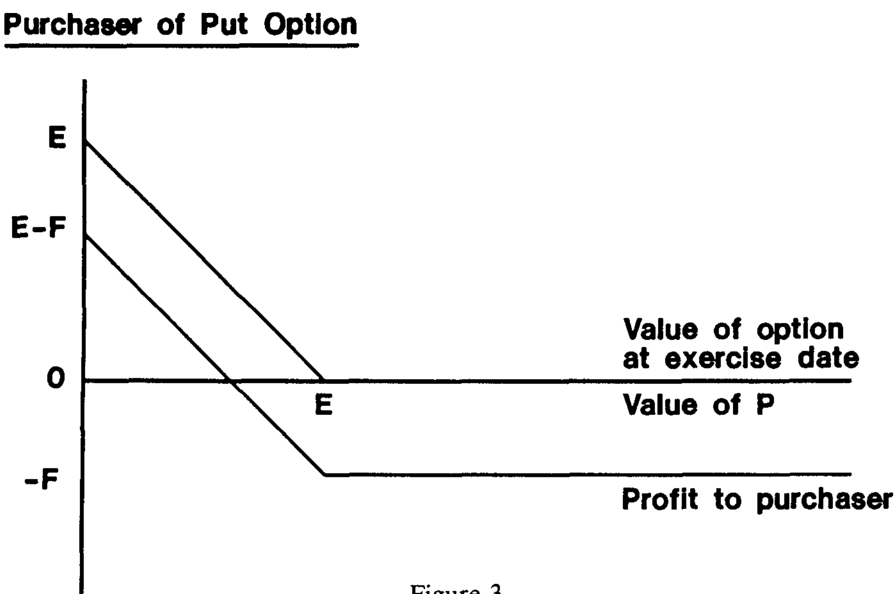

Figure 3

A.3.2.5 Likewise the writer of a put option obtains a profit of

$$\begin{array}{ll} P - E + F & \text{if } P < E, \\ F & \text{if } P \geqslant E. \end{array}$$

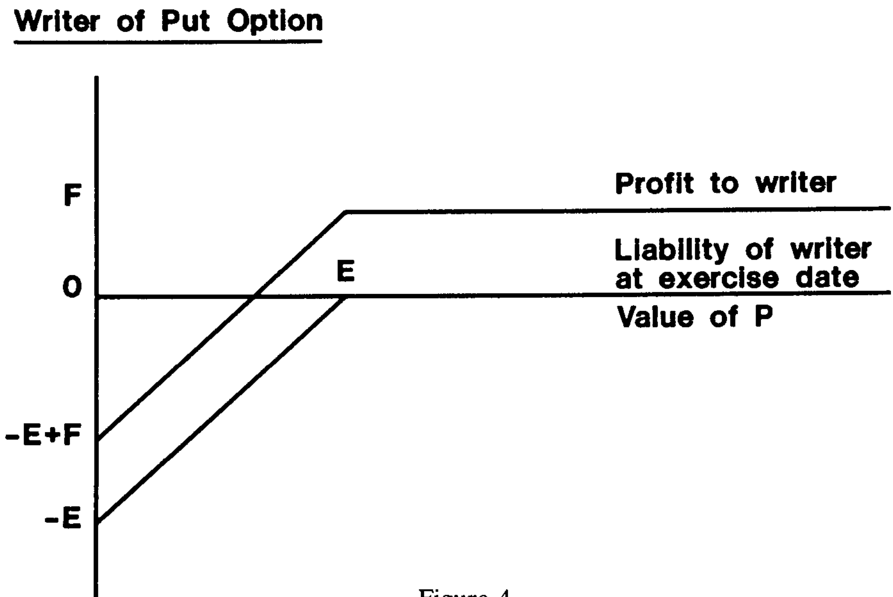

Figure 4

A.3.2.6 Combinations of options at the same exercise date, at the same or different exercise prices, can be described in similar diagrams. A useful one to note is that, for a holder:

$$\text{Call} + \text{Cash} = \text{Put} + \text{Share}.$$

The financial effect of holding a call option plus cash of E on the exercise date is the same as holding a put option for the same exercise price plus a share. The value of the combination at the exercise date is, in each case:

$$\begin{array}{l} P \text{ if } P \geqslant E, \\ E \text{ if } P \leqslant E. \end{array}$$

A.4 VALUE OF A EUROPEAN CALL OPTION

A.4.1 *Limits on the Values*

A.4.1.1 Consider first a European call option, exercisable only at time u at an exercise price E. Consider it at time t, $t \leqslant u$, when the price of the underlying security is $P = P(t)$. Let the value of the call option in these circumstances be $W = W(P,t)$. Assume, for the time being, that the underlying security is in fact an ordinary share and that there are no dividends payable on the share (strictly, it does not go ex-dividend) before the exercise date. What can we say about W?

A.4.1.2 When $t = u$ we know the value, from the profit profile above.

$$W(P,u) = \begin{cases} P(u) - E & \text{if } P(u) > E \\ 0 & \text{if } P(u) \leqslant E \end{cases}$$

This gives us one boundary.

A.4.1.3 We can also say that $W(P,t) \geqslant 0$, since the option is never a liability to the holder, whatever the price of the share.

A.4.1.4 Further, we can say that $W(P,t) \leqslant P$, since, to acquire the share by exercising the option will cost us E, and however small E is, it is never negative, so the option must be worth less than owning the share outright.

A.4.1.5 We can also say that $W(P,t) + E \geqslant P$, since the option plus the cash to exercise it must be worth at least as much as the share. In fact, if we can earn interest at a constant risk-free force δ on cash between t and u, we have:

$$W(P,t) \geqslant P - Ee^{-\delta(u-t)}.$$

A.4.1.6 This gives us limits within which W must lie:

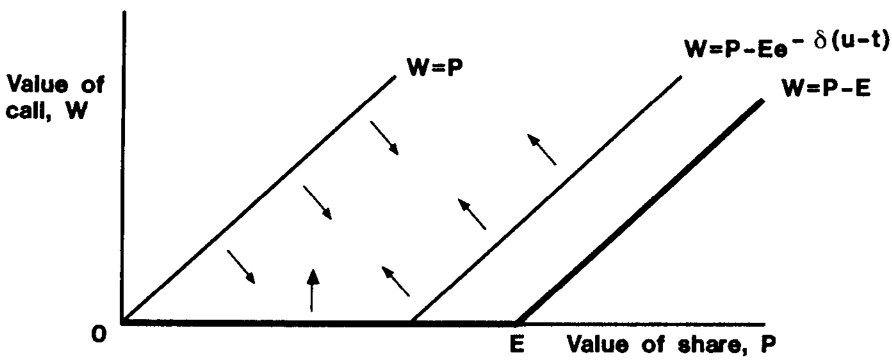

Figure 5

The bold line gives the value of $W(P,u)$. The arrows show the region within which $W(P,t)$ must lie.

A.4.1.7 If $P = 0$ we have $W(0,t) \geqslant 0$ and $W(0,t) \leqslant 0$, so $W(0,t) = 0$.

A.4.1.8 A more subtle condition we shall require is that, as P tends to infinity, W, which lies between $P - Ee^{-\delta(u-t)}$ and P, goes up smoothly, so that:

$$\frac{\partial W}{\partial P} \to 1 \text{ as } P \to \infty.$$

Strictly, since prices of both share and option are quoted in round currency units, e.g. pence, this condition is not satisfied in practice; but, ignoring this, we deny the possibility of the value of W going up in a wavy fashion.

A.4.1.9 It is also reasonable to expect that, if P increases, so also does W (or at least it doesn't reduce) so

$$\frac{\partial W}{\partial P} \geqslant 0.$$

Both these conditions are for fixed t.

A.4.1.10 We thus have a general form for $W(P,t)$

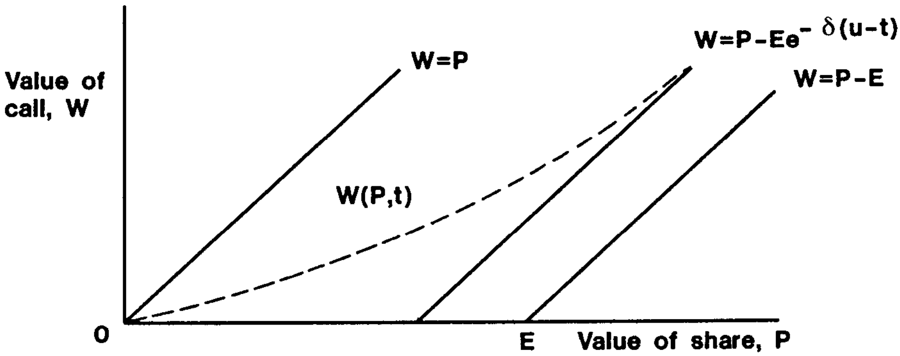

Figure 6

A.4.1.11 It now looks reasonable to expect the value of the option to reduce as t approaches u. Hence, keeping P fixed,

$$\frac{\partial W}{\partial t} \leqslant 0.$$

The dotted line in A.4.1.10 approaches the boundary (the bold line in A.4.1.6) as t approaches u.

A.4.2 *The Hedging Strategy*

A.4.2.1 Now for the hard part. Imagine that one sets up a 'portfolio' by buying or selling bits of option, share and cash, putting say amounts h_1, h_2, h_3 as follows:

h_1 into the option,
h_2 into the share,
h_3 into cash.

A.4.2.2 We do this at time t_0, when the share has price $P_0 = P(t_0)$ and the option has price $W_0 = W(P_0, t_0)$. We thus buy:

$\frac{h_1}{W_0}$ of an option at price W_0, cost h_1.

$\frac{h_2}{P_0}$ of a share at price P_0, cost h_2.

The total cost of our portfolio is

$$V_0 = \frac{h_1}{W_0} . W_0 + \frac{h_2}{P_0} . P_0 + h_3,$$

and this is its value at time t_0.

A.4.2.3 A moment later the price of the share will have changed; so will the value of the option; the cash (h_3) will have earned some interest, we assume at a risk-free force of δ. The value of our portfolio has therefore also changed. We can describe its rate of change:

$$\frac{dV}{dt} = \frac{h_1}{W_0} \cdot \frac{dW}{dt} + \frac{h_2}{P_0} \cdot \frac{dP}{dt} + h_3 \delta.$$

A.4.2.4 Now we need to make some strong assumptions. Assume that there are no taxes, no transaction costs and we can continuously buy or sell options or shares in any quantities we wish, small or large, without restriction, and without affecting the prices of the option or the share. Then we can continuously change our proportions in the portfolio in any way we wish. In particular, we can start out with $V_0 = 0$, i.e. set up a matching portfolio by buying some option, selling some of the share and balancing with borrowing or lending cash. The writer of an option could imagine doing this, by selling one option, borrowing some cash, and investing in an appropriate amount of share.

A.4.2.5 We thus postulate that $V_0 = h_1 + h_2 + h_3 = 0$. Now, *if* we can arrange the portfolio so that, whatever happens to the share price, we know what the change of the value of the portfolio will be, then

$$\frac{dV}{dt} = \text{constant},$$

and we have a wholly risk-free portfolio. But economic theory says that, in a fully competitive market, with the strong assumptions made above, you cannot make a profit out of an investment with no value unless there is some risk attached. So *if* we can arrange that

$$\frac{dV}{dt} = \text{constant},$$

that constant must be zero. There are no free lunches.

A.4.2.6 We now need to postulate a stochastic model for the change in the share price P. We shall assume that the natural logarithm of the share price follows a Gauss-Wiener process, with constant μ and σ;

$$d \ln P = \mu \, dt + \sigma dz,$$

$$\text{or} \qquad dP = \mu P dt + \sigma P dz.$$

A.4.2.7 Repeated investigations have shown that this is approximately true for share prices over a short period, such as the typical few months duration of an option. Other investigations, such as those of the Maturity Guarantees Working Party (1980) cast doubt on this model over longer periods, such as a number of years. But we shall assume here that the model holds.

A.4.2.8 We now need some more stochastic calculus. $W = W(P,t)$ is a function of P and t, and $P = P(t)$ is a stochastic function of t. By Taylor's theorem:

$$W(P+\Delta P, t+\Delta t) = W(P,t) + \Delta P \frac{\partial W}{\partial P} + \Delta t \frac{\partial W}{\partial t}$$
$$+ \frac{1}{2}\left\{ \Delta P^2 \frac{\partial^2 W}{\partial P^2} + 2\Delta P \Delta t \frac{\partial^2 W}{\partial P \partial t} + \Delta t^2 \frac{\partial^2 W}{\partial t^2} \right\}$$
$$+ \frac{1}{6}\left\{ \Delta P^3 \frac{\partial^3 W^3}{\partial P^3} + \dots \right\} + \dots$$

Usually we would let ΔP, Δt tend to zero, giving

$$\frac{dW}{dt} = \underset{\Delta P, \Delta t \to 0}{\mathrm{Lt}} \frac{W(P+\Delta P, t+\Delta t) - W(P,t)}{\Delta t} = \frac{\partial W}{\partial P}\frac{dP}{dt} + \frac{\partial W}{\partial t}.$$

But when we have stochastic functions, such as P, we need to remember that $\frac{\Delta P^2}{\Delta t}$ tends to a finite value, in this case $\sigma^2 P^2$ (cf A.2.2.3).

Thus we have

$$\frac{W(P+\Delta P, t+\Delta t) - W(P,t)}{\Delta t} = \frac{\Delta P}{\Delta t}\frac{\partial W}{\partial P} + \frac{\partial W}{\partial t} + \frac{1}{2}\frac{\Delta P^2}{\Delta t}\frac{\partial^2 W}{\partial P^2}$$
$$+ \Delta P \frac{\partial^2 W}{\partial P \partial t} + \frac{1}{2}\Delta t \frac{\partial^2 W}{\partial t^2}$$
$$+ \frac{1}{6}\frac{\Delta P^3}{\Delta t}\frac{\partial^3 W}{\partial t^3} + \dots$$

As $\Delta P, \Delta t \to 0$ and $\frac{\Delta P^2}{\Delta t} \to \sigma^2 P^2$ we get:

$$\frac{dW}{dt} = \mathrm{Lt} \frac{W(P+\Delta P, t+\Delta t) - W(P,t)}{\Delta t} = \frac{dP}{dt}\frac{\partial W}{\partial P} + \frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2 P^2 \frac{\partial^2 W}{\partial P^2}$$

which has an extra term at the end as compared with the non-stochastic formula.

This result is known as *Ito's lemma*.

A.4.2.9 We now substitute for $\frac{dW}{dt}$ in $\frac{dV}{dt}$ to get:

$$\frac{dV}{dt} = \frac{h_1}{W_0}\left\{\frac{dP}{dt}\frac{\partial W}{\partial P} + \frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2P^2\frac{\partial^2 W}{\partial P^2}\right\} + \frac{h_2}{P_0}\frac{dP}{dt} + h_3\delta$$

and substituting for $\frac{dP}{dt} = \mu P + \sigma P\frac{dz}{dt}$, we get

$$\frac{dV}{dt} = \frac{h_1}{W_0}\left(\mu P + \sigma P\frac{dz}{dt}\right)\frac{\partial W}{\partial P} + \frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2P^2\frac{\partial^2 W}{\partial P^2}$$
$$+ \frac{h_2}{P_0}\left(\mu P + \sigma P\frac{dz}{dt}\right) + h_3\delta$$
$$= \left(\frac{h_1}{W_0}\frac{\partial W}{\partial P} + \frac{h_2}{P_0}\right)\mu P + \frac{h_1}{W_0}\frac{\partial W}{\partial t} + \frac{h_1}{W_0}\frac{1}{2}\sigma^2P^2\frac{\partial^2 W}{\partial P^2} + h_3\delta$$
$$+ \left(\frac{h_1}{W_0}\frac{\partial W}{\partial P} + \frac{h_2}{P_0}\right)\sigma P\frac{dz}{dt}$$

an expression in which the first part is non-stochastic, and the final term in $\frac{dz}{dt}$ is stochastic.

A.4.2.10 Now, if we put the factor multiplying $\frac{dz}{dt}$ to zero, that is

$$\frac{h_1}{W_0}\frac{\partial W}{\partial P} + \frac{h_2}{P_0} = 0.$$

then the stochastic part of $\frac{dV}{dt}$ is zero, and the rest is non-stochastic, so at any fixed time it has a certain value. We thus have met the condition we were seeking in A.4.2.5. above.

A.4.2.11 From $V = h_1 + h_2 + h_3 = 0$, we get $h_3 = -(h_1 + h_2)$.

From $\frac{dV}{dt} = 0$ we get

$$\left(\frac{h_1}{W_0}\frac{\partial W}{\partial P} + \frac{h_2}{P_0}\right)\mu P + \frac{h_1}{W_0}\frac{\partial W}{\partial t} + \frac{h_1}{W_0}\frac{1}{2}\sigma^2P^2\frac{\partial^2 W}{\partial P^2} - (h_1+h_2)\,\delta = 0.$$

But the factor multiplying μP is the same as the one multiplying $\sigma P \dfrac{dz}{dt}$, which has been set to zero, so we get, multiplying through by W_0 ($\neq 0$)

$$h_1 \frac{\partial W}{\partial t} + h_1 \frac{1}{2} \sigma^2 P^2 \frac{\partial^2 W}{\partial P^2} - (h_1 + h_2)\, \delta W_0 = 0.$$

From above we have

$$h_2 = -h_1 \frac{P_0}{W_0} \frac{\partial W}{\partial P},$$

and substituting this gives:

$$h_1 \frac{\partial W}{\partial t} + h_1 \frac{1}{2} \sigma^2 P^2 \frac{\partial^2 W}{\partial P^2} - h_1 \delta W_0 + h_1 \delta P_0 \frac{\partial W}{\partial P} = 0$$

or dividing through by h_1 ($\neq 0$)

$$\frac{\partial W}{\partial t} + \frac{1}{2} \sigma^2 P^2 \frac{\partial^2 W}{\partial P^2} - \delta\, W_0 + \delta\, P_0 \frac{\partial W}{\partial P} = 0.$$

A.4.2.12 This last expression gives a differential equation for W at time t_0. But, provided we can change our portfolio continuously to meet the required conditions, we can keep it so that V=0 and $\dfrac{dV}{dt} = 0$ at all times. We can do this by ensuring that at time t:

$$h_2 = h_2(t) = -\frac{P}{W} h_1 \frac{\partial W}{\partial P} = -\frac{P}{W} \frac{\partial W}{\partial P} \text{ if } h_1 = 1.$$

Since one option is worth W and one share is worth P, the function $\dfrac{\partial W}{\partial P}$ gives us the number of shares the writer of one call option needs to buy to keep exactly 'hedged' ('immunised' is Redington's word), and it is known as the 'hedging ratio'.

A.4.3 *The Black-Scholes Solution*

A.4.3.1 The differential equation in A.4.2.11 looks formidable, but Black and Scholes (1973) found a solution to it that satisfies the boundary conditions noted above. The solution is:

$$W(P,t) = P \,.\, \mathrm{N}(d_1) - Ee^{-\delta(u-t)}.\, \mathrm{N}\,(d_2)$$

$$\text{where } d_1 = \frac{(\ln P/E + (\delta + \sigma^2/2)(u-t))}{\sigma\sqrt{u-t}}$$

$$d_2 = \frac{(\ln P/E + (\delta - \sigma^2/2)(u-t))}{\sigma\sqrt{u-t}}$$

and $\mathrm{N}(d)$ is the normal distribution function:

$$\mathrm{N}(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-x^2/2}\, dx,$$

A.4.3.2 It is convenient to write $T = u-t$, the time to expiry of the option, giving

$$W(P, T) = P \,.\, \mathrm{N}(d_1) - Ee^{-\delta T}\,.\, \mathrm{N}\,(d_2)$$

$$d_1 = \frac{(\ln P/E + (\delta + \sigma^2/2)T)}{\sigma\sqrt{T}} = \frac{\ln\,(P/Ee^{-\delta T})}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}$$

$$d_2 = \frac{(\ln P/E + (\delta - \sigma^2/2)T)}{\sigma\sqrt{T}} = \frac{\ln\,(P/Ee^{-\delta T})}{\sigma\sqrt{T}} - \frac{\sigma\sqrt{T}}{2}$$

Thus the calendar date disappears from the solution. Note that we assumed that μ and σ were constant, and did not change with time. If they were constant over the interval $(u-t)$, but changed from time to time, one would need to use the appropriate values in the formula. The same is true of δ.

A.4.3.3 But where has μ gone? It doesn't appear in the solution. Remember that in A.4.2.11 it was multiplied by the same factor as $\frac{dz}{dt}$, so by eliminating the stochastic part we also eliminated μ, the mean rate of change of the share price. This has the remarkable result that the value of the option, derived in this way, does not depend at all on the expected value of the change in (the logarithm of) the share price, but only on the variance σ^2. Whether you think the price of the share will rise or fall over the period to the exercise date does not affect the value of the option.

A.4.3.4 It is not difficult to verify that the solution above does in fact satisfy the boundary conditions described in A.4.1.6, A.4.1.7 and A.4.1.8, remembering that

$$\frac{dN(d)}{dd} = \frac{1}{\sqrt{2\pi}} e^{-d^2/2},$$

and noting that d_1 and d_2 both tend to $+\infty$ as $T \to 0$ if $P/E > 1$ and both tend to $-\infty$ as $T \to 0$ if $P/E < 1$.

A.4.3.5 We can also derive the hedging ratio

$$\frac{\partial W}{\partial P} = N(d_1),$$

and we can verify our postulates in A.4.1.4 and A.4.1.5 that

$$P \geqslant W \geqslant P - Ee^{-\delta T}.$$

A.4.3.6 Further analysis shows us that

$$\frac{\partial W}{\partial T} > 0,$$

the value of the option increases with time to expiry, that

$$\frac{\partial W}{\partial \sigma} > 0,$$

the value of the option increases with increasing σ, and that

$$\frac{\partial W}{\partial \delta} > 0,$$

the value of the option increases as the risk free force of interest increases.

[Hint: It helps to prove first that $Pe^{-d_1^2/2} = Ee^{-\delta T} e^{-d_2^2/2}$.]

A.5 VALUE OF A EUROPEAN PUT OPTION

A.5.1. In order to derive the value of a European put option we can either repeat the development of A.4, suitably modified, or we can use the equivalence suggested in A.3.2.6. At time t, T before the exercise date of u, consider two possible portfolios:

(a) one share at current price $P(t)$,
 plus one put option with exercise price E;

(b) one call option also with exercise price E and the same exercise date as the put option,
 plus cash of amount $Ee^{-\delta T}$ invested in a risk-free security.

By the time the exercise date is reached the cash will have earned interest at force δ so it will have risen to a value of exactly E. If the price of the share at the exercise date , $P(u)$, exceeds E, then the put option is worthless and the call option has value $P(u) - E$. Thus both portfolios have the value P. If the price of the share at the exercise date is less than E, then the call option is worthless and the put option has value $E - P(u)$, so both portfolios have the value E. The two portfolios have the same value in all circumstances, so are exactly equivalent, and have the same value at all times. Thus if $Wp(P,t)$ is the value of the put option at time t we have:

$$\text{(a)} = P(t) + Wp(P,t) = W(P,t) + Ee^{-\delta T} = \text{(b)}$$

Hence

$$Wp(P,t) = Ee^{-\delta T} + W(P,t) - P(t),$$

which simplifies readily to

$$Wp(P,t) = Ee^{-\delta T}.\mathrm{N}(-d_2) - P.\mathrm{N}(-d_1),$$

where all the terms have the same definitions as in A.4.3.

[Note that $1-\mathrm{N}(d) = \mathrm{N}(-d)$.]

A.6 OTHER CONSIDERATIONS

A.6.1 *American Calls and Puts*

A.6.1.1 The option prices derived above are those for European options which can be exercised only on the stated exercise date. It can be shown that, if no dividends are due on the share in the period during which the option may be exercised, then an American call option has the same value as a European call; it is never worth exercising it early.

A.6.1.2 The same is not true for an American put option, which it may be worth exercising early, in effect because one can earn interest on the cash received by selling the share at the exercise price. Although the differential equation and boundary conditions from which the price of an American put can be derived are known, a closed-form solution has not been found, and it is necessary to use numerical methods to find the theoretical value.

A.6.2 *Dividends*

A.6.2.1 The basic Black-Scholes formula assumes that no dividends are payable on the share during the period up to the exercise date (or strictly, the share does not go ex-dividend). Various adjustment formulae have been derived that allow for dividends, which depend on whether the amounts and the due dates of dividends are known in advance, or whether a stochastic model for them must be assumed. The way in which actual traded options are adjusted for dividends, if at all, is also relevant.

A.6.3 *Alternative stochastic models.*

A.6.3.1 In the derivation of the basic Black-Scholes model it was assumed that the risk-free force of interest, δ, was constant over the period up to the exercise date. An alternative is to assume that the risk-free force of interest is a function of time, $\delta(t)$, which itself follows some stochastic process. This was investigated by Merton (1973).

A.6.3.2 A further assumption was that the share price actually followed the stochastic process whose differential equation was given in A.4.2.6:

$$dP = \mu P dt + \sigma P dz,$$

and that μ and σ were constant over the lifetime of the option. A number of alternative models have been investigated, and are described by Jarrow and Rudd (1983) and by Cox and Rubinstein (1983). These include models where the standard deviation, σ, is a function of time, $\sigma(t)$, and also those which include a Poisson 'jump process' as well as the 'diffusion process', dz.

A.7 FURTHER READING

A.7.1 The best available book describing the mathematics of option pricing is Jarrow and Rudd (1983). This starts from first principles, and is both clear and comprehensive. Brenner (1983) has edited a collection of articles on option pricing which elaborate certain points. In both books can be found very many references to the numerous articles that have appeared since the original work by Black and Scholes (1973) and Merton (1973).

A.7.2 Many of the practical aspects of traded options are described by Chamberlin (1983), which avoids algebra entirely. At the other extreme, the least incomprehensible book on stochastic calculus is that by Malliaris and Brock (1982), which describes in a full, rigorous and yet readable way what I have tried to explain more simply above.

REFERENCES

BEENSTOCK, M. & BRASSE, V. (1986) Using Options to Price Maturity Guarantees. *J.I.A.*, **113**, 151.

BLACK, F. & SCHOLES, M. (1973) The Pricing of Options and Corporate Liabilities. *Journal of Political Economy*, **81**, 637.

BOYLE, P.P. & SCHWARTZ, E.S. (1977) Equilibrium Prices of Guarantees under Equity-linked Contracts. *The Journal of Risk and Insurance*, **44**, 639.

BRENNAN, M.J. & SCHWARTZ, E.S. (1976) The Pricing of Equity-linked Life Insurance Policies with an Asset Value Guarantee. *Journal of Financial Economics*, **3**, 195.

BRENNAN, M.J. & SCHWARTZ, E.S. (1979) Alternative Investment Strategies for the Issuers of Equity-linked Life Insurance with an Asset Value Guarantee. *Journal of Business*, **52**, 63.

BRENNER, M., Editor (1983) *Option Pricing, Theory and Applications*. Lexington Books, D.C. Heath and Company, Lexington, Mass.

CHAMBERLIN, G. (1983) *Trading in Options, Second Edition*. Woodhead-Faulkner, Cambridge.

COLLINS, T.P. (1977) *Brownian Motion*. Unpublished note.

COLLINS, T.P. (1982) An Exploration of the Immunization Approach to Provision for Unit-linked Policies with Guarantees. *J.I.A.*, **109**, 241.

COX, J.C. & RUBINSTEIN, M. (1983) A Survey of Alternative Option-pricing Models. In Brenner (1983).

FAGAN, J.C. (1977) Maturity Guarantees under Investment-linked Contracts. Presented to the Society of Actuaries in Ireland, 10 November 1977.

JARROW, R.A. & RUDD, A. (1983) *Option Pricing*. Richard D. Irwin, Homewood, Ill.

MALLIARIS, A.G. & BROCK, W.A. (1982) *Stochastic Methods in Economics and Finance*. North-Holland, Amsterdam.

Maturity Guarantees Working Party (1980) Report. *J.I.A.*, **107**, 101.

MERTON, R. (1973) Theory of Rational Option Pricing. *Bell Journal of Economics and Management Science*, Spring 1973, **141**.

MUCKART, R.D. & SMITH, W.J. (1984) Investment Options and Traded Trusts. Presented to the Faculty of Actuaries Students' Society, 6 February 1984.

WILKIE, A.D. (1985) Steps Towards a Stochastic Investment Model for Actuarial Use. *O.A.R.D.*, **36**, Institute of Actuaries.

WILKIE, A.D. (1986a) A Stochastic Investment Model for Actuarial Use. *T.F.A.*, **39**, 341.

WILKIE, A.D. (1986b) Some Applications of Stochastic Investment Models. *J.S.S.*, **29**, 25.

ABSTRACT OF THE DISCUSSION

Mr J.M. Maud (opening the discussion): Until a few months ago I was almost entirely ignorant about option pricing theory. Having been introduced to the subject, I was astonished that a theory with so many obvious uses in actuarial work is so little known to many of us.

In the Appendix to his paper the author has produced a description of options and an explanation of the Black-Scholes formula which should not present too many difficulties to actuarial readers. Nevertheless, some people may have been put off by the fearsome looking differential equations in section A4. If this applies to you, I would just draw your attention to the remarkable result noted by the author in § A4.3.3. We would intuitively expect that the higher the expected return on an equity portfolio, the lower would be the value of an option to sell the portfolio at a given fixed price at some future date. In fact the value is quite independent of the expected return. This surprising result follows naturally from the hedging strategy which is used to derive the Boundary condition satisfied by the Black-Scholes formula, and which relies on the option writer's ability to immunize himself against the effect of fluctuating prices, rather than on the need to offset the guarantee against the expectation that an equity portfolio will out-perform a risk-free investment.

Turning now to the detail of the paper, the author examines first a single premium with-profit policy and then an annual premium one. In the latter case he outlines, in § 3.2, just one possible strategy for purchasing guarantees out of a series of annual premiums. This is based on a uniform percentage of the maximum present guarantee which could theoretically be bought with each premium. Another possible approach would be by analogy with the proportionate paid-up policy values. In this example of a 20-year policy with a guaranteed pay-out of £1,000, a guarantee of £50 might be purchased in year 1 and £104, which includes £4 bonus, in year 2. This compares with £76.42 and £163.22 using the author's method. The lower level of guarantee in the earlier years seems to correspond more closely with common practice in the case of endowment assurances and I see no need for a higher guarantee.

In § 4.3 the analogy between the author's example and a real life policy breaks down when bonus cannot be maintained. He assumes that existing bonus would be cancelled if a risk-free investment could not otherwise meet the guaranteed sum at maturity. He is looking at a single policy when he does this. In practice, there might be scope for cross-subsidy between generations in these circumstances. It is also probably the case that some of the guarantees used in Table 3.2 are rather greater than a prudent life office would have offered in practice at that time. Also, of course, the office would not have invested solely in equities, so I do not think that we should read too much into these results. In § 5.1, the author reveals to us the assumptions he has used in the earlier sections: he has allowed for tax on income at appropriate rates but no capital gains tax. Although this would have been a distraction in the present paper, a suitable CGT assumption would be needed to apply the method in practice.

In § 5.7, the author describes the 7% level rate of interest in Table 5.1 as being intermediate between his original 5% assumption in Table 2.3 and the variable rates he then used in Table 5.2. In fact the variable interest rates are equivalent to a level interest rate of 7% a year over the term for a single premium investment. I do not see why the 7% interest rate is described here as an intermediate one. Certainly the results in Table 5.1 do not generally lie between those in the other two tables. The variable interest rates over the period follow a pattern which is rather like the cross-section of an escarpment: climbing up the steep slope in the first nine years and then going gently down the other side. This presumably accounts for the variation of the results in the three tables and demonstrates how volatile the results are as the interest rate assumption is varied. Similarly, the results are highly variable as the bonus rate and policy date assumptions are changed, in sections 6 and 7. I think this demonstrates what we should have known all along: if they are set at a moderate level, guarantees can be financially insignificant but, beyond a certain point, their effect can rapidly become overwhelming.

If the paper helps us to assess when that point is reached, it is certainly valuable.

I was puzzled by § 9.6. In the examples given, on average the guarantee costs the policyholder a positive amount. Remembering the hedging strategy and the derivation of Black-Scholes, why should this be so? Is the option writer getting a free lunch? I came to the conclusion that he is making a profit only if he invests wholly in equities rather than in the hedging portfolio, which will probably put some money in risk-free interest-bearing deposits. In all cases I think the expected equity return assumed by the author exceeds that on risk-free investments. This is an entirely reasonable assumption, but if the two rates of return were brought into line I suspect the profits would disappear. I would have welcomed a mention of this in the paper.

I have not commented, so far, on the assumptions which underlie the author's results. The absence of initial expenses and mortality are just details which are unlikely to alter the pattern of the results. The absence of transaction costs and the ability to deal in large amounts of securities in an infinitesimal time are necessary approximations. In practice one would aim to include a margin to allow for them. The assumptions which concern me most are those which may be good enough for most traded options, whose lifetime is usually a matter of months, but may break down if we are looking at a much longer time scale. For example, I suspect that the nature of the equity market may change over the years so that the standard deviation of the return in successive periods could vary. More seriously the assumption that risk-free investments will continue to be available at a given fixed interest rate looks unrealistic.

It could produce substantial distortions if we are looking at a period of up to 20 years. We have only to compare the results assuming 5% and 7% interest rates to imagine how critical this assumption might be. The author is aware of these aspects and in § A6.3 of the Appendix he has noted alternative models which have been developed taking into account the variability of risk-free interest rates and the changing volatility of equity markets. I have not followed up these references so I cannot tell whether they lead to workable methods. I would certainly feel more confident of the results from a model which allowed for the stochastic nature of interest rates. With this proviso I would like to see life office actuaries using and developing the methods which the author has outlined for us. Certainly any development which helps us to assess the cost of guarantees and bonus expectations must be of interest to the profession, particularly at a time when lower inflation and interest rates combined with sharper competition have put more pressure on appointed actuaries to set bonus rates at the highest level they can justify.

Professor S. Benjamin: I am a little surprised that the author managed to avoid producing stochastic bundles in order to illustrate his paper since it might have shown in a more diagrammatic way what happens in these stochastic processes.

The author uses the Black-Scholes formula for pricing yet practitioners tell me that the market does not necessarily conform to this formula. The Maturity Guarantees Working Party did not find that the random walk model, which underlies the Black-Scholes formula, was a good representation over the longer term; the author had a great part to play in coming to that conclusion. Options, as I understand it, are available in the market for only up to about nine months. Yet the author is using option pricing over 20 years, so perhaps the model is not so important for his purpose. This appears to show how the ideas behind option-pricing can be used to price the values of guaranteed bonuses, as compared with non-guaranteed bonuses, from the point of view of equity between policyholders.

I wonder if this degree of sophistication is suitable for this particular purpose since there are some important loose ends left. For example, we do not know the policyholders' utility function and also, as the author has shown, there are many different investment strategies which the policyholder could have adopted. If the purpose is an in-house exercise in equity then the model may not matter too much, but once we start trying to use it to produce *ad hoc* numerical rules of thumb, as the author does in § 9.9, then the model and the values of the parameters become much more important. The rule he gives there, in my words, is: do not declare reversionary bonuses if the value of the liabilities on a market basis and excluding future bonuses is more than 90% of the market value of the assets. I think we would all agree with that, but our agreement would be in spite of the calculations via options, rather than because of them. The author could say that his production of a result which is acceptable does not disprove what he is doing.

It is very difficult to see a simplifying pattern in the many tables and examples which the author gives. I attempted a simplification myself trying to stay within the underlying model used by the author. There seemed to be some odd consequences of assuming that there is a risk-free rate of interest. Firstly, if it is risk-free then I would say that by definition it must be known. It must be a contradiction in terms to say that it could be 4% or it could be 5%. Secondly, it must be known, not merely now, but at any future time, i.e. at all future times. Let us assume, as the author does at one point, that the risk-free rate of return is 5%. If we assume that £1 invested in units will be worth not less than £1 in 20 years time then, following the single premium example and the figures in § 2.16, we would actually invest £300 in the risk-free assets and the balance of £700 in units which seems to meet both of his requirements. The first requirement is a minimum guarantee of £1,500; in this method that would be made up of £300 accumulated to £800 at the risk-free rate of interest of 5% p.a. and the balance of £700 in units which I am assuming would not drop below their opening value at the end of 20 years. The second requirement is also satisfied; on my unchecked arithmetic, I would get a final total of £9,273, which is very close to the figure in § 2.16.

When this simple approach is applied to the annual premium case it is much more difficult to make comparisons with the figures in the paper, because they assume a particular set of varying unit prices and the desired minimum guarantee is expressed as increasing each year. I did try some simple arithmetic on the basis that the units could drop to zero value, and, of course, I found that the cost of the guarantee would be higher than the figures in the paper.

The model in the paper implies that the probability of a near-zero value is effectively nil, certainly over the longer periods. I wonder if a simple strategy would be to use the final part of the 20 years to invest in the risk-free investments and the early part to invest in units. I know that the other way round seems intuitively safer, but under the given model it does not seem to be necessary and would give a lower return.

The concept of a risk-free rate of interest as the foundation for applying a sophisticated mathematical model causes me some concern. This paper tries to bring together the old ideas of the guarantees which we give in reversionary bonuses with the new ideas of option pricing where normally we think of using the concepts of matching by government stock. If long options were available in the open market, and not guaranteed by the Government, would we expect the issuing institutions to hold maturity guarantee reserves? Suppose it were a general insurance company and an actuary had to certify the provisions – horror of horrors!

Mr E.B.O. Sherlock: I am stimulated into offering a contribution tonight by the conjunction of four events: my reading of this paper; the Presidential address one month ago, and in particular its references to the tensions between successive generations of with-profits policyholders in the early history of life assurance; the recent change in practice of illustrating with-profits policies; and some conversations within my own office. These conversations have concentrated on the usefulness of a model developed in the early 1970's as a guide to the amount of capital appreciation which ought to be regarded as available for allocation as reversionary bonus.

I described the model in closing the discussion on Redington's 'The Flock and the Sheep' (*J.I.A.* **108**, 399). It was developed at a time of rising interest rates when the running yield on the fund was substantially lower than the long-term gilt rate and the question which we had to face was how much growth in equities and properties was needed to bring the current total return on premiums paid in the past up to the level needed to justify declaring reversionary bonuses equivalent to the current gilt rates, which was what we felt the target for current premiums should be. Any further surplus was available for terminal bonuses. In our conversations we recalled that on one occasion there was no margin of capital at all, but the valuation basis was adjusted by releasing inner reserves to provide the smoothing of troughs, which is an essential part of the with-profits system, with its inevitable smoothing of peaks which the paper demonstrates so clearly.

As an aside, I would like to express my concern at the way the strength of a life office has developed recently as an important concept, because usually it is undefined. If it means that an office has been extremely effective in using the savings of its policyholders and in returning those enhanced savings to the appropriate policyholders, subject only to the smoothing of peaks and troughs, then I am all for it. If it entails a deliberate policy of withholding money from one generation, or even worse saving it for all time, then I recommend a study of your address, Mr President, and the consequences for the *Equitable*, which found itself in a not dissimilar position in 1816. The existence of a solvency margin means, I believe, that we have to be particularly careful to avoid withholding surplus and suggests to me that a terminal bonus would be an essential component of a bonus structure even if all our assets were in gilts.

The table in § 5.3 combined with the knowledge of the rate of interest underlying current levels of reversionary bonuses will confirm that, from 1983 onwards, the underlying rate is very likely to have been above the gilt rate. In our discussions that fact brought into question the usefulness of a model which assumed the reverse. My conclusion, and I stress it is my conclusion not that of my colleagues', is that the model is still valid, and that the present relationships must be seen as unstable. I see a link between that assertion and the author's suggestion in § 9.7, that his calculations may give guidance on the modification to terminal bonuses needed to take account of previously declared reversionary bonuses which could be a modification either upwards or downwards. I see this simply as a question of risk. If we declare bonuses which cannot be matched by fixed interest investments we are increasing the risk, and that after all is why a mismatching reserve is needed. We have sadly seen an exact example of this process carried to the extreme recently and any appointed actuary who has not fully explored the nature of the risk with the directors of the company which he advises, has surely failed in his duty. What decision to make after considering that advice is, however, a different matter and brings me to the other event I mentioned, namely, the change in the system of illustrations of projected benefits under with-profits policies.

Actuaries, both in their professional and in their business capacities, played a major role in this, as did several other influences. No matter how it happened, no matter that it is not perfect, the important thing is that it succeeded in its objective of bringing down excessive projections by breaking the tyranny of the link with current bonus levels – a tyranny advisers and decision makers alike could not ignore. That constraint has gone and the profession has, I suggest, an obligation to take full account of that fact. I believe it to be breaking faith with our policyholders if we withhold money from them, but also if we take undue risks, even if fully disclosed, the significance of which they are unlikely to understand fully and which anyway may be inconsistent with the basis of security behind much of our selling.

I conclude that the reversionary bonus rates must over a period of time come down to a level more consistent with the levels of gilt returns. Terminal bonuses should be used to ensure a fair return to policyholders either on a block system or even, as the author suggests, on an individual basis. If my analysis is correct, we must not fail our policyholders. You will remember that Frank Redington recalled the way the last war intervened and provided actuaries, as he expressed it, with the ideal explanation of a reduction of bonus rates or a deferment of declarations. I hope our generation does not need a similar catastrophe before taking appropriate action.

Mr R.J. Squires: I have to admit to some lingering doubt as to the validity of the option pricing model as it seems to me to make a crucial assumption: viz. that the variance of the distribution of a share price can be assumed to be constant over long periods of time. However, I accept that the model provides a useful analytical tool which can be used to consider questions of the kind we are now discussing.

My conceptual model of what happens in a with-profits life fund is that each generation of policyholders gives a guarantee to those whose policies mature earlier than their own and receives a guarantee from later generations. In return for the guarantee given they receive the benefit of investment earnings on the estate and in return for the guarantee received they make a contribution to the estate. In a proprietary office the shareholders may bear a proportion of the cost of the guarantees, or simply act as a long-stop. Provided the situation is understood, I take this to be a fair arrangement. The actuary will then wish to see the fund manager pursue an investment policy designed to minimize the cost of the guarantees. That is what matching and immunization are all about, subjects which have been well covered in the past.

The existence of traded options gives the investment manager a new weapon to deploy, and strategies involving the use of such options will be worth investigating. My impression, from a cursory investigation of the practical terms on which options can be bought, is that such strategies will involve the limitation of risk on a 1-year view rather than the elimination of risk on a 10-year view, but that they are worth considering.

The aspect of this paper which surprised me was the apparent high cost of the guarantee. I believe that part of the reason for that is the fact that the approach adopted considers a single generation of policies investing in an accumulation unit. The Maturity Guarantees Working Party found that the investment income from a portfolio is much more stable than its market value and I believe this is an important factor in matching strategies. If there are several generations in one fund the investment income of the whole fund will go a long way towards providing the maturity proceeds of each successive generation. Another slice can be matched by maturing fixed interest investments, and a final provision can be made by purchasing traded options. These arrangements all have their cost. For the option content of the fund the cost is explicit. A fixed interest content means a reduction in the expected return. The cost of applying the investment income to pay the maturities is more subtle. On a unitized basis each policyholder would expect the investment income of his share of the fund to be reinvested at current prices. If prices have fallen below the level required to support the guarantees the continuing policyholder is effectively paying more than market value for the reallocated investments. Nevertheless, I would submit that this is an efficient way of dealing with the problem of matching guarantees, and would expect to find that the effective cost was less than it would be using traded options alone.

Mr D.E. Fellows: I, too, was most interested to see the author's suggestion of a link between option pricing and bonus determination, although, as he says, offices invest in a wide range of assets, and not solely in equities as was the case in his model. The effective asset backing for most of our long-term funds, if we strip out the immediate annuity and linked business, is in fact heavily weighted by equity type investments; so the comparisons do have a good deal of relevance.

The relatively high bonus loading content of most with-profit premium scales tends to induce a sense of complacency about the worth of the implicit guarantees. Indeed, if within any one company the with-profit rates for various policies imply a reasonably consistent level of guarantee, and there are strong asset margins or other reserves, any charge deemed appropriate against a maturing policy in respect of the guarantee may be counterbalanced by that policyholder's share of miscellaneous surplus, including profits stemming from charges made on other policies. The author refers to this aspect in his paper.

However, it would be unreasonable to conclude that this circular effect reduces the need for care. In particular, the intrinsic levels of guarantee in various contracts issued by the same office may be materially different. For example, I recall an investigation made in my own office not long ago. We were looking at the possibility of introducing a new with-profit contract for a particular product line, involving a higher guaranteed element than for other with-profit business, possibly by way of an increased basic rate of interest or higher reversionary bonus rates, or both. The question we were trying to answer was: by how much should we reduce the total investment return, by adjusting terminal bonuses, on the new contract as compared with other participating business? This was on the assumption of an unchanged pattern of assets. After repeated tests based on 80% of the fund being in equities, the answer was more sobering than some of us had expected. The conclusion was that for each 1% p.a. increase in the reversionary bonus element of the new contract we should, on the whole, need to reduce the total yield, inclusive of the terminal bonus, by up to 1% p.a. This was for a single premium tax-exempt contract. An alternative solution would, of course, have been to reduce the exposure to equities in the assets backing the new policy. It would be unwise to attach undue significance to these figures since our research was somewhat limited. Moreover, the results varied by term of contract and underlying levels of guarantee. But I mention them because they do tend to reinforce the message conveyed in the paper as to the significance of guarantees, not simply in basic benefits but through reversionary bonuses. This is not to suggest that we can be cavalier about terminal bonus rates; they have become an important part of policyholder expectations.

The subject also links with the question of capital backing for guaranteed liabilities. The backing capital which is needed may be quite large, a point which Sidney Benjamin made in comments at a discussion here in 1981 (*J.I.A.* **108**, 391). Indeed, the author's work may help in the search for a more satisfactory solution to the current problems that we face in deriving sensible mismatching reserves, referred to in Temporary Practice Note 2 to GN 8, and under discussion with the Government Actuary's Department.

Mr J. Plymen: The main point of the paper is to give an idea of the risks and the problems involved in bonus distributions. However, I do not think anybody would make their bonus distribution in practice by reference to these calculations.

I feel that the question of setting a bonus distribution is made far too complicated. The first thing to decide is the degree to which capital profits are to be brought into the surplus. It was pointed out by Redington that if you invest in equities yielding 4% when gilts are yielding 10%, then it is perfectly legitimate to have an investment write-up representing the 6% difference. That was no more than proper equity in the distribution. Surely you should not do this if the equity investments have not appreciated by at least 6% over the year. What then is the amount of the reversionary bonus? That has to be determined by the usual actuarial processes; but in determining it, a degree of reserve must be put on the value of the equities. Based on my experience of following markets over some 45 years I assume that the equity index at any time is liable to shed about 30%. I developed this from my early investment experience between 1945 and 1947, when the 30 share index moved from 100 to 140, back to 100 and then up again to 140. Watching indices subsequently seems to confirm this sort of pattern. Therefore when determining a bonus rate a reserve of, say, 30% of the market value of the equities should be held. I do not think it responsible to declare a bonus where the resulting estate does not cover this equity fluctuation reserve.

The terminal bonus ought to be based on an asset share calculation. The investments have a certain appreciation over book value, surely the part that is appropriate to the maturing policyholder should be distributed to them; that seems to be equitable. I see no other way of doing it. Obviously, my method would produce fluctuations but it is ridiculous that terminal bonuses should develop an established level. They ought to go up and down each year with the market, and in line with the investment cushion of the assets over the book values.

Mr P.E.B. Ford: I welcome the ideas in this paper as a means of tackling, in a consistent manner, the problems of the risk/reward relationship for with-profit policies. In thinking how to proceed in practice to build this option pricing approach into an asset share model, I hit a few stumbling blocks and would welcome some comments about them.

Firstly, on a point of detail, in § 5.1 the standard deviation of the share price, as used in the random walk formula in § 4.2.6 of the Appendix to build up the Black-Scholes formula, has been taken at 20% p.a. However, in the author's earlier work for the Maturity Guarantees Working Party, taking table D2.15 of their report (*J.I.A.* **107**, 103) I think it can be calculated that the results of the dividend yield model approximate to a simple random walk approach with a standard deviation that starts at somewhere around 20% in the first couple of years, but reduces in the longer term to about 13%. Should we be using the short-term standard deviation of 20%, or the longer term value of 13%?

Secondly, in applying the option pricing approach to unit-linked maturity guarantees, Collins showed (*J.I.A.* **109**, 241) that as the option date approached there could be very rapid switching between the backing assets and I wonder if the same point is relevant in the current paper.

Taken in the context of a with-profit mutual office, I am a little uneasy that if the overall risk-free asset portfolio of the office is included as part of the units, as is suggested as a possibility in § 9.4, it may not be particularly appropriate for the guarantee of the put options in total for all the business. Since for a with-profit mutual life office, the risk-free reinvestments must be provided by all the with-profit policyholders, plus the free estate of course, I wonder whether that asset distribution has an optimum solution corresponding to that required under a traditional immunized portfolio. The immunization approach and the option pricing approach remind me of two systems of underground caverns believed to be linked. However the connecting tunnel has not yet been found. Considerable work has been put into exploring the immunization system, most recently by the author in his comments on Andrew Wise's Institute paper (*J.I.A.* **111**, 489). The current paper requires an apparently different set of risk-free assets for an optimum solution. I wonder if there is a connecting link between the two approaches and if so whether the author has already found it.

Mr P.S. Carroll: The paper is a considerable milestone in actuarial methodology. It makes known the advantages of using stochastic models for purposes of assessing the value of options. The more familiar actuarial approach to the value of options, such as are implicit in convertible stocks, is to use a deterministic model of increasing income on ordinary shares, with the convertible providing extra income in the early years. Just how superior is this approach using a stochastic model of random walks?

In § A4.2.6 the author tells us what the model is for the change in the share price. It is assumed that the natural logarithm of the share price follows a Gauss-Wiener process with constant μ and σ. One speaker earlier this evening was worried about a constant variance; it is the logarithm of the share price that follows the constant variance and the logarithm transformation can have some variance stabilizing effect, so the model is perhaps better than was feared.

When the traditional method of valuing a convertible is satisfactory, why take the trouble to apply this more complicated model? One answer to these questions is that ordinary shares and convertible stocks are particularly important for life offices and perhaps even more so for pension fund investment and it is dangerous to rely on a deterministic model that obscures the risks inherent in such investments. It has been suggested that about 60% or 70% of privately invested pension funds are in ordinary shares and convertible stocks; earlier this evening Mr Fellows spoke of a fund that was 80% in ordinary shares. Yet ordinary shares are not particularly well matched to the nature of pension fund liabilities in relation to the underlying demographic variables. At the same time convertible stocks are, in many ways, more suitable for pension funds. This might apply as regards tax considerations. From the point of view of stability, the convertible stocks might vary less in value and trustees, taking into account their responsibilities, might prefer to invest in convertible stocks rather than buy pure options or pure warrants; they might even not be allowed to invest in some things that are options. Convertibles with a maturity date, a loan stock, might also match the term of a pension fund. The date might coincide with a time when someone is retiring and would want to have his pension fixed, so that there could be advantages of that kind.

By using methodology, such as the paper has described, it is possible to pursue what the author calls a hedging strategy, to reduce the risks inherent in what might otherwise be a reckless enthusiasm for ordinary shares on the part of fund managers. By decomposing the value of convertible stocks held into, one, the value of the land stock as a fixed interest security and, two, the option to not convert into ordinary shares, it is possible to construct a clearer picture of how a fund is invested as between equity type investments and investments fixed in money terms.

The deterministic approach to valuing convertible stocks implies a false assumption that conversion is now possible, or is already taking place, as well as ignoring the risks and fluctuations in value of ordinary shares. It may be that this obscuring of the risks also obscures the advantages conferred by investing in convertible stocks, which can help to achieve hedging.

One reason put forward by those studying pension fund investments for the rather small proportion of funds invested in convertible stocks, is the limited supply of such stocks. If, however, the advantages for investing institutions of convertible stocks were better known, those concerned with raising new capital would make better and greater use of convertible stocks, with ultimate advantage for the members of pension funds by way of more secure investments. 'Junk bonds' is an unfair and misleading label to attach to stocks that are more secure than ordinary shares.

Mr C.N.H. Foster (a visitor): The market is rapidly providing the very products that are needed to meet the applications described. The biggest problem in providing guarantee type products using options has been the duration of options, typically less than 12 months. As a previous speaker pointed out, convertible securities have a great advantage in that respect because they have a longer duration, although the convertible securities to which he was referring rarely possess the qualities of those which initially emanated from the United States, and which are increasingly seen in the UK. We are seeing the provision of long duration options which are laying the framework for guarantee type products, which I believe the actuarial profession will particularly welcome.

Convertible bonds of AAA credit which are convertible into market indices (e.g. the FTSE 100 Index or the US Standard and Poors 500 Index) have been extremely well received. Although a recent innovation, I think one of the reasons for their popularity is their provision of market vehicles for people, like the author, to examine the cost of providing guarantees. Another development along similar lines is the growth in usage of an investment technique called portfolio insurance. Although an American innovation, interest in portfolio insurance is growing in the UK.

At its simplest, portfolio insurance is a stop-loss system using, or simulating, a put option to provide the kind of guarantee the author alludes to.

Mr G.M. Morrison: In §§ 9.1 and 9.2, the author refers to his hypothetical investment strategy, to the purchase and sale of real traded options and the need to consider actual portfolios in which assets relating to with-profit policies are invested. However, I see it as being much simpler than this. If it was possible to trade options on all the investments held by the office there would be little or no problem. However, this is not possible for two reasons. Firstly, not all the asset categories have got options on them. Not only are they not long enough, but an option on property is unobtainable. Nor are there many options on overseas securities, particularly not in the smaller markets. Even on equities, options are confined to the larger capitalization companies. We are not interested in options on individual parts of the portfolio; ideally we require a put option on the total portfolio.

In simple terms, the investor is buying put options and at the same time he is buying exactly the same number of units in the unit trust. What has he got at the end of the day? Figure 3 in § A3.2.4 shows the profit to the purchaser of taking on board these put options. As well as his put options, he has also got units, and his units are clearly going to go up and down in value. Superimposing on that graph the price performance of those units produces a diagonal line running at 45° from bottom left to top right passing through E. So the investor has these two parts of his portfolio acting together. Putting the two together produces a call option, as can be seen by comparison with the graph in § A3.2.2. This graph is horizontal and then goes up, i.e. his guarantee is a long horizontal line with up-side potential. So the effect of the call option is to cut out the down-side risk of the investment. Capital losses cannot be made and that surely is the author's guarantee – his protection. Therefore, I would have thought the interpretation for bonuses is simple. In the language of § 2.17, the guaranteed amounts correspond to the horizontal line in the diagram; the terminal bonus is paid if unit prices rise and the cost of that capital projection, over the 20 year period, is 20.3%.

Turning to the insurance company, how does it cope with these hypothetical traded options? It is possible to replicate the performance of that call option just by investing in the risk-free asset and the underlying securities. The two need to be combined but a computer can generate the changes needed in the hedge ratio between the risk-free asset and the unit prices. There is nothing special about the guaranteed amount. There is a lot of freedom as to its level. For example, a strategy could be devised which gave a guaranteed return of 5% p.a., although it might be expensive. In this way the insurance company can match its investment strategy to the expectations of its policyholders without resorting to these odd options. To that extent the methodology of pricing the options whether the standard deviation is stable or not seems to me to be largely irrelevant and obscures the central issue of the paper.

The investment strategy outlined of combining the risk-free asset and the risky asset can be adopted quite simply for UK equities, overseas equities, fixed interest investments; indeed almost any type of investment so long as it can be sold easily but that probably eliminates property. Futures' contracts make it even easier. The previous speaker referred to portfolio insurance in the US which is just a call option, or a replicate of a call option, producing the guaranteed minimum return.

In § 9.8 the author refers to the theoretical hedge position as being of interest; clearly it is. All offices should maintain sufficient assets, with appropriate solvency margins, to carry on business, but I wonder if the fact that the hedged position can be put into practice without the cost of the hypothetical options implies that less assets are required. The point about needing to change the hedge position continuously is well made. However, it is important to be clear why this is necessary: the hedge changes as time to expiry of the option reduces and as the markets change.

In section 2 the author highlights the problem of maintaining the investors wish to receive 5% reversionary bonuses with a guarantee which is set too high. It should be noted that the investor is trying to get his cake and eat it too. He is essentially mismatching his position by reviewing his strategy each year, and yet he is seeking to hedge over a 20-year period, a 19-year period, or whatever it might be. If he was to take out one year options and hedge his portfolio over just one year, rolling it over at the end, some of the problems, although certainly not all of them, would be eliminated. You could still envisage situations though where he could not meet his guarantee.

Is the author correct in suggesting that his methodology automatically produces surrender values? For these an American option, which can be exercised at any date, is more appropriate than a European option. Surely a surrender value involves selection against the office. Similarly, I suspect that some of the problems with mortality raised in section 10 are suitable for treatment in this way. I believe that the use of options, futures and similar investment vehicles should become increasingly important to life offices, both in their thinking on bonus declarations, as the author has so eloquently shown, and in investment and other areas.

Mr M. Lacroix: In § A2.2.5 I wonder if instead of the normal distribution and its fixed variance, would it not be possible, and interesting, to consider other distributions, in particular the Goldton Gibrate transform, Ludwig transform, which is very often successful in cases which have some similarities to the point dealt with in the paper.

In § A6.11 it seems to me that there are other implicit assumptions for stating that the value of an American call is identical to the value of a European call option. The further assumption, I submit, relates to the question of market conditions, marketability. It does make a difference if I have a call option to be able to exercise it at any time before the date of expiration, rather than only on that date, and I think the previous speaker made a reference to this question in the case of surrender values. There are other reasons for believing that use of the American option is one of the assumptions made. Suppose you have an option on a share which rises considerably and you have reasons to believe that after this considerable rise it may go down before the date of expiration, especially in the, not at present very realistic, case of a long-term option. Then what would you do? You could, in the American case exercise your option and sell your share. But in the European case you could, theoretically, do exactly the same by selling your option, but this supposes that the markets for option of any duration are open and freely available and can give you as many possibilities as the market and the shares themselves, which is far from being always the case. A similar point could be made about put options.

Turning to more general points, and in particular, to section 10 of the paper. I believe it is important that our profession should address itself to applications of option theory, or let us say such application of the option possibilities, including futures and warrants which some previous speakers have referred to. It may be worth mentioning here that applications of options have recently received much attention in other European Actuarial Associations. For example, in a paper by Delbenne presented on 16 September 1986 at the Belgian Association of Actuaries; in a paper by three French authors published in the Bulletin of the Institut des Actuaires Français for September/December 1984; in two papers by Steffner published in the December 1984 and June 1985 issues of the Bulletin of the Association of Actuaries. Actuaries have played a very prominent role in the preparation and operation of the newly opened market in futures for financial instruments at the Matif in Paris. One can be sure that there will be a lot to discuss and many opportunities for actuaries to contribute to the advancement of the science in the financial field, if the financial section of the International Actuarial Association is created.

I should also mention that as early as 1912 Baschulier included three chapters on speculation – and this is very much trading in futures – a very similar problem to options, in his treatise on probabilities and wrote a separate short book on this subject in the 1930's.

Mr C.J. Hairs: The simulations run by the author produced a number of interesting results, two of which I would like to highlight using the figures in Table 8.2 for mean final proceeds of the policy arising from the 101 simulations. We see that if we have a 3% normal bonus rate the mean final proceeds are 2,904, whereas if the normal bonus rate is set at 4% p.a. it drops to 2,667. That is quite a substantial drop being equivalent to a fall of .7% p.a. in the policyholder's yield; although this is not quite as high as the fall of 1% p.a. to which Mr Fellows referred earlier. This result confirms the general finding that the higher level of guarantee has a real and substantial cost. If we compare columns *e* and *g*, or alternatively columns *f* and *h*, we have a contrast between an absolutely static policy for normal bonus rates and a dynamic policy where the normal bonus rate follows the rate of increase in dividends. The mean final proceeds are similar as between the 3% flat bonus rate with .5 times dividends, and the 4% with .6 times dividends. But, looking at the standard deviations that correspond to these, the drop in variability of final proceeds between the static normal bonus rate philosophy and the dynamic bonus rate philosophy is quite substantial, being about 15% in standard deviation. When they effect with-profits business, I believe that for a given level of expected proceeds policyholders have a preference for less variability rather than more. This suggests that in these times, when thanks to the changes in future bonus illustrations, we are freed in our bonus policy, we should be aiming not to have normal bonus rates any higher than they need be and to feel free to vary them when conditions change. This would seem to contrast with the actual situation we have today in respect of normal bonus rates where there are expectations that these hardly ever change. Indeed we still have the anachronistic situation where a projection of a company's normal bonus rates is given considerable significance when determining the premium payable under low cost endowment policies used for mortgage business.

Mr R.D. Corley (closing the discussion): If anyone were to survey the trends in actuarial science over the last 140 years or so, the result would make an interesting commentary on the changes in the environment in which we operate, our services to the community, the resources at our disposal and the pressures upon us. I should add that I have not made such a survey and I am not about to launch into a history of the subject, but no actuary of my generation could have missed the complete reversal in the attitudes to handling data that has taken place in our working life. For more than a century one of the actuary's major concerns was to group data to reduce the number of calculations that had to be performed. He had to establish valuation and other formulae which gave a good approximation to the results that would have been obtained by laborious individual calculations. With the advent of the computer all that dropped away. Calculations on individual pieces of data became easy – easier than finding legitimate groupings and then adjusting for the areas of approximation – and as the science developed it became possible to compare valuations calculated on many different bases.

From this position it was a logical step, though perhaps not a particularly short one, to work with hypothetical future data generated through simulations based on a range of factors which varied with time and thus to be able to ask some 'what if?'-type questions and expect some help with the answers. In reviewing this work future historians will find much cause to remark on the industry and perspicacity of this evening's author for this and his earlier papers on stochastic processes.

It is now some 15 years since Professor S. Benjamin's trailblazing work on maturity guarantees for unit-linked contracts. His conclusion that the reserves required were very large was validated by the later work of the Maturity Guarantee Working Party which led to the withdrawal from the market of virtually all the contracts which included significant guarantees of the investment results.

A few years later using something of the same methodology, Messrs P.E.B. Ford and N.B. Masters demonstrated the high cost of the implicit guarantee in an open-ended endowment, under which the policyholder can choose, at very short notice, the date of maturity. Their work has been neither validated nor challenged and there has been little change in the construction of these contracts, but perhaps this can be explained if it is accepted that the newer flexible whole life plans have taken the pressure off this sector. The present author has tackled the more intractable problem of the cost of the increasing guarantee in a with-profit endowment assurance. Using an option pricing approach he has once again shown that the cost of a guarantee is large. Perhaps many of us listening to this discussion will be wondering whether his approach will also eventually be validated, although perhaps we have heard about too many pitfalls to believe that validation will come quickly, and, if so, whether it will lead to changes in practice.

If the cost of the guarantee of added reversionary bonus is as high as that calculated in the paper, then it might seem that the with-profit policyholders have enjoyed their guarantees without paying properly for them over the last couple of centuries. In a mutual life office this could perhaps be explained by noting that the with-profit policyholders are also the guarantors, so that any payment for guarantees is recirculated through the fund. But do we now have to wonder whether the shareholders of a proprietary office have been properly compensated for the large risks that they have unknowingly assumed?

These thoughts lead to the question of whether the parts add up to the whole. The opener and Mr Squires both raised the question of cross-subsidy between generations but Mr Fellows warned against assuming that a varied product portfolio provides sufficient lateral cross-subsidization to cover the costs of the various guarantees. Mr Plymen made a plea for keeping things simple and basing decisions on broad views and, as Mr Ford mentioned, this links in with the recent paper of Mr A.J. Wise in which he looked at the whole portfolio of liabilities from an immunization standpoint and arrived at some conclusions perhaps less threatening than those presented in the paper this evening. Despite all the reservations that have been raised, I agree with Professor S.

Benjamin that it would be difficult to dispute one conclusion of the paper which, as I rephrase it, reads 'if a realistic valuation of the benefits already guaranteed adds up to more than 90% of the market value of the assets, then watch out!' But does this statement really arise from the apparent 90% barrier which the author has found by examination of individual simulated contracts? Do we need the methodology of option pricing to recognize it as sensible advice?

For the option pricing model M. Lacroix would like to use American options, and it would be interesting to see what difference that would make. However, perhaps that would be too much finesse, for as Professor S. Benjamin pointed out the Black-Scholes formula is not used by the option market makers to determine their prices and we have no practical experience of long-period options. Underlying the theory is a risk-free rate of interest with which it is difficult to come to terms especially as to be really risk-free the rate of interest cannot change in the future, or can only change according to a pattern which is now known. It is perhaps unfortunate that the option price is quite so sensitive to the rate of interest selected.

Mr Morrison preferred to view the problems as seen by the life office, rather than as individual policyholders, and to start with a call option on the whole portfolio. This could provide Mr Ford's 'lost link' between the author and Mr Wise. The development of these ideas on paper would be very welcome.

It is understandable, in preparing a paper like this, that the author had to select one policyholder's strategy for purchasing guarantees. The opener and Mr Squires suggested different portfolios which might serve the policyholder's purposes better. Mr Carroll and Mr Foster seemed to give their main votes to a portfolio constructed from convertibles. It is perhaps a pity that no-one has demonstrated the effects of choosing different strategies, but Professor S. Benjamin has drawn attention to the need to use a strategy which reflects realistic policyholder aspirations. More work in this field should increase our understanding of what is going on in the model. However, for some of us, policyholder strategies in themselves are perhaps a little remote from reality, for in the end the only investment strategy open to the policyholder is to select initially a life office which he will support.

One fact which emerges from this paper and the current discussion is that the higher the initial guarantee the more it costs and the less likely it is to be achieved. Such a conclusion may seem so obvious as to be trivial. However, the cost of the guarantee starts very low and then suddenly escalates once a certain point is reached, so the opportunity to identify that point seems well worth while. The opener suggested that we keep our eyes on the ratio of the guarantee granted to the maximum attainable guarantee, whilst Mr Hairs gave us an example of using the figures in the paper to derive some practical guidelines. For the policyholder, the original choice of guarantee pattern has a large influence on the results. If he were given the choice he should be able to achieve results significantly varied from those of a contemporary policyholder who chooses a different guarantee pattern or a policyholder choosing the same guarantee but entering at a different time. In practice the policyholder does not have this choice; by grouping the policyholders and making the choice for them, the actuary can operate an effective cross-subsidy in such a way as to make the cost of the whole guarantee much less than the sum of its parts.

There is more work to be done in this field but it will be a pity if our increasing ability to break the portfolio of liabilities into its constituent parts and to make individual calculations leads to more differentiation in the treatment of with-profit policyholders. The potential investor has a choice between a personalized investment resulting from a unit-linked policy and a shared experience under a traditional with-profit fund. Any further erosion of the difference between these two would be a reduction in choice which is unlikely to be in the consumers' long-term interest.

Nonetheless, we have in this discussion achieved further insight into the operation of a with-profit contract. By indicating possible allocations of asset shares between reversionary bonus, terminal bonus and guarantee funds the paper adds to the knowledge on which the actuary can rest his judgement. I do not, however, believe that we have reached a stage where the need for that judgement is reduced or can in any part be replaced by increasingly powerful and detailed calculations; Mr Sherlock gave us a strong warning never to suspend our powers of disbelief.

The President (M.H. Field): The paper we have just discussed has been of particular interest to me as it contributes to the area of actuarial expertise where research is most necessary, as I mentioned in my recent Address. I have tried to draw attention to the marked shift brought about by inflation, and its effect on stock markets, in the carrying of risk by insurance companies and their policyholders. I sought a new means of introducing some discipline into the discretionary process so as to restore, in a dynamic way, the balance of risk-sharing and insurance. This paper bears on that question and particularly on the balance between the relative levels of reversionary and terminal bonus from time to time. The work has been painstakingly performed and we are most grateful for the detailed analyses and voluminous tables of results. These will stimulate, assist and direct further work on the same lines.

Of itself I fear that the paper will not immediately help to answer the question which I posed, but I am hopeful that its development will do so. At its crudest, and in relation to my question, the message I get from the paper is that caution should be exercised in declaring reversionary bonuses, because of their permanence, and safety lies in preserving the full discretions by the use of terminal bonus declarations. I, therefore, hope that the development of these concepts will lead to a generally accepted perception of a proper balance between reversionary and terminal bonuses and a curb on what is seen by the observant as too high a degree of discretion.

I have a further concern that I must voice. The high level of discretion exercisable has been noted by those working on the investor protection legislation and the associated rules. More significantly it has been noted by some politicians and some consumerists. The solution, when it is found, must bear scrutiny from such quarters. It will not, I am sure, be sufficient to publish a paper or a formula that is incomprehensible to the layman. It will, I believe, be necessary to develop an acceptably disciplined approach and to present its logic and objectives in terms that can be appreciated and accepted by the layman. Whilst this paper does not achieve this I am hopeful that it will point the way to such an approach; we can then attend to the explanations for the layman.

Professor A.D Wilkie (replying): I very much agree with the President's remarks. This paper was obviously not a paper for laymen; it was a paper for actuaries. I would be horrified at the thought of writing the Black-Scholes formula into SIB rules or an Act of Parliament. My intention was not to reduce the actuary's discretion in deciding on bonus levels, but to give him, as I put it, a handle to work with, to get a grasp of the problem, so as to get the right sort of balance between reversionary and terminal bonus. There is an awful long way to go as has been pointed out and as I admitted in the conclusion of the paper. Mr Plymen, however, suggested that terminal bonuses should be based on asset shares, and the whole point of my paper was to say that it should be less than the asset share by some amount which could be calculated, as I explained in § 9.7.

The opener drew attention to the rather curious business of μ, the mean rate of return on the share, apparently disappearing. Another way of looking at this is to say that the value of the option at any point is equal to the expected value (in probablistic terms) of the final proceeds. This does not assume any hedging. The value of the call option is equal to $P - E$ if $P > E$ and is equal to zero if $P \leq E$. Provided that the mean rate of return on shares, $\mu = \delta - \frac{1}{2}\sigma^2$ then the value of the call option calculated by the Black-Scholes model is equal to that expected value. In a sense the mean rate of return on the shares can be there if you want to bring it back in by using this straightforward expected value approach.

It was pointed out that in my simulations in § 9.6, the office appeared to make a 'profit' because the policyholder appeared to lose. That is correct because in my simulations μ is not equal to $\delta - \frac{1}{2}\sigma^2$, because I have deducted tax. I am not sure whether I should have taken tax off or not, but in my simulation model the mean rate of return on shares is very close to the mean rate on fixed interest and the $\frac{1}{2}\sigma^2$ is fairly small so that if I do the same calculations gross (as I have subsequently done after writing the paper) then the average cost of the guarantee is much closer to zero.

That is an area where others may care to make further investigations, as they might on other aspects of the model, such as whether one should use the short-term standard deviation or the long-term and what happens if μ, the mean rate of return on the shares, is varying or δ, the risk-free force of interest, is varying. The 'risk-free force of interest', by the way, is just the old actuarial constant i, that one always learns about in compound interest and life contingencies books. There is nothing new about it apart from its name.

In the derivation of the Black-Scholes formula if μ, the mean rate of return on the share, is variable and is a function either of time or of the price and of time it still drops out of the formula.

But the σ is the short-term standard deviation. If we use the Maturity Guarantees Working Party model or my own later stochastic asset model, what that actually says is that the mean rate of return on shares varies with the level of yield on the index etc. But the standard deviation in my model is constant so that the formula is applicable. Merton's (1973) paper does show how to deal with the problem of δ, the risk-free force of interest, varying. If the risk-free force of interest itself follows a random walk model with a zero drift and a standard deviation σ_δ, independent of the σ on shares, then you simply replace δ in the formula by $\sqrt{\sigma^2 + \sigma_\delta{}^2}$. If the risk-free rate and the share prices are correlated, you need to subtract the covariance from the sum of the variances.

M. Lacroix mentioned other models. I think a feature of a continuous random walk is that the distribution either has to be normal or one of the stable Paretian sort, that I do not really like the algebra of and do not want to go into. The other possibility is a discontinuous Poisson jump process, where you get occasional large jumps. A feature about the Gauss-Wiener process is that in any short period of time you cannot get extremely large jumps, or rather the probability of extremely large jumps is extremely small. A more elaborate model could be used for deriving the option prices, but I really am talking about hypothetical options which one uses as a method of pricing and not about traded options that are used as an investment policy.

Mr Morrison saw what I was doing and came back to the concept that the insurance company which has written the option needs to do the same sort of thing as the writer of the option would. But if an insurance company just goes and buys an option it throws the problem over to the writer of the option. The buck has to stop somewhere, so let it stop here. The writer of the option can price his option according to the formula because he has the choice of following the immunization or hedging strategy for his investment. I think that is very like what Professor S. Benjamin was talking about, but he was suggesting an *ad hoc* hedging strategy. I would suggest following the hedging strategy implied by the option pricing model.

I realise now that I had forgotten to refer to the paper by Kennedy and others on bonus distribution with equity backing (*J.I.A.* **103**, 11). I should have made some comparisons with the results they had obtained. I think that would have been quite interesting.

Mr Squires asked what would happen if σ itself varied. In that case we have a considerably more complicated model mathematically. We need then to define yet another model which describes the process whereby σ varies. I hope that someone else will learn about stochastic differential equations and find a way to solve that problem.