

PROPOSALS FOR AN INTERNATIONAL ACTUARIAL PUBLICATION LANGUAGE AND ITS REPRESENTATION IN COMPUTER COMPATIBLE FORM

by

CARL BOEHM and GEORG REICHEL (Germany)

in collaboration with

A. BERTSCHE, H. HÄRLEN, M. HELBIG, G. HEUBECK (Germany);
P. LEEPIN (Switzerland); G. LINDAUER, E. NEUBURGER (Germany);
B. ROMER (Switzerland); A. SCHÜTZ (Germany);
K.-H. WOLFF (Austria)

[Excerpt from the Transactions of the 18th International Congress of Actuaries, Munich 1968, Volume II, pp. 815-836, Edition: Verlag Versicherungswirtschaft e.V., D75 Karlsruhe/West Germany 1968].

1. INTRODUCTION

1.1. It has always been one of the concerns of the International Congress of Actuaries to standardize and codify the system of notation for actuarial terms. The efforts of the 2nd and 3rd Congresses (London 1898 and Paris 1900) and the work of the committee appointed by the 11th Congress (Paris 1937), which was completed at the 14th Congress (Madrid 1954), have led to a uniform system of notation for life insurance mathematics in educational establishments, in actuarial publications, in professional research, and in insurance companies.

1.2. Today pressures of various kinds are working towards a revision of this notation :

- (a) the need to extend, beyond the area of life assurance, the range of actuarial terms which have a standard definition—this therefore requires a real *extension of the actuarial "Vocabulary"* ; and
- (b) the necessity to model the notation in such a way that it can be easily and directly set up mechanically for data processing—this therefore demands more or less *machine-linked representations of the "Vocabulary"*. The strongest pressure for this has certainly come in recent years with the use of electronic computers as printers and as program-controlled calculators.

1.3. The following suggestions satisfy the need for an extension of the Vocabulary, in addition to life assurance and compound interest, into the calculations concerned with the mathematics of pension funds (including statutory and occupational retirement benefit schemes) and of sickness insurance. It is to be hoped that the 18th International Congress will, through its papers and discussions on Subject 4, give an impetus to the inclusion in the future of other areas of the mathematics of non-life insurance.

1.4. The present notation raises numerous difficulties in any machine-linked representation of the actuarial Vocabulary—such as in the stock of symbols needed—especially because of *the halo of indices around one basic letter*. For all types of printing machine (typewriters, typesetting machines . . .) as well as for computers the input and output of symbols proceeds naturally *in a linear sequence*. There are, however, many problems in translating from the previous two-dimensional representation into a linear representation, such as is used in the customary linear form of expressing a function by means of a function name and an expression in brackets in which the parameters are given in sequence.

1.5. In sections 2 to 5 we develop a machine-linked notation for an extended actuarial Vocabulary, which is directed towards all types of printing machine and so satisfies all the requirements of a “*publication language*”. In section 2 a system of general principles of notation is formulated for this purpose; the three following sections give examples of this system for the actuarial quantities of life assurance (section 3), pension fund insurance (section 4) and sickness insurance (section 5). In section 3 the force of previous practice and the attempt to maintain the existing conventions stand out especially; see also the comparisons in the Appendix.

1.6. Section 6 describes a system of rules, by means of which the “*publication language*” developed in sections 2 to 4 can be turned into a “*computer language*”, which is appropriate for certain applications for computers, such as the area of problem-oriented programming languages like ALGOL, FORTRAN, COBOL and PL/I. The many-faceted difficulties that result from this, and their solutions, are discussed.

1.7. The present proposals were worked out by a group of German-speaking actuaries. This may show itself in sections 4 and 5, which

44 *International Actuarial Publication Language*

contain the extensions to the present convention, for example in the choice of certain letters or the selection of actuarial quantities in a particular field and the naming of them ; these details can surely be settled by discussion.

The authors think that, besides the proposals presented here, the following questions should still be thoroughly discussed :

- (a) the extension into other areas of non-life insurance ;
- (b) the transformation of the publication language into a computer language which is compatible with problem-oriented programming languages requires a complex system of rules. A substantial simplification of these rules would certainly be possible by a closer assimilation of both languages, although this has as a consequence disadvantages for the publication language as a medium of communication.

2. GENERAL RECOMMENDATIONS

The following recommendations apply to the writing of *all* actuarial expressions ; in particular they apply also to quantities beyond those contained in the following sections The recommendations are aligned to the usual present-day mathematical notation—for example in the symbols for operators.

2.1. *Letters*

The expressions to be used employ an alphabet of

28 small Latin letters a, b, . . . , y, z ; ä, ã,
26 capital Latin letters A, B, . . . , Y, Z,
and the Greek letters α , β . . .

These symbols are independent of the typeface of the alphabet. When the Greek letters are not available they can be written out, for example α =alpha=ALPHA

2.2. *Digits*

The ten digits at our disposal are

0, 1, 2, . . . , 9

and decimal fractions can be written by using either a full stop or a comma.

Thus, for example 1,5=1.5 or 0,05=0.05=.05

and its Representation in Computer compatible Form 45

2.3 Arithmetic operations

Addition :	$a + b$	
Subtraction :	$a - b$	(a ./ b is not to be used)
Multiplication :	$a \cdot b$	(the point should if possible be placed half-way up)
	$= ab$	(the point can be omitted, if no ambiguity is possible)
	$= a \times b$	(this notation is possible only when a true multiplication sign $\times$ is available ; the letter x cannot be used here)
Division :	$\frac{a}{b}$	
	$= a/b$	(the notation a:b is not to be used)
Exponentiation :	$e^b = \exp b$	
	$a^b = \exp (b \cdot \ln a) = \exp (b \cdot \log_e a)$	
Equal signs :	$a = b$	(equality)
	$a \equiv b$	(identity)
	$a : = b$	(definition)

2.4. Punctuation

Comma ,	The comma is used (a) as a decimal comma (b) as a division mark between terms of the same type (e.g. : x, y gives the entry ages of joint lives)
Point .	The point serves (a) as an operator for multiplication and (b) as a decimal point in place of the decimal comma
Semicolon ;	This is used as a division mark between terms of a different type (e.g. : x;n represents entry age x and duration of contract n)
Colon :	The colon can be used in the following way : In the first Block (see 3.3.1) x:y represents the survivorship of joint lives. In the second Block x:s from age x to age s, x:n from age x for a term of n years, n:s after n years and up to age s, n:m after n years and for a further duration of m years

46 *International Actuarial Publication Language*

2.5. *Indices*

An index written above and to the right always stands for exponentiation (with the exception that multiple differentiation is indicated with the index enclosed in brackets). An index below and to the right always gives an enumeration. Indices on the left, either above or below, are not to be used.

2.6. *Brackets*

As far as possible round brackets will be used (,) ; if necessary curly brackets { , } and possibly also square brackets [,] can be used. The brackets signs < , > are not to be used.

2.7. *Other symbols*

Signs like |, $\neg$, $^\circ$, accents, bars and underlining are not to be used.

2.8. *Functions*

The present method of writing actuarial functions is by a function name (e.g. a , $\ddot{a}$) with indices attached to it (e.g. ${}_n\text{lt}a_x^{(m)}$). For the reasons given in the introduction we are now recommending a method of writing a function in which the halo of indices (= variables) will be attached to the function name by a linear representation enclosed in brackets. It is obvious that the present positioning of an index relative to the function name must be transformed into a fixed position inside the function brackets (expressed by the sequence).

Functions will be expressed by a function name, to which are attached function arguments enclosed in round brackets.

(a) The function name will be symbolized by one or more letters—possibly also with numbers following the letters. The initial letter shall be written in accordance with tradition as a small letter :

for probabilities,

for present values of periodic payments (annuity values),

and as a capital letter :

for present values of single payments,

for commutation functions, and

for monetary values other than present values (such as premiums, reserve values).

(b) the arguments placed in round brackets (functions, variables, parameters) are separated by punctuation marks. Arguments of

and its Representation in Computer compatible Form 47

the same type, provided they are not directly dependent on one another, are separated by a comma. If arguments of the same type are associated with one another (e.g. in the form "from age x to age s ") this is expressed by a colon. Arguments of the same category form a *block*. Different blocks are divided by a semi-colon. Unused blocks are indicated by a semicolon, if further used blocks follow them. If numerical values are used within an expression in brackets, then in order to avoid misunderstanding the argument can be given in addition to the numerical value with an equals sign linking them, e.g. $f(x=30)=f(30)$.

3. INTERNATIONAL ACTUARIAL NOTATION FOR LIFE ASSURANCE

3.1. *Elementary symbols*

3.1.1. Variables, parameters

Entry age of a person :	$x; x_1 \text{ or } x1, x_2 \text{ or } x2, \dots$
Entry age of a female :	$y; y_1 \text{ or } y1, y_2 \text{ or } y2, \dots$
Entry age of a child :	$z; z_1 \text{ or } z1, z_2 \text{ or } z2, \dots$
Time interval in the sense of an elapsed duration :	t
Fixed period in the sense of an agreed duration of contract or deferment period :	n
Period of payment of premiums (if different from n) or guaranteed period under an annuity :	m
Ages other than the entry age (e.g. commencing age for a payment, maturity age, expiry age or termination age) :	$s(=x+n); s_1 \text{ or } s1, s_2 \text{ or } s2, \dots$
Number of payments in a year if more frequent than one :	$k(k=1, 2, 4, 12, 52)$

3.1.2. Constants

Final age of a mortality table (previously usually ω or $\omega+1$) :	$w(l(w-1) > 0, l(w)=0)$ ¹
Date of a particular mortality	

¹ This definition of the final age of a mortality table shows its uses when, as seen in 3.2, we wish to introduce the notation $N(x;s)$, etc.

48 *International Actuarial Publication Language*

table or alternatively a fixed calendar year :	T
Expense loadings—	
initial expenses :	α
expenses related to premiums :	β
expenses related to insurance benefits :	γ, γ_1 or γ^1, γ_2 or γ^2
Zillmer loading (if different from α) :	ζ

3.1.3. Interest Calculations

Yearly rate of interest (e.g. 0.05=0.05=.05) :	i
Yearly rate of interest payable in k instalments :	$i(k)$
Accumulation factor :	$r=1+i$
Discounting factor :	$v=(1+i)^{-1}$
Discounting factor for payment in k instalments :	$v(k)$
Discounting for x years :	v^x
Yearly rate of discount :	$d=1-v$
Yearly rate of discount for payment in k instalments :	$d(k)=1-v(k)$
Force of discount :	δ or vc
Continuous discounting factor :	$e^{-\delta x}$ or $\exp(-vc \cdot x)$
Present value and final value of a yearly term annuity of 1 for n years payable in arrear :	$a(;n)$ and $s(;n)$
Present value and final value of a yearly term annuity of 1 for n years payable in advance :	$\ddot{a} (;n)$ and $\ddot{s} (;n)$
Present value and final value of a term annuity of 1 per year for n years payable k times a year in arrear :	$a (;n;k)$ and $s (;n;k)$
Present value and final value of a term annuity of 1 per year for n years payable k times a year in advance :	$\ddot{a} (;n;k)$ and $\ddot{s} (;n;k)$

and its Representation in Computer compatible Form 49

3.2. Mortality tables and commutation factors

	Present	Proposed
Number of living	$l_x, l_{[x]+t}$	$l(x), l((x) + t)$
Number of deaths	d_x	$d(x)$
Probability of survival for one year	p_x	$p(x)$
Probability of a life aged x surviving for a term of n years ²	$n p_x$	$p(x; n)$
to the attainment of age ² s	$s-x p_x$	$p(x; s)$
Probability of two lives (x) and (y)		
surviving jointly for n years	$n p_{xy}$	$p(x, y; n)$
surviving jointly till (x) reaches age ³ s	$s-x p_{xy}$	$p(x, y; s)$
Probability of death within one year	q_x	$q(x)$
Probability of a life aged x dying within a term of n years ²	$n q_x$	$q(x; n)$
before attaining age ² s	$s-x q_x$	$q(x; s)$
Force of mortality	μ_x	$\mu(x)$ or $qc(x)$
Curtate expectation of life	e_x	$e(x) = \frac{1}{l(x)} \sum_{t=x}^{w-1} l(t) - \frac{1}{2}$
Complete expectation of life	$\overset{\circ}{e}_x$	$ec(x) = \frac{1}{l(x)} \int_0^{w-x} l(x+t) dt$

² It seems to be useful to be able to use n and s as alternatives. For particular numerical values (especially where $n > x$) they must be expressly written as $p(30; n=35)$ and $p(30; s=35)$. Also

$$p(x; n=1) = p(x; s=x+1) = p(x)$$

³ The notation $p(x, y; s)$ means here and later that the first given termination age (here s) is always attached to the first given entry age (here x). One can also write $p(x, y; s(x))$.

Commutation functions :

Present	Proposed	Present	Proposed
Discontinuous, one life			
D_x	$D(x) = v^x \cdot l(x)$	C_x	$C(x) = v^{x+1} \cdot d(x)$
N_x	$N(x) = \sum_{t=x}^{w-1} D(t)$	M_x	$M(x) = \sum_{t=x}^{w-1} C(t)$
$N_x - N_s$	$N(x; s) = N(x) - N(s)$	$M_x - M_s$	$M(x; s) = M(x) - M(s)$
$N_x - N_{x+n}$	$N(x; n) = N(x) - N(x+n)$	$M_x - M_{x+n}$	$M(x; n) = M(x) - M(x+n)$
S_x	$S(x) = \sum_{t=x}^{w-1} N(t)$	R_x	$R(x) = \sum_{t=x}^{w-1} M(t)$
$S_x - S_s$	$S(x; s) = S(x) - S(s)$	$R_x - R_s$	$R(x; s) = R(x) - R(s)$
$S_x - S_{x+n}$	$S(x; n) = S(x) - S(x+n)$	$R_x - R_{x+n}$	$R(x; n) = R(x) - R(x+n)$

Continuous, one life

$Dc(x) = l(x) \cdot \exp(-vc \cdot x)$	$Cc(x) = qc(x) \cdot l(x) \exp(-vc \cdot x)$
$Nc(x) = \int_0^\infty Dc(x+t) dt$	$Mc(x) = \int_0^\infty Cc(x+t) dt$
$Nc(x; s) = \int_0^{s-x} Dc(x+t) dt$	$Mc(x; s) = \int_0^{s-x} Cc(x+t) dt$
$Sc(x) = \int_0^\infty Nc(x+t) dt$	$Rc(x) = \int_0^\infty Mc(x+t) dt$
etc.	etc.

Payable in the middle of the year, one life

$\bar{C}_x$	$Cm(x) = v^{-\frac{1}{2}} \cdot C(x)$	$\bar{R}_x$	$Rm(x) = \sum_{t=x}^{w-1} Mm(t)$
$\bar{M}_x$	$Mm(x) = \sum_{t=x}^{w-1} Cm(t)$	$\bar{R}_x - \bar{R}_s$	$Rm(x; s) = Rm(x) - Rm(s)$
$\bar{M}_x - \bar{M}_s$	$Mm(x; s) = Mm(x) - Mm(s)$	etc.	etc.

Discontinuous, joint lives

l_{xy}	$l(x, y) = l(x) \cdot l(y)$	d_{xy}	$d(x, y) = l(x, y) - l(x+1, y+1)$
D_{xy}	$D(x, y) = v^x \cdot l(x, y) = D(x) \cdot l(y)$	C_{xy}	$C(x, y) = v^{x+1} \cdot d(x, y)$
N_{xy}	$N(x, y) = \sum_{t=0}^{w-x} D(x+t, y+t)$	etc.	etc.

3.3. Actuarial values

The following principles apply (see section 2.8) :

- present values of periodically repeated payments begin with a small letter ;
- present values of single payments begin with a capital letter ;
- monetary values, other than present values, begin with a capital letter.

The present notation writes for example the present value of a temporary life annuity payable in advance by the symbol $\ddot{a}_{x:\overline{n}|} = \ddot{a}_{x:\overline{x-x}|}$. This notation leads in many formulae (for example for reserve values) to the situation where one must write in them the expression $\ddot{a}_{x+t:\overline{n-t}|}$. This and the fact that in many tables actuarial values are arranged according to a fixed terminal age led the authors to seek a solution whereby only one argument needs to be changed, which in the example given is simply the attained age. This works when one is making use not of the duration of assurance, but of the terminal age. The authors could not, however, decide to exclude an expression of the traditional form. They decided therefore in favour of the following compromise, which has already been given in 3.2 with the introduction of probabilities of surviving for or of dying within a number of years.

It should also be the aim of proposals for a new notation to be able to express actuarial values in as general a form as possible. Therefore the following proposals for generally comprehensive symbols for values lead by a system of rules of abbreviation to the short formulae essential for practical use.

3.3.1. Present values of life annuities

The following values all apply to a yearly payment of 1. Present values of life annuities payable in advance are denoted by $\ddot{a}$, and annuities payable in arrear by a . The variables that are necessary to define the present value are divided into blocks, which are separated one from another by a semicolon (see 2.8 b) :

$$a(\textcircled{1}; \textcircled{2}; \textcircled{3}; \textcircled{4}; \textcircled{5}).$$

The blocks have the following contents :

- ① : gives the age(s) of the person(s), e.g. x or x, y , during whose life or joint lives the annuity is payable.
- ② : gives information about the time of maturity or the period of deferment or commencement age of the annuity, and about the

52 *International Actuarial Publication Language*

maximum duration or expiry age of the annuity, given in the same order as the persons stated in ①. The sequence of the information about ages and durations in the formula corresponds with those values that occur in the progression through time of an insurance contract.

- ③ : indicates, where appropriate, payment of the annuity more frequently than yearly.
- ④ : gives the rate of interest.
- ⑤ : gives the mortality table. If a generation mortality table is intended, this can be represented by the argument $T+x$ or $T+x, y$ (T being the year of birth of (x)).

So for example

$$\ddot{a}(x, y; n_1:s_1, s_2:n_2; k; i; T)$$

represents a life annuity payable k times a year in advance, which relative to the life (x) is deferred n_1 years and ends at the latest when (x) attains age s_1 . Relative to the life (y) the annuity begins at age s_2 and is payable for a maximum of n_2 years. The annuity is payable only so long as both lives still survive. The value is calculated at rate of interest i and according to the mortality table for the year T . Either the durations n , or the ages s , can be set as variables.

For simplifying the expression the following *abbreviation rules* apply :

1. If that part of block ② that refers to a life x , contains this age in the form $x_v:n_v$ or $x_v:s_v$ or else $0:n_v$ or $0:s_v$ —that is in respect of life (x_v) it is an immediate annuity—this part can be replaced by n_v or s_v .

2. If that part of block ② that refers to a life x , contains the terminal age w of the mortality table in the form $n_v:w$ or $s_v:w$, this part can be written in the form $n_v:$ or $s_v:$. In respect of life (x_v) it is therefore an annuity for the whole of life, though possibly deferred.

3. If block ② contains simply expressions of the form $x:w$, $y:w \dots$ or $0:w-x$, $0:w-y, \dots$ then this block can be omitted.

4. Blocks ④ and ⑤ in general do not need to be given, so that normally a life annuity (in advance) is written in the form $\ddot{a}(\textcircled{1}; \textcircled{2}; \textcircled{3})$.

5. If $k=1$, block ③ can be omitted.

According to these rules we have, for example

Present	Proposed
$\ddot{a}_{x:n}^{(4)} = \ddot{a}_{x:s-x}^{(4)}$	$= \ddot{a}(x; x:n; 4) = \ddot{a}(x; x:s; 4)$ $= \ddot{a}(x; n; 4) = \ddot{a}(x; s; 4),$
$n a_x^{(12)} = s-x a_x^{(12)}$	$= a(x; n:w; 12) = a(x; s:w; 12)$ $= a(x; n; 12) = a(x; s; 12),$
$n_1 n_2 \ddot{a}_x^{(2)} = s_1-x s_2-s_1 \ddot{a}_x^{(2)}$	$= \ddot{a}(x; n_1:n_2; 2) = \ddot{a}(x; s_1:s_2; 2).$

With these expressions we have further a reversionary annuity to (y) after the death of (x)

$$\begin{aligned} \ddot{a}_{x|y} &= \ddot{a}_y - \ddot{a}_{xy} &= \ddot{a}(y; y:w; 1) - \ddot{a}(x, y; x:w, y:w; 1) \\ & &= \ddot{a}(y) - \ddot{a}(x, y). \end{aligned}$$

For this reversionary annuity we propose the abbreviation

$$\ddot{a}(x:y) = \ddot{a}(y) - \ddot{a}(x, y).$$

Further we have for a joint life and last survivor annuity

$$\ddot{a}_{\overline{xy}} = \ddot{a}_x + \ddot{a}_y - \ddot{a}_{xy} = \ddot{a}(x) + \ddot{a}(y) - \ddot{a}(x, y) = \ddot{a}(x) + \ddot{a}(x:y)$$

for which we can also write $\ddot{a}(x, x:y) \equiv \ddot{a}(y, y:x).$

3.3.2. Present values of single assurance payments

The following present values relate in every case to a sum assured of amount 1.

For present values of single assurance payments the same division of the arguments into blocks applies as for annuities. It is necessary only to replace

start of annuity by start of period of liability
and
end of annuity by end of period of liability.

The same rules for simplifying the expressions apply as are given in 3.3.1.

The common types of benefit are indicated as follows :

Present value of the expectation of a payment on survival :	$E()$
Present value of the expectation of a payment on death :	$A()$
Present value of the expectation of a payment on death or on survival (endowment assurance) :	$AE()$

54 *International Actuarial Publication Language*

In particular we have

	Present	Proposed
Present value of the expectation of a person aged x of a payment on survival to age $s = x + n$:	${}_nE_x$	$E(x; s) = E(x; n)$
Present value of the expectation of a person aged x of a payment on death deferred for n_1 years and running for a duration of n_2 years ; payment at the end of the year of assurance :	${}_{n_1 n_2}A_x$	$A(x; n_1:n_2)$
the same without a deferred period :	${}_nA_x$	$A(x; 0:n) = A(x; n)$
the same with a deferred period but for the whole of life ($n_2 = w - x$) :	${}_n A_x$	$A(x; n:w-x) = A(x; n:)$
the same without a deferred period ($n_1 = 0$) but for the whole of life ($n_2 = w - x$) :	A_x	$A(x; 0:w) = A(x; 0:) = A(x)$
Present value of the expectation of a person aged x of a payment on death for the whole of life ; payment to be made in the middle of the year :	$\bar{A}_x$	$Am(x)$
Present value of the expectation of a payment on death for the whole of life ; payment at the end of stated intervals of length $1/k$:	$A_x^{(k)}$	$A(x; k)$

	Present	Proposed
Present value of the expectation of a person aged x of a payment on death for the whole of life ; payment on death :	$\bar{A}_x$	$Ac(x)$
Present value of an endowment assurance for a person aged x for the duration of n years at a maturity age of s :	$A_{x:\overline{n} } = A_{x:\overline{s-x} }$	$AE(x; 0:n) = AE(x; n)$ $= AE(x; x:s) = AE(x; s)$

3.4. Premiums, reserve values, surrender values and paid-up policy amounts

In order to express these values the concept of block divisions used so far will be extended so that the present values of 3.3. are introduced as blocks, as it were, of a higher order.

3.4.1. Premiums

Premiums can be written out expressly so that the arguments (i.e. blocks) of the premium symbols (e.g. $P()$) are, as the first block, the present value of the expectation of the sum assured (e.g. $A()$) and, as the second block, the present value of a premium payment of 1 (e.g. $\ddot{a}()$), thus giving, e.g. $P(A(); \ddot{a}())$.

In particular we have :

Type of premium	Expense loading included	Symbol
1. yearly premiums		
net premium	—	$P()$
reserve premium	γ	$P\gamma()$
Zillmer premium	ζ	$P\zeta()$
office premium	α, β, γ	$Pb()^4$
2. single premiums		
net premium	—	$PU()$
reserve premium	γ	$PU\gamma()$
office premium	α, β, γ	$PUb()^4$

⁴ As an abbreviation $B()$ or $BU()$ may be written in place of $Pb()$ or $PUb()$ respectively.

56 *International Actuarial Publication Language*

For commonly occurring types of assurance such as an endowment assurance with limited premium term, certain simplifications can be made, e.g.

$$P(AE(x; n); \ddot{a}(x; m)) = PAE(x; n; m)$$

wherein also $PAE(x; n; n)$ can be written as $PAE(x; n)$ if $m = n$. General abbreviation rules for these will not be determined here.

3.4.2. Reserve values, surrender values and paid-up policy amounts

For these concepts the already extended block system is further extended by a third block, which gives the expired duration of the assurance, t . They will be shown expressly as :

Reserve value :	$V(A(); P(); t)$
Surrender value :	$U(A(); P(); t)$
Paid-up policy amount	$W(A(); P(); t)$

As with premiums (3.4.1) these expressions can be shortened when referring to commonly occurring types of assurance. For an endowment assurance we have simply, e.g.

Net reserve after t years :

$$V(AE(x; n); PAE(x; n); t) = VAE(x; n; t)$$

Net surrender value after t years :

$$U(AE(x; n); PAE(x; n); t) = UAE(x; n; t)$$

Paid-up policy amount after t years :

$$W(AE(x; n); PAE(x; n); t) = WAE(x; n; t)$$

3.5. *Variable payments*

At present, for example, an increasing payment is indicated by a letter I placed in front of the function name, the function being enclosed in brackets. The following proposal seems to us to be more appropriate and more adaptable. In it, variable payments are described by an extension of block ②.

- I indicates a payment of benefit or annuity *increasing* by 1 per year ;
- D indicates a payment of benefit or annuity *decreasing* by 1 per year ;
- $Ix_1:Ix_2$ indicates a benefit or annuity that increases first at age x_1 and finally at age x_2 (in total therefore there are $x_2 - x_1 + 1$ increases each of an amount of 1) ;
- $Dx_1:Dx_2$ the same for decreasing payments.

and its Representation in Computer compatible Form 57

These abbreviations are placed in block ②. So, for example,

$$\ddot{a}(x; x_1:Ix_2:Ix_3:x_4)$$

indicates the present value of a life annuity payable in advance to a person aged x , which begins at age x_1 , and has a yearly payment of 1 from age x_1 to age x_2-1 . At age x_2 the amount is 2, then at age x_2+1 the amount is 3, etc. Finally at age x_3 the amount of annuity is $1+(x_3-x_2+1)$, and the annuity continues to be paid unaltered at this amount, with a final payment at age x_4-1 , provided the annuity has not ceased earlier by death.

The following simplifications are allowed :

$x_1:I(x_1+1)$: the first payment of annuity at age x_1 , then increasing from age x_1+1 onwards, abbreviated as Ix_1 :

$:Ix_2:x_2$ the last increase and the last payment of annuity both at age x_2 , abbreviated as $:Ix_2$.

Further, payments that first increase then decrease can be written as

$$x_1:Ix_2:Ix_3: Dx_4: Dx_5: x_6$$

and so on.

If we are dealing with an amount that increases immediately, this can be transferred to block ①, e.g.

$$(I\ddot{a})_x = \ddot{a}(x; Ix) = \ddot{a}(Ix)$$

3.6. *Examples: see Appendix*

4. INTERNATIONAL ACTUARIAL SYMBOLS FOR PENSION FUNDS (INCLUDING BOTH STATE AND OCCUPATIONAL RETIREMENT SCHEMES)

There has as yet been no international agreement on a notation for pension funds. Therefore we have generally chosen as our "present notation"—cf. (1.7)—that used in "Richttafeln für die Pensionsversicherung" by G. Heubeck and K. Fischer (especially the "Explanations", 2nd edition 1959, Weissenburg).

For the same reason the following proposal is limited only to the basic values.

4.1. *Basic rules*

The subsequent definitions of function names conform to the following rules :

1. The individual actuarial values will be symbolized by the basic letters introduced in the preceding sections (e.g. q , l , a , . . .).

58 *International Actuarial Publication Language*

2. To the basic letter are attached one or more letters with the following meanings :

The first letter indicates the status of the person concerned at the starting point in time (entry status). The second and if necessary the third letters indicate in the sequence of the letters the status after a change in status, on entry to which payments fall due.

The correspondence between statuses and letters is given in the following table :

Status	Letter
active	a
invalid	i
pensioner	r
widow	w
orphan	o
full orphan	u

3. Thereafter follows the closer definition of the relevant actuarial values by arguments inside function brackets. The significance and sequence of the arguments follow the principles of section 3.

4.2. *Basic probabilities*

	Present	Proposed
Probability of death of an active life aged x :	q_x^a	$qa(x)$
Probability of an active life aged x dying while active :	q_x^{aa}	$qaa(x)$
Probability of an active life aged x dying as an invalid :	q_x^{ai}	$qai(x)$
Probability of death of an invalid aged x :	q_x^i	$qi(x)$
Probability of an invalid aged x dying as an invalid :	q_x^{ii}	$qii(x)$
Probability of an invalid aged x dying as an active life :	q_x^{ia}	$qia(x)$
Probability of death of a pensioner aged x :	q_x^{Lrm}	$qr(x)$

	Present	Proposed
Probability of death of a widow aged y :	$q_y^{Lrf} = q_y$	$qw(y)$
Probability of an active life aged x becoming an invalid :	i_x	$ia(x) = i(x)$
Probability of an invalid aged x become active again :	r_x	$ri(x) = r(x)$
Probability of an active life, invalid or pensioner being married on his death at age x :	h_x	$ha(x), hi(x), hr(x)$ or generally $h(x)$
Probability of a widow aged x remarrying :	w_y	$w(y)$

4.3. Decrement tables

For active lives	l_{x+1}^{aa}	$la(x+1) = la(x)(1 - qaa(x) - i(x))$
For invalids	l_{x+1}^l	$li(x+1) = li(x)(1 - qii(x) - r(x))$
For pensioners	l_{x+1}^{Lrm}	$lr(x+1) = lr(x)(1 - qr(x))$
For widows	l_{y+1}	$lw(y+1) = lw(y)(1 - qw(y) - w(y))$

4.4. Commutation functions

D_x^{aa}	$Da(x) = la(x)v^x$
D_x^l	$Di(x) = li(x)v^x$
D_x^{Lrm}	$Dr(x) = lr(x)v^x$
D_y	$Dw(y) = lw(y)v^y$
N_x^{aa}	$Na(x; s) = \sum_{t=x}^{s-1} Da(t)$
N_x^l	$Ni(x; s) = \sum_{t=x}^{s-1} Di(t)$
N_x^{Lrm}	$Nr(x) = \sum_{t=x}^{w-1} Dr(t)$
N_y	$Nw(y) = \sum_{t=y}^{w-1} Dw(t)$

4.5. Present values

The rules set out in 3.3.1 define the arrangement of arguments in blocks.

		(assuming an annuity payable in advance)
Annuity to an active life	$a_{\dots}^{aa}$	$\ddot{a}a()$
Annuity to an invalid :	$a_{\dots}^i$	$\ddot{a}i()$
Annuity to a pensioner :	$a_{\dots}^{Lrm}$	$\ddot{a}r()$
Annuity to a widow :	$a_{\dots}^{Lrf}$	$\ddot{a}w()$
Expectation of an active life for pension annuity :	$..a_{..}^{aA}$	$\ddot{a}ar()$
Expectation of an active life for invalidity annuity :	$a_{..}^{ai}$	$\ddot{a}ai()$
Expectation of an active life for invalidity and pension annuity :	$a_{..}^{aiA}$	$\ddot{a}air()$
Expectation of an active life for widow's pension :	$a_{..}^{aw}$	$\ddot{a}aw()$
Expectation of an active life for orphan's pension :	$a_{..}^{ak}$	$\ddot{a}ao()$
Expectation of an invalid for pension annuity :	$a_{..}^{iA}$	$\ddot{a}ir()$
Expectation of an invalid for widow's pension :	$a_{..}^{iw}$	$\ddot{a}iw()$
Expectation of an invalid for orphan's pension :	$a_{..}^{ik}$	$\ddot{a}io()$
Expectation of a pensioner for widow's pension :	$a_{..}^w$	$\ddot{a}rw()$
Expectation of a pensioner for orphan's pension :	$a_{..}^k$	$\ddot{a}ro()$

4.6. Examples

(a) Expectation of an active life aged x for an invalidity pension of 1 per annum payable quarterly in advance at the latest to the attainment of age $s = x + n$:

present expression :

$${}^{(4)}a_x^{ai(s)} = \frac{1}{D_x^{aa}} \sum_{t=x}^{s-1} D_t^{aa} i_t \cdot v^{\frac{1}{4}} \cdot {}^{(4)}a_{t+1/2:s-t-1/2}^1$$

proposed expression :

$$\begin{aligned} \ddot{a}ai(x; 0:s; k=4) &= \ddot{a}ai(x; s; 4) = \\ &= \frac{1}{Da(x)} \sum_{t=x}^{s-1} Da(t) i(t) v^{\frac{1}{4}} \cdot \ddot{a}i(t + \tfrac{1}{2}; s; 4) \end{aligned}$$

and its Representation in Computer compatible Form 61

(b) Collective expectation of an invalid aged x for a widow's pension of 1 per annum payable quarterly in advance for the whole of life :

$$\begin{aligned} \text{present : } {}^{(4)}a_x^{iw} &= \\ = \frac{1}{D_x^i} &\left\{ \sum_{t=x}^{s-1} D_t^i q_t^i h_{t+\frac{1}{2}} v^{\frac{1}{2}} \cdot {}^{(4)}a_{y(t)+\frac{1}{2}} + D_s^i \frac{1}{D_s^{Lrm}} \sum_{t=s}^w D_t^{Lrm} q_t^{Lrm} h_{t+\frac{1}{2}} v^{\frac{1}{2} (4)} a_{y(t)+\frac{1}{2}} \right\} \\ \text{proposed : } \ddot{a}_{iw}(x; k=4; h) &= \end{aligned}$$

$$\begin{aligned} &= \frac{1}{D_i(x)} \left\{ \sum_{t=x}^{s-1} D_i(t) q_i(t) h_i(t + \frac{1}{2}) v^{\frac{1}{2}} \cdot \ddot{a}_w(y(t) + \frac{1}{2}; 4) \right. \\ &\quad \left. + \frac{D_i(s)}{D_r(s)} \sum_{t=s}^w D_r(t) q_r(t) h_r(t + \frac{1}{2}) v^{\frac{1}{2}} \ddot{a}_w(y(t) + \frac{1}{2}; 4) \right\} \\ &= \frac{1}{D_i(x)} \sum_{t=x}^{s-1} C_{mi}(t) \cdot h_i(t + \frac{1}{2}) \ddot{a}_w(y(t) + \frac{1}{2}; 4) + \\ &\quad + \frac{D_i(s)}{D_i(x)} \cdot \frac{1}{D_r(s)} \sum_{t=s}^w C_{mr}(t) h_r(t + \frac{1}{2}) \ddot{a}_w(y(t) + \frac{1}{2}; 4) \end{aligned}$$

5. INTERNATIONAL ACTUARIAL SYMBOLS FOR SICKNESS INSURANCE

There has as yet been no international agreement on a notation for sickness insurance. Therefore we shall proceed generally—cf. 1.7—from the notation set out by the German *Verband der privaten Krankenversicherer* 1958 (association of private sickness insurance funds). In so far as the ideas of life assurance are used in sickness insurance, they are not expressly set out in what follows.

5.1. Fixed amounts (expense loadings and contingency loadings)

Initial expenses in proportion to premium : α

Administrative expenses in proportion to premium : β

and those *not* in proportion to premium : γ

Loading for premium refund independent of results, in proportion to premium : δ

Zillmer loading in proportion to premium : ζ

Contingency loading in proportion to premium : λ

and that *not* in proportion to premium : η

5.2. Decrement tables

Probability of withdrawal of a life aged x within one year : $r(x)$

62 *International Actuarial Publication Language*

Number of living where a withdrawal rate is explicitly introduced :	$l_r(x)$
where there is no confusion about the use of a withdrawal rate :	$l(x)$
Correspondingly there are :	$D(x), Dr(x), N(x), Nr(x)$

5.3. *Claim concepts*

Number of insured persons aged x in a population :	$L(x)$
Number of claims for a population in one year :	H
Number of claims for a population aged x :	$H(x)$
and if a distinction must be made : for sickness expenses insurance, etc. (see 5.6) :	$Hkk(x)$
Claim frequency of a person aged x :	$h(x) = H(x)/L(x)$
Total amount of claims of a population in one year :	S
Total amount of claims for a population aged x :	$S(x)$
and if a distinction must be made : for sickness expenses insurance etc.	$Skk(x)$
Amount of one claim :	$s = S/H$
Claims per head = claim amount per year per person insured :	$K = S/L$
Claims per head for a person aged x :	$K(x) = S(x)/L(x)$
and if a distinction must be made : for sickness expenses insurance etc.	$Kkk(x)$
Normalized claims per head (= age-relative of a person aged x ; say $x_0 = 28$) :	$k(x) = K(x)/K(x_0)$
Basic claims per head :	$G = G(x) = S/(k(x) \cdot L(x) + k(x+1) \cdot L(x+1) + \dots)$
and if a distinction must be made : for sickness expenses insurance, etc.	Gkk
Claim frequency parameter :	p
Daily rate :	T
Reduction factor in insurance of a daily rate during sickness :	R

and its Representation in Computer compatible Form 63

5.4. Commutation factors and present values of benefits

Discounted number aged x , appropriate to the middle of the year :

$$Dm(x) = 0.5 \times (D(x) \cdot v^{\frac{1}{2}} + D(x+1) \cdot v^{-\frac{1}{2}})$$

$$Om(x) = Dm(x) \cdot K(x)$$

$$Um(x) = \sum_{j=x}^{w-1} Om(j)$$

Present value of benefit = net single premium :

$$Am(x) = Um(x)/D(x)$$

(If there is no doubt about the use of factors appropriate to the middle of the year the letter " m " can be omitted)

if a distinction must be made :

for sickness expenses insurance, etc.

$$Amkk(x) \text{ or } Akk(x)$$

5.5. Premiums, reserve values

Net premium :

$$P(\dots)$$

for example : net premium for a life aged x for a sickness benefit insurance for n years, payable monthly :

$$Pkk(x; n; 12) \\ = Amkk(x; n)/a(x; n; 12)$$

Zillmer premium

$$P\zeta(\dots)$$

Gross premium

$$Pb(\dots) \text{ or } B(\dots)$$

Tariff premium, rounded gross premium

$$Pt(\dots) \text{ or } Bt(\dots)$$

Reserve value after t years for entry age x :

$$V(x; t)$$

Zillmerized :

$$V\zeta(x; t)$$

5.6. Types of insurance

To indicate the type of insurance or claim payment two small letters are placed between the large letter of the symbol and the function brackets :

Sickness expenses insurance : kk

Hospital expenses insurance : hk

Hospital daily rate insurance : ht

Operation charges insurance : op

Sickness daily rate insurance : tg

Death payment insurance : st

Doctor : ar

Medicaments : md

64 *International Actuarial Publication Language*

6. THE TRANSFORMATION OF THE PUBLICATION LANGUAGE INTO A NOTATION APPROPRIATE TO PROBLEM-ORIENTED COMPUTER LANGUAGES

6.1. The notation developed in the preceding sections is not yet appropriate for the representation of actuarial values and functions in a program for a computer, not even when this program is written in a problem-oriented programming language. The most important reasons for this will be given in 6.2. In 6.3 we formulate a system of rules, on the basis of which the publication language can be transformed into a representation which is at present compatible with the notation prescribed for the programming languages ALGOL, FORTRAN, COBOL and PL/I. Obviously we cannot take account here of all "dialects" of these languages and their implementation for particular computer installations; for this purpose the rules of 6.3 must be extended in any particular case.⁵

6.2. The most important reasons for such a system of transformation rules being necessary are :

6.2.1. The set of letters of Latin capital and small letters, as well as Greek letters, must be reduced to an alphabet of 26 letters.

6.2.2. The use of punctuation marks (and generally of special symbols) is strictly laid down in the programming languages; the rules laid down in sections 2 to 5 are not applicable here. In particular this is true for the division of function arguments into blocks and the indication of the meaning by three different punctuation marks.

6.2.3. The use of equality signs inside expressions in brackets is not allowed in the programming languages.

6.2.4. Function arguments, as for example the rate of interest, more often come in not as current variables, but as constant values. In this case the thought occurs to extend the function name in a suitable way. In the extreme case, that is if all function arguments are single-valued, we reach a function name whose representation corresponds to the representation of a general variable in the programming language.

6.3. *The system of transformation rules*

6.3.1. The rules and definitions of present-day programming languages, including their special implementations for particular

⁵ For this see *E. Neuburger* in : *Blätter der DGVM*, Heft 4, Band VIII, 1968.

and its Representation in Computer compatible Form 65

computers, dominate everything in the rules laid down below; in the event of these leading to an expression which is not compatible with the rules of present-day programming languages, the rules must be altered so as to produce an error-free program.

6.3.2. *Letters*

6.3.2.1. Small letters without a trema are represented by the corresponding large letter with the addition of a P; e.g. instead of a we write AP.

6.3.2.2. Small letters with a trema are represented by the corresponding large letter with the addition of a T; e.g. instead of ä we write AT.

6.3.2.3. Greek letters are written with their names in capitals; e.g. instead of β we write BETA. Abbreviations are naturally permitted, so long as ambiguities are avoided.

6.3.3. *Functions*

The idea mentioned in 6.2.4. to extend the function *name* in the event of single-valued function arguments, is generally used in what follows, so that the meanings of arguments contained in blocks are brought into the expression. A quick look at the diagram gives a schematic representation.

6.3.3.1. The *function name* consists of a *nucleus* and one or more *specifications*.

6.3.3.2. The nucleus consists simply of letters, or in other cases it is followed by digits followed by a P (which has a significance for the preceding letter) or by a Z (which is a pure separator, in the event of there being digits in the nucleus). In general function names are made up according to sections 2 to 5.

6.3.3.3. The specification begins with a digit, followed by digits and/or letters, in which P and Z directly following the first digit of the specification are not allowed; then the beginning of the specification is clearly recognizable.

6.3.3.4. In the first digit specification the number of digits corresponds to the number of variables in the first block. The meaning of the digits is according to Table 1.

Diagram : the notation for functions and constants in the “computer language”

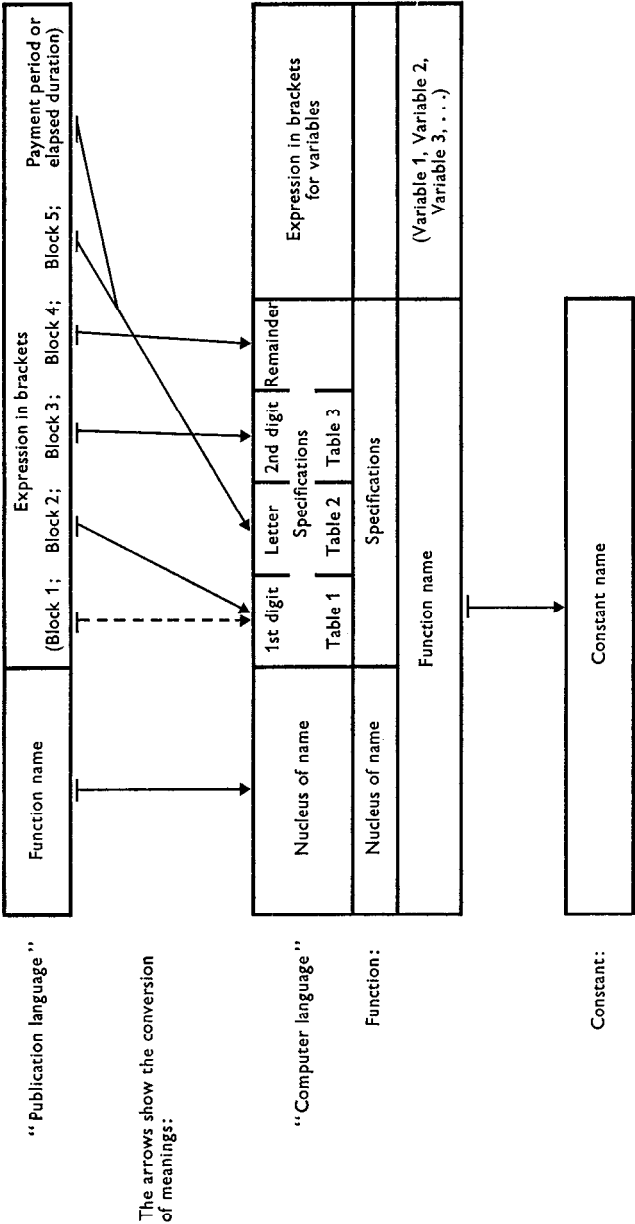


TABLE 1

1st digit specification	Arrangement in the second block
0	nothing appears in the 2nd block
1	$s_1 : s_2$ i.e. from age to age
2	$s_1 :$ (corresponding to $s_1 : w$)
3	s_2 (corresponding to $x : s_2$)
4	$s_1 : n_2$ i.e. from age to duration
5	$n_1 : n_2$ i.e. from duration to duration
6	$n_1 :$ (corresponding to $n_1 : w$)
7	n_2 (corresponding to $0 : n_2$)
8	$n_1 : s_2$ i.e. from duration to age
9	$P()$ the first block contains a present value and the second a premium, cf. 3.4.2

6.3.3.5. In the immediately following letter specification the letters A to D are available ; letters E to Y (except P) are available for special specifications that must be explained in the program documentation. The meaning of the first four letters is given below

TABLE 2

Letter specification	Year of birth	Elapsed duration or premium paying term
A	—	—
B	+	—
C	—	+
D	+	+

+ The variable is present as an argument.

— The variable is *not* present as an argument.

If there is no variable, A has the meaning of a separator ; if both variables exist, then the year of birth comes into the expression in brackets before the elapsed duration or premium paying term.

6.3.3.6. The meaning of the second digit specification is determined as follows :

TABLE 3

Second digit specification	Method of payment
0	indicated by a current variable
1	yearly or not applicable
2	half-yearly
3	quarterly
4	monthly
5	weekly

The digits 6 to 9 are available for use as special specifications explained in the program documentation.

68 *International Actuarial Publication Language*

6.3.3.7. After the second digit specification may follow further letter or digit specifications ; they must be defined at will.

6.3.3.8. Example as an illustration :

$\ddot{a}(x, y; n_1:s_1, s_2:n_2; k; T) = AT84BO (X, Y, N1, S1, S2, N2, K, T)$

6.3.4. *Special case* : variable payments

Variable payments are indicated by putting *before* the nucleus the letter I for increasing, or the letter J for decreasing, payments. In the first digit specification the number of digits is as a rule double the number of variables in the first block. According to 3.5 in the case of variable payments dependent on the life of one person the second block has the structure

$$x_1:Ix_2:Ix_3:x_4$$

The first digit of the two-position specification indicates (according to Table 1) the arrangement of the values not indicated by an I, here x_1 and x_4 ; the second digit indicates the arrangement of the values indicated by an I, here Ix_2 and Ix_3 . However the sequence of values in the expression in brackets remains as before, here x_1, x_2, x_3, x_4 .

6.3.5. *Functions of functions*

In certain actuarial functions, such as reserve values, the type of policy and possibly further functions enter as function arguments. These functions of functions can if necessary be expressed as functions of the type of 6.3.3, in which the type of policy is incorporated into the nucleus of the chief function.

6.3.6. *Examples* : See Appendix

APPENDIX
Comparison of representations of more important actuarial values

Present (XIV ICA)	Publication language	Proposal for	Computer representation
$i; i^{(m)}$	$i, i(k)$		I, IP; IP(K)
$r = 1 + i$	r		R, RP
$v = (1 + i)^{-1}; v^{(m)}$	$v; v(k)$		V, VP; VP(K)
v^x	v^x		V**X (not in ALGOL)
$d = 1 - v; d^{(m)}$	$d; d(k)$		D, DP; DP(K)
$\delta = \ln(1 + i)$	δ, v_c		DELTA, VC
$a_{\overline{n}}; \ddot{a}_{\overline{n}}$	$a(; n); \ddot{a} (; n)$		APN(N); ATN(N)
$s_{\overline{n}}; \ddot{s}_{\overline{n}}$	$s(; n); \ddot{s} (; n)$		SPN(N); STN(N)
$a_{\overline{n}}^{(m)}; \ddot{a}_{\overline{n}}^{(m)}$	$a (; n, k); \ddot{a} (; n, k)$		APN0A0(N, K) = APN0V(N, K)
			ATN0A0(N, K) = ATN0V(N, K)
$l_x; l_{x t+t}$	$l(x); l((x) + t)$		L(X); LOC(X, T)
d_x	$d(x)$		DP(X), DP0(X)
$p_x; n p_x; s-x p_x$	$p(x); p(x, n); p(x, s)$		PP(X); PP7(X, N); PP3(X, S)
$n p_{xy} = s-x p_{xy}$	$p(x, y, n) = p(x, y, s)$		PP30(X, Y, N), PP70(X, Y, S)
$q_x; n q_x; s-x q_x$	$q(x); q(x, n); q(x, s)$		Q(X); Q7(X, N), Q3(X, S)
μ_x	$\mu(x), q_c(x)$		MY(X), QC(X)
e_x, e_x	$e(x), ec(x)$		EP(X); EPC(X)
$D_x; N_x; S_x$	$D(x); N(x); S(x)$		D(X); N(X); S(X)

Present (XIV.ICA)	Publication language	Proposal for	Computer representation
$C_x; M_x; R_x$	$C(x); M(x); R(x)$		$C(X); M(X); R(X)$
	$De(x); Ne(x); Se(x)$		$DC(X); NC(X); SC(X)$
	$Ce(x); Me(x); Re(x)$		$CC(X); MC(X); RC(X)$
$\bar{C}_x; \bar{M}_x; \bar{R}_x$	$Cm(x); Mm(x); Rm(x)$		$CM(X); MM(X); RM(X)$
$l_{xy}; d_{xy}$	$l(x, y); d(x, y)$		$L(X, Y); DP(X, Y)$
$D_{xy} = v^{t(x+y)} \cdot l_{xy}; N_{xy}$	$D(x, y) = v^x \cdot l(x, y) = D(x) \cdot l(y); N(x, y)$		$D(X, Y); N(X, Y)$
$a_x; \ddot{a}_x$	$a(x); \ddot{a}(x)$		$AP(X); AT(X)$
$a_x^{(k)}; \ddot{a}_x^{(k)}$	$a(x; k); \ddot{a}(x; k)$		$AP0A0(X, K) = AP0V(X, K)$
			$AT0A0(X, K) = AT0V(X, K)$
$\ddot{a}_{x:\overline{n}} = \ddot{a}_{x:s-\overline{x}}^{(4)}$	$\ddot{a}(x; n; 4) = \ddot{a}(x; s; 4)$		$AT7A3(X, N) = AT7I(X, N),$
			$AT3A3(X, S) = AT3I(X, S)$
$n a_x; n _{t^2}a_x$	$a(x; n); a(x; n:t)$		$AP6(X, N), AP5(X, N, T)$
$n a_x^{(12)} = s-x a_x^{(12)}$	$a(x; n; 12) = a(x; s; 12)$		$AP6A4(X, N) = AP6M(X, N),$
			$AP2A4(X, S) = AP2M(X, S)$
$n _{\log} \ddot{a}_x$	$\ddot{a}(x; n_1; n_2; 2)$		$AT5A2(X, N1, N2) = AT5E(X, N1, N2)$
$a_x _y = \ddot{a}_y - \ddot{a}_{xy}$	$\ddot{a}(x;y) = \ddot{a}(y) - \ddot{a}(x, y)$		$AT2Z(Y) - AT3Z(X, Y)$
$a_{\overline{xy}} = \ddot{a}_x + \ddot{a}_y - \ddot{a}_{xy}$	$\ddot{a}(x, x;y) = \ddot{a}(y, y;x) = \ddot{a}(x) + \ddot{a}(y) - \ddot{a}(x, y) =$ $= \ddot{a}(x) + \ddot{a}(x;y)$		$AT1Z(X) + AT2Z(Y) - AT3Z(X, Y)$
nE_x	$E(x; n)$		$E7(X, N)$
$n1 n_2A_x$	$A(X; n_1; n_2)$		$A5(X, N1, N2)$

$ nA_x; n A_x$	$A(x, n); A(x, n);$	$A7(X, N); A6(X, N)$
$A_x; A_x^{(t)}$	$A(x); A(x; k)$	$A(X); A0A0(X, K) = A0V(X, K)$
$\overline{A_x}$	$A_m(x) \text{ or } A_c(x)$	$AM(X), AC(X)$
$A_{x:\overline{m}}$	$A_E(x; n)$	$AE7(X, N)$
$P_x = P(A_x)$	$PA(x)$	$PA(X)$
$P_{x:\overline{m}} = P(A_{x:\overline{m}})$	$PAE(x; n)$	$PAE7(X, N)$
$n\overline{P(A_x)}$	$PAC(x; w; n)$	$PAC0C(X, N)$
$P_{xy}(A_x)$	$P(A(x); \ddot{a}(x, y))$	$P(A(X), AT(X, Y))$
$n\overline{P(n A_x)}$	$P(a(x; n); a(x; n))$	$PAP7(X, N)$
$m\overline{P^{(t)}(A_{x:\overline{m}})}$	$PAE(x; n; m; k)$	$PAE7C0(X, N, K, M) =$ $= PAE7X(X, N, K, M)$
${}_tV_x$	$V(A(); P(); t)$	$V9C(A(), P(), T)$
${}_tV_{x:\overline{m}}$	$VAE(x; n; t)$	$VAE7C(X, N, T)$
$U \text{ (surrender value)}$	$U(A(); P(); t)$	$U9C(A(), P(), T)$
${}_tW_{x:\overline{m}}$	$WAE(x; n; t)$	$WAE7C(X, N, T)$
$(Ia)_x; (IA_x)$	$a(Ix); A(Ix)$	$IAP(X); IA(X)$
$(D_{\overline{n}} A)_{x:\overline{m}}^I$	$A(Dx; n)$	$JA07(X, N) = JA7(X, N)$