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RESERVING FOR DEFERRED CAPITAL GAINS TAX (AN APPLICATION OF OPTION PRICING THEORY)

BY M. SHERRIS, B.A., M.B.A., F.I.A., F.I.A.A., A.S.A.

ABSTRACT

This paper sets out a framework, based on option pricing theory, that can be used to assess the value of deferred unrealised capital gains tax. In the U.K. and Australia, capital gains tax is paid on realisation of assets and the basis for determining the tax allows for inflation indexation of the cost base of the asset. Capital gains tax payments under these circumstances are shown to resemble those of a complex option. A number of theoretical approaches to the valuation of this option are discussed in the paper.

KEYWORDS

Options; Reserves; Tax; Valuation

1. INTRODUCTION

1.1 In most countries investors are taxed on capital gains (and losses) which are determined and paid on the realisation of assets rather than on an accruals basis. This is so for investors in the United Kingdom, United States of America, Canada and Australia. In Australia, assets are subject to capital gains tax on such a basis, including those held by pension and superannuation funds, as a result of taxation changes in 1988. Differences in capital gains tax between different countries arise from the income that capital losses can be offset against and, in the U.K. and Australian case, from the indexation of the cost of the asset to inflation for the determination of the capital gain.

1.2 Institutional investors, such as life insurance companies, general insurance companies, pension and superannuation funds, and unit trusts or mutual funds, hold assets subject to capital gains tax. These institutions need to establish reserves for tax on unrealised capital gains/losses prior to the realisation of these assets. Because of the deferral of the capital gains tax in these situations a number of issues arise:

- (a) how to determine reserves for deferred capital gains tax liabilities and the appropriate investment policy for the assets backing the reserve,
- (b) how to determine the optimal investment strategy to minimise capital gains tax payments, and
- (c) how to determine theoretically the adjustment to be made for capital gains tax for claims backed by assets where accrued gains/losses are not realised, but effective ownership of the assets is transferred.

1.3 An approach to the determination of these reserves would be to estimate

the capital gains tax by assuming an average holding period for the different classes of assets, and then projecting the asset value to the end of the holding period using a fixed rate of capital growth. This asset value would be used to determine the tax, which would then be discounted to the current date to determine the reserve. The problem with this approach is that it does not allow for stochastic growth in the asset values, nor does it provide a theoretical justification for the appropriate interest rate to use in discounting the future tax.

1.4 The aim of this paper is to set out a framework that can be used to consider these issues using option pricing theory, basing the analysis on the Australian taxation rules. In a sense the U.K. and Australian case is more general than for other countries, since the capital gains tax rules in these countries provide for indexation of the cost of assets. Other tax regimes typically have a zero or no-indexation of the cost base for capital gains tax. The benefit of such an approach is that it is based on a stochastic model, and the discount rate used to calculate the present value of the deferred tax has a valid theoretical basis for its determination.

1.5 The paper begins with a brief outline of the capital gains tax rules in Australia. The option features of the capital gains tax are then examined for the case where the investor is assumed to buy and hold a single asset for a fixed holding period. A number of techniques to determine the theoretical value of the capital gains tax payoffs are considered.

1.6 The option pricing approach that allows for the tax effects is based on the construction of a hedging portfolio which replicates the capital gains tax payoffs. The value of the assets in this hedging portfolio is the reserve for future capital gains tax that would be established for the deferral of the accrued capital gain. The hedging portfolio is, in theory, continuously rebalanced over time. The resulting dynamic investment strategy, based on option pricing theory, provides the appropriate investment composition for the assets backing the tax reserve. The model for asset returns assumed in this paper is most suited for equity assets such as ordinary shares. Alternative models can be used for fixed-interest assets.

1.7 The framework used to determine reserves for deferred capital gains tax can be used to consider the optimum investment strategy to minimise capital gains tax payments when investors are assumed to hold a portfolio of assets and do not adopt a buy and hold investment strategy. This problem would be better considered in terms of maximising the after tax returns of an investment portfolio, with due allowance for the deferral of capital gains tax.

1.8 This paper also covers and uses developments in option pricing theory that have wider applications in insurance and investment than the determination of capital gains tax reserves. These techniques can be used to value insurance and pension benefits, which are the maximum of two future values with a fixed guaranteed minimum. Option pricing theory has been used in a wide variety of insurance applications. The paper by Wilkie (1987) covers one such application, and Appendix 4 of the paper by Daykin & Hey (1990) surveys insurance applications of the option pricing model.

2. CAPITAL GAINS TAX RULES

2.1 Capital gains tax (CGT) in Australia is calculated using a cost base for the asset which is the purchase cost or deemed purchase cost. It is paid on realisation of the asset, except in the case of deep discount or zero coupon securities, where an accruals basis applies. Some investors, such as life insurance funds, are taxed on realised gains and losses as income, the gain or loss being the difference between the sale proceeds and the purchase cost.

2.2 For most other investors the capital gains tax aims to tax only 'real' gains by adjusting for the inflationary increases in the asset value. To do this, an inflation index is applied to the cost base to determine an indexed cost base. Provided the indexed cost base is higher than the actual cost base, this is used for determining capital gains tax. If the indexed cost base is lower than the actual cost base, then the actual cost base is used to determine the tax. The value of the cost base used to determine the tax is, therefore, the maximum of the actual cost base and the indexed cost base.

2.3 If the sale price exceeds the cost base, adjusted for indexation as outlined above, then the investor is subject to capital gains tax on the difference. The tax rate differs between investors taxed on a concessional basis, such as pension and superannuation funds, and fully taxable investors.

2.4 A capital loss arises whenever the sale price is lower than the actual cost base. Capital losses can be offset against capital gains. Net losses in any year are carried forward.

2.5 For assets disposed of within 12 months of purchase the cost base is not indexed. As mentioned above, this is also the case for life insurance funds where capital gains and losses are taxed as income with no indexation allowed.

3. THE CAPITAL GAINS TAX OPTION PAYOFFS

3.1 It is assumed that assets cannot be held for an infinite holding period, so that a capital gain or loss will eventually arise on realisation of the asset. The payments (or receipts) on the sale of the asset in respect of the capital gains tax are shown to be equivalent to those of an option.

3.2 The capital gains tax can be considered for two distinct cases, depending on whether or not the cost base is indexed for the purposes of determining the amount of tax.

The No-Indexation Case

3.3 Assume that the investor purchased, or is deemed to have purchased, a single asset at time 0, which has an actual cost base of E for capital gains tax purposes and is held for a fixed time period of T years. The asset is assumed to pay all of its return in the sale price, so that shares do not pay dividends. On realisation of the asset a capital gain or loss will occur, equal to the difference between the sale price and the cost base.

3.4 If it is assumed that capital losses cannot be offset against taxable income or capital gains, and hence have no tax benefit, then the tax on the capital gain will be given by:

$$\begin{array}{ll} -\tau (S_T - E) & \text{if } S_T > E \\ 0 & \text{otherwise} \end{array}$$

where S_T is the asset price at time T , τ is the tax rate and the negative sign indicates a payment to the government.

3.5 If losses can be offset against other income, so that they will reduce total tax payable at time T , then the tax on the capital gain/loss will be given by:

$$\begin{array}{ll} -\tau (S_T - E) & \text{if } S_T > E \\ \tau (E - S_T) & \text{if } S_T < E. \end{array}$$

3.6 To specify the nature of the option feature involved in the capital gains tax, note that the buyer (owner) of a European call option with exercise price X receives the following payoffs on the exercise date:

$$\begin{array}{ll} S - X & \text{if } S > X \\ 0 & \text{otherwise} \end{array}$$

which is equivalent to $\max\{S - X, 0\}$. The buyer (owner) of a European put option with exercise price X , receives the following payoffs on the exercise date:

$$\begin{array}{ll} X - S & \text{if } X > S \\ 0 & \text{otherwise.} \end{array}$$

3.7 These options can also be considered as options to exchange an asset (cash) of value X for an asset (share) with value S at the exercise date. For more details of this equivalence see Margrabe (1978).

3.8 The payoffs to the holder of the asset in respect of the capital gains tax are equal to the tax rate times the payoffs arising if the investor has sold a call option on the asset to the government with exercise price E . If the capital loss can be offset against other income, then the capital gain/loss payoffs are equivalent to the tax rate times the payoffs arising if the investor had sold a call option to the government and bought a put option from the government, both with exercise price E .

The Indexation Case

3.9 For the indexation case, denote the indexed cost base by C , the asset value by S and the inflation index by F . The asset is purchased at time 0 for a cost base of E and the current time is t years. The indexed cost base at time t is given by C_t , the asset price at time t by S_t and the index value at time t by F_t . For the time being, assume that losses are not carried forward, so that at time T , when the

asset is assumed to be realised, any capital loss results in no tax benefit to the holder of the asset. Only capital gains result in tax payments.

3.10 If the cost base for determining the capital gain is denoted by V_T , then we have that V_T is the maximum of C_T and E (written as $\max\{C_T, E\}$) where:

$$\begin{aligned} C_T &= \frac{F_T}{F_0} \cdot E \\ &= \frac{F_T}{F_t} \cdot C_t \end{aligned}$$

3.11 The cost base E will equal S_0 if there are no purchase costs allowed in determining the cost base for tax purposes.

3.12 On realisation, the holder of the asset will receive the following proceeds net of the capital gains tax:

$$\begin{array}{lll} \text{(capital gain)} & S_T - \tau (S_T - V_T) & \text{if } S_T > V_T \\ & S_T & \text{otherwise} \end{array}$$

where τ is the capital gains tax rate. The tax effect of the capital gain is:

$$\begin{array}{ll} -\tau (S_T - V_T) & \text{if } S_T > V_T \\ 0 & \text{otherwise} \end{array}$$

where the minus sign indicates a payment to the government.

3.13 If the investor held other assets which generated capital profits at time T , against which any loss on this asset could be offset, the tax benefit on the loss would then have a value. The loss would be given by an effective payment from the government to the seller of:

$$\begin{array}{ll} \tau (E - S_T) & \text{if } S_T < E \\ 0 & \text{otherwise.} \end{array}$$

3.14 The payoffs to the holder of the asset in respect of the indexed capital gains tax are equivalent to the tax rate times the payoffs under a complex option. The gain payoffs are equivalent to those arising if the investor had sold a call option to the government on the asset with an exercise price V_T , where V_T is the maximum of the indexed cost and the base cost E , or bought an option from the government to exchange the maximum of C and E for the asset S at time T . The value of C is not equivalent to the value of an indexed zero coupon bond unless the real rate of return on such bonds is zero.

3.15 If the loss payoffs could be offset against capital gains on other assets, then they would be equivalent to those arising if the investor had bought a put option from the government on the asset S with an exercise price E .

3.16 Because the asset is assumed to be held for a fixed holding period, the capital gain and loss options are European options with maturity at time T , the assumed realisation date.

3.17 Having specified the payoffs in the option pricing framework, it is possible to determine a theoretical reserve for the deferred capital gains tax on claims to assets with unrealised gains/losses. The option pricing valuation of the tax option payoffs will also provide the investment strategy appropriate for the assets backing any capital gains tax reserve. The investment strategy issue is briefly discussed in a later section.

4. VALUING THE CAPITAL GAINS TAX OPTION

The No-Indexation Case

4.1 Where no indexation of the cost base applies in determining the capital gain, the value of the capital gains tax payoff can be determined assuming that the logarithm of the rate of return on the asset is normally distributed using the Black-Scholes option pricing formula. Adjustments can also be made for dividend payments. The formula for the capital gain is minus the tax rate times $G(S, E, T-t) = S_t N(d_1) - E e^{-r(T-t)} N(d_2)$ and for the capital loss it is the tax rate times $L(S, E, T-t) = E e^{-r(T-t)} N(-d_2) - S_t N(-d_1)$ where:

- t is the current time in years from the original purchase date of the asset,
- T is the assumed date of realisation of the asset in years from the date of original purchase,
- S_t is the current asset value at time t , future values of which are assumed to be log-normally distributed,
- σ^2 is the p.a. instantaneous variance of the continuously compounding return on asset S ,
- r is the p.a. risk-free interest rate (continuously compounded),
- S_0 is the current asset price,
- $d_1 = (\ln(S_0/E) + (r + \sigma^2/2)(T-t)) / \sigma \sqrt{(T-t)}$,
- $d_2 = d_1 - \sigma \sqrt{(T-t)}$, and
- $N(\cdot)$ is the standard normal cumulative distribution function.

The details of these formulac can be found in Jarrow & Rudd (1983).

4.2 Whenever the capital loss can be offset against other taxable income or gains, the value of the total capital tax effect is minus the tax rate times $G(S, E, T-t) - L(S, E, T-t)$ which, from the put-call parity relationship (Jarrow & Rudd, Chapter 4), is equal to $(S_t - E e^{-r(T-t)})$ if there are no dividends paid. To allow for dividends the current share price is replaced with a price net of the value of dividends payable between the current date and the assumed realisation date of the asset.

4.3 It is worth noting that this value is not the current capital gain or loss, $(S_t - E)$. The adjustment to the current gain/loss that arises as a result of the deferral of its realisation is that the current asset price is reduced by the value of the dividend payments to the actual realisation date, and the cost base is discounted for interest to the actual realisation date. It should also be noted that

this value is the value of the tax which accrues over the total holding period of the asset. It concludes a component that has accrued and a component that is yet to accrue over the remaining holding period. In establishing capital gains tax reserves it is only the accrued component that is valued.

The Indexation Case

4.4 Assuming that any capital loss cannot be absorbed and cannot be carried forward, the value of the capital gains tax allowance is equal to minus the tax rate times $G(C, E, S, T - t)$, where $G(C, E, S, T - t)$ is the value of a European call option on S , with exercise price equal to the maximum of C and E and maturity $(T - t)$. This option can be valued as a European option to exchange an asset whose value is given by the maximum of C_T and E for the asset with value S_T at time T .

4.5 Options on the maximum of two assets have been analysed by Stulz (1982) and also by Johnson (1987). A formula for their value can be derived in a number of ways. The conventional approach is to construct the hedge portfolio for the option, and use this to derive the partial differential equation governing the option price. Solving this differential equation subject to the relevant boundary conditions provides the option value. Stulz uses a risk-neutral valuation approach, where the expected value of the option payoffs are discounted at the risk free rate to determine the option value. The Johnson approach is based on probabilistic arguments to determine the form of the option value. These approaches are considered in turn.

Hedge Portfolio Valuation Approach

4.6 The hedge portfolio that will replicate the indexed capital gains tax option payoffs will contain the asset S , the risk-free asset, and an asset to hedge the effects of the indexation of the cost base on the option value. Denote the value of the latter asset by H . This asset will be an inflation-indexed zero coupon bond providing a real rate of return of r_f . The risk free asset is a nominal default free bond.

4.7 The derivation of the partial differential equation governing the option value using the hedging portfolio approach is identical to that set out in Stulz (1982, 163-164). Assume that the asset prices of S and H , and the indexed value C , follow the stochastic differential equations given by:

$$\frac{dS}{S} = \mu_s dt + \sigma_s dZ_s$$

$$\frac{dH}{H} = \mu_h dt + \sigma_c dZ_c$$

$$\frac{dC}{C} = \mu_c dt + \sigma_c dZ_c$$

$$dZ_c dZ_s = \rho_{cs} dt$$

where:

- μ_s, μ_h and μ_c are the expected rates of change (or drift terms) in the asset values S and H and the index C , respectively,
 σ_s and σ_c are the instantaneous variances of the continuously compounding rate of change in the asset value S and the index value C , respectively,
 Z_s and Z_c are standard Weiner or Brownian motion processes for the asset value S and the index value C , respectively, and
 ρ_{cs} is the instantaneous correlation between these Brownian motion processes.

4.8 Assume that these dynamics are adjusted for tax, and that capital gains on the assets in the hedging portfolio are not deferred. This avoids the need to allow for capital gains tax effects on the hedging portfolio. In effect, all assets in the portfolio are assumed to be sold and repurchased continuously, so that the drift terms in the differential equations are assumed to be adjusted for tax. If this is not the case, then the formula will need to be modified to reflect the actual timing of payment of the capital gains tax on the hedging portfolio.

4.9 Asset H is assumed to provide a perfect hedge against stochastic changes in the index, so that the stochastic component of the return on H , given by $\sigma_c dZ_c$, is the same as that for the change in the index. The instantaneous drift for $d \ln H$, the rate of return on H , ($\mu_h - \frac{1}{2}\sigma_c^2$), is assumed to be equal to the instantaneous drift for $d \ln C$, the rate of change in the index, ($\mu_c - \frac{1}{2}\sigma_c^2$), plus the real rate of return on H , r_f , so that $\mu_h = \mu_c + r_f$. The hedge security is assumed to be default free so that μ_h contains no risk premium.

4.10 The value of the capital gains tax option will be a function of S , H and time to maturity. Itô's Lemma is then used to determine the dynamics of the option value, in terms of the value of the indexed hedge security H . Denote this value by $G = G(S, H, t)$ so that:

$$dG = G_s dS + G_h dH - G_t dt + \frac{1}{2}\{G_{ss}(dS^2) + G_{hh}(dH^2) + 2G_{sh}dSdH\} \\ = G_s dS + G_h dH - G_t dt + \frac{1}{2}\{G_{ss}S^2\sigma_s^2 + G_{hh}H^2\sigma_c^2 + 2G_{sh}SH\rho_{sc}\sigma_s\sigma_c\}dt$$

where subscripts on the option value G denote partial differentiation. For more details on Itô's Lemma see the Appendix of Wilkie (1987) or Sherris (1989a).

4.11 Construct the hedge portfolio to replicate the option payoffs so that it is self-financing, with fractions x of asset S , y of asset H and $(1 - x - y)$ of the risk free asset. If the risk free rate of return is r then the return on the hedge portfolio will be given by:

$$\frac{dG}{G} = x \frac{dS}{S} + y \frac{dH}{H} + (1 - x - y)r dt$$

so that:

$$dG = x \frac{dS}{S} G + y \frac{dH}{H} G + (1 - x - y)rG dt.$$

4.12 A comparison of the dynamics of the hedge portfolio, given in § 4.11, with that of the option value dynamics, given in § 4.10, gives the values for x and y as $x = G_s S / G$ and $y = G_h H / G$.

4.13 Now eliminate the stochastic terms in the differential of the option value and, after dividing by dt , obtain:

$$-G_t = rG - rG_s S - rG_h H + \frac{1}{2} \{G_{ss} S^2 \sigma_s^2 + G_{hh} H^2 \sigma_c^2 + 2G_{sh} SH \rho_{sc} \sigma_s \sigma_c\}.$$

This partial differential equation is solved subject to the appropriate boundary conditions to obtain the option value. The terminal boundary condition for the option value is that the option value at time T is equal to:

$$\max\{(S_T - \max\{C_T, E\}), 0\}.$$

4.14 The equation can also be expressed in terms of the dynamics of the index C to get $G = G(S, C, t)$. To do this we have:

$$C_t = C_0 \exp\{(\mu_c - \frac{1}{2}\sigma_c^2)t + \sigma_c t Z\}$$

and

$$H_t = H_0 \exp\{(\mu_h - \frac{1}{2}\sigma_c^2)t + \sigma_h t Z\}$$

where Z is a standard normal random variable. Hence:

$$C_t / C_0 = H_t / H_0 \exp\{(\mu_c - \mu_h)t\}.$$

This gives partial differentials for:

$$G_h = G_c C_h = G_c \exp\{(\mu_c - \mu_h)t\}$$

$$G_{hh} = [G_c \exp\{(\mu_c - \mu_h)t\}]_h = G_{cc} \exp\{2(\mu_c - \mu_h)t\}$$

$$G_t = [G(S, H \exp\{(\mu_c - \mu_h)t\}, t)]_t$$

$$= G_t + (\mu_c - \mu_h) G_c C$$

and

$$G_{sh} = G_{sc} C_h = G_{sc} \exp\{(\mu_c - \mu_h)t\}.$$

4.15 Substituting these expressions into the partial differential equation in terms of H in § 4.13 gives:

$$-(G_t - (\mu_c - \mu_h)G_c C) = rG - rG_s S - rG_c \exp\{(\mu_c - \mu_h)t\} H + \frac{1}{2} \{G_{ss} S^2 \sigma_s^2 + G_{cc} \exp\{2(\mu_c - \mu_h)t\} H^2 \sigma_c^2 + 2G_{sc} \exp\{(\mu_c - \mu_h)t\} SH \rho_{sc} \sigma_s \sigma_c\}$$

which simplifies to:

$$-G_t = rG - rG_s S - (\mu_h - \mu_c + r)G_c C + \frac{1}{2} \{G_{ss} S^2 \sigma_s^2 + G_{cc} C^2 \sigma_c^2 + 2G_{sc} SC \rho_{sc} \sigma_s \sigma_c\}.$$

Risk-Neutral Valuation Approach

4.16 The indexed capital gains tax option payoffs can be valued as the expected value of the payoffs discounted at the risk free rate, on the assumption that an appropriate hedging portfolio can be constructed.

4.17 The capital gain option payoffs are $\max\{(S - \max\{C, E\}), 0\}$ and the value of the option is $\exp\{-RT\}EV[\max\{(S - \max\{C, E\}), 0\}]$ where R is the risk

free rate on a non-indexed bond and EV is the expected value operator with respect to the appropriate risk-neutral probability distribution.

4.18 Appendix 1 derives the expected value of the option payoffs on the assumption that the rate of return on S and the rate of change in C are bivariate normally distributed. The evaluation of the expected value requires the substitution of the means or drift terms of the relevant bivariate distribution with the risk-free rates of return. The details are provided in Appendix 1. The resulting option value is:

$$\begin{aligned} & SN_2\{\alpha_1 + \sigma_s\sqrt{T}, (\ln(S/C) + (r_f - \frac{1}{2}\sigma^2)T)/\sigma\sqrt{T}, (\sigma_s - \rho_{cs}\sigma_c)/\sigma\} \\ & - C\exp\{-r_f T\}N_2\{(\ln(S/C) + (r_f - \frac{1}{2}\sigma^2)T)/\sigma\sqrt{T}, \beta_1 + \sigma_c\sqrt{T}, (\rho_{cs}\sigma_s - \sigma_c)/\sigma\} \\ & - E\exp\{-RT\}N_2\{\alpha_1, \beta_1, \rho_{cs}\} \end{aligned}$$

where $N_2\{\alpha, \beta, r\}$ is the bivariate cumulative standard normal distribution with upper limits of integration α , and β and correlation coefficient ρ ,

$$\alpha_1 = \{\ln(S/E) + (R - \frac{1}{2}\sigma_s^2)T\}/\sigma_s\sqrt{T},$$

$$\beta_1 = \{\ln(C/E) + (R - r_f - \frac{1}{2}\sigma_c^2)T\}/\sigma_c\sqrt{T},$$

T = time to realisation in years,

r_f = p.a. continuously compounding real rate of return on risk-free indexed bond,

R = p.a. continuously compounding risk free rate, and

$$\sigma^2 = \sigma_c^2 + \sigma_s^2 - 2\rho_{sc}\sigma_s\sigma_c.$$

Probability Based Valuation Approach

4.19 The probability approach to option pricing, as given by Johnson (1987), involves transforming the option value into a different numeraire and recognising that the term multiplying the numeraire in any option price is the probability, based on the transformed probability distribution, that the option will be exercised. This technique is set out and used in Appendix 3 in the case of a conventional European put option on a non-dividend paying stock. This approach was developed for the case of options on the maximum or minimum of several assets by Johnson.

4.20 In order to use the Johnson technique to value the capital gains tax option, the option value is expressed using the values for S and C as numeraire. This is necessary in order to determine the terms multiplying S and C in the capital gains tax option pricing formula. These terms are the probabilities that options based on the transformed dynamics are exercised.

4.21 The term multiplying the share price in the capital gains tax option value is determined by expressing the option value with the share price as numeraire. Doing this gives an option value of $G(C, E, S, T)/S$, where S is the current share price, which equals the value $G((C/S), (E/S), 1, T)$ with a risk free rate of zero. The risk free rate is zero since, in units of the asset S , a loan that is repaid with the

asset provides a zero return. The term required is the risk neutral probability that this transformed option will be exercised.

4.22 Risk neutral probabilities are determined by setting the instantaneous drift term for the transformed values to the relevant risk free rate. Appendix 4 sets out a non-rigorous derivation of the instantaneous drifts, variances and correlations.

4.23 The term multiplying the current share price in the capital gains tax option value will be:

$$\begin{aligned} \Pr[\max\{C/S, E/S\} \geq 1] &= 1 - \Pr[C/S \leq 1, E/S \leq 1] \\ &= N_2\{(\ln(S/E) + (R + \frac{1}{2}\sigma_s^2)T)/\sigma_s\sqrt{T}, (\ln(S/C) \\ &\quad + (r_f - \frac{1}{2}\sigma^2)T)/\sigma\sqrt{T}, (\sigma_s - \rho_{cs}\sigma_c)/\sigma\}. \end{aligned}$$

4.24 The term multiplying C in the capital gains tax option is determined by expressing the option value with C as the numeraire. The option value divided by C is a mix of options with payoffs given by $\max\{(S/C - \max\{1, E/C\}), 0\}$. These payoffs are the same as those from a call option on S/C with unit exercise price provided $E/C \leq 1$, plus an option to exchange E/C for unity provided $E/C \geq 1$ and $E/C \leq S/C$. The term multiplying (minus) the current value of C , discounted at the real rate of return, in the capital gains tax option value will therefore be the probability:

$$\begin{aligned} \Pr[S/C \geq 1, E/C \leq 1] &= N_2\{(\ln(S/C) + (r_f - \frac{1}{2}\sigma^2)T)/\sigma\sqrt{T}, \\ &\quad (\ln(C/E) + (R - r_f + \frac{1}{2}\sigma_c^2)T)/\sigma_c\sqrt{T}, (\rho_{cs}\sigma_s - \sigma_c)/\sigma\}. \end{aligned}$$

4.25 The term multiplying the value of E , discounted at the risk free rate, in the original option value can be determined by dividing the option value by E and noting that the resulting option is a mixture of a call option on S/E with unit exercise price, provided $C/E \leq 1$, and an option to exchange C/E for unity if $C/E \geq 1$ and $C/E \leq S/E$. This term is:

$$\begin{aligned} \Pr[S/E \geq 1, C/E \leq 1] &= N_2\{(\ln(S/E) + (R - \frac{1}{2}\sigma_s^2)T)/\sigma_s\sqrt{T}, (\ln(C/E) + (R - r_f - \frac{1}{2}\sigma_c^2)T)/\sigma_c\sqrt{T}, \rho_{cs}\} \end{aligned}$$

4.26 The resulting option value agrees with that derived using the risk neutral valuation approach.

4.27 The capital loss option value is not indexed, so that, in the case where it can be assumed that there are profits against which the losses can be offset, the value is the same whether or not indexation applies.

5. CAPITAL GAINS TAX OPTION VALUES

5.1 Values of the capital gain/loss options are readily obtained where no indexation applies using the Black-Scholes formula. Dividends can be allowed

for using an appropriate modification of this formula. The indexed capital gains option is more difficult. A formula allowing for indexation has been derived in this paper. The indexation case, as for the Australian tax rules, involves an asymmetric treatment of capital gains and losses, since losses are not indexed whereas gains are indexed. The non-indexation case is typically symmetric. This symmetry means that, assuming capital losses are deductible for tax purposes, that the value of the deferred capital gains tax is given by the put-call parity result of §§ 4.2 and 4.3.

5.2 In order to determine values for the effect of the capital gains tax using a bivariate log-normal distribution for S and C in the indexation case, it is necessary to evaluate the bivariate cumulative standard normal distribution. Generally this will require numerical techniques. Abramowitz & Stegun (1965) and Cadwell (1951) provide details on the evaluation of the bivariate normal integral. A technique to determine approximate values was given and used to determine illustrative values for the indexed capital gain option in Sherris (1989b).

5.3 The actual reserve required for the capital gains tax reserve is based on the accrued component of the total value of the capital gain/loss value. This is determined as the total value of the gain/loss to the current holder based on cost base E and original date of purchase minus the current value of the gain/loss to a new purchaser at the current asset value. Both of these values are determined using the approach set out in this paper. More details of this can be found in Hall *et al.* (1990).

6. CAPITAL GAINS TAX FOR A PORTFOLIO OF ASSETS

6.1 The above analysis considered only a single asset. It can be extended to a portfolio of assets, assuming that the portfolio is held for a fixed holding period. In this case capital losses will have value at the end of the fixed holding period, since they will be offset against capital profits on other assets in the portfolio. To determine the value of the capital gains tax option for the portfolio, denote the price of asset i at time t by S_{it} . Capital gains for the total portfolio at the end of the holding period (time T) will amount to:

$$G = \sum_i \max\{S_{iT} - V_{iT}, 0\}$$

where $V_{iT} = \max\{C_{iT}, E_i\}$, and total losses will amount to:

$$L = \sum_i \max\{E_i - S_{iT}, 0\}.$$

6.2 The capital gains tax payments at the end of the holding period will then be determined as:

$$\begin{array}{ll} \tau(G - L) & \text{if } G > L \\ 0 & \text{otherwise.} \end{array}$$

6.3 The capital gains tax effects for the portfolio can be valued using the risk neutral valuation approach for a complex European option whose payoffs are given above. Numerical techniques will be required to value this option. This value will be lower than that derived in the previous section, since the offsetting of capital losses on some assets against capital gains on other assets will reduce the total tax liability.

6.4 Assuming multivariate log-normality for asset values, the capital gain/loss option values for a portfolio containing a large number of assets will require the evaluation of a multivariate normal integral which will be numerically complex. Gupta (1963a) provides details on the evaluation of this integral, and Gupta (1963b) gives further references on this problem. The value of the option could also be approximated using simulation techniques as discussed by Boyle (1977).

6.5 Where net losses are realised in any period, the analysis can be extended to recognise that these can be carried forward and offset against future realised gains.

7. INVESTMENT STRATEGY FOR TAX RESERVE

7.1 The valuation of the indexed capital gains tax effects using option pricing theory provides the investment strategy to replicate the payoffs under the tax. If a reserve is created for deferred capital gains tax payments, then the investment composition of the assets backing the tax liability can be determined from the option pricing hedge portfolio.

7.2 For the capital gains tax payoffs, the proportions invested in the asset S , the hedge asset H and the risk free asset (nominal risk free bond) are given by $x = G_s S/G$, $y = G_h H/G = G_c C/G$, and $(1 - x - y)$.

7.3 In order to determine the partial differentials of the option value, note that the option value is linear homogenous in both S and C and also in S and H (Geske, 1979, 71, and Johnson, 1987, 281). The general form of the option value can therefore be written as $G = G_s S + G_c C + G_e E$.

7.4 Hence G_s is the term multiplying the current asset price in the indexed capital gains tax option value and G_c is the term multiplying the price of the indexed zero coupon bond H (or the term multiplying the indexed cost C) in the option value. From the formula for the option given in § 4.18 the value of G_s will be positive and the value of G_c will be negative. This means that the investment reserve for the capital gain will contain the asset S in positive amounts and the hedge security, an inflation index linked bond, in negative amounts (short sold). The net funds required to construct the investment reserve are financed using a nominal (risk free) bond.

8. CONCLUSION

8.1 This paper has examined capital gains tax and provided a framework to evaluate the reserve for its deferral for assets with unrealised gains/losses. The

analysis has been based on option pricing theory, which allows the simultaneous determination of the value of the deferred capital gains tax and the investment strategy for deferred tax reserves. The assumptions used would most suit returns on equity assets such as ordinary shares. Alternative return models for fixed-interest assets could be used to develop similar formulae for these assets.

8.2 Some issues arising from the deferral of capital gains tax have not been addressed in any detail. The portfolio effects arising from the offsetting of losses against gains have not been examined, and will be an important factor in practice. The problem of maximising expected after tax returns allowing for capital gains tax has also not been addressed. This latter problem can be analysed using stochastic programming techniques along the lines of that developed by Merton (1969) and Samuelson (1969). Papers which address this problem based on the United States tax rules are Constantinides (1983) and Stiglitz (1983).

REFERENCES

- ABRAMOWITZ, M. & STEGUN, I. A. (1965). *Handbook of Mathematical Functions*. Dover Publications, New York, 936-940.
- BOYLE, P. P. (1977). Options: A Monte Carlo Approach. *Journal of Financial Economics*, **4**, 323-338.
- CADWELL, J. H. (1951). The Bivariate Normal Integral. *Biometrika*, **38**, 475-479.
- CONSTANTINIDES, G. M. (1983). Capital Market Equilibrium with Personal Tax. *Econometrica*, **51**, 611-636.
- DAYKIN, C. D. & HEY, G. B. (1990). Managing Uncertainty in a General Insurance Company. *J.I.A.*, **117**, 173.
- GESKE, R. (1979). The Valuation of Compound Options. *Journal of Financial Economics*, **7**, 63-81.
- GUPTA, S. S. (1963a). Probability Integrals of Multivariate Normal and Multivariate t . *Annals of Mathematical Statistics*, **34**, 792-828.
- GUPTA, S. S. (1963b). Bibliography on the Multivariate Normal Integrals and Related Topics. *Annals of Mathematical Statistics*, **34**, 829-838.
- HALL, M., GEUE, A., RUSH, D. & SIMMONS, P. (1990). Capital Gains Tax Reserves. *Quarterly Journal of The Institute of Actuaries of Australia*. September, 11-15.
- JOHNSON, H. (1987). Options on the Maximum or the Minimum of Several Assets. *Journal of Financial and Quantitative Analysis*, **22**, 277-283.
- JARROW, A. J. & RUDD, A. (1983). *Option Pricing*. Richard D. Irwin, Homewood, Illinois.
- MARGRABE, W. (1978). The Value of an Option to Exchange One Asset for Another. *Journal of Finance*, **33**, 177-186.
- MERTON, R. C. (1969). Lifetime Portfolio Selection under Uncertainty: The Continuous Time Case. *The Review of Economics and Statistics*, **51**, 247-257.
- SAMUELSON, P. A. (1969). Lifetime Portfolio Selection by Dynamic Stochastic Programming. *The Review of Economics and Statistics*, **51**, 239-246.
- SHERRIS, M. (1989a). Options: Pricing and Applications. Paper presented to The Institute of Actuaries of Australia Biennial Convention.
- SHERRIS, M. (1989b). Capital Gains Tax; An Application of Option Pricing Theory. *Quarterly Journal of The Institute of Actuaries of Australia*, December, 35-49.
- STIGLITZ, J. E. (1983). Some Aspects of the Taxation of Capital Gains. *Journal of Public Economics*, **21**, 257-294.
- STULZ, R. M. (1982). Options on the Minimum or the Maximum of Two Risky Assets. *Journal of Financial Economics*, **10**, 161-185.
- WILKIE, A. D. (1987). An Option Pricing Approach to Bonus Policy. *J.I.A.*, **114**, 21-77.

APPENDIX 1

EVALUATION OF EXPECTED VALUE OF THE INDEXED CAPITAL GAINS TAX OPTION

The expected value of the option payoffs is given by:

$$\begin{aligned} & \int_0^{\infty} \int_0^{\infty} \max\{s - E - \max\{c - E, 0\}, 0\} f(c, s) dsdc \\ &= \int_E^{\infty} \int_0^{\infty} \max\{s - c, 0\} f(c, s) dsdc + \int_0^E \int_0^{\infty} \max\{s - E, 0\} f(c, s) dsdc \\ &= \int_E^{\infty} \int_c^{\infty} sf(c, s) dsdc - \int_E^{\infty} \int_c^{\infty} cf(c, s) dsdc + \int_0^E \int_E^{\infty} sf(c, s) dsdc - \int_0^E \int_E^{\infty} Ef(c, s) dsdc \end{aligned}$$

where $f(c, s)$ is the joint density function of C and S .

To evaluate the integrals, assume S and C are bivariate log-normal (so that rates of return on S and C are bivariate normal). The assumed dynamics of S and C are given in Appendix 4. Appendix 2 provides some useful normal and bivariate normal distribution results used to evaluate the integrals.

Consider the integrals involving S . We have:

$$\begin{aligned} & \int_E^{\infty} \int_c^{\infty} sf(c, s) dsdc + \int_0^E \int_E^{\infty} sf(c, s) dsdc \\ &= \int_E^{\infty} \int_E^s sf(c, s) dc ds + \int_0^E \int_E^s sf(c, s) dc ds \\ &= \int_E^{\infty} \int_0^s sf(c, s) dc ds. \end{aligned}$$

Set $S = S_0 \exp\{X\}$ and $C = C_0 \exp\{Y\}$, so that X, Y are bivariate normal with means $(\mu_s - \frac{1}{2}\sigma_s^2, \mu_c - \frac{1}{2}\sigma_c^2)$, variances (σ_s^2, σ_c^2) and correlation ρ_{cs} . This gives:

$$\int_{\ln(E/S)}^{\infty} \int_{-\infty}^{x + \ln(S/C)} \text{Sexp}\{x\} f(x, y) dy dx$$

where $S = S_0$ and $C = C_0$ for ease of notation.

Use the first result in Appendix 2 to get:

$$\text{Sexp}\{\mu_s\} \int_A^{\infty} \int_{-\infty}^{x + \ln(S/C)} f(z, y) dy dz$$

where $A = \ln(E/S) - \sigma_s^2$ and $Z = X - \sigma_s^2$.

Apply the transformation $U = Y - Z$ and $V = Z$ to get:

$$S \exp\{\mu_s\} \int_A^\infty \int_{-\infty}^{\ln(S/C)} f(u, v) dudv$$

where U, V are bivariate normal with:

$$\begin{aligned} \text{means} & \quad (\mu_c - \frac{1}{2}\sigma_c^2 - \mu_s + \frac{1}{2}\sigma_s^2, \mu_s - \frac{1}{2}\sigma_s^2) \\ \text{variances} & \quad (\sigma_s^2 + \sigma_c^2 - 2\rho_{cs}\sigma_c\sigma_s, \sigma_s^2) = (\sigma^2, \sigma_s^2) \\ \text{and correlation} & \quad (\rho_{cs}\sigma_c - \sigma_s)/\sigma. \end{aligned}$$

We require the risk neutral probabilities, so that the drift terms for the variables U and V (see Appendix 4 for details of the derivation of these terms) are replaced with the relevant risk free rates to get $\mu_s = R$ and $\mu_c - \mu_s + \sigma_s^2 - \rho_{cs} = (\mu_c - \frac{1}{2}\sigma_c^2 - \mu_s + \frac{1}{2}\sigma_s^2) + \frac{1}{2}\sigma^2 = -r_f$.

Substituting and reducing the integral to the standard bivariate normal gives:

$$S \exp\{R\} N_2[\alpha, \beta, \beta]$$

where:

$$\begin{aligned} \alpha &= -\{\ln(E/S) - \sigma_s^2 - (R - \frac{1}{2}\sigma_s^2)\}/\sigma_s = (\ln(S/E) + (R - \frac{1}{2}\sigma_s^2) + \sigma_s^2)/\sigma_s \\ \beta &= \{\ln(S/C) + (r_f - \frac{1}{2}\sigma^2)\}/\sigma, \text{ and} \\ \rho &= (\sigma_s - \rho_{cs}\sigma_c)/\sigma. \end{aligned}$$

A similar procedure is applied to evaluate the integral containing C . In this case we have:

$$\int_E^\infty \int_c^\infty cf(c, s) dsdc = \int_{\ln(E/C)}^\infty \int_{y + \ln(C/S)}^\infty C \exp\{y\} f(x, y) dx dy$$

where $S = S_0 \exp\{X\}$ and $C = C_0 \exp\{Y\}$ and S_0 and C_0 are replaced with S and C for notational convenience.

Use the first result in Appendix 2 to get:

$$C \exp\{\mu_c\} \int_A^\infty \int_{y + \ln(C/S)}^\infty f(x, z) dx dz$$

where $Z = Y - \sigma_c^2$ and $A = \ln(E/C) - \sigma_c^2$.

Apply the transformation $U = X - Y$ and $V = Z$ to get:

$$C \exp\{\mu_c\} \int_A^\infty \int_{\ln(C/S)}^\infty f(u, v) dudv$$

where U, V are bivariate normal with:

$$\begin{aligned} \text{means} & \quad (\mu_s - \frac{1}{2}\sigma_s^2 - \mu_c + \frac{1}{2}\sigma_c^2, \mu_c - \frac{1}{2}\sigma_c^2) \\ \text{variances} & \quad (\sigma_s^2 + \sigma_c^2 - 2\rho_{cs}\sigma_c\sigma_s, \sigma_s^2) = (\sigma^2, \sigma_s^2) \\ \text{and correlation} & \quad (\rho_{cs}\sigma_s - \sigma_c)/\sigma. \end{aligned}$$

The risk neutral probabilities are needed so that the drift terms for the variables U and V (see Appendix 4 for details of the derivation of these terms) are replaced with the relevant risk free rates to get $\mu_c = R - r_f$ and $\mu_s - \mu_c + \sigma_c^2 - \rho_{cs}\sigma_c\sigma_s = (\mu_s - \frac{1}{2}\sigma_s^2 - \mu_c + \frac{1}{2}\sigma_c^2) + \frac{1}{2}\sigma^2 = r_f$.

Substituting and reducing the integral to the standard bivariate normal gives $C \exp\{R - r_f\} N_2[\alpha, \beta, \rho]$ where:

$$\begin{aligned} \alpha &= -\{\ln(E/C) - \sigma_c^2 - (R - r_f - \frac{1}{2}\sigma_c^2)\}/\sigma_c \\ &= (\ln(C/E) + (R - r_f - \frac{1}{2}\sigma_c^2) + \sigma_c^2)/\sigma_c \\ \beta &= -\{\ln(C/S) - (r_f - \frac{1}{2}\sigma^2)\}/\sigma = \{\ln(S/C) + (r_f - \frac{1}{2}\sigma^2)\}/\sigma, \text{ and} \\ \rho &= (\rho_{cs}\sigma_s - \sigma_c)/\sigma. \end{aligned}$$

The integral containing E is evaluated in a similar fashion. We have:

$$\int_0^{E\infty} \int_E^{\infty} E f(c, s) ds dc = E \int_{-\infty}^{\ln(E/C)} \int_{\ln(E/S)}^{\infty} f(x, y) dx dy$$

which simplifies to $EN_2[\alpha, \beta, \rho]$ where:

$$\begin{aligned} \alpha &= \{\ln(E/C) - (R - r_f - \frac{1}{2}\sigma_c^2)\}/\sigma_c \\ \beta &= -\{\ln(E/S) - (R - \frac{1}{2}\sigma_s^2)\}/\sigma_s = \{\ln(S/E) + (R + \frac{1}{2}\sigma_s^2)\}/\sigma_s \\ \rho &= -\rho_{cs}. \end{aligned}$$

APPENDIX 2

SOME USEFUL NORMAL AND BIVARIATE NORMAL DISTRIBUTION RESULTS

If Y is Normal (μ, σ^2) , N is the cumulative standard normal distribution function, $B = (b - (\mu + \sigma^2))/\sigma$, and $A = (a - (\mu + \sigma^2))/\sigma$, then:

$$\int_a^b e^{yf(y)} dy = \exp(\mu + \frac{1}{2}\sigma^2) \{N(B) - N(A)\}.$$

To show this, we have:

$$\int_a^b e^{yf(y)} dy = \int_a^b \exp(y) \frac{1}{\sqrt{(2\pi)\sigma}} \exp\left[-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2\right] dy.$$

Now collect the exponential terms and complete the square to get:

$$\int_a^b \exp(\mu + \frac{1}{2}\sigma^2) \frac{1}{\sqrt{(2\pi)\sigma}} \exp\left[-\frac{1}{2}\left(\frac{y - (\mu + \sigma^2)}{\sigma}\right)^2\right] dy.$$

Substitute $Z = (y - (\mu + \sigma^2))/\sigma$, $B = (b - (\mu + \sigma^2))/\sigma$, and $A = (a - (\mu + \sigma^2))/\sigma$, to get:

$$\exp(\mu + \frac{1}{2}\sigma^2) \int_A^B \frac{1}{\sqrt{(2\pi)}} \exp\left[-\frac{1}{2}(Z^2)\right] dZ = \exp(\mu + \frac{1}{2}\sigma^2) \{N(B) - N(A)\}.$$

The following results are adapted from Abramowitz & Stegun (1965). The standard bivariate normal density function is given by:

$$g(x, y, \rho) = [2\pi\sqrt{(1 - \rho^2)}]^{-1} \exp\left[-\frac{1}{2}\left(\frac{x^2 - 2\rho xy + y^2}{1 - \rho^2}\right)\right]$$

which can be expressed as:

$$g(x, y, \rho) = (1 - \rho^2)^{-\frac{1}{2}} n(x) n\left[\frac{y - \rho x}{(1 - \rho^2)^{\frac{1}{2}}}\right]$$

where $n(\)$ is the standard normal density.

The cumulative standard bivariate normal density function is given by:

$$\begin{aligned} L(h, k, \rho) &= \int_{-\infty}^h \int_{-\infty}^k g(x, y, \rho) dy dx \\ &= \int_{-\infty}^h \int_{-\infty}^w n(x) n(w) dw dx \end{aligned}$$

where:

$$w = \left(\frac{k - \rho x}{(1 - \rho^2)^{\frac{1}{2}}}\right).$$

We also have:

$$L(-h, -k, \rho) = \int_h^{\infty} \int_k^{\infty} g(x, y, \rho) dy dx$$

$$L(-h, k, -\rho) = \int_h^{\infty} \int_{-\infty}^k g(x, y, \rho) dy dx$$

$$L(h, -k, -\rho) = \int_{-\infty}^h \int_k^{\infty} g(x, y, \rho) dy dx$$

$$L(h, k, \rho) = L(k, h, \rho).$$

If the random variables X, Y are said to be bivariate normal with means (μ_x, μ_y) , variances (σ_x^2, σ_y^2) and correlation ρ , then the probability that X is less than or equal to h and Y is less than or equal to k is given by:

$$L((h - \mu_x)/\sigma_x, (k - \mu_y)/\sigma_y, \rho).$$

APPENDIX 3

PROBABILITY BASED VALUATION OF EUROPEAN PUT OPTION

The probability based approach recognises that the term multiplying the value of the (certain) exercise price, for any option, is the probability that the option is exercised in a risk-neutral framework. For a put option, assuming that the rate of return on the asset is normally distributed, so that the asset value is log-normal with instantaneous mean μ and instantaneous variance σ^2 , this is $\Pr[S_T \leq X | S_0 = S] = \Pr[Z \leq \{\ln(X/S) - \mu T\} / \sigma \sqrt{T}] = N(-d_2)$ where Z is Normal (0, 1) and $d_2 = (\ln(S/X) + (r - \sigma^2/2)T) / \sigma \sqrt{T}$, since $\mu = r - \frac{1}{2} \sigma^2$ in a risk neutral framework.

Note that if $dS = \mu S dt + \sigma S dZ$, then, using Itô's Lemma, we have $d \ln(S) = (\mu - \frac{1}{2} \sigma^2) dt + \sigma dZ$ since, if $f(s) = \ln(S)$ then $f_s = 1/S$ and $f_{ss} = -1/S^2$.

Now, express all values using the asset price as numeraire. In this framework the option value divided by S is the value of a call option on the exercise price, expressed in units of the asset price S with unit exercise price and zero risk free interest rate. The current value of the asset is $x = Xe^{-rt}/S$.

Itô's Lemma gives the dynamics of x as:

$$dx = x_t dt + x_s dS + \frac{1}{2} x_{ss} (dS)^2 = -rx dt + - (x/S)(\mu S dt + \sigma S dZ) + \frac{1}{2} (2x/S^2) \sigma^2 S^2 dt$$

$$dx/x = (r - \mu + \sigma^2) dt - \sigma dZ.$$

The rate of return dynamics for x are given by:

$$d \ln(x) = ((r - \mu + \sigma^2) - \frac{1}{2} \sigma^2) dt - \sigma dZ.$$

The drift term for x is set to zero in the risk neutral framework so that the mean value for $\ln(x)$ is $(-\frac{1}{2} \sigma^2) T$ and the variance is $\sigma^2 T$.

The probability that the transformed call option will be exercised is:

$$\begin{aligned} \Pr(x \geq 1 | x_0 = Xe^{-rT}/S) &= \Pr[x/x_0 \geq 1/x_0] \\ &= \Pr[\ln(x/x_0) \geq -\ln(x_0)] \\ &= \Pr[Z \geq (-\ln(x_0) - (-\frac{1}{2} \sigma^2)T) / \sigma] \\ &= \Pr[Z \leq (\ln(x_0) - \frac{1}{2} \sigma^2)T / \sigma] \\ &= N(d_2') \end{aligned}$$

where $d_2' = (\ln(x_0) - \frac{1}{2} \sigma^2) / \sigma$.

For the call option the value using the Black-Scholes formula is $(Xe^{-rT}/S)N(d_1') - 1 N(d_2')$ where, since $r=0$, $d_1' = (\ln(Xe^{-rT}/S) + \frac{1}{2} \sigma^2 T) / \sigma \sqrt{T}$ and $d_2' = (\ln(Xe^{-rT}/S) - \frac{1}{2} \sigma^2 T) / \sigma \sqrt{T}$.

The term multiplying the asset price in the original put option formula is the term multiplying the exercise price for this call option (the transformed put option). Hence it is minus the probability that this call option is exercised. This is $-N(-d_1)$ since:

$$\begin{aligned}
 -d_1 &= -(\ln(S/X) + (r + \sigma^2/2)T)/\sigma\sqrt{T} \\
 &= (\ln(Xe^{-rT}/S) - \sigma^2/2)T/\sigma\sqrt{T} \\
 &= d_2'.
 \end{aligned}$$

This gives the Black-Scholes put option formula, which is set out below for completeness, and confirms the probability approach to deriving the option formula.

Black-Scholes formula for European put option on asset S with exercise price X and maturity T .

$$P(S, X, T) = Xe^{-rT}N(-d_2) - S_0N(-d_1)$$

where S is the asset value which is assumed to be lognormally distributed with $d\ln(S) = \mu dt + \sigma dZ$ so that:

$$\frac{dS}{S} = (\mu + \frac{1}{2}\sigma^2)dt + \sigma dZ$$

where:

- σ^2 is the instantaneous variance of the logarithm of the return on asset S ,
- r is the risk-free interest rate (continuously compounded),
- S_0 is the current asset price,
- $d_1 = (\ln(S_0/X) + (r + \sigma^2/2)T)/\sigma\sqrt{T}$,
- $d_2 = d_1 - \sigma\sqrt{T}$,
- $N(\cdot)$ is the standard normal cumulative distribution function.

APPENDIX 4

DISTRIBUTION OF TRANSFORMED VARIABLES

In order to use the probability approach to determining the capital gains tax option value, it is necessary to specify the joint distribution of C/S , E/S and of S/C , E/C . These distributions are derived here in a non-rigorous manner using Itô's Lemma.

Assume that the dynamics of C and S are given by:

$$\frac{dC}{C} = \mu_c dt + \sigma_c dZ_c$$

$$\frac{dS}{S} = \mu_s dt + \sigma_s dZ_s$$

$$dZ_c dZ_s = \rho_{cs} dt$$

so that they are bivariate lognormally distributed.

Applying Itô's Lemma gives:

$$\frac{d(C/S)}{(C/S)} = (\mu_c - \mu_s + \sigma_s^2 - \rho_{cs}\sigma_s\sigma_c)dt + \sigma_c dZ_c - \sigma_s dZ_s$$

$$\frac{d(S/C)}{(S/C)} = (\mu_s - \mu_c + \sigma_c^2 - \rho_{cs}\sigma_c\sigma_s)dt + \sigma_s dZ_s - \sigma_c dZ_c$$

$$\frac{d(E/S)}{(E/S)} = (\sigma_s^2 - \mu_s)dt - \sigma_s dZ_s$$

$$\frac{d(E/C)}{(E/C)} = (\sigma_c^2 - \mu_c)dt - \sigma_c dZ_c$$

We then have that:

$$\log(C_t/c_0) \text{ is distributed as } N((\mu_c - \frac{1}{2}\sigma_c^2)t, \sigma_c^2 t),$$

$$\log(S_t/s_0) \text{ is distributed as } N((\mu_s - \frac{1}{2}\sigma_s^2)t, \sigma_s^2 t),$$

and hence:

$$E[C_T|c_0] = c_0 \exp\{\mu_c T\} \text{ and}$$

$$E[S_T|s_0] = s_0 \exp\{\mu_s T\}.$$

It follows that $(C_t/c_0)/(S_t/s_0)$ is lognormal since $\log((C_t/c_0)/(S_t/s_0)) = \log(C_t/c_0) - \log(S_t/s_0)$ is the difference between two normal random variables. Hence $\log(C_t/c_0)/(S_t/s_0)$ is distributed as $N((\mu_c - \mu_s + \sigma_s^2 - \rho_{cs}\sigma_s\sigma_c - \frac{1}{2}\sigma^2)t, \sigma^2 t)$ where $\sigma^2 = \sigma_c^2 + \sigma_s^2 - 2\rho_{cs}\sigma_c\sigma_s$.

It also follows that E/S_t is lognormal since $\log(E/(S_t/s_0))$ is distributed as $N((\sigma_s^2 - \mu_s - \frac{1}{2}\sigma_s^2)t, \sigma_s^2 t)$.

If we denote $H = \log(C_t/S_t)$ and $V = \log(E/S_t)$, then the instantaneous correlation coefficient of H and V is given by:

$$\rho_{hv} = \frac{\text{cov}(H, V)}{(\text{var}(H)\text{var}(V))^{1/2}}$$

where:

$$\begin{aligned} \text{cov}(H, V) &= \text{cov}(\log(C_t/S_t), \log(E/S_t)) \\ &= \text{cov}(\log(C_t) - \log(S_t), \log(E) - \log(S_t)) \\ &= -\text{cov}(\log(C_t), \log(S_t)) + \text{cov}(\log(S_t), \log(S_t)) \\ &= \sigma_s^2 - \rho_{cs}\sigma_c\sigma_s \end{aligned}$$

so that:

$$\begin{aligned} \rho_{hv} &= \frac{\sigma_s^2 - \rho_{cs}\sigma_c\sigma_s}{\sigma_s (\sigma_c^2 + \sigma_s^2 - 2\rho_{cs}\sigma_c\sigma_s)^{1/2}} \\ &= \frac{\sigma_s - \rho_{cs}\sigma_c}{(\sigma_c^2 + \sigma_s^2 - 2\rho_{cs}\sigma_c\sigma_s)^{1/2}} \end{aligned}$$

In a similar fashion we have:

$\log(S_t/s_0)/(C_t/c_0)$ is distributed as $N((\mu_s - \mu_c + \sigma_c^2 - \rho_{cs}\sigma_c\sigma_s - \frac{1}{2}\sigma^2)t, \sigma^2 t)$, $\log(E/(C_t/c_0))$ is distributed as $N((-\mu_c + \frac{1}{2}\sigma_c^2)t, \sigma_c^2 t)$, and

$$\rho_{xy} = \frac{\sigma_c - \rho_{cs}\sigma_s}{(\sigma_c^2 + \sigma_s^2 - 2\rho_{cs}\sigma_c\sigma_s)^{1/2}}$$

where $X = \log(S_t/C_t)$ and $Y = \log(E/C_t)$.