ON THE DERIVATION OF APPROXIMATE ACTUARIAL FORMULAE

BY V. C. P. DRASTIK, B.A., A.I.A.

This paper has been written to present a new approach to the determination of approximate actuarial formulae. The method employed involves using certain desirable conditions to find the values of arbitrary parameters in a general formula. It is considered that the best way to introduce the new method is by example, consisting of applications of the new technique to some practical problems in the theory of compound interest.

1. THE ANNUITY EQUATION \( a_m = k \)

1.1. Bizley's series\(^1\) may be written as:

\[
i = f(\phi) = \phi - \frac{n-1}{3} \cdot \phi^2 + \frac{(n-1)(2n+1)}{9} \cdot \phi^3 - \ldots\]  (1.1)

where

\[
\phi = \frac{2}{n+1} \left( \frac{n}{k} - 1 \right).
\]

Worger\(^2\) makes use of the theory of Padé approximations, and notes an important feature of these functions; namely, that their accuracy is substantially better than that of the equivalent power series expansion. This is true only for convergent power series, and hence the number of terms of Bizley's series which may be used is limited for certain values of \( n \) and \( i \). Karpin\(^3\) notes that for these values of \( n \) and \( i \), the series given by Bizley diverges. It is found empirically that the maximum number of terms which may be used without allowing the divergent nature of the series to influence the result is three.

Let

\[
R_N(\phi) = \frac{\sum_{t=0}^{n} a_t \phi^t}{\sum_{t=0}^{m} b_t \phi^t} \quad (1.2)
\]

where \( b_0 = 1, N = n + m \).

This is known as an \( N \)th degree Padé approximation for \( f(\phi) \). The most useful of the Padé approximations are with the degree of the numerator equal to or one greater than the denominator\(^4\).
1.2. Let \( n = 2, m = 1 \).

Hence,

\[
R_3(\phi) = \frac{a_0 + a_1 \phi + a_2 \phi^2}{1 + b_1 \phi}.
\]  

(1.3)

Now,

\[
f(\phi) - R_3(\phi) = \phi - \frac{n - 1}{6} \phi^2 + \frac{(n - 1)(2n + 1)}{36} \phi^3 \ldots - \frac{a_0 + a_1 \phi + a_2 \phi^2}{1 + b_1 \phi}
\]

\[
= -\frac{(\phi - \frac{n - 1}{6} \phi^2 + \frac{(n - 1)(2n + 1)}{36} \phi^3)(1 + b_1 \phi) - (a_0 + a_1 \phi + a_2 \phi^2)}{1 + b_1 \phi}.
\]

(1.4)

1.3. Since \( R_3(\phi) \) approximates \( f(\phi) \), the coefficients of \( \phi^K \) \((K = 0, 1, 2, 3)\) in the numerator must be 0.

Therefore,

\[
a_0 = 0 \\
a_1 = 1 \\
a_2 = \frac{n + 2}{6} \\
b_1 = \frac{2n + 1}{6}
\]

Therefore

\[
R_3(\phi) = \frac{\phi + \frac{n + 2}{6} \phi^2}{1 + \frac{2n + 1}{6} \phi}.
\]

(1.5)

1.4. This may be written as:

\[
R_3(\phi) = \lambda = \phi \left[ 1 - \frac{(n - 1) \phi}{6 + (2n + 1) \phi} \right].
\]

(1.6)

This appears to be a new formula, but it is really a particular case of the general formula:

\[
i = \phi \left[ 1 - \frac{(n - 1) \phi}{6 + (2n + \alpha) \phi} \right].
\]

(1.7)

For example, using the substitution \( P = [(n + 1)/2] \phi \) in Karpin’s formula\(^{(3)}\) enables it to be written as:

\[
i = \phi \left[ 1 - \frac{(n - 1) \phi}{6 + 2n \phi} \right].
\]

(1.8)
Substituting similarly into Fallon’s first formula \(^{(5)}\) gives:

\[
i = \phi \left[ 1 - \frac{(n-1)\phi}{6 + (2n-2)\phi} \right].
\]  

(1.9)

The reason for the difference is in the treatment of Bizley’s series. The new formula given above equates the first three derivatives of Bizley’s series to 0, hence leaving a positive error in the rate of interest for all interest rates and terms. The other two make some adjustment in the third derivative error to allow for the rest of the terms in the series.

1.5. The relative accuracy of the three formulae is shown in Tables 1–3, and the error pattern discussed may be clearly observed.

<table>
<thead>
<tr>
<th>Table 1. Percentage error for ( \lambda )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n )</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>8</td>
</tr>
<tr>
<td>16</td>
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</tr>
<tr>
<td>64</td>
</tr>
<tr>
<td>128</td>
</tr>
<tr>
<td>256</td>
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</table>

* Less than .0005.

<table>
<thead>
<tr>
<th>Table 2. Percentage error for Karpin’s formula</th>
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<tr>
<td>( n )</td>
</tr>
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</tr>
<tr>
<td>8</td>
</tr>
<tr>
<td>16</td>
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<td>64</td>
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<tr>
<td>128</td>
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<td>256</td>
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<table>
<thead>
<tr>
<th>Table 3. Percentage error for Fallon’s first formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n )</td>
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<tr>
<td>4</td>
</tr>
<tr>
<td>8</td>
</tr>
<tr>
<td>16</td>
</tr>
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<td>32</td>
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<tr>
<td>64</td>
</tr>
<tr>
<td>128</td>
</tr>
<tr>
<td>256</td>
</tr>
</tbody>
</table>
1.6. It is possible to make $\alpha$ the subject of equation (1.7):

$$\alpha = \frac{1}{\phi} \left[ \frac{(n-1)\phi}{1-(i/\phi)} - 6 \right] - 2n. \quad (1.10)$$

This may be tabulated as a function of $n$ and $\phi$ as in Table 4.

<table>
<thead>
<tr>
<th>$n$</th>
<th>0.005</th>
<th>0.01</th>
<th>0.02</th>
<th>0.04</th>
<th>0.08</th>
<th>0.16</th>
<th>0.32</th>
<th>0.64</th>
<th>1.28</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>12.873</td>
<td>997</td>
<td>931</td>
<td>971</td>
<td>950</td>
<td>906</td>
<td>830</td>
<td>716</td>
<td>568</td>
</tr>
<tr>
<td>4</td>
<td>2.223</td>
<td>805</td>
<td>975</td>
<td>933</td>
<td>871</td>
<td>767</td>
<td>608</td>
<td>403</td>
<td>196</td>
</tr>
<tr>
<td>8</td>
<td>4.85</td>
<td>903</td>
<td>892</td>
<td>797</td>
<td>630</td>
<td>376</td>
<td>54</td>
<td>2.45</td>
<td>3.92</td>
</tr>
<tr>
<td>16</td>
<td>9.53</td>
<td>797</td>
<td>644</td>
<td>351</td>
<td>096</td>
<td>655</td>
<td>1-138</td>
<td>1.264</td>
<td>973</td>
</tr>
<tr>
<td>32</td>
<td>6.31</td>
<td>334</td>
<td>210</td>
<td>1045</td>
<td>2080</td>
<td>2931</td>
<td>3004</td>
<td>2205</td>
<td>1.290</td>
</tr>
<tr>
<td>64</td>
<td>-2.81</td>
<td>-1.333</td>
<td>-2.943</td>
<td>-4.933</td>
<td>-6.522</td>
<td>-6.484</td>
<td>-6.658</td>
<td>-2.694</td>
<td>-1.432</td>
</tr>
<tr>
<td>1024</td>
<td>-114.266</td>
<td>-110.838</td>
<td>-78.153</td>
<td>-44.758</td>
<td>-23.697</td>
<td>-12.166</td>
<td>-6.161</td>
<td>-3.100</td>
<td>-1.555</td>
</tr>
</tbody>
</table>

1.7. Table 4 may be used in conjunction with equation (1.7) to obtain an accurate value for $i$. Linear interpolation gives best results over the whole range of the table.

For example, let $i = 0.06$, $n = 48$. \( a_{48} = 15.65 \). \( \phi = 0.0844 \)

By interpolation, $\alpha = -4.323$.

Using equation (1.7), $i = 0.060012$, which is an error of $0.02\%$.

1.8. A general formula for the rate of interest may be written:

$$g(i) = \frac{T_0 + T_1k + T_2k^2}{B_0 + B_1k + B_2k^2}. \quad (1.11)$$

In this formula, $B_0$ should be 0, for otherwise there is a maximum value for $g(i)$, which is not acceptable.

There are certain properties which it is desirable for this general formula to have:

(a) $\lim_{n \to \infty} g(i) = i$

(b) $\lim_{n \to 0} g(i) = i$

(c) $\lim_{i \to 0} g(i) = i$

(d) $\lim_{i \to \infty} g(i) = i$
To these may be added some practical modifications:

(e) Property (b) may be modified to \( n \to 1 \) since \( n \to 0 \) is of no practical significance.

(f) For large \( n \) and \( i \), the best approximation for \( i \) is \( 1/k \), since \( v^n \) becomes less than the limit of accuracy of the instrument of calculation.

\( \lambda \) satisfies the conditions proposed, except for (b).

1.9. Iterative improvement

\( \lambda \) gives a value for \( i \) with an error which is normally much less than 5\%, and using Table 4 gives an accuracy far greater than this. However, for certain values of \( n \) and \( i \), a precise estimate of the error is difficult, and hence one calculates the annuity value using the estimated rate of interest in order to determine whether or not the error is sufficiently small. This annuity value may be used to find a much improved estimate.

Let \( k \) be the given annuity value. Hence, we require the solution of the equation

\[
f(i) = a_m - k = 0\tag{1.12}
\]

Now,

\[
\frac{d}{di} f(i) = \frac{nv^{n+1} - a_m}{i}.
\]

Using Newton's method,

\[
i_{n+1} = i_n - \frac{f(i_n)}{f'(i_n)}.
\]

Thus,

\[
i' = i - i \cdot \frac{(a_m - k)}{(nv^{n+1} - a_m)}
= i \left[ 1 + \frac{a_m - k}{a_m(1 + i \cdot nv) - nv} \right].
\]

Once the approximate interest rate \( i \) is found, it may be substituted into the above formula to find an improved value \( i' \). Newton's method has quadratic convergence in the neighbourhood of the root i.e. the error in the \((n + 1)\)th iterate is proportional to the square of the error in the \(n\)th iterate, and hence \( i' \) will be very accurate. The accuracy may be gauged from the bracketed factor above. If it is close to 1, then the present estimate is good, and the improved one will be excellent.

2. REDEEMABLE SECURITIES

2.1. All equations of value concerned with the payment of a price in return for a
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series of equal payments in the future for a predetermined period, followed by a
different final payment may be written in the form

\[ s(i) = g a_m + v^n. \]  

(2.1)

Using the general method expounded in the previous section, the formula for
the rate of interest may be written as:

\[ f(i) = \frac{T_0 + T_1 S + T_2 S^2}{B_0 + B_1 S + B_2 S^2} \ldots \]  

(2.2)

where the arbitrary parameters \( T_i, B_i (i=0, 1, 2) \) should be chosen so as to make
\( f(i) \) approximate \( i \) as closely as possible over the required range of \( n \) and \( i \).

If \( B_0 \neq 0 \), then \( f(i) \) is bounded above. This is not acceptable. Setting \( B_0 \) to 0,
dividing through by \( B_1 \) and redefining the parameters, equation (2.2) becomes:

\[ f(i) = \frac{T_0 + T_1 S + T_2 S^2}{S(1 + B_1 S)} \ldots \]  

(2.3)

The conditions available are the ones used in §1, namely,

\[ f(0) = 0, f'(0) = 1, f^{(j)}(0) = 0, \ (j > 1) \]

and, in addition, the requirement afforded by the nature of redeemable securities,
viz., \( S(g) = 1 \), hence \( f(g) = g \).

2.2. Letting \( T_2 = B_1 = 0 \), we obtain a function of \( S \) analogous to \( \phi \) in §1. By
equating \( f \) to 0, and its derivative to 1, we obtain:

\[ f(i) = \phi = \frac{1 + ng}{n\left(1 + \frac{n+1}{2} g\right)} \left(1 + \frac{ng}{S} - 1\right) \]  

(2.4)

since \( S(0) = 1 + ng \)
\[ S'(0) = -n\left(1 + \frac{n+1}{2} g\right) \]
\[ S''(0) = n(n+1)\left(1 + \frac{n+2}{3} g\right). \]

This form of \( f(i) \) gives adequate results for small \( i \), but is generally unsatisfactory.

2.3. Letting \( T = B = 0, f(0) = 0, f'(0) = 1 \), we obtain

\[ f(i) = \epsilon = \frac{(1 + ng)^2}{S} - S \]  

(2.5)

\[ 2n\left(1 + \frac{n+1}{2} g\right) \]

Table 5 shows the accuracy with which \( \epsilon \) determines \( i \) (using \( g = 0.12 \)).
On the derivation of approximate actuarial formulae

Table 5. Percentage error for $\varepsilon$

<table>
<thead>
<tr>
<th>$n$</th>
<th>0.01</th>
<th>0.02</th>
<th>0.04</th>
<th>0.08</th>
<th>0.16</th>
<th>0.32</th>
<th>0.64</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>-0.60</td>
<td>-1.16</td>
<td>-2.14</td>
<td>-3.65</td>
<td>-5.14</td>
<td>-4.14</td>
<td>5.26</td>
</tr>
<tr>
<td>8</td>
<td>-0.88</td>
<td>-1.63</td>
<td>-2.78</td>
<td>-3.93</td>
<td>-3.03</td>
<td>4.41</td>
<td>18.70</td>
</tr>
<tr>
<td>16</td>
<td>-1.51</td>
<td>-2.65</td>
<td>-4.06</td>
<td>-5.42</td>
<td>-4.86</td>
<td>5.96</td>
<td>9.29</td>
</tr>
<tr>
<td>32</td>
<td>-2.62</td>
<td>-4.24</td>
<td>-5.48</td>
<td>-4.20</td>
<td>-4.00</td>
<td>1.72</td>
<td>2.20</td>
</tr>
<tr>
<td>64</td>
<td>-4.22</td>
<td>-5.81</td>
<td>-5.47</td>
<td>-2.64</td>
<td>-0.96</td>
<td>0.06</td>
<td>0.06</td>
</tr>
<tr>
<td>128</td>
<td>-5.80</td>
<td>-5.78</td>
<td>-3.21</td>
<td>-1.15</td>
<td>-0.52</td>
<td>-0.37</td>
<td>-0.33</td>
</tr>
<tr>
<td>256</td>
<td>-5.80</td>
<td>-3.29</td>
<td>-1.16</td>
<td>-0.49</td>
<td>-0.32</td>
<td>-0.28</td>
<td>-0.27</td>
</tr>
</tbody>
</table>

2.4. Alternatively, let $T_1 = B_1 = 0, f(0) = 0, f(g) = g$. This gives:

$$f(i) = \beta = \frac{g}{S} + \frac{1}{n(2 + ng)} \left( \frac{1}{S} - S \right).$$

(2.6)

The accuracy of $\beta$ is shown by Table 6 ($g = 0.12$).

Table 6. Percentage error for $\beta$

<table>
<thead>
<tr>
<th>$n$</th>
<th>0.01</th>
<th>0.02</th>
<th>0.04</th>
<th>0.08</th>
<th>0.16</th>
<th>0.32</th>
<th>0.64</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>4.21</td>
<td>3.62</td>
<td>2.59</td>
<td>1.02</td>
<td>-0.55</td>
<td>-0.49</td>
<td>10.35</td>
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<tr>
<td>8</td>
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<td>23.51</td>
</tr>
<tr>
<td>16</td>
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</tr>
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<td>3.81</td>
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<tr>
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<td>1.30</td>
</tr>
<tr>
<td>128</td>
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<td>-5.13</td>
<td>-2.54</td>
<td>-0.47</td>
<td>-0.16</td>
<td>-0.32</td>
<td>-0.36</td>
</tr>
<tr>
<td>256</td>
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<td>-2.94</td>
<td>-0.79</td>
<td>-0.12</td>
<td>-0.04</td>
<td>-0.09</td>
<td>-0.10</td>
</tr>
</tbody>
</table>

2.5. There is a limit on the number of arbitrary parameters that may be determined by this technique since the series expansion analogous to Bizley's expansion in §1 has even more serious divergence problems, and hence equating higher derivatives of $f(i)$ to 0 will not necessarily produce good results. However, formulae better than those previously proposed do exist.

Let $B_1 = 0, f(0) = 0, f'(0) = 1, f(g) = g$. This gives:

$$f(i) = \frac{(1 + ng) \left[ \frac{(n+1)(1+ng)}{S} - 2 \right] - (n-1)S}{2n^2 \left( 1 + \frac{n+1}{2} g \right)}$$

(2.7)

This, in effect, combines the two previous formulae. The accuracy of the new formula is shown in Table 7 ($g = 0.12$).
2.6. This process may be carried to one further step. Two main problems prevent formulae of any greater accuracy being produced:

(a) The more accurate a formula becomes, the more complicated is its form, and hence the more time-consuming is its evaluation.

(b) Only a limited number of conditions exist to allow one to determine parameters. This situation is caused by the divergence of the power series expansion of \( i \) as a function of \( \phi \).

2.7. If the substitution

\[
S = \frac{1 + ng}{n(1 + \frac{n+1}{2} g)} \frac{1}{1 + \frac{1}{1 + \frac{n}{g}} \phi}
\]

is made in equation (2.3), then, by redefining the arbitrary parameters, it may be written:

\[
f(i) = \frac{T_0 + T_1 \phi + T_2 \phi^2}{1 + B_1 \phi} \ldots
\]  

i.e. in the form of a third degree Padé approximation. However, to mitigate the effects of the divergent nature of the equivalent series expansion, we use the conditions:

\[
f(0) = 0, f'(0) = 1, f''(0) = 0, f(g) = g
\]

to find the values of the parameters, using the result that:

\[
\phi(0) = 0
\]

\[
\phi'(0) = 1
\]

\[
\phi''(0) = \frac{2n(1 + \frac{n+1}{2} g)}{1 + \frac{n}{g}} \frac{(n+1)(1 + \frac{n+2}{3} g)}{1 + \frac{n+1}{2} \cdot g}
\]
After the parameters have been found and substituted into equation (2.8), it may be rewritten as:

\[ f(i) = \phi \left[ 1 - \frac{\phi''(0) \cdot \phi}{2 \left( 1 + \phi \cdot \frac{\phi''(0)}{2 \left( 1 - \frac{g}{\phi(g)} \right)} \right) \phi(g)} \right] \]  

(2.9)

This may again be rewritten as:

\[ f(i) = \lambda = \phi \left[ 1 - \frac{\phi}{B_0 + B_1 \phi} \right] \]  

(2.10)

where

\[ B_0 = \left[ \frac{n \left( 1 + \frac{n+1}{2} g \right)}{1 + ng} \frac{(n+1) \left( 1 + \frac{n+2}{3} g \right)}{2 \left( 1 + \frac{n+1}{2} g \right)} \right]^{-1} \]

\[ B_1 = \frac{1}{g} \left[ \frac{2(1+ng)}{n-1} \cdot \frac{B_0 \left( 1 + \frac{n+1}{2} g \right)}{g} \right] \]

This formula is remarkably accurate over the whole range of \( n \) and \( i \), as shown by the following table (\( g = 0.12 \)).

<table>
<thead>
<tr>
<th>( n )</th>
<th>0.01</th>
<th>0.02</th>
<th>0.04</th>
<th>0.08</th>
<th>0.16</th>
<th>0.32</th>
<th>0.64</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>-0.001</td>
<td>-0.003</td>
<td>-0.009</td>
<td>-0.017</td>
<td>-0.062</td>
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<td>-0.015</td>
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<td>-0.083</td>
<td>-0.251</td>
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</tr>
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</tr>
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<td>-0.123</td>
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<td>-4.207</td>
</tr>
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<td>-0.489</td>
<td>-6.55</td>
<td>-2.346</td>
<td>-3.531</td>
</tr>
<tr>
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<td>-0.535</td>
<td>1.339</td>
<td>-0.797</td>
<td>-0.552</td>
<td>1.580</td>
<td>-2.188</td>
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<td>-0.780</td>
<td>1.913</td>
<td>1.743</td>
<td>-0.596</td>
<td>-0.339</td>
<td>-0.902</td>
<td>-1.207</td>
</tr>
</tbody>
</table>

Table 8. Percentage error for \( \lambda \).
On the derivation of approximate actuarial formulae

2.8. To give some idea of the relative accuracy of these formulae, they may be compared with the widely used formula:

\[
g \frac{\frac{k}{n}}{i = \frac{n + 1}{1 + \frac{k}{2n} \cdot k}} \tag{2.11}
\]

where \( k = S - 1 \), quoted by many textbooks, including that of Donald.\(^6\) The percentage error pattern of formula (2.11) is given below, and shows that it is inferior to all of the new formulae.

Table 9.
Percentage error for Donald's formula \((g = \cdot12)\)

<table>
<thead>
<tr>
<th>(n)</th>
<th>(\cdot01)</th>
<th>(\cdot02)</th>
<th>(\cdot04)</th>
<th>(\cdot08)</th>
<th>(\cdot16)</th>
<th>(\cdot32)</th>
<th>(\cdot64)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>(\cdot11)</td>
<td>(\cdot19)</td>
<td>(\cdot30)</td>
<td>(\cdot29)</td>
<td>(\cdot55)</td>
<td>(-4\cdot82)</td>
<td>(-18\cdot03)</td>
</tr>
<tr>
<td>8</td>
<td>(\cdot37)</td>
<td>(\cdot67)</td>
<td>(1\cdot06)</td>
<td>(1\cdot02)</td>
<td>(-1\cdot83)</td>
<td>(-13\cdot68)</td>
<td>(-37\cdot80)</td>
</tr>
<tr>
<td>16</td>
<td>(1\cdot15)</td>
<td>(2\cdot07)</td>
<td>(3\cdot23)</td>
<td>(2\cdot97)</td>
<td>(-4\cdot66)</td>
<td>(-26\cdot23)</td>
<td>(-53\cdot07)</td>
</tr>
<tr>
<td>32</td>
<td>(3\cdot12)</td>
<td>(5\cdot56)</td>
<td>(8\cdot38)</td>
<td>(6\cdot92)</td>
<td>(-8\cdot46)</td>
<td>(-35\cdot67)</td>
<td>(-60\cdot90)</td>
</tr>
<tr>
<td>64</td>
<td>(7\cdot52)</td>
<td>(13\cdot08)</td>
<td>(18\cdot06)</td>
<td>(12\cdot06)</td>
<td>(-11\cdot30)</td>
<td>(-40\cdot59)</td>
<td>(-64\cdot70)</td>
</tr>
<tr>
<td>128</td>
<td>(16\cdot40)</td>
<td>(26\cdot57)</td>
<td>(30\cdot52)</td>
<td>(15\cdot91)</td>
<td>(-12\cdot79)</td>
<td>(-43\cdot03)</td>
<td>(-66\cdot57)</td>
</tr>
<tr>
<td>256</td>
<td>(32\cdot03)</td>
<td>(43\cdot95)</td>
<td>(39\cdot96)</td>
<td>(17\cdot95)</td>
<td>(-13\cdot54)</td>
<td>(-44\cdot25)</td>
<td>(-67\cdot50)</td>
</tr>
</tbody>
</table>

2.9. Iterative improvement

Formulae produced in the past for the purpose of improving estimates of the rate of interest have generally been awkward to handle or slow in convergence. An iterative technique based on Newton's method is developed below.

Let

\[
y(i) = S - ga_m - v^n = 0
\]

\[
y'(i) = -g \cdot v \cdot (1a_m - . - n.v^{n+1} = \frac{g}{i} a_m + n v^{n+1} \left(1 - \frac{g}{i}\right)
\]

By Newton's method

\[
i' = i - \frac{y(i)}{y'(i)}
\]

\[
= i - \frac{i(S - ga_m - v^n)}{ga_m + n v^{n+1}(i - g)}
\]
Now, \( v^n = 1 - ia_m \)
\[
\therefore \, ga_m + nv^{n+1}(i-g) = (i-g)nv + a_m[g - i \cdot nv(i-g)]
\]
\[
S - ga_m - v^n = S - 1 + a_m(i-g)
\]
\[
\therefore \quad i' = i \left[ 1 - \frac{a_m + \frac{S-1}{i-g}}{a_m \left[ \frac{g}{i-g} - i(nv) \right] + nv} \right]
\]

This is now in the most convenient form for manual computation, since some of the quantities appear more than once, and may be calculated beforehand and used when needed.

Newton's method has quadratic convergence in the neighbourhood of the root, and if this iterative improvement formula is used in conjunction with the formula \( \lambda \) as an initial approximation to the rate of interest, then at most two iterations will produce accuracy adequate for most practical purposes.

3. THE GENERAL COMPOUND INTEREST EQUATION

3.1. The ideas expounded above may be extended to the solution of the general compound interest problem which may be stated as:

Given \( P \) and \( \{C_i\} \), find \( i \) in the equation
\[
P = \sum_{i=1}^{n} C_i v^i
\]

\( P \) is a function of \( i \), and may be written as a power series which may be reverted as in the previous sections.

Let the function
\[
f(i) = \frac{\alpha}{\beta} \left[ \left( \sum_{i=1}^{n} \frac{C_i}{p} \right)^{\beta} - 1 \right]
\]

approximate \( i \).

Now,
\[
p(0) = \Sigma C
\]
\[
p'(0) = -\Sigma t \cdot C
\]
\[
p''(0) = \Sigma t(t+1) \cdot C
\]

The parameters in the formula may be found by application of the conditions:
\[f(0) = 0, f'(0) = 1, f''(0) = 0.\]
This gives:

\[ \alpha = \frac{\Sigma C}{\Sigma t \cdot C} \quad (3.2) \]

\[ \beta = \frac{\Sigma C(\Sigma t^2 \cdot C + \Sigma t \cdot C)}{(\Sigma t \cdot C)^2} - 1. \quad (3.3) \]

3.2. For an example of the application of this general formula, let \( C_t = 1 \) \((t = 1, n)\)

\[ \Sigma C = n \]
\[ \Sigma t \cdot C = \frac{n(n + 1)}{2} \]
\[ \Sigma t^2 \cdot C = \frac{n(n + 1)(2n + 1)}{6} \]

\[ \therefore \quad \alpha = \frac{2}{n + 1}, \quad \beta = \frac{n + 5}{3(n + 1)} \]
\[ f(i) = \frac{6}{n + 5} \left[ \left( \frac{n}{P} \right)^{\frac{n+5}{3(n+1)}} - 1 \right]. \quad (3.4) \]

Table 10 shows the accuracy of this formula.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \cdot 01 )</th>
<th>( \cdot 02 )</th>
<th>( \cdot 04 )</th>
<th>( \cdot 08 )</th>
<th>( \cdot 16 )</th>
<th>( \cdot 32 )</th>
<th>( \cdot 64 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>-0.002</td>
<td>-0.007</td>
<td>-0.026</td>
<td>-0.099</td>
<td>-0.370</td>
<td>-1.289</td>
<td>-3.978</td>
</tr>
<tr>
<td>8</td>
<td>-0.006</td>
<td>-0.025</td>
<td>-0.099</td>
<td>-0.380</td>
<td>-1.385</td>
<td>-4.550</td>
<td>-12.321</td>
</tr>
<tr>
<td>16</td>
<td>-0.025</td>
<td>-0.097</td>
<td>-0.377</td>
<td>-1.410</td>
<td>-4.809</td>
<td>-13.513</td>
<td>-28.278</td>
</tr>
<tr>
<td>32</td>
<td>-0.096</td>
<td>-0.374</td>
<td>-1.415</td>
<td>-4.928</td>
<td>-14.145</td>
<td>-47.781</td>
<td>-126.397</td>
</tr>
<tr>
<td>64</td>
<td>-0.371</td>
<td>-1.416</td>
<td>-4.985</td>
<td>-14.471</td>
<td>-30.558</td>
<td>-63.077</td>
<td>-166.304</td>
</tr>
<tr>
<td>128</td>
<td>-1.415</td>
<td>-5.012</td>
<td>-14.637</td>
<td>-30.954</td>
<td>-48.608</td>
<td>-63.600</td>
<td>-75.009</td>
</tr>
</tbody>
</table>

This is much less accurate than the formulae produced in §1, but is still adequate considering that it was produced from a general formula.

3.3. Now, let \( C_t = g \) \((t = 1, n - 1)\), \( C_n = 1 + g \). This is the case of the redeemable securities as in §2.
Here,
\[ \Sigma C = 1 + ng \]
\[ \Sigma t \cdot C = n \left( 1 + \frac{n+1}{2} g \right) \]
\[ \Sigma t^2 \cdot C = n^2 + \frac{n(n+1)(2n+1)}{6} g. \]

Hence,
\[ \alpha = \frac{1 + ng}{n \left( 1 + \frac{n+1}{2} g \right)} \quad \beta = \frac{(n+1)(1+ng) \left( 1 + \frac{n+2}{3} g \right)}{n \left( 1 + \frac{n+1}{2} g \right)^2} - 1. \quad (3.5) \]

Table 11 shows the accuracy of this formula \((g = \cdot 12)\).

<table>
<thead>
<tr>
<th>(n)</th>
<th>(\cdot 01)</th>
<th>(\cdot 02)</th>
<th>(\cdot 04)</th>
<th>(\cdot 08)</th>
<th>(\cdot 16)</th>
<th>(\cdot 32)</th>
<th>(\cdot 64)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>(-\cdot 001)</td>
<td>(-\cdot 005)</td>
<td>(-\cdot 021)</td>
<td>(-\cdot 084)</td>
<td>(-\cdot 326)</td>
<td>(-1\cdot 224)</td>
<td>(-4\cdot 266)</td>
</tr>
<tr>
<td>8</td>
<td>(-\cdot 008)</td>
<td>(-\cdot 031)</td>
<td>(-\cdot 125)</td>
<td>(-\cdot 491)</td>
<td>(-1\cdot 883)</td>
<td>(-6\cdot 664)</td>
<td>(-19\cdot 102)</td>
</tr>
<tr>
<td>16</td>
<td>(-\cdot 035)</td>
<td>(-\cdot 139)</td>
<td>(-\cdot 546)</td>
<td>(-2\cdot 090)</td>
<td>(-7\cdot 305)</td>
<td>(-20\cdot 231)</td>
<td>(-39\cdot 139)</td>
</tr>
<tr>
<td>32</td>
<td>(-\cdot 130)</td>
<td>(-\cdot 512)</td>
<td>(-1\cdot 951)</td>
<td>(-6\cdot 777)</td>
<td>(-18\cdot 735)</td>
<td>(-36\cdot 616)</td>
<td>(-54\cdot 087)</td>
</tr>
<tr>
<td>64</td>
<td>(-\cdot 457)</td>
<td>(-1\cdot 739)</td>
<td>(-6\cdot 067)</td>
<td>(-17\cdot 048)</td>
<td>(-34\cdot 221)</td>
<td>(-51\cdot 719)</td>
<td>(-66\cdot 142)</td>
</tr>
<tr>
<td>128</td>
<td>(-1\cdot 588)</td>
<td>(-5\cdot 580)</td>
<td>(-15\cdot 959)</td>
<td>(-32\cdot 779)</td>
<td>(-50\cdot 370)</td>
<td>(-65\cdot 059)</td>
<td>(-76\cdot 129)</td>
</tr>
<tr>
<td>256</td>
<td>(-5\cdot 312)</td>
<td>(-15\cdot 380)</td>
<td>(-32\cdot 046)</td>
<td>(-49\cdot 719)</td>
<td>(-64\cdot 562)</td>
<td>(-75\cdot 778)</td>
<td>(-83\cdot 782)</td>
</tr>
</tbody>
</table>

3.4. For example, consider a security at a price \(p\) which guarantees the payment of £2 in 2 years, and £5 in 5 years. What is the yield afforded (ignoring tax)?

\[ t \quad C_t \quad t \cdot C_t \quad t^2 \cdot C_t \]
\[ 2 \quad 2 \quad 4 \quad 8 \quad \alpha = \frac{203}{841} \]
\[ 5 \quad 5 \quad 25 \quad 125 \quad \beta = \frac{293}{841} \]

\[ \therefore \quad f(i) = \frac{203}{293} \left( \frac{e^{293/841}}{p} - 1 \right) \]

This formula may be tested at various rates of interest.

| Actual yield | \(\cdot 0100\) | \(\cdot 0200\) | \(\cdot 0400\) | \(\cdot 0800\) | \(\cdot 1600\) | \(\cdot 3200\) | \(\cdot 6400\) |
| Formula yield| \(\cdot 0100\) | \(\cdot 0200\) | \(\cdot 0400\) | \(\cdot 0799\) | \(\cdot 1591\) | \(\cdot 3138\) | \(\cdot 6012\) |
| Percentage error | \(\cdot 002\) | \(\cdot 009\) | \(\cdot 037\) | \(\cdot 143\) | \(\cdot 542\) | \(\cdot 1934\) | \(\cdot 6065\) |
As in §§1 and 2, the estimates may be improved by an iterative method.

Let

\[ h(i) = P - \sum_{t=1}^{n} C_t \cdot v^t \]

\[ h'(i) = - \sum_{t=1}^{n} C_t \cdot tv^{t+1} \]

\[ P - \sum_{t=1}^{n} C_t \cdot v^t = \sum_{t=1}^{n} t \cdot C_t \cdot v^{t+1} \]

\[ \therefore \quad i' = i - \frac{\sum_{t=1}^{n} t \cdot C_t \cdot v^{t+1}}{\sum_{t=1}^{n} C_t \cdot v^t} \]

If the values of \( \{c_t\} \) implicit in previous sections are substituted into this general formula, then, as expected, the iterative formulae derived there emerge.

REFERENCES

(3) Karpin, H. Simple Algebraic Formulae for Estimating the Rate of Interest, J.I.A. 93, 297.
(4) Gerald, Applied Numerical Analysis, p. 300.