IMPLICATIONS OF MODERN PORTFOLIO THEORY FOR LIFE ASSURANCE COMPANIES

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(A paper presented to the Society on 19 January 1982)

1. INTRODUCTION

The general literature on the topic of Modern Portfolio Theory (M.P.T.) is now quite copious but in order to make this paper more self sufficient than it might otherwise be I make no apologies for repeating what is available outside United Kingdom actuarial literature. There are not very many actuarial papers advocating the use of M.P.T. which might suggest that many actuaries practising the techniques of M.P.T. have not been convinced that their work is conclusive. Moore's paper (1) in 1972 laid the groundwork for discussion of the models of M.P.T. by the profession. In 1977 Holbrook (6) discussed in his more general paper on pension fund performance the relevance of risk and return by summarizing the work of Treynor, Sharpe and Fama. There have been two recent papers from north of the border. The 1980 paper by Pountain and Fitzgerald (12) is the earlier. Clarkson's paper (16) to the Faculty contains a particularly interesting section in which he compares his own model for managing an ordinary share portfolio with the methodology of M.P.T.

M.P.T. is not formally covered in the actuarial syllabus and this paper aims to be a 'plain' actuary's guide to M.P.T. and will, I hope, be useful to those employed outside the investment scene. In an effort to convey the ideas I have sacrificed a 'rigorous' mathematical treatment for which I hope I may be forgiven. If after reading this paper the reader understands the distinctions between the related M.P.T. topics of portfolio theory, the market model, the diagonal model, the capital asset pricing model and the efficient markets hypothesis, I shall have achieved my primary objective.

The paper is based on an earlier one produced jointly with Ian Henderson for a symposium on "M.P.T. and Financial Institutions" at Exeter University in 1980. The theory in that paper was largely based upon Sharpe's excellent textbook (8). I have taken the opportunity in this paper of drawing upon the recent textbook by Elton and Gruber (18) which is a model of clarity and logical development of the theories.

I owe a great debt of thanks to several people who have advised me on aspects of this paper.

2. MODERN PORTFOLIO THEORY

2.1 Portfolio Theory

The idea of assigning probabilities to events is a concept familiar to all
actuarial students. Extending the idea to assets rather than liabilities is thus a reasonable step. Portfolio theory is the process of determining the properties of portfolios of assets given the properties of the individual assets, delineating the characteristics of portfolios that make them preferable to others and, finally, showing how the composition of the preferred portfolios can be determined.

In order to reach a solution to a portfolio problem two aspects need to be explored. It is necessary to know the choices available to an investor—called the ‘opportunity set’—and it is necessary to know the investor’s tastes or preferences—called ‘indifference or utility curves’. By aggregating across all investors and all capital markets, and assuming equilibrium conditions, it is possible to construct mathematical models. It is the uncertainty of future returns which necessitates the complexity of these models which are then used to assist in the choice of assets. Inevitably, assumptions are made to reduce the mathematical complexity and these will be discussed fully later.

Portfolio theory assumes that the returns are normally distributed. In most portfolio literature the variance of the distribution is used to measure by how much observations differ from the average return. The standard deviation is also often used as the measure of dispersion. The average return and variance of an individual asset are calculated by assigning different probabilities to differing outcomes. If two assets have the same variance most investors would prefer the asset with the higher expected return. Similarly, if two assets have the same expected return most investors would choose the one with the lower variance because the investor is more certain of obtaining the expected return and will have fewer poor outcomes to contend with.

If an investor holds more than one asset and if the rate of return per cent per period on an asset, \( i \), is \( R_i \) then the return per cent per period of a portfolio of \( n \) assets \( R_p \) is the weighted average of the individual returns.

\[
R_p = \sum_{i=1}^{n} X_i R_i
\]

(1)

where \( X_i \) is the proportion of the portfolio invested in asset \( i \). Thus the mean return per cent per period of the portfolio is

\[
\bar{R}_p = \sum_{i=1}^{n} X_i \bar{R}_i
\]

(2)

The variance of a portfolio, \( \sigma_p^2 \), is

\[
\text{Ex} (R_p - \bar{R}_p)^2
\]

(3)

Expression (3) is defined statistically as

\[
\sigma_p^2 = \sum_{i} X_i X_j \sigma_{ij} + \sum_{i} X_i^2 \sigma_i
\]

(4)

where \( \sigma_{ij} \) is the covariance between the return per cent per period on security \( i \) and the return per cent per period on security \( j \). By reducing formula (4) to the case of...
just two securities some examination of the effect of diversification can be undertaken.

\[ \sigma_p^2 = X_1^2\sigma_1^2 + 2X_1X_2\sigma_{12} + X_2^2\sigma_2^2 \]  
(5)

Substituting for \( \sigma_{12} \) by using the correlation coefficient \( r_{12} = \sigma_{12}/(\sigma_1\sigma_2) \) in expression (5) we have

\[ \sigma_p^2 = X_1^2\sigma_1^2 + 2X_1X_2r_{12}\sigma_1\sigma_2 + X_2^2\sigma_2^2 \]  
(6)

If there is perfect positive correlation between the two securities \( r_{12} = +1 \) and formula (6) becomes

\[ \sigma_p^2 = X_1^2\sigma_1^2 + 2X_1X_2\sigma_1\sigma_2 + X_2^2\sigma_2^2 \]  
(7)

whence

\[ \sigma_p = X_1\sigma_1 + X_2\sigma_2 \]  
(8)

In other words, expression (8) says that the standard deviation of the rate of return of a fund comprising two perfectly correlated securities is the weighted average of the standard deviations of the two securities. Diversification in this case has merely averaged the standard deviations of return. For an example of two positively correlated securities think of two South African Gold Shares.

If there is perfect negative correlation between the two securities \( r_{12} = -1 \) and formula (6) becomes

\[ \sigma_p^2 = X_1^2\sigma_1^2 - 2X_1X_2\sigma_1\sigma_2 + X_2^2\sigma_2^2 \]  
(9)

whence

\[ \sigma_p = X_1\sigma_1 - X_2\sigma_2 \]  
(10)

If the proportionate holdings are inversely proportional to the standard deviations of the returns of the two securities we have

\[ \frac{X_1}{X_2} = \frac{\sigma_2}{\sigma_1} \]  
(11)

or

\[ X_1 = \frac{\sigma_2X_2}{\sigma_1} \]  
(12)

Substituting for \( X_1 \) in formula (10) shows \( \sigma_p = 0 \) and demonstrates the basis of ‘hedging’ strategies. If the two securities are perfectly negatively correlated it is possible to arrange their proportions such that the standard deviation of returns on a mix of the two is zero.

If the two securities are uncorrelated \( r_{12} = 0 \) and formula (6) becomes

\[ \sigma_p^2 = X_1^2\sigma_1^2 + X_2^2\sigma_2^2 \]  
(13)

The nature of equation (13) is such that \( \sigma_p \) will always be less than \( \sigma_1 \) or \( \sigma_2 \) because
$X_1 + X_2$ equals unity. This demonstrates the key point of portfolio theory that the standard deviation of return of a portfolio is less than the standard deviation of individual assets and is the principle of diversification.

Reverting to the general expression (4) for the variance of a portfolio and assuming that all assets are independent and $\sigma_{ij} = 0$ for all $i, j$ we have

$$\sigma_p^2 = \sum_{i=1}^{n} X_i^2 \sigma_i^2$$

if $X_i = 1/n$ then

$$\sigma_p^2 = \frac{1}{n} \sum_{i=1}^{n} \left[ \frac{\sigma_i^2}{n} \right]$$

Equation (15) says that if the $n$ securities of a fund are equally weighted by market value, the variance of returns of the portfolio is the average of the individual variances divided by $n$. Clearly, as $n$ becomes extremely large, the variance of the return on the portfolio of assets approaches zero.

In the more usual case where assets are correlated (either positively or negatively) the arithmetic is not as simple. Assume, again, equal investments by market value. Equation (4) becomes

$$\sigma_p^2 = \frac{1}{n^2} \sum_{i,j=1}^{n} \sigma_{ij}$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} \sigma_i^2 + \frac{1}{n^2} \sum_{i,j=1, i \neq j}^{n} \sigma_{ij}$$

$$= \frac{1}{n} \sum_{i=1}^{n} \left[ \frac{\sigma_i^2}{n} \right] + \frac{n-1}{n} \sum_{i,j=1, i \neq j}^{n} \left[ \frac{\sigma_{ij}}{n(n-1)} \right]$$

The terms in brackets in expression (18) are averages. There are $n$ values of $i$ and $(n-1)$ values of $j$. In total there are $n(n-1)$ covariance terms. The second term in expression (18) is the summation of the covariances divided by the number of covariances. Thus

$$\sigma_p^2 = \frac{1}{n} \bar{\sigma}_i^2 + \frac{n-1}{n} \bar{\sigma}_{ij}$$

In a portfolio of correlated assets the variance of returns of the portfolios is the sum of two items. One item reduces to zero as $n$ gets large. The other item reduces to the average value of the individual covariances as $n$ gets large. Rearranging expression (19) we get

$$\sigma_p^2 = \frac{1}{n} \left( \bar{\sigma}_i^2 - \bar{\sigma}_{ij} \right) + \bar{\sigma}_{ij}$$
The minimum variance of a portfolio is obtained for very large portfolios and is equal to the average covariance between all stocks in the portfolio.

One further special case should be mentioned before moving on and that is if $\sigma_1 = 0$ in formula (13). We then get

$$\sigma_p = X_2 \sigma_2$$  \hspace{1cm} (21)

In order to introduce the next section I shall for the first time use the word risk instead of variance or standard deviation of returns per cent per period. Equation (21) says that the risk of the combination of a risk-free investment and a risky security (or portfolios) is proportional to the amount invested in the latter. The next section discusses the terms risk and risk-free in more detail. This opening section has described the concept of asset returns and variability assuming a normal distribution of rates of return per period.

2.2 Efficient Portfolios

At this point I introduce the concept of risk-return space. This I propose to use to demonstrate the concepts of portfolio optimization.

From equation (2) in the case of two securities we have

$$\bar{R}_p = X_1 \bar{R}_1 + X_2 \bar{R}_2$$ \hspace{1cm} (22)

If the portfolio is fully invested $X_1$ plus $X_2$ must equal one. In this case equation (8) becomes

$$\sigma_p = X_1 \sigma_1 + (1 - X_1) \sigma_2$$ \hspace{1cm} (23)

and

$$X_1 = \frac{\sigma_p - \sigma_2}{\sigma_1 - \sigma_2}$$ \hspace{1cm} (24)

Substituting for $X_1$ in equation (22) gives

$$\bar{R}_p = \left( \frac{\sigma_p - \sigma_2}{\sigma_1 - \sigma_2} \right) \bar{R}_1 + \left( 1 - \frac{\sigma_p - \sigma_2}{\sigma_1 - \sigma_2} \right) \bar{R}_2$$ \hspace{1cm} (25)

whence

$$\bar{R}_p = \left[ \bar{R}_2 - \left( \frac{\bar{R}_1 - \bar{R}_2}{\sigma_1 - \sigma_2} \right) \sigma_2 \right] + \left( \frac{\bar{R}_1 - \bar{R}_2}{\sigma_1 - \sigma_2} \right) \sigma_p$$ \hspace{1cm} (26)

This expression is similar to $y = a + bx$ and is the equation of a straight line in risk-return space.
In the case of two perfectly correlated assets \( r_{12} = +1 \) both the return and risk of the portfolio are weighted averages of the returns and risks of the two underlying assets.

Without labouring the arithmetic a similar treatment of the situation when \( r_{12} = -1 \), i.e. the case of perfectly negative correlation produces a graph as follows:

This shows graphically how it is possible under these assumptions to arrange the assets such that \( \sigma_p = 0 \), i.e. all risk is eliminated in the portfolio.

If \( r_{12} = 0 \) in equation (6) the expression for risk of the portfolio becomes

\[
\sigma_p = \left[ X_1^2 \sigma_1^2 + (1 - X_1)^2 \sigma_2^2 \right]^{\frac{1}{2}}
\]  

(27)

The following graph shows the curve which can be derived from equation (27) to express the relationship between \( \bar{R}_p \) and \( \sigma_p \).
Figure 3

At this point it is worth considering the portfolio that gives minimum risk. By differentiating equation (27) with respect to $X_1$, equating the derivative to 0 and solving for $X_1$ we obtain for the present case in which $r_{12} = 0$ the value of $X_1$

$$X_1 = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

This defines the minimum-risk portfolio.

Figure 4

The figure above shows that:

(a) The closer the correlation to $-1$, the greater will be the benefits of diversification, all other factors remaining constant, as is intuitively obvious.

(b) The maximum risk from combinations of two assets is never more than that represented by the straight line in equation (26).
(c) It is possible to produce an expression for finding the minimum variance when two assets are combined in a portfolio.

If we move on to discuss more generalized portfolios of various assets, and there are of course an infinite number, a scatter diagram can be constructed as follows.

![Figure 5](image)

Point A is the portfolio with minimum variance—Point B is the portfolio (usually a single security) that offers the highest expected return of all portfolios. The line AB is called the efficient frontier. The simple analysis earlier of two securities intuitively suggests that the line is concave. It is an efficient frontier because any portfolio selected below the line will have a lower expected return for the same level of risk. If short sales are permitted (i.e. selling assets one does not own) a curve similar to the following is produced.

![Figure 6](image)
The efficient set of portfolios is the upper half of the hyperbola and has no upper limit.

At this point we introduce the idea of a risk-free asset. It is generally assumed that short-term government bills are such risk-free assets. Borrowing is regarded as selling such a security short and we can now move to the concept of lending or borrowing at the risk-free rate. Because the return is certain the standard deviation of return is zero. The expected return on a combination of risk-less asset and risky portfolio is given by

\[ \bar{R}_p = (1 - X)R_f + X\bar{R}_2 \]  

(28)

where \( R_f \) is the return on the risk-free asset which is of course certain and \( \bar{R}_2 \) is the expected return on the risky portfolio. From equation (21) we have

\[ \sigma_p = X\sigma_2 \]  

(29)

thus

\[ \bar{R}_p = \left(1 - \frac{\sigma_p}{\sigma_2}\right)R_f + \frac{\sigma_p}{\sigma_2}\bar{R}_2 \]  

(30)

or

\[ \bar{R}_p = R_f + \left(\frac{\bar{R}_2 - R_f}{\sigma_2}\right)\sigma_p \]  

(31)

This is a straight line. It says that all combinations of risk-less lending or borrowing with portfolio 2 lie on a straight line in risk-return space. The intercept on the return axis is \( R_f \) (the risk-less rate) and the slope is \((\bar{R}_2 - R_f)/\sigma_2\)

Combining the concepts graphically represented in the last two figures we can note the following
There is an infinite number of straight lines IGH representing combinations of different portfolios with the risk-free asset. We already know that ABC is the efficient frontier. Point G is the tangent. Point G is the risky portfolio offering the highest attainable reward per unit of risk. The line IG is the combination of the risk-free asset and portfolios that offers the highest expected return per unit of risk. Line GH is for risky investors who are prepared to borrow funds to invest in portfolio G. Every 'efficient' investor invests in portfolio G with a proportion of the risk-free asset from some negative figure to 100%.

The optimization process for portfolios consists of finding the portfolio with the greatest ratio of excess return (expected return minus risk-free rate) per unit of risk. In other words

\[ \frac{\bar{R}_p - R_f}{\sigma_p} \text{ is to be maximized} \]

subject to

\[ \sum_{i=1}^{n} X_i = 1 \]

where \( R_f \) is the risk-free rate.

The calculations required in this process are enormous and simplified ways of dealing with the problem have been devised by using mathematical models.

2.3 The Market and Diagonal Models

Equations (2) and (4) provide the necessary input data to perform portfolio analysis. The efficient frontier can be determined by examining the expected
return and standard deviations of return on a portfolio. We need estimates of the expected return and variance of each security in the portfolio and estimates of the correlation between each possible pair of securities within the portfolio. The number of correlation coefficients that have to be estimated in a portfolio of \( n \) stocks is \( n(n-1)/2 \). The estimate is made by using linear regression techniques on historic returns.

In order to make the problem easier the market model regresses each stock against a market proxy instead of every other stock. In the U.K. the suitable market proxy is the Financial Times–Actuaries All-Share Index. A linear regression is assumed. If \( R_i \) is the rate of return per period per cent for stock \( i \), \( R_m \) is a random variable representing the rate of return on the market index; \( a_i \) is the component of return on security \( i \) that is independent of the market's performance and is also a random variable; and \( \beta_i \) is a constant that measures the expected change in \( R_i \) given a change in \( R_m \) then the model is

\[
R_i = a_i + \beta_i R_m
\]  

The constant \( \beta_i \) implies that if the market produces a return, positive or negative, of say 5\% then the stock \( i \) will produce a return, positive or negative, of \((a_i + 5 \times \beta_i)\%\).

\( a_i \) is independent of the return on the market. It is useful to split \( a_i \) into two components. Let \( \alpha_i \) be the expected value of \( a_i \) and let \( e_i \) represent the random element of \( a_i \). Then

\[
a_i = \alpha_i + e_i
\]  

where the expected value of \( e_i \) is zero.

The market model is

\[
R_i = \alpha_i + \beta_i R_m + e_i
\]  

This equation is the basis of M.P.T. It is assumed that \( e_i \) is uncorrelated with \( R_m \) which means that the market index is unrelated to any unique return.

The model if developed produces a mean return of

\[
\bar{R}_i = \alpha_i + \beta_i \bar{R}_m
\]  

The variance of a security’s return is

\[
\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{(e_i)}^2
\]  

A stock’s risk is the sum of market risk \((\beta_i^2 \sigma_m^2)\) and residual risk \((\sigma_{(e_i)}^2)\).

If we now assume that securities are related only through their common response to the market, i.e. \( \sigma_{(e_i)} = 0 \) then it is possible to simplify the portfolio optimization process. This is called the diagonal model. The diagonal model still assumes relationships between stocks because

\[
\sigma_{ij} = \beta_i \beta_j \sigma_m^2
\]  

The simplicity of this model is that all covariances can be estimated from the \( \beta_i \)’s
of the market model. Instead of \( n(n-1)/2 \) correlation coefficients only \( n \) are required.

The complicated procedures required to produce the optimization process are much reduced.

If we now consider a portfolio of \( n \) stocks and define the \( \beta \) of a portfolio \( \beta_p \) as a weighted average of the individual \( \beta_i \)'s on each stock in the portfolio where the weights are the fraction of the portfolio in each stock then

\[
\beta_p = \sum_{i=1}^{n} X_i \beta_i
\]  

(38)

If \( \alpha_p \) is defined as

\[
\alpha_p = \sum_{i=1}^{n} X_i \alpha_i
\]  

(39)

then the expected return on the portfolio can be derived as

\[
\bar{R}_p = \alpha_p + \beta_p \bar{R}_m
\]  

(40)

Further, the risk of a portfolio can be derived as

\[
\sigma_p^2 = \beta_p^2 \sigma_m^2 + \sum_{i=1}^{n} X_i^2 \sigma_{(e)}^2
\]  

(41)

In other words the residual risk of a portfolio is a weighted average of the residual risks of the individual stocks.

If equal amounts of money are placed in \( n \) stocks the risk is

\[
\sigma_p^2 = \beta_p^2 \sigma_m^2 + \frac{1}{n} \left( \sum_{i=1}^{n} \frac{1}{n} \sigma_{(e)}^2 \right)
\]  

(42)

The last term in equation (42) diminishes to zero as \( n \) gets larger. Thus as \( n \) gets large the risk of the portfolio approaches

\[
\sigma_p = \beta_p \sigma_m = \sigma_m \left[ \sum_{i=1}^{n} X_i \beta_i \right]
\]  

(43)

This equation does not diminish as \( n \) gets large and \( \beta_i \) is the measure of a security's non-diversifiable risk. Betas are estimated by regression techniques using historical data. Residual risk (non-market risk) can be diversified away.

2.4 The Capital Asset Pricing Model

The previous sections have discussed the theory of portfolios and the basis of selecting optimum portfolios from sets of estimates. By aggregating the actions of all investors equilibrium models may be constructed which show the measure of risk for any asset and the relationship between expected return and risk. A number of assumptions are made.
(a) All assets are perfectly divisible and marketable and there are no transaction costs.
(b) There are no taxes.
(c) All investors can borrow or lend an unlimited amount at the risk-free rate and there are no restrictions on short sales of any asset.
(d) All investors have identical subjective estimates of the means, variances and covariances of return among all assets.
(e) The total quantity of all assets is known.
(f) All investors make portfolio decisions based on their expectations of mean and variance of return which maximize their expected utility of ‘terminal wealth’.

These clearly unrealistic assumptions should not prohibit us from looking at the theory further.

Figure 8 showed the efficient frontier of efficient portfolios where short sales were permitted and unlimited lending or borrowing.

The portfolio held by investors is irrelevant to investors’ risk preferences. It lies at the tangency point M and risk preferences are accommodated by combining the risky portfolio at M with lending or borrowing. Every investor faces the same diagram. Every investor holds the same risky portfolio. Therefore, in equilibrium, it must be the market portfolio.

Thus all investors have portfolios somewhere along line $R_fMC$ and they are necessarily efficient. By adapting equation (31) which was the line connecting a risk-less asset and a risky portfolio we obtain the capital market line.

$$R_e = R_f + \frac{\bar{R}_m - R_f}{\sigma_m} \sigma_e$$

where subscript $e$ denotes an efficient portfolio.

All strategies other than those employing the market portfolio and borrowing or lending lie below the capital market line because the market is efficient. The rate of interest on a risk-free asset is the reward for waiting or the price of time.
The slope of the line \((R_m - R_f)/\sigma_m\) can be regarded as the reward per risk borne. Thus the expected return is the sum of two components—the price of time and the price of risk times the amount of risk. This formula relates only to efficient portfolios.

A further line, the *security market line*, is defined as

\[
R_i = R_f + \beta_i(R_m - R_f)
\]  

(45)

The expected return on any asset, efficient or otherwise, can be determined from this relationship. The difference between expected returns on any two assets is related simply to the difference in beta. This line can be derived intuitively as has equation (44).

2.5 *The Efficient Markets’ Hypothesis*

The basic tenet of the E.M.H. is that security prices reflect fully all available known information. This is clearly a most stringent requirement and the hypothesis has been subdivided into three categories.

*Weak form* tests have looked at whether the information contained in historical prices is fully reflected in current prices.

*Semi-strong* tests are tests of whether publicly available information is fully reflected in current stock prices.

*Strong-form* tests examine whether all information, public or private, is fully reflected in security prices and whether any type of investor can make an excess profit.

The difference in the U.S. and U.K. markets is sufficient to explain why more academic and other research has been carried out to test these theories in the U.S. I do not propose to explore the differences or similarities between the markets. Suffice it to say that the case for making the hypothesis is probably stronger in the U.S.

2.6 *The Real World*

The theoretical basis for M.P.T. lies on a bed of many assumptions. The crucial assumption in many critics’ eyes is the use of the variance of return of a security as a measure of risk. M.P.T. exponents assume a normal probability distribution for analysing returns on diversified portfolios over short periods of up to, say, three months. For longer periods of observation it is assumed that the portfolio’s continuously compounded rate of return is normally distributed, i.e. the actual return is distributed ‘log-normally’. For short holding periods the actual return differs little from the continuously compounded return and thus a normal distribution is assumed reasonable. Thus the mean of the distribution is regarded
as the expected rate of return and the use of the standard deviation, as well as the variance, is often adopted (ambiguously) as the measure of risk.

Lever (3) has made the point that trustees of pension funds would rarely define risk as variability which in his view provides a fine battleground for academic debate because, *inter alia*, it can be subjected to complex mathematical treatment. He refers to a paper by Machol and Lerner in which they argue that risk should be defined as the probability of achieving less than some specified threshold level of return which *must* be achieved and *which will be different for each investor* and that the probability should be measured over a period of time which is also different for different investors. Clarkson in his paper is also critical of M.P.T.'s measure of risk and presents a simple numerical example to illustrate the possibilities of M.P.T. leading to what he regards as absurd buy-and-sell decisions. A reviewer of Clarkson's paper (17), however, does not feel that the possibilities look absurd and iterates the point that "M.P.T. leads to perfectly logical choices as long as investors are risk averse (surely a reasonable assumption for an institutional investor?) and equity returns are multi-variate normal (or log-normal or any other distribution parametered by mean and variance)—equity returns do appear to be so distributed".

The risk-free rate adopted is clearly unrealistic. In this country one could argue convincingly that either call money, seven-day money, three-month money, or 15-year gilts are all risk-free. Whilst this tends to upset the theory I find the concept of a short-term risk-free rate easy to understand even if I cannot provide tangible proof of its existence.

What is clear is that academics by now have developed their work to try to cope with the real world. The C.A.P.M. has been amended to meet the criticism of one or more of its quite stringent assumptions having no real basis in fact. The proprietary services available for portfolio optimization are sold to users who must have a clear idea of their limitations and the assumptions implicit in the techniques used. Different services use slightly different techniques but the underlying theory is essentially as described earlier. The challenge for actuaries is deciding how and to what extent to use the methodology of M.P.T.

2.7 Quotable Quotes

The reader who has struggled this far may like to read some quotations which help clarify the concepts of the previous six sections.

*Portfolio theory*: "Portfolio theory is a rational and consistent theory of investment behavior under uncertainty and *does not* in any way depend on the invalidity of the C.A.P.M.—a misapprehension which underlines the arguments put forward in 'Is Beta Dead?'" (FUNG) (15).

*C.A.P.M.*: "The truth of the matter is, that it was the pioneering work of
Markowitz [on portfolio theory] that formed the very foundation of the C.A.P.M. and certainly not the other way around" (FUNG) (15).

C.A.P.M.: “It has long been recognized that the C.A.P.M. in its original form simply cannot hold if practical restrictions such as short sales of assets, taxes and transaction costs are imposed” (FUNG) (15).

Portfolio theory: “Contrary to popular belief the use of the statistical quantification of risk, which is the essence of M.P.T., presumes absolutely nothing about the ‘efficiency’ of the market” . . . “Similarly, use of M.P.T. does not presume that any particular theory of share prices such as the Dividend Capitalization Model or the Capital Asset Pricing Model is true” (STAPLETON) (14).

Market model and diagonal model: “The Market Model and its special case, the Diagonal Model, are statistical models relating the rate of return on a stock over a period to the rate of return on a market index over the same period. These models say nothing about the pricing of securities, except in so far as the price is loosely related to the price of securities in general” (STAPLETON) (14).

C.A.P.M.: “The C.A.P.M. is a theory of pricing securities” (STAPLETON) (14).

E.M.H.: “Similarly, the Efficient Markets’ Hypothesis is logically distinct the Market and Diagonal Models and the CAPM. . . . It is not a theory of the pricing of securities in relation to risk but more a statement that markets are competitive with respect to information” (STAPLETON) (14).

Humour: “MPT allows us to do clever sums (or to get computers to do them for us) and put numbers on these two items of Risk and Return. Roughly speaking, these sums can be either fairly complicated (Sharpe—gives money managers a headache) or very complicated (Markowitz—gives computers a headache). They can also be somewhere in between, or quite complicated (Rosenberg—gives the Institutional Investor a headache)” (MACQUEEN) (13).

3. APPLICATIONS OF MODERN PORTFOLIO THEORY BY LIFE ASSURANCE COMPANIES

3.1 Performance Measurement

Life assurance is a service industry where the quality of service provided to policyholders and agents is an important but unquantifiable factor in the acquisition of new business. Apart from this, competition between the different companies is on premium and bonus rates. The emergence of investment-linked
policies in recent years has brought investment performance under the microscope but it has always been true that profitable investment is an essential attribute of a successful life office.

Any form of measurement implies the existence of a suitable yardstick and the measurement of investment performance is no exception. The total return of any portfolio over a period can be compared with that of a model fund, for example Financial Times-Actuaries All-Share Index or Standard and Poor’s 500, or with other actual or notional portfolios. If risk is defined as the variability of the expected total return the performance of different portfolios carrying different degrees of risk can be compared using the techniques of M.P.T.

The concept of performance measurement is easy to understand. The problems arise when the portfolio is a mixed one containing different types of investment such as gilts or property as well as equities. Property investment raises particular difficulties due to the impossibility of establishing universally accepted market values and betas for any individual property. The overall measurement of a portfolio may be constrained by guidelines externally imposed such as, in the case of a life office, the need to ‘match’ assets and liabilities. The practical difficulties involved in measurement form the basis of the paper by Kingston (5).

Various external services exist for the measurement of overall investment performance by pension funds but most life offices, for their non-linked business, content themselves with the comparison, sector by sector, of their performance against an appropriate published index or an internally constructed model fund. Much study has been undertaken of the performance of linked-life assurance and comprehensive league tables are produced in the insurance press.

3.2 Role of M.P.T. in Assisting Investment Decisions

The techniques of M.P.T. should enable an investment manager to maximize the expected total return from his portfolio for any given level of risk or alternatively to minimize the risk for any given level of total return. Proprietary systems of using M.P.T. have been developed in the U.S. and it is claimed that the use of these systems has resulted in very much improved investment performance. At the very least the systems have been successful in attracting considerable numbers of subscribers.

It must be pointed out that the foundations of the theory have recently been questioned and a great debate is proceeding in the U.S. Suffice it to say that any system of selecting shares using M.P.T. must be self-defeating if one accepts the efficient market theory. As soon as a stock is identified as having an expected total return either higher or lower than it should according to its risk this information becomes part of the input known to the efficient market and the price is automatically adjusted. In the practical world markets are not efficient and money can still be made or lost by making forecasts of total return which are more or less accurate than the market consensus.
The popularity of M.P.T. in the U.S. is at least partly due to the considerable pressure on fund managers to improve their performance following the debâcle in the early 1970s when the stock prices of many of the institutionally favoured stocks collapsed. The Employee Retirement Income Securities Act, widely known as 'ERISA', which introduced the possibility of legal action by pension fund members against trustees and fund managers in connection with their management of the investments, has added to the pressures. Nevertheless, it is possible these fund managers may again have fallen victim of the desire to follow fashion. In the 1960s they were content to buy stocks at prices of 60, 80 or even 100 times earnings. When these stocks crashed they sought the protection of a theory which could be quoted to their boards of trustees. Indexed funds (which are dealt with in the next section) also became popular about the same time.

In the U.K. there has been more scepticism of the practical value of the theory. It is generally regarded as a useful addition to the investment manager’s armoury but by no means his sole weapon. One reason for the different attitude on this side of the Atlantic is that the pressure to perform, or more correctly, the pressure not to under-perform is less overwhelming. Also the potential benefits of using the technique in managing a portfolio depend on the efficiency of the market and minimal dealing costs. In both these respects the U.K. stock market provides a less favourable environment for M.P.T. than that in the U.S.

3.3 Indexed Funds

The need to avoid under-performance against a chosen yardstick, usually a published index, has given rise to the concept of an indexed portfolio—one which actually contains all the index constituents in the appropriate proportions or a sufficiently close approximation thereto to ensure that investment performance is virtually the same. Indexing should cut costs. There is no need to maintain an expensive investment team to analyse companies and recommend investments if the allocation of new money can be made automatically to maintain the indexed posture of the fund. Indexing removes the danger of under-performance but it also rules out the possibility of out-performance.

The use of an indexed fund is the harassed fund manager’s answer to fear of under-performance. That statement could be read as a complete criticism of a defeatist attitude and while the author would not recommend the use of total indexing in the sense of mirroring the chosen market index, the concept can also be used constructively. If a fund manager lacks the resources to follow and select individual companies his time is best spent in analysing sectors. When he has decided, for instance, that the banking sector is attractive, he can adopt an overweight position in banks and select shares according to their weighting in the index.

Alternatively, if his fund is of medium to large size the fund manager may decide to treat a certain part of the portfolio say, for example, 75% by value, as a core which will not be dealt in. This core portion can then be indexed leaving the
remaining 25% free to be actively managed. Partial indexing in this way will mean that the actual investment performance will only deviate from that of the index to a limited degree.

As with other aspects of M.P.T. indexed funds have never become as popular in the U.K. as in the U.S. as far as is known.

3.4 U.K. Attitudes to M.P.T.

Many investment managers of U.K. life offices have shown some interest in the theoretical background of M.P.T. Some of them would probably feel that M.P.T. has some value in adding to their knowledge of the risk in equity portfolios and that it can be a tool to assist in selecting shares. In the latter respect it can only be one factor in the decision-making process. An analogy can be drawn with the use of charts which many fund managers consult without becoming dedicated chart analysts. This analogy can also be taken a step further by suggesting that the validity of M.P.T. rests on the continuation of historical trends (particularly with regard to the degree of risk) in the same way as charts do. There are more fundamental reasons for scepticism and these are discussed in the next section.

3.5 The Segmented Structure of Life Assurance Funds

The segmented nature of life assurance funds is well known. The difficulties of applying M.P.T. to such funds is evident by examining the assumptions underlying the work of Treynor, Sharpe, Lintner, Mossin and Fama.

It is a long cry from the assumptions of the C.A.P.M. noted in paragraph 2.4 to the circumstances of an insurance company. Whilst with certain caveats such an investor might believe in the use of M.P.T. for managing his equity portfolio there is clearly difficulty in determining action for special classes of asset, for example preference shares, unquoted securities of all types, mortgages. If the wider market suggested by Roll (7) existed under the assumptions it would clearly be prudent to spend time on the theory. A great deal of success has been achieved in gilt research which together with the use of M.P.T. would enable some 50% of life assurance assets to be examined in some detail. Management of the remainder, including mortgages and property, is by and large an unresolved problem and an overall, coherent theory for determining the total strategy for a life assurance company does not yet exist. There are several reasons for this.

The first is in regard to the liabilities. An investor which is a ‘single period expected utility of terminal wealth maximizer’ is rather remote from a multi-period investor which has to give thought to its emerging liabilities under life assurance contracts. Actuaries traditionally match liabilities and assets using a technique which in its most perfect form ‘immunizes’ the fund from profit or loss. In an office with a heavy preponderance of with-profit contracts rigid adherence to ‘immunization’ is not observed but there are other constraints. The
marketing of life assurance policies in the U.K. relies heavily upon the tradition of stable reversionary bonuses and the trade-off between reversionary bonuses and terminal bonuses is quite important. These considerations do, however, confuse the issue from the point of view of maximizing investment return.

Taxation is a feature of life assurance investment which cannot be ignored whether in relation to the treatment of expenses; the role of unfranked versus franked income; the preponderance of gross or net funds and so on.

Mortgages are a particular problem. In a sense they are totally unmarketable because a one-way option against the office usually exists. Moreover, although the rate of interest at the time of effecting the loan may be close to the ‘risk-free rate’ there is a rather unusual variation to the concept of total return. Most mortgages are guaranteed by life assurance contracts which in themselves may produce a surplus or deficit as a result of variance from the assumptions made in the premium basis for mortality, interest, inflation and expenses. The question arises of whether the total return for mortgages should include any surplus emerging from associated contracts, because the two are closely linked in the office’s mind.

Overseas investment helps to confuse the issue further. Wide unhedged overseas investment creates a well-diversified portfolio, but the riskiest aspect of overseas investment tends to be the fluctuation in currencies. Why then invest in anything other than an overseas ‘risk-free’ security? Investment in sectors and industries not well represented domestically (e.g. oil and energy stocks) in themselves create a badly diversified overseas portfolio unless the investment is hedged and can be regarded as part of the domestic portfolio.

Mayers in his paper (2) on non-marketable assets presents a single-period model of capital-asset pricing which includes the effects of non-marketable assets under conditions of uncertainty. Unlike the traditional model his expanded model implies that not all maximizing investors hold the identical (except for scale) portfolio of marketable assets. It implies that each investor holds a portfolio of marketable assets that solves his personal (and possibly unique) portfolio problems.

An article in the Institutional Investor entitled “Is Beta Dead” was written following publication of certain articles by Roll. The essence of Roll’s argument was that the two-parameter asset pricing theory was untestable. He concluded that, inter alia, the theory is not testable unless the exact composition of the true market portfolio is known and used in the tests. This implies that the theory is not testable unless all individual assets are included in the sample.

Thus the ‘M.P.T. risk’ of an insurance company’s stratified portfolio is at first sight difficult to obtain because several (if not all) of the assumptions in the C.A.P.M. are breached. M.P.T. is a generic term including the market model, the capital asset pricing model and the efficient markets hypothesis. Whilst Roll’s critique casts doubt on the use of the C.A.P.M. in segmented markets the market model may still be usable. Certainly M.P.T. could still be used for the equity content of portfolios. The problems are intriguing and deserve further explor-
In particular, the use of the market model, as distinct from the C.A.P.M., for other segments of the market needs to be investigated.

A recent book by Dodds (11) provides a great deal of insight into the behaviour of insurance companies. The life company is characterized as having a desired or optimum balance sheet determined by the preference of the investment manager, the product mix of the business, expenses of the office and the nature of the assets themselves with respect to their yield and other attributes. Adjustment towards the optimum balance sheet is achieved by direction of net new funds and by reviewing existing holdings. It appears from his study "that relatively sophisticated techniques have been applied to the investment and portfolio management of financial institutions without due regard for the nature of the institution concerned". In looking at life offices he found it virtually impossible to test for a matching or immunization strategy in his model. The models adopted by Dodds focus on strategic choices among asset classes and his work suggests several areas for future research.

4. THE FUTURE

M.P.T. in the sense of applying mathematical and statistical concepts to the processes of investment management is clearly here to stay. It is one of the ways in which what was once regarded as an art shrouded in mystique is becoming more and more scientific.

It has already been mentioned that some of the work of the M.P.T. school has been questioned. Is any market efficient in the sense that the theory requires? Is beta meaningless if its value for any stock depends not only on the characteristics of the stock itself but also on the index against which the stock is measured? These doubts do not necessarily invalidate the theory as an academic exercise but they may affect the degree of reliance that one can accord to its application to the practical problems of investment.

The extent to which an individual fund manager will rely on any particular aspect of M.P.T. will be partly a matter of personal judgement, but some knowledge of the techniques, and their limitations, should be an essential part of the professional expertise he possesses. If as I suspect the issues are not as widely appreciated as they should be then there is an educational gap which should be filled by airing the topic further at professional meetings and seminars. To some extent this has already happened. M.P.T. should be incorporated in the syllabus of any serious course on investment. In particular, as an actuary involved in the investment of life office funds the author would like to see the investment sections of the actuarial examinations including some study of M.P.T.

ACKNOWLEDGEMENT

REFERENCES