A MATHEMATICAL MODEL FOR THE GILT-EDGED MARKET

by

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1. Introduction

1.1. The market in British Government stocks in many ways resembles what might be called a perfect market. In particular, very large amounts can be dealt in, dealing expenses are low, powerful statistical techniques are used to identify price anomalies, and frequent switching operations take place to exploit these anomalies. Given these features, prices of stocks will vary in a highly regular manner, and it should be possible, by the application of graduation methods, to represent the price structure in mathematical terms. If such a descriptive framework can be developed solely from the principle that prices are in equilibrium under the switching action of all participants in the market, it will provide the basis for a mathematical model that could be used for many functions of portfolio management.

1.2. This paper investigates what general form of price model is required to represent a market that is in equilibrium under switching action and describes how a special case of this general price model can be applied to the gilt-edged market, with particular emphasis on long-dated stocks. Applications of the mathematical model are then developed in a form that could be used by a life office or pension fund for the management of a portfolio of long-dated British Government stocks. Finally, a comparison is made between conventional yield curve methods of analysis and methods based on this mathematical model. The paper therefore falls into four parts:

Part I. Theoretical derivation of a general price model.
Part II. Detailed price model for long-dated stocks.
Part III. Applications.
Part IV. Comparison with yield curve methods.
2. Outline of Approach

2.1. The opening paragraph of the paper refers to the application of graduation techniques. The most common actuarial use of graduation is the derivation from observed data of a smooth series for $q_x$, the rate of mortality at age $x$, as the basis of a life table. From data involving actual deaths and exposed to risk statistics for various age ranges, a series is obtained which represents the most satisfactory fit of a particular class of function chosen as being a suitable portrayal of the main properties of $q_x$. Any such representation of observed data by a smooth mathematical function must strike a balance between too slavish a following of the observed data, which can allow random fluctuations to distort the general pattern, and too sweeping an approximation, which can result in salient features of the underlying data being suppressed.

2.2. There are obvious parallels between the life table application and the fitting of a mathematical model to prices of gilt-edged stocks. In the graduation of a life table, observed data, often involving quinquennial groups, are replaced by a smooth mathematical function. The experimental data are therefore embedded in a more complete structure which then gives values at any age as required for practical applications. In the case of the gilt-edged market, the aim is to embed actual stocks in a mathematical structure in such a way that this structure can be used both for descriptive purposes in terms of general market features and as a frame of reference which will identify whether the prices of individual stocks appear too high or too low in the context of the whole market. Again, a balance must be struck in the graduation process. Too flexible a fit may result in an unstable price model which has no predictive properties as regards the relative cheapness or dearness of individual stocks, whereas the use of arbitrary mathematical functions in any part of the price model may result in certain important features of the market not being reflected in the price model. The construction of a general price model is therefore equivalent to identifying what class of function must be used for the graduation process to achieve this correct balance. Since the two most important attributes affecting the price of a stock are coupon and outstanding term to redemption (hereafter referred to simply as "term"), the problem is two-dimensional in contrast to the one-dimensional case of $q_x$, and the choice of the appropriate class of function is accordingly
very much more difficult. The remainder of Part I deals only with the question of the required class of function, as the stability element cannot be tackled until the general graduation method has been chosen.

2.3. We require in the first instance to derive the most general model possible. Any graduation function which implies properties other than those which can be deduced from the hypothesis that the market is in equilibrium under switching action is therefore inadmissible in the present context. The application of this hypothesis requires great care, since any erroneous assumptions regarding investor behaviour will lead to loss of generality. As will be seen later, it is fortunately not necessary to obtain a comprehensive definition of admissible investor behaviour. The crux of the problem is to use the weakest possible assumptions regarding investor behaviour as are necessary to determine a class of graduation function which is sufficiently explicit to be of practical value as a price model, and then to justify that these assumptions do not impair the generality of the price model. It may, of course, be the case that stronger assumptions have to be made before an adequate price model can be derived, in which event a general price model cannot be constructed solely from the hypothesis that the market is in equilibrium under switching action.

2.4. This completes the preliminary discussion of the principal features of a general price model. In the next section it is shown that certain very weak assumptions regarding investor behaviour lead to conditions that are both necessary and sufficient for prices to be in equilibrium under switching action.

3. Mathematical Formulation

3.1. In order to avoid for the moment as many of the practical complications as possible, we assume that the gilt-edged market consists entirely of homogeneous dated stocks which differ on account of only two attributes, the outstanding term to maturity \( n \) and the coupon \( g \). All stocks are assumed to be redeemable at 1 on a fixed redemption date, and dividends are assumed to be paid continuously. The price of a general stock is defined as

\[
P = P(n, g)
\]

where \( P \) is a function which can take any positive value, rational or irrational. Dealing expenses are assumed to be zero.

3.2. If two stocks differ only very slightly in coupon and term, we expect their prices also to differ only very slightly, as otherwise
a switching opportunity would exist between two stocks which are, for all practical purposes, identical. This argument can be made more precise to show that if \( P \) represents a price structure which is in equilibrium under switching action then \( P \) is a continuous function of the two variables, term and coupon. For convenience in later work, we assume that \( P \) is also smooth in the sense that partial derivatives of all orders exist.

3.3. The above assumptions are made purely to facilitate the mathematical development, and can all be relaxed later where necessary to reflect observed market features. One further important assumption made in deriving necessary conditions for a price structure to be in equilibrium under switching action is that the market contains an infinite number of stocks throughout the ranges of term and coupon. Since this assumption is not in accord with the very sparse nature of the gilt-edged market as regards distribution of stocks by term and coupon, this point is discussed in Part II when dealing with special cases of the general price model.

3.4. In 2.3 it was explained why we wish to use the weakest possible assumptions regarding investor behaviour as are necessary to specify a practical mathematical model. The difficulty in attempting any formal description of investor behaviour as regards switching is that different investors use widely different criteria for assessing the relative cheapness or dearness of individual stocks. To circumvent this difficulty, we return to first principles and examine the relative attractiveness of stocks in terms of the entitlements that result from ownership of unit value of each stock.

3.5. For unit investment in a stock, the only three attributes that affect its relative attractiveness are:

(i) the term, \( n \)
(ii) the capital amount on maturity, \( C \)
and (iii) the annual income, \( i \).

Each investor has his own decision rules for determining the relative attractiveness of each stock in the light of these three attributes. We therefore postulate the existence, for each investor, \( Z \), of a function

\[
M^Z = M^Z (n, C, i)
\]

which defines the measure of attractiveness that he places on each stock, with a higher value of \( M^Z \) corresponding to the stock being more attractive. If we have two stocks with the same term and the same income per unit investment but different capital amounts
on maturity per unit investment, all investors will place a higher measure of attractiveness on the stock giving the higher capital amount on maturity, except possibly in extreme cases where tax complications are involved. A similar argument applies where two stocks differ only on account of the income per unit investment. We can therefore in general assume that $M^z$ is strictly increasing with respect to both $C$ and $i$.

3.6. For any fixed value of $n$, the main properties of $M^z$ can be illustrated in a capital-income diagram as shown in Figure 1 below. The function $M^z$ assigns a value to each point $X$ in the capital-income diagram, and, by joining up all points which are assigned the same value, smooth contours are obtained. Points $X_1$, $X_2$ and $X_3$ have the same value of $C$ and increasing values of $i$, while points $X_1$, $X_4$ and $X_5$ have the same value of $i$ and increasing values of $C$. Although the diagram shows what measure of attractiveness the investor assigns at a particular time to every combination of capital and income purchased by an investment of unit amount, it is not necessarily the case that each combination $X$ corresponds to a stock which forms part of the price structure at that time.

![Figure 1](image)

**Fig. 1.** Measure of attractiveness $M^z(n, C, i)$ for fixed $n$

3.7. A similar diagram can be produced to represent the same investor’s assessments for any other fixed value of term, and in general a different pattern of contours will result. In line with the requirement to use the weakest possible assumptions regarding investor behaviour, we make no attempt to specify the sensitivity with respect to term of either the location or gradient of these contours. This in effect restricts our definition of switching action to operations involving only stocks of the same term.

3.8. Consider now the case of any two stocks $S_1$ and $S_2$ of the same term, where the gradient of the straight line joining $S_1$ and $S_2$ in the
A Mathematical Model for capital-income diagram is negative. In the absence of further assumptions regarding investor behaviour, it is possible that there may exist two investors $Z_1$ and $Z_2$ such that one considers $S_1$ to be more attractive than $S_2$ while the other considers $S_2$ to be more attractive than $S_1$. The position is illustrated in Figure 2 below. In such circumstances, it cannot be proved that the prices of stocks $S_1$ and $S_2$ are not in equilibrium under switching action.

![Figure 2. Relative attractiveness of stocks $S_1$ and $S_2$](image)

3.9. If the gradient of $S_1$ $S_2$ in the capital-income diagram is not negative, every investor will agree about which stock is the more attractive and holders of the less attractive stock will switch out of that stock into the other until the anomaly is removed. In this case, the prices of stocks $S_1$ and $S_2$ are not in equilibrium under switching action.

3.10. The position regarding price equilibrium of any two stocks of the same term is summarised in Figure 3 below. If stock $S_2$ lies on VW or XY or in quadrants WS$_1$Y or VS$_1$X, the prices of stocks $S_1$ and $S_2$ cannot be in equilibrium under switching action.

![Figure 3. Price equilibrium of stocks $S_1$ and $S_2$](image)
If stock S$_2$ lies in quadrants XS$_1$W or YS$_1$V, it cannot be proved that the prices of stocks S$_1$ and S$_2$ are not in equilibrium under switching action.

3.11. For any given term, it cannot without loss of generality be assumed that an investor holds more than one stock of that term, and hence in generalising the above arguments to multiple switches we need only consider the case of unit value of any stock being sold and reinvested in any combination of stocks of the same term as that of the stock sold. If it is found that sufficient investors hold more than one stock at any particular term, it is likely that prices will vary in a manner consistent with this behaviour. Stronger results could then be derived from considerations of investor behaviour, and the resultant general price model would be a very much simpler special case of the model derived below.

3.12. We now assemble these arguments to give the following definition of price equilibrium under switching action:

**Definition 1.** Prices are in equilibrium under switching action if and only if no switch exists from any one stock into any combination of stocks of the same term as that stock which results in:

(i) a higher capital amount at maturity and maintained income,

or (ii) higher income and a maintained capital amount at maturity,

or (iii) higher income and a higher capital amount at maturity.

3.13. From Definition 1 we deduce the following theorem:

**Theorem 1.** A smooth positive function $P$ of the two independent variables, term $n$ and coupon $g$, represents a price structure which is in equilibrium under switching action if and only if:

$$\frac{\partial}{\partial i} \frac{1}{P} < 0$$

and

$$\frac{\partial^2}{\partial i^2} \frac{1}{P} \leq 0$$

(3.1)

where

$$i = \frac{g}{P}.$$

The proof is set out in detail in the Appendix.

3.14. In the next section, the general price model is obtained as the most general class of function satisfying conditions 3.1 of Theorem 1.
4. **General Price Model**

4.1. The conditions 3.1 of Theorem 1 can be represented in the capital-term diagram as shown in Figure 4 below. Consider any price structure which is in equilibrium under switching action. By Lemma 3 of the proof of Theorem 1, \( \frac{1}{p} \) is a smooth function of the two variables, term and running yield. The locus of all stocks with a given value of running yield is therefore a smooth contour in the capital-term diagram. For any fixed value \( i_0 \) of running yield and any small positive quantity \( \Delta i \), the contour for \( i_0 - \Delta i \) lies above the contour for \( i_0 \) at all values of term since

\[
\frac{\partial}{\partial i} \frac{1}{p} < 0,
\]

and similarly the contour for \( i_0 \) lies above the contour for \( i_0 + \Delta i \).

Also, since

\[
\frac{\partial^2}{\partial i^2} \frac{1}{p} \leq 0,
\]

the vertical distance between the contours for \( i - \Delta i \) and \( i_0 \) is, for all values of term, less than or equal to the corresponding value for the contours for \( i_0 \) and \( i_0 + \Delta i \).

![Fig. 4. Constant running yield contours.](image-url)
4.2. Consider now a general stock in Figure 4 with term \( n \) and coupon \( g \). Then

\[
\frac{1}{P} = AB + BC
\]

\[
= AB + \int_{i_0}^{g} \frac{\partial}{\partial i} \frac{1}{P} \, di.
\]

Let

\[ h(n) = AB - 1 \]

and

\[ f(n) = -\left[ \frac{\partial}{\partial i} \frac{1}{P} \right]_{i=i_0} \]

By Theorem 1, \(-\frac{\partial}{\partial i} \frac{1}{P}\) is positive and is an increasing function of \( i \). Hence there exists a positive smooth function \( \lambda(n, t) \) such that

\[
\frac{\partial}{\partial i} \frac{1}{P} = -f(n) \lambda(n, i_0 - i)
\]

where

\[ \lambda(n, 0) = 1 \]

and

\[ \frac{\partial}{\partial t} \lambda(n, t) \leq 0. \]

Since \( \frac{1}{P} \) is a smooth function of term and running yield, both \( h(n) \) and \( f(n) \) are continuously differentiable.

4.3. This completes the derivation of the general price model, and the result is stated as a theorem.

**Theorem 2.** The smooth positive function \( P \) of the two independent variables, term \( n \) and coupon \( g \), represents a price structure which is in equilibrium under switching action if and only if

\[
\frac{1}{P} = 1 + h(n) + f(n) \int_{i_0}^{g} \lambda(n, t) \, dt \quad (4.1)
\]

where

(a) \( i_0 \) is a positive constant,

(b) \( h(n) \) is a continuously differentiable function of \( n \),

(c) \( f(n) \) is a positive continuously differentiable function of \( n \),

(d) \( \lambda(n, t) \) is a positive smooth function of \( n \) and \( t \) with:

(i) \( \lambda(n, 0) = 1 \)

and (ii) \( \frac{\partial}{\partial t} \lambda(n, t) \leq 0 \quad (4.2) \)
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Proof. If \( P \) is such that the integral equation 4.1 and conditions 4.2 are satisfied, it is easily verified that conditions 3.1 are satisfied. Hence, by Theorem 1, the price structure is in equilibrium under switching action. The above derivation of equation 4.1 and conditions 4.2 provides the proof in the other direction.

PART II: DETAILED PRICE MODEL FOR LONG-DATED STOCKS

5. Preliminary Considerations

5.1. The application of the general mathematical model derived in Part I to any given set of price data is equivalent to using equation 4.1 as a graduation formula. This involves three distinct stages:

(a) The graduation functions \( i_0, h(n), f(n) \) and \( \lambda(n, t) \) have to be defined.

(b) An optimisation criterion is required to determine which special case of the graduation formula gives the best fit.

(c) In view of the complicated nature of the graduation formula, it has to be demonstrated that a convenient method of solution exists.

Before even the first of these stages can be tackled, certain practical features of the market (accrued interest and optional redemption dates) have to be accommodated.

5.2. In 3.1 it is assumed that interest is paid continuously, whereas for long-dated stocks interest payments are made twice yearly and quoted prices therefore include an element of accrued interest. Comparisons at various dates of simultaneous "specially ex dividend" and "cum dividend" prices for long-dated stocks show that the difference is fairly stable at approximately 0.9 times the relevant interest payment. We therefore take the "true" price, \( P \), as the quoted middle market price less 0.9 times the gross accrued interest. This definition of \( P \) removes the increasing trend in price due to interest accrual and in particular ensures that no significant discontinuity occurs when a stock is first quoted in the ex dividend form.

5.3. A further assumption in 3.1 is that each dated stock has a fixed redemption date. There are, however, certain stocks which can be redeemed, at the option of the Government, at any time between two specified dates. The market convention in this regard is to assume that redemption will take place at the earliest or latest possible date depending on whether the present price, less gross
accrued interest, is above or below par respectively. For the moment we adopt a similar convention, but using 0.9 times the gross accrued interest.

5.4. Stocks with special features such as a variable interest rate, a conversion option or a sinking fund, are excluded from consideration, since they clearly cannot be expected to fit readily into any simple price pattern.

6. Graduation Method

6.1. It is explained in 2.2. that a balance must be struck between goodness of fit and stability of variable parameters. Accordingly, the graduation method must be such that the number of variable parameters is the smallest that is consistent with the satisfactory representation of the principal features of price structure.

6.2. The general price model described by equation 4.1 and conditions 4.2 provides a highly satisfactory graduation technique in that the four arbitrary functions represent the following distinct features in the capital-term diagram:

\[ i_0 \] — an arbitrary value of running yield;
\[ h(n) \] — the contour of \( \frac{1}{P} \) for running yield \( i_0 \);
\[ f(n) \] — the variation with respect to running yield of the vertical spacing of contours at \( i_0 \);
\[ \lambda(n, t) \] — the ratio of the vertical spacing of contours at \( i_0 - t \) to that at \( i_0 \).

It is therefore possible to isolate the effect of each component function and to determine the appropriate number of variable parameters in each case to ensure both stability and adequate goodness of fit.

6.3. The complexity of the general problem of defining these four component functions is illustrated by Chart 1, which shows the scatter diagram obtained by plotting running yield against \( \frac{1}{P} \) and term at 31st December 1970 for dated stocks of term 15 years and over. The fitting of the general price model is equivalent to the determination of the pattern of contours of \( \frac{1}{P} \) for fixed values of running yield. Simple inspection shows that for values of term between 15 and 30 years the contours for running yields up to 9.6% have positive gradient, with the gradient decreasing as running yield
6.4. For certain choices of \( i_0 \), \( h(n) \) is an increasing function of \( n \). Since \( f(n) \) is an increasing function of \( n \), it will improve the stability of the price model if \( i_0 \) can be defined in such a way that \( h(n) \) is small in absolute value for all values of term. Inspection of price data at various past dates shows that a stock tends to stand at a price close to par if its coupon is approximately equal to the yield on irredeemables, and accordingly this yield is chosen for \( i_0 \). This definition has the further advantage that irredeemable stocks are incorporated in the price model as the limiting case for large values of term.

6.5. The definition of the yield on irredeemables (hereafter denoted by \( I \)) must be such that potential sources of systematic bias are eliminated. Since different investors have widely differing tax positions, price distortion is most likely to occur around ex dividend periods. Accordingly, the yield on irredeemables is defined as the average running yield, after adjustment for accrued interest as described in 5.3, of Consols 4\%, Treasury 2\( \frac{1}{2} \)%, and War Loan 3\( \frac{1}{2} \)%.

The dividend dates of these stocks are the first days of February and August, April and October, and June and December respectively, so that any distortion from the incidence of ex dividend periods is minimised by this even distribution over the year. These stocks are also the three largest irredeemable issues in terms of market capitalisation.

6.6. The value of \( I \) at 31st December 1970 is 9.85\%, and from Chart 1 it can be seen that, as a first approximation, \( h(n) \) may be taken as zero for all values of \( n \) above 15 years. Considering only stocks where

\[
I - i > 0.004,
\]

this approximation gives the values for the ratio

\[
\frac{1}{P} - \frac{h(n)}{I - i} - 1
\]

shown in Table 1 as a function of \( n \) and \( I - i \):

These results and similar calculations at almost all dates before December 1976 demonstrate that this ratio has the general properties shown in Figure 5 below. A smooth curve, with increasing gradient and passing through the origin, can be found such that, for each
neighbourhood of term, the distance that a stock lies below the curve increases with $I-i$.

![Fig. 5](image)

Since

$$\frac{1}{P} = 1 + h(n) + f(n)\int_0^{I-i} \lambda(n, t)dt,$$

$$\frac{1}{P} - h(n) - 1 = \frac{f(n)\lambda(n, t_0)}{I-i}$$

for some $t_0$ with $0 < t_0 < I-i$, and

$$f(n) = \lim_{1-i \to 0} \frac{1}{P} \frac{h(n) - 1}{I-i}.$$

Hence the curve in Figure 5 represents $f(n)$, and at 31st December 1970

$$\lambda(n, t) < 1$$

for positive values of $t$.

6.7. A suitable definition for $f(n)$ is therefore

$$f(n) = b_0 n + b_1 n^2.$$
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Since both the gradient and the convexity of the curve in Figure 5 must be allowed to vary freely, any definition of \( f(n) \) involving only one variable parameter is unlikely to be satisfactory in terms of goodness of fit.

6.8. If the vertical spacing of contours is allowed to vary in a complicated manner, the price structure will be too flexible to be of use in comparisons of the cheapness and dearness of individual stocks. It is therefore desirable to have only one variable parameter to specify \( \lambda(n, t) \) for a particular value of term, and a convenient basis is to assume that successive vertical distances between contours at incremental values of running yield are in geometric progression. This gives

\[
\lambda(n, t) = e^{-c_n t}
\]

where \( c_n \) is a function of \( n \). If it is assumed that \( c_n \) is constant for all values of \( n \), only one parameter is required to define \( \lambda(n, t) \). There is, however, no reason to expect that such a rigid definition will give a satisfactory description of the price structure, and accordingly a second parameter is introduced by assuming linear variation. Hence

\[
\lambda(n, t) = e^{-(c_0 + c_1 n)t}.
\]

6.9. The conditions 4.2 require that

\[
c_0 + c_1 n \geq 0.
\]

This result is derived from the assumption that investors have a very large number of stocks to choose from, whereas the actual distribution of stocks is exceptionally sparse. Distortion from the highly regular price structure of the model could result from the absence in particular maturity areas of either a high coupon stock or a low coupon stock. Also, the model assumes that no switching opportunities exist between stocks of similar term. It is possible that this assumption might be invalid in certain circumstances, in which case the best fit would correspond to

\[
c_0 + c_1 n < 0.
\]

To accommodate these possibilities, no restriction is placed on the value of \( c_0 + c_1 n \).

6.10. From inspection of the general behaviour of \( h(n) \) at various dates, it is found that at certain dates both positive and negative values exist for different values of term and that either a maximum or a minimum turning point can occur. In view of these features, it is unlikely that a definition involving two or less variable para-
meters will be satisfactory. A suitable definition involving three variable parameters is

\[ h(n) = a_0 + \frac{a_1}{n} + \frac{a_2}{n^2}. \]

This gives either one or no turning points, and a limit of \( a_0 \) as \( n \) tends to infinity.

6.11. The special model for long-dated stocks is therefore

\[
\frac{1}{P} = 1 + a_0 + \frac{a_1}{n} + \frac{a_2}{n^2} + (b_0n + b_1n^2) \left[ 1 - \exp \left\{ -(c_0 + c_1n) \left( 1 - \frac{g}{P} \right) \right\} \right]^{-1} \tag{6.1}
\]

where \( I \) is the yield on irredeemables and \( a_0, a_1, a_2, b_0, b_1, c_0 \) and \( c_1 \) are variable parameters.

6.12. For short-dated or medium-dated stocks, a suitable price model is equation 6.1 with the definition of \( h(n) \) changed to

\[ h(n) = a_0 + a_1n + a_2n^2. \]

7. Optimisation Criterion and Method of Solution

7.1. Since it is clearly the proportionate price error rather than the absolute price error that is relevant, the error contribution for each stock is

\[ e_r = \frac{P_r - \hat{P}_r}{P_r} - 1 \]

where \( P_r \) is the true price defined in 5.2 and \( \hat{P}_r \) is the price given by the graduation formula.

The two most obvious choices of weights are to have equal weights for all stocks or to have weights proportional to the market capitalisation of each stock. As discussed in 2.2, we wish the mathematical model to provide a frame of reference which will identify whether the prices of individual stocks appear high or low in the context of the whole market. If weighting by market capitalisation is used, stocks with a large market capitalisation will be dominant in the fitting of the graduation formula, with the result that these stocks will, on average, deviate less from the graduated price structure than will stocks with a small market capitalisation. This would result in a serious bias whereby price anomalies of stocks with a large market capitalisation would be suppressed and stocks with a small market capitalisation would appear to be very much more volatile in terms of short-term price movements relative to the market. For descriptive purposes, on the other hand, the details
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of the graduation process are of secondary importance to considerations of consistency. We therefore use equal weights for all stocks, so that the function to be minimised is

$$E = \sum_{r=1}^{m} \left( \frac{P_r}{\bar{P}_r} - 1 \right)^2.$$  

7.2. Since the graduation formula is a transcendental equation in $\frac{1}{P}$, it is appropriate to show in detail that convenient methods exist for obtaining the values of the variable parameters and for solving for $\hat{P}$. We show first that equation 6.1 can be solved by an iterative process for given values of the variable parameters.

7.3. Equation 6.1 can be rearranged as

$$\hat{P} = \frac{1 + gf(n) Y(\hat{P})}{1 + h(n) + If(n) Y(\hat{P})}$$  \hspace{1cm} (7.1)

where

$$f(n) = b_0 n + b_1 n^2$$

$$h(n) = a_0 + \frac{a_1}{n} + \frac{a_2}{n^2}$$

and

$$Y(\hat{P}) = \frac{1 - \exp\left\{- (c_0 + c_1 n) \left( 1 - \frac{g}{\hat{P}} \right) \right\}}{(c_0 + c_1 n) \left( 1 - \frac{g}{\hat{P}} \right)}.$$  

Then the sequence $P_{[t]}$ defined by

$$P_{[t]} = \frac{1 + gf(n)}{1 + h(n) + If(n)},$$

$$P_{[t+1]} = \frac{1 + gf(n) Y(P_{[t]})}{1 + h(n) + If(n) Y(P_{[t]})},$$

converges geometrically to $\hat{P}$. In practice it is found that convergence is very rapid.

7.4. From equation 6.1, we have

$$E = \sum_{r=1}^{m} P_r^2 \left( \frac{1}{\hat{P}_r} - \frac{1}{P_r} \right)^2$$

$$= \sum_{r=1}^{m} P_r^2 \left[ 1 + a_0 + \frac{a_1}{n_r} + \frac{a_2}{n_r^2} + (b_0 n_r + b_1 n_r^2) X_r - \frac{1}{P_r} \right]^2$$

where

$$X_r = Y(\hat{P}_r) \left( 1 - \frac{g}{\hat{P}_r} \right).$$
For particular values of \( c_0 \) and \( c_1 \), we first obtain the values of \( a_0, a_1, a_2, b_0 \) and \( b_1 \) which minimise \( E \), and denote this minimum by \( E(c_0, c_1) \). Then we show that \( E(c_0, c_1) \), regarded as a function of the two variables \( c_0 \) and \( c_1 \), has a unique minimum which corresponds to the required solution.

7.5. Consider \( c_0 \) and \( c_1 \) as constants. Since the relative error \( \left( \frac{P_r}{\hat{P}_r} - 1 \right) \) is expected to be small, it is clear from the power series expansion of \( Y(P_r) \) that \( Y(P_r) \) provides a very good approximation to \( Y(\hat{P}_r) \).

Also, from equation 7.1, we note that \( \hat{P}_r \) is relatively insensitive to changes in \( Y(\hat{P}_r) \). Thus if we minimise

\[
E^1 = \sum_{r=1}^{m} P_r^2 \left[ 1 + a_0 + \frac{a_1}{n_r} + \frac{a_2}{n_r^2} + (b_0 n_r + b_1 n_r^2) X_r^1 - \frac{1}{\hat{P}_r} \right]^2
\]

with

\[
X_r^1 = Y(P_r) \left( 1 - \frac{g}{\hat{P}_r} \right),
\]

the resulting values of the parameters \( (a_0^{[1]}, a_1^{[1]}, a_2^{[1]}, b_0^{[1]} \text{ and } b_1^{[1]} \), say) will be very close to the values which minimise \( E \). Since the partial derivatives of \( E^1 \) with respect to each of the five variable parameters are zero, \( a_0^{[1]} \) etc. can be calculated directly by inverting the symmetric \( 5 \times 5 \) matrix obtained by equating each of these partial derivatives to zero. Using the iterative process of 7.3 we substitute \( a_0^{[1]} \) etc. in equation 6.1 to give \( \hat{P}_r^{[1]} \), a first approximation to \( \hat{P}_r \).

Various iterative methods can be constructed to obtain successively more accurate values for each of the five parameters. For most practical purposes, however, the first approximations \( a_0^{[1]} \), etc., can be taken as the values which minimise \( E \).

7.6. It only remains to demonstrate how to find the values of \( c_0 \) and \( c_1 \) which minimise \( E(c_0, c_1) \). Fitting the mathematical model is equivalent to finding, for successive increments of running yield, a series of contours in the capital-term diagram where the vertical distances between successive contours are in geometric progression. An analogous but much simpler problem is finding a geometric series to represent the increasing sequence

\[
z_1, z_2, z_3, \ldots, z_m.
\]

If we put

\[
z_1 = z + a
\]
\[
z_2 = z + a + ak
\]
\[
z_3 = z + a + ak + ak^2
\]
\[
\vdots
\]
\[
z_m = z + a + ak + ak^2 + \ldots + ak^{m-1}
\]
with $a > 0$ and $k > 0$

then $z$, $a$ and $k$ can be interpreted in a similar way to $h(n)$, $f(n)$ and $\lambda(n, t)$ in the graduation formula. Taking the error function as

$$E = \sum_{r=1}^{m} (z_r - \hat{z}_r)^2,$$

we can show that for each value of $k$ there are unique values of $a_k$ and $z_k$ which give the minimum value of $E$, which we denote by $E(a_k, z_k)$. Moreover, it can be shown that for each value of $k$

$$\frac{\partial^2}{\partial k^2} E(a_k, z_k) > 0,$$

so that there is a unique value of $k$ which minimises $E(a_k, z_k)$ and hence unique values of the three parameters $a$, $z$ and $k$ which minimise $E$.

Returning now to the function $E(c_0, c_1)$, we can extend these arguments to show that

$$\frac{\partial^2}{\partial c_0^2} E(c_0, c_1) > 0$$

and

$$\frac{\partial^2}{\partial c_1^2} E(c_0, c_1) > 0,$$

so that unique values of $c_0$ and $c_1$ exist which minimise $E(c_0, c_1)$ and hence that there exist unique values of the seven parameters $a_0$, $a_1$, $a_2$, $b_0$, $b_1$, $c_0$ and $c_1$ which minimise $E$. Furthermore, it is obvious from these convexity properties that $c_0$ and $c_1$ can be found by an elementary search procedure. A convenient method is to put

$$c_{15} = c_0 + 15c_1$$

and

$$c_{40} = c_0 + 40c_1,$$

so that

$$c_0 + c_1 n = \frac{40 - n}{25} c_{15} + \frac{n - 15}{25} c_{40},$$

and to restrict both $c_{15}$ and $c_{40}$ to be multiples of some suitable small quantity.

8. Numerical Results

8.1. Table 2 shows the results at various dates of using equation 6.1 as a price model for dated stocks of term 15 years or over. To illustrate the improvement in fit from allowing non-zero values of
Table 2
31st December

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Stocks</td>
<td>12</td>
<td>12</td>
<td>9</td>
<td>12</td>
<td>15</td>
<td>14</td>
<td>19</td>
</tr>
<tr>
<td>Minimum coupon %</td>
<td>2.5</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>Maximum coupon %</td>
<td>3.5</td>
<td>5.5</td>
<td>6</td>
<td>9</td>
<td>9</td>
<td>12.75</td>
<td>15.5</td>
</tr>
<tr>
<td>Yield on irredeemables %</td>
<td>3.73</td>
<td>5.96</td>
<td>6.60</td>
<td>9.85</td>
<td>9.90</td>
<td>17.63</td>
<td>14.60</td>
</tr>
</tbody>
</table>

\[ a_0 = -0.6194, a_1 = 0.648, a_2 = -7.79, b_0 = 2.435, b_1 = 0.1140, \]
\[ c_{18} = 130, c_{40} = 390 \]

Root mean square

\[ \text{Best fit} = 0.44, 0.43, 0.34, 0.40, 0.37, 0.50, 0.41 \]
\[ \text{price error %} \]
\[ c_{18} = c_{40} = 0 = 0.63, 0.56, 0.39, 1.10, 0.76, 1.35, 0.57 \]
\[ \frac{\partial^2}{\partial t^2} \frac{1}{\bar{P}} \] the root mean square error is shown also for the fit obtained with \( c_{15} = 0 \) and \( c_{40} = 0 \).

8.2. The most interesting feature is that, with the exception of 31st December 1976, the best fit always corresponds to positive values of both \( c_{15} \) and \( c_{40} \). Where the yield on irredeemables is low and the range of coupons is small, the best fit represents only a modest improvement over that for \( c_{15} = 0 \) and \( c_{40} = 0 \). At 1970, 1972 and 1974 the improvement in fit is substantial.

8.3. The graphical interpretations used to determine the component functions in equation 6.1 ensure the stability of the parameters \( a_0, a_1, a_2, b_0 \) and \( b_1 \). The definition of \( \lambda(n, t) \) is, however, highly arbitrary, and careful examination of the stability of \( c_{15} \) and \( c_{40} \) is necessary before the model can be used to identify the cheapness or dearness of individual stocks.

8.4. As expected from the discussion in 7.6, the identification of the minimum value of \( E(c_{15}, c_{40}) \) presents no difficulty. The second difference of the error function is always positive along any straight line in the \( c_{15} - c_{40} \) plane.

8.5. The stability of the values of \( c_{15} \) and \( c_{40} \) over time is demonstrated by Table 3, which shows a summary of the results for dated stocks of 15 years and over at weekly intervals from 8th October 1976 to 30th September 1977.

8.6. The existence of negative values of \( c_{40} \) shows that

\[ \frac{\partial^2}{\partial t^2} \frac{1}{\bar{P}} > 0 \]

for values of term greater than \( \frac{40c_{15} - 15c_{40}}{c_{15} - c_{40}} \).

Either the price structure is such that prices are not in equilibrium under switching action or the functions chosen to represent \( f(n) \), \( h(n) \) and \( \lambda(n, t) \) are unsatisfactory. Extensive tests were carried out using different functions for \( f(n) \) and \( h(n) \) and various transformations of the term axis, but the resulting price structures were little different from those using equation 6.1. The positive values for \( \frac{\partial^2}{\partial t^2} \frac{1}{\bar{P}} \) must therefore be accepted as a genuine feature of the price structure at recent dates. Possible causes of this surprising result are discussed in the following section.

8.7. Table 3 also shows that the goodness of fit varies rapidly over time, with a very low value of root mean square price error being
<table>
<thead>
<tr>
<th>Date</th>
<th>$c_{15}$</th>
<th>$c_{40}$</th>
<th>I (%)</th>
<th>Root mean square price error (%)</th>
</tr>
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<tbody>
<tr>
<td>8:10:76</td>
<td>16</td>
<td>8</td>
<td>15.87</td>
<td>0.311</td>
</tr>
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<td>16</td>
<td>15.76</td>
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<tr>
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<td>16</td>
<td>8</td>
<td>15.75</td>
<td>0.252</td>
</tr>
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<td>29:10:76</td>
<td>16</td>
<td>4</td>
<td>16.00</td>
<td>0.368</td>
</tr>
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<td>4</td>
<td>15.52</td>
<td>0.259</td>
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<td>12</td>
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<td>0.272</td>
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<td>14</td>
<td>8</td>
<td>15.28</td>
<td>0.302</td>
</tr>
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<td>18</td>
<td>15.34</td>
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<td>0</td>
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<td>24:12:76</td>
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<td>0.717</td>
</tr>
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</tr>
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<td>12.84</td>
<td>0.746</td>
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<td>10.93</td>
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</tr>
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<td>30:9:77</td>
<td>7</td>
<td>-104</td>
<td>10.14</td>
<td>0.636</td>
</tr>
</tbody>
</table>
achieved at certain dates. Examination of daily results shows that major changes in the market level tend to result in a worsening of the fit, and that as the market stabilises at the new level the fit gradually improves again. This suggests that equation 6.1 gives a highly satisfactory price model in terms of description and that very short-term price anomalies represent the most important source of discrepancies between actual and graduated prices.

9. Causes of Non-Linearity

9.1. Although a rigorous proof cannot be given without using very much stronger assumptions of investor behaviour than are contained in Definition 1, it can be argued that the differing tax positions of investors in the gilt-edged market will lead to the condition

$$\frac{\partial^2}{\partial t^2} \frac{1}{P} < 0$$

instead of the weaker condition

$$\frac{\partial^2}{\partial t^2} \frac{1}{P} \leq 0.$$

which is derived in Theorem 1.

9.2. An illustration of this argument is given for the special case where, for a fixed value of term, investors assess stocks as cheap or dear and switch accordingly depending on the net redemption yield at their particular rate of income tax. Suppose that there are three stocks with the same term and with running yields which may be classified as low, medium and high. Suppose also that all three stocks have the same net redemption yield at an average market rate of tax. The price structure is then such that

$$\frac{\partial^2}{\partial t^2} \frac{1}{P} = 0.$$

If it is assumed that investors subject to either a higher or a lower tax rate than the market average invest only in the two most attractive stocks on their respective bases, the relative attractiveness of each stock can be summarised as follows:

<table>
<thead>
<tr>
<th>Running yield</th>
<th>Low</th>
<th>Average</th>
<th>High</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low</td>
<td>(Too dear to hold)</td>
<td>Neutral</td>
<td>Cheap</td>
</tr>
<tr>
<td>Medium</td>
<td>Dear</td>
<td>Neutral</td>
<td>Dear</td>
</tr>
<tr>
<td>High</td>
<td>Cheap</td>
<td>Neutral</td>
<td>(Too dear to hold)</td>
</tr>
</tbody>
</table>

The low rate tax payer will therefore sell only the medium running yield stock and buy only the high running yield stock, whereas the
high rate tax payer will sell only the medium running yield stock and buy only the low running yield stock. Such a price structure is unstable, and the medium running yield stock must fall in price relative to both other stocks before equilibrium is restored. These relative price movements result in the price structure satisfying condition 9.1. At the equilibrium position, the medium running yield stock is more attractive to the average tax rate investor than the other two stocks, but switching by this investor into the medium running yield stock does not invalidate the argument.

9.3. The argument can be generalised to cover other more complicated hypotheses of investor behaviour. The essential feature is always that the medium running yield stock is unattractive to both high and low tax rate investors if price is assumed to be a linear function of coupon.

9.4. The expected strict inequality of condition 9.1 is confirmed by the positive values of $c_{15}$ and $c_{40}$ which correspond to the best fit of equation 6.1 for long-dated stocks at almost every date tested up until December 1976.

9.5. The sudden emergence of negative values of $c_{40}$ at December 1976 is extremely interesting, since it implies that the market is not in equilibrium under switching action. During November and December 1976, prices of long-dated stocks rose sharply as a result of restored confidence in the U.K. economic situation. For the first time ever, high coupon stocks rose to prices well above par, and some stockbrokers and institutional investors appeared reluctant to recommend investment in these stocks at prices which would result in a significant capital loss if held till maturity. Accordingly, reluctance to suffer a capital loss, or the expectation that others would take this view, led to an opinion in some quarters of the market that the performance potential was poorer for high coupon stocks than for low coupon and medium coupon stocks.

9.6. If such an opinion resulted in price rises of high coupon stocks being appreciably less than would otherwise be expected, the relative spacing of contours at high running yields in the capital-term diagram would decrease and the value of $c_0 + c_4 n$ corresponding to the best fit would also decrease. If $c_0 + c_4 n$ became negative, corresponding to

$$\frac{\partial^2}{\partial t^2} \frac{1}{P} > 0,$$

switches of all three types referred to in Definition 1 could be carried out by selling medium coupon stocks and buying a combination of
low coupon and high coupon stocks. Provided that investors identified and acted upon these switching opportunities, the negative value of \( c_0 + c_1 n \) would be a short-lived phenomenon.

9.7. The weekly results set out in Table 3 are consistent with this scenario of investor behaviour. Since \( c_{15} \) changes very little over the period, \( c_{40} \) alone can be compared with the yield on irredeemables to show how \( \frac{\partial^2}{\partial i^2} P \) varies with changes in the market level. From 3rd December 1976 when the yield on irredeemables was 15.34%, both \( c_{40} \) and the yield on irredeemables decreased each week until 21st January, when the values were –86 and 13.30% respectively. Thereafter the value of \( c_{40} \) stabilised and then increased to –50 at 4th March. The very strong price rises the following week took the yield on irredeemables below 13% for the first time in several years, and \( c_{40} \) decreased sharply to –88, after which it again stabilised and then increased. The most rapid price rises during the period occurred over the 4 weeks from 2nd September; \( c_{40} \) decreased steadily to the minimum value of –132 on 23rd September and then increased.

9.8. The regular pattern is that \( c_0 + c_1 n \) increases after attaining a minimum value during a period of sharply rising prices. However, \( c_0 + c_1 n \) has remained negative since December 1976 for values of term above about 20 years. This suggests that investors in general are inefficient at identifying switching opportunities, perhaps on account of the sparse distribution of stocks.

9.9. Although the continuing existence of positive values of \( \frac{\partial^2}{\partial i^2} P \) is unexpected, it does not invalidate the use of equation 6.1 as a graduation formula but instead highlights the suitability of the model in current conditions.

PART III: APPLICATIONS

10. Description of Market Price Structure

10.1. The detailed model of Part II allows the graduated price of any dated stock, actual or hypothetical, to be calculated directly. Description of market price structure is therefore a matter of summarising the results of the fitted price model in some convenient form. Since the price model was derived from considerations of investor choice between capital and income, a useful starting point is to construct a capital-income diagram from the graduation formula. Figure 6 below illustrates for a fixed value \( n_0 \) of term the
capital amount at maturity corresponding to each value of $i$. The lines $XX^1$ and $YY^1$ correspond to $i = I$ and $\hat{P} = 1$ respectively. The distance $AB$ represents $h(n_0)$, and the gradient at $B$ is $-f(n)$. The general value of the gradient is

$$\frac{\partial}{\partial i} \frac{1}{\hat{P}} = -f(n) \exp\{- (c_0 + c_2) (1 - i)\}.$$ 

As a direct result of the choice of $\lambda(n, t)$ discussed in 6.8, this gradient (ignoring sign) increases or decreases geometrically with $i$ according to whether $c_0 + c_2 n$ is negative or positive respectively. The value $g_{n_0}$ corresponding to the point $C$ represents the running yield and hence the coupon of the unique stock with term $n_0$ which stands at par.

10.2. If we require an indicator of the general market level at a certain value of term, the obvious measure is $g_n$ as defined above. The graph of this function against term gives the par yield curve, which has the useful property that there is no capital appreciation element in the yield. Burman and White also consider this curve to be the most satisfactory representation of the term structure of yield.
10.3. The equation of the par yield curve is

\[ g_n = 1 + \frac{h(n)}{f(n) X(g_n)} \]

where

\[ X(g_n) = \frac{1 - \exp\left\{ -(c_0 + c_1n)(1 - g_n) \right\}}{(c_0 + c_1n)(1 - g_n)}. \]

The sequence \( g_n^{[t]} \) defined by

\[ g_n^{[1]} = 1 + \frac{h(n)}{f(n)}, \]
\[ g_n^{[t+1]} = 1 + \frac{h(n)}{f(n) X(g_n^{[t]})}, \]

converges geometrically to \( g_n \), and convergence is very rapid. As a result of the definitions of \( h(n) \) and \( f(n) \), \( g_n \) tends to 1 as \( n \) tends to infinity.

10.4. Chart 2 shows the par yield curve for the long-dated section of the market at weekly intervals from 10th December 1976 to 28th January 1977. The shape as well as the general level can change significantly over a few weeks. Abrupt changes in shape can often be explained in terms of tap stock operations.

10.5. An illustration of the many varied shapes assumed by the par yield curve in the past is given by Chart 3, which shows the curves for long-dated stocks at 31st December 1950, 1955, 1960 and 1965. Although the par yield curves in Charts 2 and 3 span more than a quarter of a century and exhibit great variety in shape, they are all derived from equation 6.1. This consistency represents a significant advance over conventional yield curve methods, where the particular mathematical formula fitted to gross redemption yield is generally suitable for only a limited period of time.

10.6. Since many policy decisions regarding gilt-edged investment take into account the present and past relationships between the yield levels of the short-dated and long-dated sections of the market, we now extend the par yield curve to include all values of term above one year. For both short-dated stocks (terms of 1 to 5 years) and medium-dated stocks (terms of 5 to 15 years) we use the model for long-dated stocks with the definition of \( h(n) \) changed to

\[ h(n) = a_0 + a_1n + a_2n^2. \]

Applying the price model separately to each of these three sections, we obtain the par yield curve for 17th August 1977 shown in Chart 4. The Bank of England yield curve for the same day is shown for comparison.
The weight function
\[ w(t) = \frac{1}{4}(t^3 - 3t + 2) \]
is used for splicing bands of 3 to 7 years and 13 to 17 years to remove the small discontinuities of 0.24% at 5 years and 0.11% at 15 years. The extrapolation of each curve for splicing purposes is based on the tangent at the end point of the range. The dotted lines show the original curves within the splicing bands. Two very interesting features are highlighted in this par yield curve. Firstly, there is an unexpected dip in the region from 5 to 10 years. Burman and Page draw attention to the same feature in the Bank of England yield curve and make minor modifications to the model to ensure that the curve does not slope downwards in this region. Their explanation for this feature is that gross investors, in the absence of a high coupon stock maturing in the mid-eighties, bid up the prices of medium coupon stocks in this maturity area beyond the normal equilibrium levels and so reduce the yields. The second interesting feature is the rapid change of gradient around 15 years. The Burman and White yield curve is constructed in such a manner that no sharp change of gradient can occur in this neighbourhood.

10.7. Although the point is of little practical significance at present, the par yield curve can be used to determine the outstanding term to redemption of a stock with an optional redemption period. For example, it may be assumed that the stock will be redeemed at the earliest date if the value of the par yield curve at the term corresponding to the length of the optional redemption period is less than the coupon of the stock. This basis, which assumes that the price structure will remain unchanged, is more consistent than the rule described in 5.3.

10.8. The most commonly used method of description is the gross redemption yield curve, which may be defined as the representation in graph form of the relationship between redemption yield and term to maturity of comparable stocks. The widening range of coupon over the past 10 years has caused redemption yield to become much more dependent on coupon, with the result that the conventional type of yield curve has become less satisfactory as a descriptive device.

10.9. To overcome this lack of homogeneity, we may calculate, for several fixed values of coupon, the yield curve obtained from the gross redemption yield of hypothetical stocks at all maturities with the same fixed coupon. Chart 5 shows these fixed coupon yield curves for 27th May 1977 using coupon values of 3%, 6%,
9%, 12% and 15%. This example demonstrates the existence of simultaneous positive and negative gradients for yield curves corresponding to different coupons.

10.10. The conventional yield curve can be considered as a weighted average of fixed coupon yield curves, the weighting being by coupon in each neighbourhood of term.

10.11. The capital interest diagram, the par yield curve and the fixed coupon yield curve all portray the variation of some attribute with respect to term at a particular time. To portray the variation in market level over time, the value of any of these attributes for some fixed value of term is immediately available as an index. For a 20-year government stock index, for example, one of the following could be used, depending on the purpose for which it is required:

(a) the par yield curve value at 20 years;
(b) the price of a 20-year stock for some fixed value of coupon;
(c) the gross redemption yield corresponding to (b);
(d) a net redemption yield corresponding to (b).

11. Cheapness and Dearnness of Individual Stocks

11.1. The aim is to identify short-term price aberrations from the price structure. These short-term anomalies can be considered as random noise superimposed on the price structure.

11.2. Since the optimisation criterion of minimising the sum of the squares of the proportionate price errors was chosen with this application in mind, the proportionate price error is immediately available as a suitable measure of the short-term cheapness or dearness of individual stocks. However, this measure can be used directly only if it is unbiased in the sense that it is centred on zero. There are two principal reasons why this is unlikely always to be the case. Firstly, the functions selected in Section 6 to represent $h(n), f(n)$ and $\lambda(n, t)$ are to a large extent arbitrary, and cannot be expected to reflect all features of the price structure. In order to ensure the stability of the variable parameters at the possible expense of goodness of fit, these functions were defined using as few parameters as were absolutely essential to reflect their main properties. Secondly, the model assumes that investors have a very large number of stocks to choose from in the neighbourhood of each value of term and coupon. In practice there is a very sparse distribution of stocks, and in particular the absence of a high coupon stock maturing in the mid-eighties is referred to in 10.6 as a possible source of serious distortion in the par yield curve.
11.3. In view of these factors, a suitable statistical approach is to use the Mean Absolute Deviation (MAD) methods described by Plymen and Prevett\textsuperscript{4}. The detailed steps for an individual stock are:

(i) The proportionate price error $e_t$ for day $t$ is calculated.

(ii) The exponential moving average for $e_t$ is $A_t$ where $A_t = \alpha e_t + (1-\alpha)A_{t-1}$ with $0 < \alpha < 1$.

(iii) The absolute deviation is $d_t$ where $d_t = |e_t - A_t|$.

(iv) The MAD is $M_t$ where $M_t = \alpha d_t + (1-\alpha)M_{t-1}$.

(v) A fixed multiple, $k$, of $M_t$ determines the control limit.

(vi) If $e_t < A_t - kM_t$—"buy" signal;
    if $e_t > A_t + kM_t$—"sell" signal.

11.4. The results are graphed daily, with one graph for each stock. Chart 6 shows the graph for Treasury 9\%\% 1999 for the first 9 months of 1977 using $\alpha = 0.027$ and $k = 1.6$, these values having been found suitable from practical experience.

11.5. This approach not only reduces the identification of anomalies to the consideration of just one measure for each stock but also provides a largely mechanical rule for deciding when extremes of cheapness or dearness have been reached. Experience of the system shows that the number of stocks lying outside the control limits is highly sensitive to market movements. When the general market level changes, the root mean square error and the number of buy and sell signals both tend to increase. As the market stabilises at the new level, the root mean square error gradually decreases and switching opportunities become fewer in number.

11.6. When a stock is extreme in the sense that its term or coupon is a maximum or a minimum and there are no stocks with fairly similar values of term or coupon, it will have a significant effect on the statistical fit, and both the magnitude and variability of its price error will be smaller than average. This is particularly noticeable with British Gas 3\% 1990-95 (lowest coupon) and Treasury 7\%\% 2012-15 (longest term). For stocks such as these, methods of the type discussed in the following section must be used to supplement MAD techniques.

12. Medium Term Price Projection

12.1. A mathematical model can be applied to the forecasting of likely future price trends only if the future behaviour of all the
parameters involved in the model can be predicted with a satisfactory degree of confidence. We show that the present model resolves the total return from a stock (including both income and change in capital value) into five statistically stable components.

12.2. To avoid complications involving accrued interest and the reinvestment of income, we use a holding period of half a year starting on a dividend date. For a stock with coupon $g$ and term $n$ years at the beginning of the period, let $tP(n, g)$ be the market price at the start of the period, $t\hat{P}(n, g)$ be the graduated price at the start of the period, and $k$ be the investor's rate of income tax. Then the total return to that investor over the period is $R$ where

$$R = \frac{t_{i+1}P(n, g)}{tP(n, g)} + \frac{(1-k)^g}{2} - 1$$

in an obvious notation. Each of $R^p$, $R_3$ and $R_4$ is a component corresponding to the proportionate price change from:

- $R^p$—change in the price structure over the period
- $R_3$—passage of time
- $R_4$—change in short term cheapness,

and $R_5$ is the interest component of half the commencing running yield net of tax. Since $R_3$ and $R_5$ are known at the beginning of the period and $R_4$ is discussed in Section 11, the problem reduces to the estimation of the component $R^p$.

12.3. A special case is where it is assumed that the price structure at the required future date will be identical to the price structure at some date in the past. Given the highly volatile nature of the market in recent years, this approach is often helpful. Price projection in this case is purely arithmetical, as no further estimates are required once a suitable past date has been selected.

12.4. Including the yield on undated stocks, there are eight parameters that have to be specified to determine the price structure at any point in time. Since certain of these parameters are clearly interdependent, it would be quite impracticable to specify each
the Gilt-edged Market

parameter individually in an attempt to represent expected future price structures and thereby predict the properties of \( R^p \). Instead, historical price data will have to be summarised in such a manner that the behaviour of \( R^p \) can be expressed in terms of one or more statistically stable components.

12.5. Since the expected future price structure will in the first instance be described in terms of the general level and shape of the par yield curve, it is necessary to find a statistic which relates the graduated price of a stock to the value of the par yield curve at that term. A suitable statistic is the ratio of the value of the par yield curve to the running yield based on the graduated price. For a general stock with term \( n \) and coupon \( g \) at time \( t \), this function is 
\[
\tau C(n, g) = \frac{t g_n}{g} \tau \hat{P}(n, g),
\]
and \( t g_n \) is the par yield curve value. Then
\[
1 + R^p = \frac{t+\frac{1}{2} \tau \hat{P}(n-\frac{1}{2}, g)}{\tau \hat{P}(n-\frac{1}{2}, g)} = \frac{t g_{n-\frac{1}{2}}}{t+\frac{1}{2} g_{n-\frac{1}{2}}} \frac{t+\frac{1}{2} C(n-\frac{1}{2}, g)}{t+\frac{1}{2} C(n-\frac{1}{2}, g)}
\]
where \( R_1 \) and \( R_2 \) are the proportionate changes in the values of the par yield curve at term \( n-\frac{1}{2} \) and of \( C(n-\frac{1}{2}, g) \) respectively.

The total return \( R \) is therefore given by
\[
R = (1+R_1)(1+R_2)(1+R_3)(1+R_4)+R_5-1.
\]

12.6. There is a striking similarity between this result and the approach described by Pepper². Using a mathematical model based on gross redemption yield, Pepper shows that the following factors need to be considered when comparing two stocks’ likely performances:

1. Possible changes in the general level of interest rates (i.e. the prospects for the market as a whole).
2. Possible changes in the shape of the gross redemption yield curve (i.e. the outlook for the different sections of the market).
3. Stocks’ fluctuations about the yield curve (i.e. the cheapness and dearness of individual stocks).
4. The effect of the passage of time (including the difference in income).
12.7. Variations in $\frac{\partial^2 1}{\partial t^2 \hat{P}}$ are reflected in the component $R_2$, and, as shown in the following section, the essential feature of the present model as compared with yield curve methods is that $\frac{\partial^2 1}{\partial t^2 \hat{P}}$ is allowed to be non-zero. The component $R_2$ therefore depends on $\frac{\partial^2 1}{\partial t^2 \hat{P}}$ as well as on yield level, whereas with the yield curve model the variation of price with yield level is a matter of compound interest only. In all other respects the two analyses of relative price movements are identical. Pepper's model can therefore be considered as a special case of the present model where, for description purposes, $\frac{\partial^2 1}{\partial t^2 \hat{P}}$ is assumed to be zero and where, for relative performance purposes, variations in $\frac{\partial^2 1}{\partial t^2 \hat{P}}$ are ignored.

12.8. The results for 31st December 1965 described in Table 2 show that the improvement in fit by allowing non-zero values of $\frac{\partial^2 1}{\partial t^2 \hat{P}}$ is so small in absolute terms that Pepper's model and the present model give virtually identical results around that time in the two principal areas of application, namely the identification of anomalies and the comparison of relative performances.

12.9. More recently, and in particular after December 1976 when $\frac{\partial^2 1}{\partial t^2 \hat{P}}$ became positive for certain values of term, the variations of this function have been so large that conventional yield curve methods have become much less useful for comparisons of stocks of differing coupon.

12.10. The practical application of the model is demonstrated in the following detailed example of a double switch from Treasury 8% 2002-06 into a combination of Treasury 15½% 1998 and Treasury 5½% 2008-12. When the value of $c_{40}$ became strongly negative in December 1976 and January 1977, yield curve methods indicated that medium coupon stocks were very expensive on anomaly considerations, whereas the present model explained most of this relative price movement in terms of the component $R_2$ which reflects changes in the value of $\frac{\partial^2 1}{\partial t^2 \hat{P}}$. From Table 3 it is seen that the value of $c_{40}$ had stabilised by the end of January, and on 31st January the MAD techniques described in Section 11 indicated a short-term
sell signal for Treasury 8% 2002-06 and short-term buy signals for Treasury 15½% 1998 and Treasury 5½% 2008-12. Selling Treasury 8% 2002-06 and reinvesting in equal amounts by value in the other two stocks gave a double switch which could be justified both on considerations of short-term dearness and cheapness and also on the grounds that the price structure had reached an extreme point at which medium coupon stocks were expensive against a combination of high coupon and low coupon stocks.

12.11. To avoid distortions from ex dividend periods, the price movements of each of these three stocks are analysed over the period of 6 months to 29th July 1977. The calculations are summarised in Table 4 below. The component $R_5$ allows for the reinvestment of the dividend payment on each stock at a short-term deposit rate of 10%, and the income tax rate is taken as 20%.

<table>
<thead>
<tr>
<th></th>
<th>Treasury 15½%</th>
<th>Treasury 5½%</th>
<th>Treasury 8%</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(n, g)$</td>
<td>1.0713</td>
<td>0.4224</td>
<td>0.6088</td>
</tr>
<tr>
<td>$t+4P(n-\frac{1}{2}, g)$</td>
<td>1.1147</td>
<td>0.4378</td>
<td>0.6253</td>
</tr>
<tr>
<td>$P(n_2, g)$</td>
<td>1.0762</td>
<td>0.4282</td>
<td>0.6052</td>
</tr>
<tr>
<td>$t+4P(n-\frac{1}{2}, g)$</td>
<td>1.1178</td>
<td>0.4418</td>
<td>0.6244</td>
</tr>
<tr>
<td>$t+4P(n-\frac{1}{2}, g)$</td>
<td>1.0725</td>
<td>0.4289</td>
<td>0.6059</td>
</tr>
<tr>
<td>$e$</td>
<td>-0.0045</td>
<td>-0.0135</td>
<td>0.0059</td>
</tr>
<tr>
<td>$t+4e$</td>
<td>-0.0028</td>
<td>-0.0090</td>
<td>0.0014</td>
</tr>
<tr>
<td>$i$</td>
<td>0.1447</td>
<td>0.1302</td>
<td>0.1314</td>
</tr>
<tr>
<td>$d(n-\frac{1}{2})$</td>
<td>0.1415</td>
<td>0.1391</td>
<td>0.1401</td>
</tr>
<tr>
<td>$t+4d(n-\frac{1}{2})$</td>
<td>0.1360</td>
<td>0.1348</td>
<td>0.1355</td>
</tr>
<tr>
<td>$C(n-\frac{1}{2}, g)$</td>
<td>0.9791</td>
<td>1.0847</td>
<td>1.0611</td>
</tr>
<tr>
<td>$t+4C(n-\frac{1}{2}, g)$</td>
<td>0.9808</td>
<td>1.0828</td>
<td>1.0576</td>
</tr>
<tr>
<td>$R_1$</td>
<td>0.0404</td>
<td>0.0320</td>
<td>0.0339</td>
</tr>
<tr>
<td>$R_2$</td>
<td>0.0018</td>
<td>-0.0018</td>
<td>-0.0033</td>
</tr>
<tr>
<td>$R_3$</td>
<td>-0.0034</td>
<td>0.0016</td>
<td>0.0012</td>
</tr>
<tr>
<td>$R_4$</td>
<td>0.0017</td>
<td>0.0045</td>
<td>-0.0045</td>
</tr>
<tr>
<td>$R_5$</td>
<td>0.0594</td>
<td>0.0537</td>
<td>0.0539</td>
</tr>
</tbody>
</table>

12.12. Since the average term of the two stocks purchased is very similar to the term of the stock sold, the switch is to a large extent immunised against changes in the shape of the yield curve. The average value of the component $R_1$ for the two stocks purchased is thus only slightly higher than the value for Treasury 8% 2002-06.

12.13. Having shown signs of stabilising at the end of January, the value of $c_{40}$ was $-68$ on 29th July compared with $-94$ on 31st
A Mathematical Model for January. The element of switching profit from this change in the price structure is reflected in the value of the component $R_2$ being lowest for Treasury 8% 2002-06.

12.14. The components $R_3$ and $R_5$ represent the effects of the passage of time on capital and interest returns respectively. The much higher interest return on Treasury 15½% 1998 is partly offset by the expected small fall in capital value. Since the average value of $(R_3 + R_5)$ is increased, the switch can be regarded as an upgrading of basic holdings and hence need not be reversed.

12.15. The changes in the short term cheapness or dearness of all three stocks are in the expected directions, thus confirming the short-term buy and sell signals given by the MAD techniques.

12.16. The profitability of the switch over the period can therefore be summarised as follows:

<table>
<thead>
<tr>
<th>Component</th>
<th>Profit %</th>
</tr>
</thead>
<tbody>
<tr>
<td>Change in par yield curve</td>
<td>0.23</td>
</tr>
<tr>
<td>Change in price structure relative to par yield curve</td>
<td>0.33</td>
</tr>
<tr>
<td>Passage of time</td>
<td>0.06</td>
</tr>
<tr>
<td>Change in short term cheapness</td>
<td>0.76</td>
</tr>
<tr>
<td>Total</td>
<td>1.38</td>
</tr>
</tbody>
</table>

PART IV: COMPARISON WITH YIELD CURVE METHODS

13. Historical Development of Yield Curve Methods

13.1. The conventional yield curve approach has long served as an extremely useful practical tool. Since the present model differs radically from this approach, it is instructive to trace the development of yield curve methods and to show where they diverge from the approach described in Part I.

13.2. The first comprehensive discussion of the concept of the yield curve is given by Marshall. He establishes that the gross redemption yield curve at the time of writing (1951) is remarkably smooth, so that gross redemption yield may be taken as the market's valuation basis. The deviation of a stock from the yield curve is the primary indicator of whether it is dear or cheap. Once a potential switch is identified, the history of gross redemption yield differences of that pair of stocks gives a measure of the likely profit on the switch, since this yield difference is largely independent of both time and the market level, and the comparison is valid for stocks of differing coupon. Although there are reasons for expecting a better fit at a tax rate between zero and the life office rate, it is noted that gross redemption yield has for several years produced the smoothest curve.
13.3. Pepper\textsuperscript{2} discusses the background to gilt-edged investment and describes statistical methods of analysing the four components that determine a stock’s relative performance. Pepper, too, remarks that it is surprising that the gross redemption yield curve (in 1962) is always smoother than the net redemption yield curve at a low rate of tax. Detailed consideration is given to fluctuations about the yield curve, and in particular the methods of Marshall are taken a stage further by using a computer to fit a mathematical equation on a consistent basis to give the yield curve. The current position of a stock relative to the yield curve can then be compared with the stock’s past history, thereby reducing the problem to the consideration of just one measure.

13.4. In the notation of 3.1, both Marshall and Pepper use a price model of the form:

\[ P(n, g) = g \frac{[1 - (1 + j_n) - n]}{j_n} + (1 + j_n)^{-n} \]

where \( j_n \) is the value of the yield curve at term \( n \). Both authors explain why it is more logical to expect the market to follow a net redemption yield basis at some low rate of tax. If the tax rate is allowed to vary with term, a net redemption yield basis gives:

\[ P(n, g) = g \frac{[1 - \{1 + (1 - t_n)j_n\} - n]}{j_n} + (1 + j_n)^{-n} \]

with \( 0 < t_n < 1 \).

This can be rearranged to

\[ \frac{1}{P} = (1 + j_n)^n - \frac{[1 + (1 + j_n)^n]}{j_n} \frac{1 - \{1 + (1 - t_n)j_n\} - n}{j_n} \]

Since \( t_n \) and \( j_n \) are independent of \( i \), differentiating twice with respect to \( i \) gives

\[ \frac{\partial^2}{\partial i^2} \frac{1}{P} = 0. \]

This immediately highlights the significant difference between yield curve methods and the mathematical model derived in Part I, namely that yield curve methods imply

\[ \frac{\partial^2}{\partial i^2} \frac{1}{P} = 0, \]

whereas Theorem 1 only requires that

\[ \frac{\partial^2}{\partial i^2} \frac{1}{P} \leq 0. \]
It is explained in 6.9 that a positive value of \( \frac{\partial^2}{\partial t^2} \frac{1}{P} \) is not impossible, and the examples of Section 8 show that at certain dates the best fit corresponds to a strongly negative value of \( \frac{\partial^2}{\partial t^2} \frac{1}{P} \) and at others to a strongly positive value. It is therefore apparent that yield curve methods cannot be expected in present or recent market conditions to give an efficient description of price structure.

13.5. Brew\textsuperscript{3} describes a theoretical expectations model based on a prospective yield and a horizon yield. The price model can be expressed as

\[
P(n, g) = \frac{1 + g \frac{j_2}{s_{j_3}}}{1 + \left( \frac{j_1}{j_2} - 1 \right) + j_1 \frac{j_2}{s_{j_3}}}
\]

where \( j_1 \) is the prospective yield
\( j_2 \) is the horizon yield
and \( j_3 \) is an arbitrary yield normally chosen to be approximately equal to the yield on irredeemables.

Both \( j_1 \) and \( j_2 \) are determined by a curve-fitting process. By comparison with equation 7.1, it can be seen that this price model is a special case of the general price model of equation 4.1 with

\[
i_0 = j_1 \\
\lambda(n, t) = 1 \\
h(n) = \frac{j_1}{j_2} - 1 \\
f(n) = \frac{j_2}{s_{j_3}}
\]

13.6. As highlighted in Table 2, the transition from the gross redemption yield market basis to the present more complicated price structure took place during a period when interest rates followed a strongly rising trend. As a result of the rising yield levels, successive tap stocks were issued with ever increasing coupons, and the coupon range on dated stocks, having been 2\( \frac{1}{2} \)% to 4\( \frac{1}{2} \)% in 1952 and 2\( \frac{1}{2} \)% to 5\( \frac{1}{2} \)% in 1962, increased dramatically to the present range of 3% to 15\( \frac{1}{2} \)%.

13.7. By the late sixties it was apparent from the large deviations of many stocks from the yield curve that gross redemption yield was no longer a satisfactory description of the market valuation basis. Both the shape and the location of the yield curve had become highly dependent on the type of mathematical equation used and
on the details of the method of fit. As a practical expedient, stocks with low coupons were sometimes omitted in the fitting process to improve the goodness of fit for the remaining stocks. Even then the deviation of a stock from the yield curve was often so large that this measure could no longer be assumed to be independent of the yield level, and it was therefore of only limited value in the identification of cheap and dear stocks. A further disadvantage was that the value of the yield curve at a particular term tended to vary with the average coupon in that neighbourhood, so that the introduction of new tap stocks could change the slope of the yield curve and render invalid any comparisons of slope at different dates.

13.8. Despite these serious shortcomings, the gross redemption yield curve approach continued to dominate the practical management of gilt-edged portfolios, and surprisingly little research was carried out to develop methods better suited to the changed conditions.

13.9. Pepper and Salkin describe research work involving net redemption yields, and conclude that the variable rate of tax net redemption yield curve is excellent for descriptive purposes but is not suitable for assessing the short term cheapness or dearness of individual stocks because the average market rate of tax is too unstable. The present author explained at the meeting at which was presented that, as shown in 13.4, a net redemption yield basis implied

$$\frac{\partial^2}{\partial t^2} \frac{1}{P} = 0$$

and showed that the present mathematical model, with a negative value of \(\frac{\partial^2}{\partial t^2} \frac{1}{P}\), gave a significantly better description of market structure at that time.

13.10. Burman and White describe work carried out at the Bank of England to construct a yield curve whose main applications are to assist in the pricing of new issues and to determine certain interest rates. This yield curve, together with later modifications following correspondence with the present author, is discussed in detail in the following section.

13.11. Feldman describes a variation of the net redemption yield approach. Using a price model of the form

$$P(n, g) = gA(n) + V(n)$$
A Mathematical Model for

where $A(n)$ and $V(n)$ are arbitrary functions of $n$ that can be interpreted in compound interest terms, he produces a graph of the implied force of interest against term on 25th March 1977 that is remarkably similar to the par yield curve for 17th August 1977 shown in Chart 4.


14.1 Burman and White suggest that the yield curve should be derived from a priori theories about the working of the market rather than from arbitrary mathematical formulae. Their analysis starts from the view that certain types of investor have different "preferred habitats" and that within each preferred habitat all investors, having taken a view about the level of future yields at some relatively short planning horizon, expect the yield level to remain unchanged beyond the horizon in the face of virtually complete uncertainty about developments in the distant future. On this theory, say Burman and White, price equilibrium implies that within each preferred habitat all stocks have the same expected return over the holding period up to the horizon and have the same expected yield at the horizon. The influences of taxation and risk aversion are discussed, and the model is constructed to take these into account. It is found after experimentation that the most satisfactory fit results from the assumption that there are two (overlapping) segments in the market, with horizons of 1 year and 4 years and with expectations that need not be the same. Because of the difficulty of distinguishing between the effects on the yield structure of various factors such as the length of horizon, the expected level of future yields and the risk premium, certain of these variables have to be fixed at plausible but arbitrary values. Since two of the main applications are to assist in the pricing of new issues and to determine interest rates on certain loans, the curve that is calculated is the par yield curve discussed in 10.3. The new curves, say Burman and White, give a much better fit than their predecessors, and in particular yield differences due to coupon are very largely explained.

14.2. The philosophy behind the Bank of England yield curve is remarkably similar to the approach discussed in Part I. Burman and White define market equilibrium in terms of the variation of
price with term and allow the variation of price with coupon to be such as to give the best statistical fit for the chosen mathematical formula. The present model defines market equilibrium in terms of the absence of arbitrage possibilities between stocks of the same term and allows the variation of price with term to be such as to give the best statistical fit for the selected graduation formula.

14.3. The assumptions of investor behaviour underlying the general price model

\[ \frac{1}{P} = 1 + h(n) + f(n) \int_0^{n_0} \lambda(n, t) dt \]  

(14.1)

derived in Part I are so weak that all other price models suggested for the gilt-edged market can be considered as special cases of this model. We now show that this is true of the Burman and White model.

14.4. Ignoring complications such as half-yearly compounding and risk premium, the Burman and White price model is

\[ P(n, g) = (1-k)g[i_0(n_0) + (1+j_1)^{-n_0}g^2(n_0-n_0)] + (1+j_1)^{-n_0}(1+j_2)^{-n_0-n_0} \]

where \( k \) is the effective rate of income tax

\( n_0 \) is the planning horizon

\( j_1 \) is the expected return for the period up to the horizon

and \( j_2 \) is the expected return at the horizon.

This price basis is a variation of the fixed tax rate net redemption yield basis with an effective interest rate of \( j_1 \) for the first \( n_0 \) years and \( j_2 \) thereafter.

It can easily be verified that this price basis is equivalent to equation 4.1 with

\[ \lambda(n, t) = 1 \]

\[ f(n) = (1-k)[(1+j_2)^{n-n_0}g^{j_1(n_0)} + s^{j_1(n_0)}] \]

\[ h(n) = \frac{(j_1-j_2)s^{j_1(n_0)}}{1+j_2\frac{s^{j_1(n_0)}}{n_0}} \]

and

\[ i_0 = g_\infty \]

where \( g_\infty \) is the limit of the par yield curve as \( n \) tends to infinity.

14.5. This original Burman and White model differs in two important respects from the special model for dated stocks derived in Part II. Firstly, the model is such that

\[ \frac{\partial^2}{\partial i^2} \frac{1}{P} = 0 \]
so that the recent more complicated relationship between price and coupon cannot be allowed for properly. Secondly, the more restrictive definitions of $f(n)$ and $h(n)$ result in each segment of the yield curve being monotonic, whereas Chart 2 demonstrates the existence of par yield curves with a maximum turning point around 20 years.

14.6. Following correspondence with the present author, Burman\(^8\) describes modifications to the model to allow non-linearity of price with respect to coupon. The revised model, says Burman, results in only a slightly better fit using quarterly data for 1971 and 1972 but gives a significantly better fit using data for 1973. Accordingly, the Bank of England yield curve model was altered from the beginning of 1973 to allow

$$\frac{\partial^2}{\partial t^2} \frac{1}{P} \leq 0.$$  

14.7. The effects on the respective par yield curves of the different definitions for $f(n)$ and $h(n)$ are obvious from Chart 4, and much of the difference in shapes can be explained by the quite different nature of their intended applications. The par yield curve obtained by splicing together the three maturity segments is intended only as a rough guide to the positions of the various market segments. Comparison of such curves at different dates is often helpful in the identification of likely changes in the equilibrium position. The Burman and White curve, on the other hand, is designed to assist in judging the appropriate terms for new government issues. The monotonic nature within each segment and the need to eliminate any dip between the segments follow from certain practical aspects of new issues.

15. The F.T.-Actuaries Yield Indices

15.1. In recent years the Joint Investment Research Committee of the Faculty and the Institute of Actuaries, which is responsible for the design of the Financial Times-Actuaries Indices, had felt that the published indices for Fixed Interest stocks, in particular the 20-year government stocks index, were no longer wholly satisfactory. After careful consideration of the purpose and possible uses of fixed interest indices, the Committee constructed a new series of indices\(^11\). The series includes nine yield indices for redeemable stocks, giving the gross redemption yields for terms of 5, 15 and 25 years, for each of three coupon bands consisting of low, medium and high coupon stocks.
15.2. A separate graduation process is carried out for each of the three coupon bands to determine these yield indices. The formula

\[ y(n) = A + Be^{-Cn} + De^{-Fn} \]  

(15.1)
is used to define each of the three yield curves. Ignoring the complication of half-yearly compounding, the price model implicit in equation 15.1 is

\[ P(n, g) = g \frac{1 - [1 + y(n)]^n}{y(n)} + [1 + y(n)]^{-n} \]  

(15.2)

where \( y(n) \) is given by equation 15.1. Since equation 15.2 can be expressed as

\[ \frac{1}{P} = 1 + (y(n) - A) \frac{[1 + y(n)]^n - 1}{y(n)} + \left( A - g \right) \frac{[1 + y(n)]^n - 1}{y(n)} \],

the yield indices for each coupon band are based on the general price model of equation 4.1 with

\[ i_0 = A \]

\[ \lambda(n, t) = 1 \]

\[ f(n) = \frac{[1 + y(n)]^n - 1}{y(n)} \]

and

\[ h(n) = (Be^{-Cn} + De^{-Fn}) \frac{[1 + y(n)]^n - 1}{y(n)} \],

where

\[ y(n) = A + Be^{-Cn} + De^{-Fn}. \]

15.3. In terms of the capital-term diagram, the F.T.-Actuaries yield indices are based on an underlying price model where the contours for incremental values of running yield are equally spaced within each of the three coupon bands. Since only three values on each curve are used, the shape of the curve is of little importance provided that it is sufficiently flexible to represent the yield level correctly at the chosen values of term. Extensive experimentation by the Committee has shown that equation 15.1 is satisfactory in this respect.

16. Conclusion

16.1. By returning to first principles and comparing entitlements of capital and interest obtained by purchasing unit value of different stocks, a general price model has been constructed from considerations of equilibrium under very weak assumptions of investor behaviour. Unlike all other models suggested for bond markets, the formulation does not involve compound interest functions.
16.2. The special case of the model derived for long-dated stocks satisfies the two essential requirements of giving an accurate representation of the price structure at various dates and of being sufficiently stable to allow projections of future relative price movements to be made.

16.3. The par yield curve emerges as a natural application of the model. As a description of the term structure of yields, this curve, being independent of any capital appreciation element, is much more satisfactory in concept than the conventional yield curves obtained by fitting arbitrary mathematical formulae to gross redemption yields.

16.4. It is suggested in the opening paragraph of the paper that the gilt-edged market possesses many of the attributes of a "perfect" market. Although the results show that prices vary, as expected, in a regular manner, highly profitable switching opportunities remain. Control theory techniques indicate the frequent existence of short-term anomalies where the potential profits are well in excess of the transaction costs, and the parameters corresponding to the best fit provide an immediate and very powerful means of identifying double switches.

16.5. A central feature of the model is that price is not restricted to being a linear function of coupon. Until the late 'sixties, the coupon range was sufficiently narrow for the divergence from linearity to be of little practical significance, and yield curve methods of the type described by Pepper² gave results virtually identical to those obtained using the present model in the identification of anomalies and the comparison of relative price performances. More recently, however, the degree of divergence from a linear relationship has been significant and has varied sharply over very short periods of time. With the exception of the modified Burman and White model, all other models suggested for the gilt-edged market assume that price is a linear function of coupon.

16.6. The recent positive values of \( \frac{\partial^2}{\partial t^2} \frac{1}{P} \) and the frequent changes in the shape of the par yield curve suggest that the equilibrium position is more complex than is generally appreciated and that it often depends on supply and demand factors affecting stocks in particular areas of coupon or term. The resulting price structure, though stable over a reasonable period of time, may be totally inconsistent with even the most elementary hypotheses of investors' expectations of future returns. In such conditions, a model of the
present type, which seeks only to identify the equilibrium position rather than to explain why it arises, appears to offer the most satisfactory approach.

16.7. In conclusion, I wish to acknowledge the invaluable assistance I have received at various stages in the preparation of the paper. My thanks are due in particular to J. P. Burman, of the Bank of England, for giving helpful advice and making available detailed information on the progress of his own researches, to C. G. Thomson, B.Sc., F.F.A., for designing the main computer program, and to N. J. McChesney, B.Sc., for developing numerous special computer programs and carrying out all the numerical work.

REFERENCES

6. CLARKSON, R. S. Discussion of (5). (The Institute of Mathematics and its Applications, October 1972.)
Theorem 1. A smooth positive function $P$ of the two independent variables, term $n$ and coupon $g$, represents a price structure which is in equilibrium under switching action if and only if:

$$\frac{\partial}{\partial i} \frac{1}{P} < 0$$

and

$$\frac{\partial^2}{\partial i^2} \frac{1}{P} \leq 0$$

(3.1)

where $i = \frac{g}{P}$.

Proof. All stocks referred to below have the same term. A switch is referred to as being of type (i), (ii) or (iii) according to the cases enumerated in Definition 1.

Suppose first that prices are in equilibrium. We use a series of lemmas to show that the above conditions 3.1 are necessary.

Lemma 1. $P(n, g)$ is a strictly increasing function of $g$.

Proof of Lemma 1. If not, there exist values $g_1$ and $g_2$ with $g_2 > g_1$, such that $P(n, g_2) < P(n, g_1)$. Then a holder of the stock with coupon $g_1$ could sell his holding and buy the other stock at a price less than or equal to the price at which he sells the first stock. This gives an equal or higher nominal amount of the second stock, and a higher annual income since the coupon is higher. On this switch, income is increased and the capital amount on maturity is at least maintained, so that a switch of type (ii) or (iii) exists. This contradicts the hypothesis that prices are in equilibrium under switching action, and the result follows.

Lemma 2. Define the running yield, $i$, as $\frac{g}{P(n, g)}$. Then $i$ is a strictly increasing function of $g$.

Proof of Lemma 2. If not, there exist values $g_1$ and $g_2$ with $g_2 > g_1$ such that $i_2 \leq i_1$. Then a holder of unit value of the stock with coupon $g_1$ is entitled to an annual income of $i_1$ and a capital amount on maturity of $\frac{1}{P(n, g_1)}$. Similarly a holder of unit value of the stock with coupon $g_2$ is entitled to an annual income of $i_2$ and a capital amount on maturity of $\frac{1}{P(n, g_2)}$. By Lemma 1, $P(n, g_2) > P(n, g_1)$, so that by switching from the stock with coupon $g_2$ into the stock with
coupon $g$, the capital amount on maturity is increased and the income is at least maintained. A switch of type (i) or (iii) therefore exists. Since this contradicts the hypothesis that prices are in equilibrium under switching action the result follows.

Lemma 3. $P$ and $\frac{1}{P}$ are smooth functions of the two independent variables $n$ and $i$.

Proof of Lemma 3. By Lemma 2 there is a one-to-one correspondence between $g$ and $i$, so that $P$ may be considered as a function of the two independent variables $n$ and $i$. Since $i$ is the quotient of two non-zero smooth functions of $n$ and $g$, any partial derivative of $P$ with respect to $n$ and $i$ can, by the rule for differentiating a function of a function, be expressed in terms of partial derivatives with respect to $n$ and $g$, and these are known to exist. $P$ is therefore a smooth function of the two independent variables $n$ and $i$. The corresponding result for $\frac{1}{P}$ follows immediately since $P$ is positive.

Lemma 4. \[ \frac{\partial}{\partial i} \frac{1}{P} < 0. \]

Proof of Lemma 4. By Lemma 3, $\frac{\partial}{\partial i} \frac{1}{P}$ exists, so that it is only necessary to show that $\frac{1}{P}$ decreases strictly with respect to $i$. By Lemma 1, $\frac{1}{P}$ is a strictly decreasing function of $g$, and by Lemma 2 $i$ is a strictly increasing function of $g$, so that $\frac{1}{P}$ decreases strictly with respect to $i$ and the result follows.

Lemma 5. \[ \frac{\partial^2}{\partial i^2} \frac{1}{P} < 0. \]

Proof of Lemma 5. Consider unit value of the stock with running yield $i$ being sold and reinvested in equal amounts by value in the stocks with running yields $i - \delta i$ and $i + \delta i$ so that the income is unchanged.

Put \[ W(n, i) = \frac{1}{P(n, i)}. \]

By Lemma 3, $W$ is a smooth function of $n$ and $i$, so that we can apply Taylor's Theorem to give

\[ W(n, i - \delta i) = W(n, i) - \delta i \frac{\partial}{\partial i} W(n, i) + \frac{(\delta i)^2}{2} \frac{\partial^2}{\partial i^2} W(n, i - \theta_1 \delta i) \]

with $0 \leq \theta_1 \leq 1$.
and

\[ W(n, i + \delta i) = W(n, i) + \delta i \frac{\partial}{\partial i} W(n, i) + \frac{(\delta i)^2}{2} \frac{\partial^2}{\partial i^2} W(n, i + \theta_2 \delta i) \]

with \(0 \leq \theta_2 \leq 1\).

After the switch, the capital amount on maturity is

\[ \frac{1}{2} [W(n, i - \delta i) + W(n, i + \delta i)] \]

Since prices are in equilibrium under switching action, this cannot exceed \(W(n, i)\) as otherwise a switch of type (i) would exist. Hence

\[ \frac{1}{2} [W(n, i - \delta i) + W(n, i + \delta i)] - W(n, i) \leq 0, \]

i.e.

\[ \frac{\partial^2}{\partial i^2} W(n, i - \theta_2 \delta i) + \frac{\partial^2}{\partial i^2} W(n, i + \theta_2 \delta i) \leq 0. \]

On letting \(\delta i\) tend to zero, each term on the left hand side tends to \(\frac{\partial^2}{\partial i^2} W(n, i)\), and the result follows.

This completes the proof that the conditions 3.1 are necessary for prices to be in equilibrium under switching action.

Suppose now that conditions 3.1 apply, and consider any switch of unit value of a stock with price \(P_0\) and running yield \(i_0\) into:

- \(t_1 \) units of value of a stock with price \(P_1\) and running yield \(i_1\),
- plus \(t_2 \) units of value of a stock with price \(P_2\) and running yield \(i_2\),

\[ \ldots \]

- plus \(t_m \) units of value of a stock with price \(P_m\) and running yield \(i_m\) where

\[ m \geq 1 \]

\[ 0 < t_r < 1 \]

and

\[ \sum_{r=1}^{m} t_r = 1. \]

Since \(\frac{\partial}{\partial i} \frac{1}{P}\) exists, \(\frac{1}{P}\) is a function of the independent variables \(n\) and \(i\).

Let

\[ W(n, i) = \frac{1}{P(n, i)} \]

and put

\[ W(n, i_r) = W_r. \]

For each \(r\),

\[ i_r = i_0 + (i_r - i_0), \]

and by Taylor’s Theorem :

\[ W_r = W_0 + (i_r - i_0) \frac{\partial}{\partial i} W_0 + \frac{(i_r - i_0)^2}{2} \frac{\partial^2}{\partial i^2} W[n, i_0 + \theta_r (i_r - i_0)] \]

where \(0 \leq \theta_r \leq 1\).
Then \[ W_r \leq W_0 + (i_r - i_0) \frac{\partial}{\partial i} W_0 \text{ since } \frac{\partial^2}{\partial i^2} W \leq 0. \]

Hence
\[
t_1 W_1 + t_2 W_2 + \ldots + t_m W_m \\
\leq t_1(i_1 - i_0) \frac{\partial}{\partial i} W_0 + t_2(i_2 - i_0) \frac{\partial}{\partial i} W_0 + \ldots + t_m(i_m - i_0) \frac{\partial}{\partial i} W_0 \\
= W_0 + (t_1i_1 + t_2i_2 + \ldots + t(mi_m - i_0) \frac{\partial}{\partial i} W_0
\]
since
\[
\sum_{r=1}^{m} t_r = 1.
\]

We now use this inequality to identify whether any switches exist which contradict the definition of price equilibrium under switching action.

If the switch results in unchanged income,
\[
t_1i_1 + t_2i_2 + \ldots + t_mi_m - i_0 = 0,
\]
and hence
\[
t_1 W_1 + t_2 W_2 + \ldots + t_m W_m \leq W_0,
\]
so that the capital amount at maturity is not increased and no switch of type (i) exists.

If the switch results in an unchanged capital amount at maturity,
\[
t_1 W_1 + t_2 W_2 + \ldots + t_m W_m = W_0,
\]
and hence
\[
0 \leq (t_1i_1 + t_2i_2 + \ldots + t(mi_m - i_0) \frac{\partial}{\partial i} W_0.
\]

Since \( \frac{\partial}{\partial i} W_0 < 0 \), this gives
\[
t_1i_1 + t_2i_2 + \ldots + t(mi_m \leq i_0
\]
so that the income is not increased and no switch of type (ii) exists.

If the switch results in increased income,
\[
(t_1i_1 + t_2i_2 + \ldots + t(mi_m - i_0) > 0
\]
and
\[
(t_1i_1 + t_2i_2 + \ldots + t(mi_m - i_0) \frac{\partial}{\partial i} W_0 < 0,
\]
so that
\[
t_1 W_1 + t_2 W_2 + \ldots + t_m W_m \leq W_0.
\]

Hence the capital amount at maturity is not increased and no switch of type (iii) exists.

By Definition 1, the price structure is therefore in equilibrium under switching action.
A Mathematical Model for

Chart 1
Running yield % against $\frac{1}{p}$ and term at 31st December 1970.
CHART 2
Par yield curves.
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**Chart 3**

*Par yield curves.*
A Mathematical Model for

**Chart 5**

CHART 6
Treasury 9½% 1999.
SYNOPSIS

This paper investigates what general form of price model is required to represent a market that is in equilibrium under switching action and describes how a special case of this general price model can be applied to the gilt-edged market, with particular emphasis on long-dated stocks. Applications of the mathematical model are then developed in a form that could be used by a life office or pension fund for the management of a portfolio of long-dated British Government stocks. Finally, a comparison is made between conventional yield curve methods of analysis and methods based on this mathematical model.
DISCUSSION

Mr. R. S. Clarkson, introducing the paper, said:—I should first of all like to thank the Faculty for giving me this opportunity to put forward my ideas on this very interesting and very complex subject.

The methods developed in the paper arose from work I carried out some years ago to examine the assumptions implicit in yield curve methods. At that time, the yield curve approach appeared to offer the best practical tool for the management of a gilt-edged portfolio, and my only aim was to reassure myself that these methods had a sound mathematical foundation. Having decided that the critical underlying assumption was that price was a linear function of coupon, I tested market data along the lines described in Section 6.6 and was somewhat surprised to find that systematic departures from linearity occurred. I then set about developing the alternative, and very much more general, approach described in the paper. Since this approach necessitates a return to first principles, much of the paper is inevitably taken up with establishing the mathematical foundations. I trust, however, that sufficient indication has been given of the potential for practical applications.

Although the final price model described by equation 6.1 appears to be somewhat intractable, the computational aspects present no insuperable problems, and Section 7 sets out the principal steps involved in deriving a numerical solution.

It must be emphasised that Part II represents only one particular special case of the general model. There were numerous stages in its construction at which I had to make a choice between two or more possible courses of action, and a full discussion of these was clearly impracticable on grounds of space. But I am confident, Mr. President, that these alternative methods of development will be adequately covered in the discussion, to which I now look forward with interest.

Mr. W. G. Knox, opening the discussion, said:—Mr. President, Mr. Clarkson, Ladies and Gentlemen—My first introduction to this paper was hardly reassuring. In a telephone conversation with the author I think I understood him to say that only one person in the country would truly understand his paper. We were quick to agree that that person was not I. However, we do have before us tonight a paper which has been described by one eminent actuary in the stockbroking community as “an instrument to replace forever gross redemption yields”—but I wonder.

Let me start by thanking Mr. Clarkson very much for tackling a subject of great financial importance to life assurance companies and pension funds, and for coming up with something fresh in a subject already greatly scrutinised by our profession. Normally an opener deals with the paper in some considerable detail, but I shall largely pass over the mathematical parts and leave to the more distinguished mathematicians among you the scope for more esoteric comments.

I shall, however, begin with some gentle points of detail as is the privilege afforded to the opener before trying to relate the author’s model assumptions and conclusions to my own experience of British Government Securities.

I enjoyed part I—I even think I understood it. From time to time I rose to the bait of the weakness of the assumptions regarding investment behaviour only to be assuaged by the generality of the derivation. When
asked to consider contour shapes in the income capital diagrams however I notice that paragraph 3.11 insists that only one stock is held for each term. This I would have thought was taking generality too far, for if we are to accept the concept of “measure of attractiveness” M² introduced in paragraph 3.5 then some historic accident could certainly leave an investor with two different stocks of the same attractiveness and term. Of course this would invalidate the convex shape of the contours in Figure 1 for the stockholder would surely be happier to replace these two stocks with a single stock lying on the curve giving in total more capital and/or income and yet having the same attractiveness. I therefore wondered why the lines weren’t straight. Whilst on the subject of contours I was amused to consider the shape of the lines in Figure 4 under the conditions commonly ruling today where i₀, the yield on irredeemables, is exceeded by the gross redemption yield of many long-dated stocks.

Part II deals with the flesh and bones of graduation and I shall only touch on a few points. In Section 5.2 the adjustment to accrued interest of 0·9 should in my experience be in some way related to coupon and perhaps term. Then in Section 6.4 we come to a great mistake commonly made by life office actuaries. They will insist on attributing some real significance to irredeemables. The market capitalisation of the whole undated sector is less than the capitalisation of many single issues made last year, and the securities themselves are an unmarketable rat-bag of oddities held by very small funds, very old ladies, a miscellany of foreigners who would prefer to remain anonymous, and wide guys who actually believe they can—if the Faculty will excuse market jargon—“screw the sinking fund” in 3½% Conversion. Should we return to the days of new irredeemable issues however—that would be another story! Thus any assumptions based on undated stocks are likely to prove wrong—in the passing, to how many people will we have to explain the F.T. Actuaries yield on irredeemables? In paragraph 6.4 there are therefore two mistakes—stocks with a coupon of i₀ do not stand close to par. The other day when I considered this, i₀ was just over 10½ and the two stocks with that coupon were respectively 6 points premium and 7 points discount to par. Secondly, irredeemable stocks are not the limiting case for large values of term for all the special reasons given. After what I have said it would therefore be churlish to nit-pick on how i₀ is derived in Section 6.5.

I now take a great leap forward to section 7.1 where the text once more re-emerges from the symbols. Here I must go for weights by market capitalisation for that is what big funds must by and large hold and they are what the market is all about. In Section 8.6 the author comes to his first “surprising result”, namely that the second partial differential with respect to i of \( \frac{1}{i} \) is positive. This is apparently a demonstration that investor behaviour was at that time such that the market was not in equilibrium under switching conditions—but I shall return to this. In the next paragraph he makes the discovery that sudden changes in market levels produce for want of a better word “anomalies”. This is hardly surprising in that the market is made by jobbers to whom like Scotsmen “a half is a half” no matter whether it’s a low coupon whisky or a high coupon beer. There’s no point in telling them that price adjustments should be roughly proportional or even be up-dated to take account of accrued interest. A market is a market.

I found Section 9.5 of great interest. The author is referring to high coupon stocks above par that apparently looked cheap in November/December 76. In practice, because they lacked good old-fashioned volatility they turned out to be about the poorest performing longs of 1977
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in relative terms, and this underlined the terrible danger of market models used predictively. But I shall be returning to that theme as well. What the section does refer to is, however, a vital investment consideration, namely the expectations of other investors.

I have chosen merely to flirt with the main part of the paper and from my comments you may feel I disapprove of the model. On the contrary—I don't. Provided it finds its way on to my desk by 9.45 each morning, and I am able to interpret the deviations and what is more important the general shape of the surface—for three dimensional it most certainly is—it will be a useful improvement on existing tools. But other conditions will always dominate in the life office, particularly volatility, tax and marketability.

If I thought the pay-off as it were should come in Section 3 then I should have realised that industrial or perhaps commercial espionage will still be required. Instead we had a nice discussion on the largely familiar ground of par yield curves, which I always think can most easily be demonstrated by the simple process of arbitrage. And then of course we discuss anomaly switching. This I find is seldom possible in a life office these days. On this subject I have long held a simple view. A switch is two separate transactions—a purchase and a sale. You should buy a recently purchased or new holding that by whatever method used seems the cheapest within your overall view. If it happens to be too volatile buy less of it and vice versa. Sell what you can for tax reasons when the stock is not cheap and the sale does not conflict with your view. Personally I never close switches. I cannot believe that stock A which I bought within the last few weeks while cheap is now my dearest holding.

Whilst on the subject of holdings a further point might amuse. Contrary to the spirit of Section 9.2 I believe that sometimes net funds should naturally hold so-called gross stocks, and to a lesser extent pension funds hold net ones. This is simply carrying out the idea of investor expectations. Generally we take a year and a day view on sections of the portfolio and are very careful when dealing in the cum- and ex-periods. If a net fund buys gross stocks it will buy ex-dividend when the stock is unattractive to its natural holders, or in other words cheap, and will sell when in demand, i.e. cum-dividend. But I digress.

Section 4 of the paper gives it an historic context which I think highlights the great strides the author has taken. In Section 16 we reach the conclusion. The author points out that he has dispensed with compound interest functions—of course we are all aware of compound interest's failings, particularly if discounting the same payments at different rates and the whole nonsense of re-investment assumptions. But I am not sure I can forego yields yet. In paragraph 16.2 a great claim for the model is made, namely the projection of future relative price movements, but I would suggest that in the absence of knowledge of scarcity factors affected by tap policy, general interest levels and funding requirements—such projections are fraught with danger. In 16.3 par yield curves are referred to but somehow I think the acid test to the use of the model for pricing new stock issues, would be found wanting. What would happen for example if the authorities were now to issue a 1986 tap stock in that hitherto uncharted and untapped area. My guess is that the surface of the market would change out of all recognition from its pre-tap configuration, and the issue would most probably be a flop. But I think the author would acknowledge this.

So it is at this point that I should like to consider the more general impact of the paper. What does any model for the gilt market do? Firstly it describes a lot of stocks that do not exist. If they were to exist in line
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with the prices forecast for them, then the all important balance between supply and demand—the ultimate arbiter in this near-perfect market—would be upset.

Secondly, models largely ignore the passage of time. Long-term investors do not buy stocks to hold till maturity. I shall not beguile you with the nonsense of running down non-existent yield curves but we must always consider how stocks are likely to be priced at some date in the future. For several reasons, not the least of which is life office taxation, a year suggests itself as an appropriate length of view, and a simple but useful technique is to project each stock from today’s price 1 year forward to give the same rate of return and hence to produce a new price structure with all stocks 1 year shorter. This can be done for several different rates of return and the resulting structures can be examined in the light of experience (or even the author’s model) to determine today’s buys and sells. The point is that this method incorporates volatility and the passage of time.

In another way the assumptions underlying the author’s model do not reflect the passage of time. Investors do not seek the same balance of capital and income in the short term as is implied by holding a stock till maturity. Thus the capital/income preferences are distorted. For example that doyen of volatile net stocks 3½% Funding 1999/04s although a net stock in theory behaves much more like a gross stock than say 5% Treasury 86/89 both shorter, lower in net yield and higher in coupon. Stocks change their nature with respect to time, for example a medium-dated stock becoming specially ex-dividend for the last time prior to being a short is a well-known metamorphosis, but there exist other subtle changes like the crossing of a building society range and the sort of change that 3% Transport has undergone over the last few years. The passage of time is fundamental to gilt activity.

Thirdly, the model does not reflect scarcity. I have already mentioned this in terms of undated stocks, but a further case was the change in status that 5¼% Treasury 08/12 suffered in 1972 when it forfeited its status as the longest-dated stock. The same sort of effect in reverse has been seen with the creation of ultra-high coupon stocks in abundance leaving the old high coupons of 7/10% at a premium rating. It is important not necessarily to call this an anomaly but to recognise that there can be a premium on scarcity. It is all very well to identify cheap stock but if all it ever does is get cheaper because others of its type are continually being created, it is of little consolation to a holder contemplating a tax loss deadline.

Fourthly, in rising markets tax will be a great inhibitor of anomaly switching and here the length of view is dominated by more fundamental characteristics than any yield curve or model can supply—’I have already referred to the author’s attraction to stocks above par one year ago—that was a prime example.

Finally it must be said that nowadays the market is not often likely to be in equilibrium under switching conditions.

Of course the foregoing is largely critical—and that is very unfair for an opening contributor to a paper. But I wouldn’t like, and I am sure the author would agree, any budding actuary to think that there are any predictive panaceas in security markets. In the paper the author says supply and demand are important—surely he is a master of litotes—in perfect markets supply and demand are everything. What we have in front of us today is a delightful and sophisticated model of a market that tomorrow will be quite different. As far as I can tell it does better than its predecessors in what it sets out to do but that is by and large impossible
in the context of a multi-dimensional problem. In fact it does so much better than previous tools that I shall gladly subscribe to the author's daily sheet, of course if the price is right. So perhaps until then I will continue to stumble along with redemption yields and until available theory becomes universal practice I shall merely thank the author once more for his fine contribution to investment thinking.

Dr. K. S. Feldman:—May I begin by congratulating Mr Clarkson on his excellent and original paper. The arguments are persuasive, the analysis rigorous, the examples clear and the research comprehensive. On first reading, the mathematical precision of the paper is so impressive that I was left with the feeling that there is little more that can be said, that virtually all features of the market have been explained, and that gilt-switching can almost be carried out by the computer itself—leaving little for the actuaries or stockbrokers to do.

On further reflection, however, a few nagging doubts began to trouble me. In my recent note in the Journal of the Institute I showed that, for equal performance throughout their lives, the prices of stocks with the same term must vary linearly with coupon. Mr. Clarkson devotes a fair amount of space to the illustration of why tax considerations result in non-linearity, although the rigour of his analysis does not approach that of many of the other sections of the paper. It is this non-linearity aspect of the work which differs from all previous models and, therefore, on which I wish to focus attention. It is clear that non-linearity is, indeed, present in the data, although it would have been helpful to see a table of deviations for each stock, and a statistical argument justifying, in Table 2 (Section 8.1) the improvement in the root mean square error, bearing in mind that the number of degrees of freedom has to be increased from 5 to 7 and there were only a dozen or so stocks considered.

What, I believe, should concern us is that these non-linearities may sometimes present switching opportunities for someone. The author admits this in the last sentence of Section of 9.2. I would contend that by accepting the non-linearity, the author is, for the purposes of identifying anomalies, under-graduating, and the method discussed in Section 11 will not work unless, as is pointed out in Section 8.3, the coefficients $c_{15}$ and $c_{40}$ are stable.

Consider, for example, the scenario of Section 9.2 with the three stocks having the same net redemption yield at some average tax rate. The error term referred to in Section 11.3 is, therefore, zero—as it would be in the linear model.

Let us suppose that the three stocks remain in switching equilibrium, but that some degree of non-linearity gradually begins to develop. This can happen in such a way that the error term remains zero, and this is illustrated by the trace in Chart 6 moving along the horizontal time axis. The average tax-payer—which could mean the life office—will only appreciate that the medium running yield stock is becoming relatively more attractive if he uses the usual price ratios or net-redemption yield models. He is, therefore, back with linear models as the standard. In fact the control theory approach seems to work best with very simple models of yield structure—even a horizontal yield-line can be satisfactory.

It is, in fact, quite easy to construct an example of stocks in equilibrium according to the paper's rules and yet admit switches which are suitable for all classes of investor. If we solve the equation in $1/P$ which has only a constant and a negative $i^2$ term, we obtain a quadratic equation for $P$, and taking some reasonable values for the parameters, we might have a
5% stock at par and a 3% stock at 97. All the conditions for equilibrium are satisfied, and yet all classes of tax-payers will prefer the 3% stock if the term is somewhat less than 2 years. Again one has to return to the conventional performance models in order to exploit such a situation. I admit that this is a rather unsatisfactory example but it was the best I could construct over the week-end. Where the model scores over the alternatives, especially at the longer end of the market, is in its abandonment of notions of performance, thereby eliminating the need to wrestle with embarrassing boundary conditions which lead to linearity. In particular I should appreciate some assistance in the verification that the limit of equation 6.1 tends to something familiar as $n$ tends to infinity.

These arguments do not, of course, invalidate the model. What I am saying is that the degree of departure from linearity should be looked at rather carefully to see if any switching opportunities are present. I still need to be convinced that, as far as switching techniques are concerned, it is worth making an adjustment for non-linearity—indeed, for the reasons already outlined, it could result in switches being missed. The benefits of the simplicity of the linear model should, perhaps, not be discarded too readily, especially when one accepts that, in practice, there will always be error terms, as is mentioned in Section 11.2. Using the linear model on the example in Table 4, it is only necessary to interpolate between the prices of the 5½% and 15½% stocks, in order to see that the 8% 02/06 is relatively expensive.

As far as explaining the structure of market prices, or in order to establish the terms of a new issue, the non-linearity present in the data is of considerable importance, and this has, as the author points out, now been incorporated in the methods used by the Bank of England.

In conclusion I should like to extend my congratulations, once again, to the author for his imaginative work, and trust that, notwithstanding these minor difficulties, there will be agreement that this is one of the most important papers on bond market theory that has appeared for many years.

Mr. J. P. Burman:—It gives me great pleasure to welcome Mr. Clarkson as a competitor in the construction of models to produce par curves for the gilt-edged market. I should like to comment on the generality of his model and to point out some important differences between it and the Bank of England model. I shall also ask Mr. Clarkson to clarify certain points.

He says in Section 14.3 that equation 14.1 is a very general model which includes the Bank’s as a special case. This is true if the functions $h(n)$, $f(n)$ and $\lambda(n,t)$ are completely general, but not, of course, when the first two are quadratic functions and the third is exponential—see equation (6.1).

One major difference between the two models is that ours is formulated in terms of a theory in which the parameters (hopefully) represent expectations and the relative importance of different types of investors, whereas Mr. Clarkson’s is a representation of the data using simple functions. So he has had to decide somewhat arbitrarily how many parameters to use in order to obtain a good fit.

As Mr. Clarkson says, he introduced the valuable idea of indifference curves in 1972 and we adopted it for our model in 1973. But the two models define quite differently the family of indifference curves at varying maturities corresponding to one par curve. Both of us have defined the curves at each maturity in terms of two parameters. He uses an exponential function over the whole length of the curve, and he takes the position and slope of the indifference curve as being given at a particular point,
where the running yield is equal to \( I \) (the irredeemable yield). His two parameters \( c_0 \) and \( c_1 \) define the curvature of the indifference curve and the way in which this changes with maturity. In our model, the indifference curve looks like a walking stick, concave towards the origin for low to medium coupons, but becoming a straight line as the par curve is approached. The two parameters which determine the position and shape of our curve at all maturities are—

(i) the relative weight of net versus gross investors for zero coupon, that is discount bonds; and

(ii) their relative weights on the par curve.

For convenience, net investors are defined in the program as those paying the standard rate of tax.

Since December 1976, Mr. Clarkson’s indifference curve has flipped over, from concave to convex towards the origin, at maturities longer than 17-20 years. Our indifference curve still shows a slight concave shape for low coupons, though it has been almost straight since September 1977. What has happened in the long segment is that stocks with coupons of 12-15½% stand at high premiums and seem to offer yields that are too high in relation to the medium coupons (9-10%). Hence, the part of the indifference curve from 6% upwards has become convex. Our model does not allow this to happen, but the wide spread of yields between medium and high coupons causes it to produce an implausibly high value for one of the parameters.

However, the very high coupon stocks range in maturity from 13 to 21 years. So why does not Mr. Clarkson’s curve for the long segment of the market start by being convex at 15 years and flip over to concave above 20 years? Indeed, there there are so few stocks above 20 years that I doubt whether anyone could make confident statements about the shape of the indifference curve in that region. I would also like to ask whether there is any evidence of flipping over in the 5-15 year segment.

Mr. Clarkson says in Section 12.10 that his model has “explained” the anomaly. But all that has happened is that it has caused certain parameters to take strange values, as is the case also in the Bank model. And we just have to admit that the market has not been in equilibrium for coupon switching since December 1976. This seems to result from an aversion by gross investors to high premiums.

Turning now to the question of the number of parameters Mr. Clarkson’s model has seven in each segment and 22 altogether, while ours has two overlapping segments with four in each, which are spliced together. His solution method is specifically adapted to the problem, but is obviously efficient. Our program uses a general search method for non-linear Least Squares, which is also very efficient: for the eight parameters, it only requires 30 to 40 function evaluations before finding the minimum. I am not sure whether it is a good idea to have two segments or three. Mr. Clarkson’s long segment slopes downwards, perhaps because of the influence of War Loan, which currently yields 2% less than the long-dated stocks; and the Bank curve is unable to pick this up. But his medium segment has only 16 stocks even now, so it seems excessive to use seven parameters to fit it. Incidentally, should I, the irredeemable yield, be regarded as an independent parameter? Could Mr. Clarkson tell us whether changing \( I \) has any effect on the final estimated curve?

My next point concerns the splicing of the three segments together. Mr. Clarkson does not seem to regard this problem as important, though he would admit that the kinks in chart 4 are probably unreal. However, there is a practical problem: in my experience, these curves tend to be
A Mathematical Model for rather unstable towards the end of the segment they are used to fit, unless anchored in some way. For example, if one is studying the yield deviations of a stock with a life of 5½ years, using the 5-15 year curve, one will have to make an abrupt switch to a new set of deviations on the 1-5 year curve when the stock crosses the boundary.

The kinks can be avoided by a device that we have adopted, which is explained in an article in the Bank of England Quarterly Bulletin for June 1976. In this case, it would involve fitting the 1-5 year curve using stocks up to, say, 7 years; then fitting the medium curve, constraining it to be close to the short curve at 5 years, and including stocks up to, say, 17 years to anchor the other end. Then the long curve is constrained to be close to the medium curve at 15 years. Finally, the three curves are spliced together in the way described by Mr. Clarkson. In practice, this avoids kinks. It is, of course, possible to start with the long segment and work back through the medium one to the short one, if preferred.

One final point: Mr. Clarkson mentions in Section 5.2 that most stocks fall in price when they go ex-dividend by, on average, about 90% of the gross dividend. From some independent analysis we have carried out recently on the deviations of stock yields from their calculated values, I can confirm this observation, even for low-coupon short-dated—though there is more variation for the latter. I have enquired in the market about the reason for this, and was told that there are only a limited number of net investors, mainly life funds, who regularly sell the dividend; and that the buyers are either jobbers or discount houses. The figure of 90% would be explained by the difference between the income tax for life funds of 37½% and the capital gains rate of 30%.

In spite of his introductory remarks about gilt-edged being a perfect market, he has drawn attention to several important irrationalities: the coupon anomaly, the yield on 3½% War Loan and the ex-dividend behaviour of low coupon shorts.

In conclusion, may I say again how much I have enjoyed Mr. Clarkson's paper and express the hope that it will encourage the wider use of par curve estimation and a better understanding in the market of the influence of coupon on yield.

Mr. G. T. Pepper:—In the early 1960's there was a technical breakthrough in the mathematical and statistical techniques used in the gilt-edged market. Prior to then the broking techniques had been inaccurate. Accidental anomalies were occurring. For the first 2 years or so after the new techniques were introduced, money could be made by exploiting these accidental anomalies. But then the average level of sophistication in the market caught up and the accidental anomalies no longer occurred. The optimum investment tactics became understanding and forecasting genuine anomalies.

In my opinion the situation is currently rather similar to that in the early 1960's. Broking techniques are again insufficiently accurate. The flaw in redemption yields was pointed out by F. S. Jamieson in a Faculty paper in the early 1960's. A redemption yield assumes compounding at the rate of the redemption yield. If two stocks have the same term but different yields, it is inconsistent to compound interest at different rates. If two stocks have the same running yields but different redemption dates, it can also be inconsistent. Therefore, the use of redemption yields involves an approximation.

In the 1960's, as Clarkson observes in his paper, the gross redemption yield curve was reasonably smooth; the gradient for medium and longer-
term stocks was also not steep. Redemption yields did not differ very much and, therefore, the inherent approximation in their use was acceptable.

In recent years the gross redemption yield curve has become anything but smooth. With redemption yields differing by large amounts, the inherent approximation has become unsatisfactory. Broking techniques are inaccurate in the same way as they were at the start of the 1960's.

Mr. Knox has mentioned that life offices are constrained by capital gains tax. He also stated that every switch consisted of two elements, the sale and the purchase. He stated that he would rather consider the sale and the purchase as separate transactions. He is right—gilt-edged switching is at present unfashionable. The fact that life offices are constrained by capital gains tax has two additional consequences. One, there are fewer people to kick accidental anomalies back into place. Two, the life offices themselves are creating anomalies. Further, the gilt-edged market consists now of a much wider range of coupons and a greater number of stocks. All this should encourage anomaly switching. The stage would appear to be set for a replay of the early 1960's. Investment often consists of fashions with people moving on from one fashion to the next fashion and the next until a full circle has been completed.

The yield curve was an attempt to describe the structure of interest rates in two dimensions. Clarkson uses three—he fits a surface rather than a curve. I have experimented several times with fitting surfaces but I have not succeeded in finding a method which provides a sufficiently stable base against which to measure the cheapness or dearness of individual stocks. Clarkson refers to the need for stability in paragraph 2.2 of his paper. It appears that he has succeeded where I have failed; this will be important for the practical usefulness of his method.

The other main use of Clarkson's method will be monitoring the changing structure of interest rates by observing his parameters. A major problem will be interpretation—how to translate Clarkson's parameters into everyday, practical, investment language.

A lot of work is being carried out in the U.S. which is based on theories about expectations. The original Bank of England yield curve, by Burman and White, was of this family. I am suspicious about this approach because I have observed in our gilt-edged market how the structure of interest rates can often be contrary to that suggested by market expectations of the general direction in which interest rates are moving. The explanation is segmentation, to use the academic term. Different financial institutions dominate the markets in short, medium and long-dated stocks. These institutions often carry out gilt-edged transactions according to their own internal flow of funds, including sales of tap stocks by the Government Broker. Therefore, I am suspicious of techniques based on theories about expectations. I am more attracted by the approaches of Clarkson and Feldman.

I would sum up Clarkson's paper by saying that the time is now ripe for a second technical breakthrough in mathematical and statistical techniques in the gilt-edged markets. Seventeen years have elapsed since the one in the early 1960's. Clarkson is right in the van of the new breakthrough. In particular, the generality of his approach is fascinating. He deserves our many congratulations.

Mr. K. E. Ayres:—There is a remarkable correlation between what Mr. Knox has said and what I intended to say. I shall try to remove some of the repetition, but it is necessary for me to cover some of the same ground in order to make some different points.
I regret that the time available for studying the paper has not allowed me to delve deeply into the mathematics. I am relieved to discover, however, that much of this produces results which can be justified intuitively. This I think applies particularly to Part I of the paper. The lack of deep involvement in the mathematics, on my part, forces me to restrict my comments to one or two aspects of the bases and some practical aspects of the conclusions, relating particularly to work that I have done in the field.

On bases first, I feel that the use of 0.9 times the gross accrued interest as a deduction from the price to obtain the true price, is a rather broad generalisation, particularly since the stock is being considered throughout the year, and not just around the dividend date. I would suggest that there should really be a further variable introduced here which would be particularly difficult, since I suggest that it would need to be dependent on coupon term and non-quantifiable aspects, such as changes in taxation bases for different types of investor and changes in the market dominance by the various classes of investor. I think this factor is probably impossible to allow for, but it should be possible to develop a variable which is dependent at least upon term and coupon, and to use this rather than the arbitrary assumption that 0.9 is always valid.

I would like to digress for a moment on the subject of the weights to be applied to constituent stocks as discussed in paragraph 7.1 since the method used is clearly at variance with the basis used for the new F.T. Actuaries Indices. The point here is that the method of weighting to be used should be dependent upon the uses to which the results are to be put. If an attempt is being made to assess market levels, there is some case for weighting according to the size of issue, but if the results are to be used to assess relative attractiveness of stocks, the weighting should be a function of reasonable marketability. Thus I would certainly accept a weighting of 1 for all stocks for which a market of reasonable size exists and a total exclusion from the data of those where the size of issue is so small as to limit the marketability. New tranches of existing stocks should be ignored and the irredeemables should probably be treated as one stock.

Returning to the paper and paragraph 10.7, I think that the author is only correct where the borrower’s option is to redeem on one of two defined dates, namely the first date and last date. In practice, the borrower can repay on any dividend date between the two dates, and I think that the statement in the middle of that paragraph should in fact read “it may be assumed that the stock will be redeemed before the final date at any time when the value of the par curve is less than the coupon of the stock, within the term corresponding to the length of the optional redemption period”.

In paragraph 11.5 the author points out that when the general market level changes, the root mean square error and the number of buy and sell signals both tend to increase, only to decrease again as the market stabilises. This surely is simply due to the actions of investors taking advantage of an imperfect market in terms of relationships between stocks, which is subsequently restored to nearer perfection as the equilibrium level is approached and thus, as the author says, switching opportunities become fewer in number. This suggestion contradicts, to some extent, the statements made in paragraph 9.2 since there may well be justification for an investor holding a theoretically unsuitable stock, if it can be proved to him that there is a sufficiently large departure from the equilibrium situation to justify his holding such a stock, in the hope of profit as equilibrium is restored.
Finally, whilst I support the concept of the par yield curve for particular purposes, and indeed regard it as invaluable in assessing the true yield level of the market as a whole at a particular term, considerable problems arise in using such a curve to assess the value of particular stocks, and the measurement of deviations from a par yield curve would not work as a relative value tool. The objections to the use of deviations from a gross redemption yield curve set out in paragraph 13.7 are strong, and I would support them, but I have found considerable value over a number of years in applying a linear multiplier to the gross redemption yield which is dependent solely upon coupon, and plotting a curve of the adjusted data, then measuring the deviation of the adjusted yield on the stock from the adjusted yield curve, and comparing that deviation with the history of that deviation. In this way I think the objections raised in paragraph 13.7 are overcome and the comparison of deviations with the history of such deviations goes a long way to neutralise the rough and ready nature of the curve.

I would like to end with my thanks to the author for a stimulating, valuable and detailed contribution to a debate which is likely to continue for as long as there are free markets.

Mr. T. Grimes:—The yield curve is dead. As a model of the gilt market it has been inadequate for some time, and now other models are being suggested to supersede it. The need for these new models has arisen with the increasing range of coupons available on stocks in the gilt market. This fact is noted by the author in Section 13.4 where he justifies it by demonstrating that \( \frac{\partial^2}{\partial t^2} \frac{1}{P} \neq 0 \). I suggest that inspection of any yield sheet or plot of yields against period to maturity would achieve the same result with less difficulty.

Having decided that a model of the market is required which allows for different coupons as well as yields, we have to decide what model to use and how to fit it.

It is worth noting that Mr. Clarkson attempts to derive his model from economic first principles. Whilst such attempts are to be commended, I am not prepared to accept their results without checking. This is especially true when the only economic conditions are inequalities in partial derivatives of \( 1/P \) and where the author feels free to relax even these conditions once his eventual model fails to fit the market.

I therefore look upon equation 6.1 as a possible model for gilt prices and take no account of the 14 pages of preceding discussion which have been used to derive it. What are its most outstanding features? It is complicated, but not more than many similar models. The most important feature to my mind is that it has seven parameters, which is too many. The only factors that are considered by this model are level of the market, term and coupon. Even if we allow two parameters each for term and coupon and one for inter-action, we still have only six and I would be surprised if a step-wise multiple regression would show all of them to be significant.

All of which brings me to my alternative suggestion which is that we use multiple regression on gross redemption yields. This procedure is inherently much simpler, but its simplicity enables the user to bring in only the variables that are actually required. He can look at his fitted model and see, for example, when the curvature of coupon relationship is becoming more important. More importantly other factors can be allowed for. Recently, anomalies have arisen in the market which can be explained by bringing in an extra factor for stocks which are tax free to overseas investors.
This feature has been extremely significant recently, affecting gross redemption yield by 20p or more and must account for a lot of the residual errors remaining after Mr. Clarkson's model is fitted. It is also possible for an investor to concentrate his attention on the part of the surface that is inherently attractive to him. As a pension fund investor, I would hesitate to take into account in any decisions a model that relies at all on the price performance of very low coupon stocks. A model incorporating such stocks may be useful to anyone studying the whole market, but they would have to be monumentally cheap before most gross investors would take any interest in them at all.

To sum up, I find Mr. Clarkson's approach very intriguing and his approach to studying the changes in fitted parameters and residuals over a time wholly commendable, but I feel that to do the job which a gilt-edged investor requires of it, his model is more complicated than is needed, and at the same time, it ignores some features that are important in the real market.

Mr. J. D. Campbell: Robert Clarkson seems to have produced a paper stemming from fresh thinking and a new approach as has been acknowledged by Mr. Pepper and other earlier speakers, which is commendable, even a little exciting, and perhaps a forerunner of further interesting developments.

I wish to take up one aspect of the paper. I was surprised to note early in the paper a reference to \( q_x \), the rate of mortality at age \( x \), because it is not immediately clear how \( q_x \) relates to a model of the gilt-edged market. The reason for the reference emerged when the author writes in 2.2, "There are obvious parallels between the life-table application and the fitting of a mathematical model to prices of gilt-edged stocks". I could not see that there was such an obvious parallel and I then wondered if it was not positively misleading, even dangerous, to suggest such a parallel especially to those with little experience or knowledge of Stock Exchange investing.

The rate of mortality, \( q_x \), is compiled for direct practical use from a very large body of data and is based on averages. Graduated tables of \( q_x \) are "ideals" that are used in practice. They are not "ideals" that are not used. I recall that tables of \( q_x \) should be smooth otherwise there would be too much irregularity in some functions. You cannot err in the use of \( q_x \) as everyone else also uses it. Tables of \( q_x \) can therefore be used with maximum confidence.

Consider now the gilt-edged market. While gilts do conform to something resembling a perfect market—and "perfect" sounds like another word for "ideal"—the body of data on gilts is sparse, as the author says, as opposed to very large for \( q_x \). That is one difference. But the main difference is, I feel, in the use of the "perfect" or "ideal" tables or model. In gilts the purpose is to spot the aberration from the ideal structure, as opposed to mortality tables where you directly use the "ideal" itself. You cannot make use of a gilt-edged model, i.e. to detect an aberration from the ideal, with nearly as much confidence as you can use tables of \( q_x \). The degree of confidence which you can have is, of course, perhaps what this paper is essentially about. The gilt-edged model needs the application of subjective judgment in looking for only what is likely to be a very small aberration from a "norm". Also the aberration in any particular situation will probably not be the only factor for which judgment will be required in coming to a decision.

So I feel that the uninitiated should be wary of believing that a gilt-edged model can be applied in the confident manner of graduated tables of \( q_x \), which might be suggested as possible by the author's comparisons. Even
Mr. G. J. Titford:—Speaking as a practitioner, not a mathematician, in anomaly switching it is necessary to make forecasts about the future shape of the market and its level. The paper seems to provide a beautiful smoothing technique, but will be very difficult to use for forecasting investors' behaviour in the future. The formula is very flexible, too flexible for this purpose.

In paragraph 9.2 I think that there is a slip of the pen in that if the medium running yield stock is dear to both high and low-tax payers, and only neutral to the average tax payer, the price must fall not rise, and the yield must rise to attain the equilibrium position. But in fact such stocks tend to be too dear in practice, and this in spite of the fact that the Bank of England issues its tap stocks at that coupon, around par. Perhaps they do so because those stocks are too dear.

I think the explanation for this dearness is a psychological one—lay investors prefer a stock which gives a good yield without the expectation of a capital loss.

Lastly, in paragraph 9.6 it is suggested that double switches could have been made out of medium running yield stocks in December 1976 into a pair of high and low running yield stocks. I refrained from doing so at that time and do not think I have been worse off in 1977.

Mr. J. Plymen:—I must congratulate Clarkson on a most impressive and fascinating paper. It is notable not only for the elegance of its mathematics and its wide coverage of its special subject but also for its very considerable practical value to all holders of gilt-edged stocks.

In the first sentence Clarkson refers to the gilt-edged market as a perfect market; well perfect is normally a technical term meaning that the analysts follow the market so closely that anomalies are quickly eradicated. The gilt-edged market is also perfect in tactical terms, that is to say the market-ability is very high and huge amounts of stock can always be dealt in at close prices. It must be appreciated that this perfect government security market is a purely U.K. phenomenon; elsewhere in the world where there are corresponding markets in long-term government stocks, America, Canada, South Africa, Australia, Japan etc. the secondary market is of negligible dimensions. Institutions once in the government market are stuck there for good. The reason why the gilt-edged market is so perfect is due to the jobbing system and to the fact that the switchers reinforce the jobbers; in fact the switchers are the master jobbers, stepping in whenever prices are moving out of line. This means that there is an unlimited amount of capital behind the overall jobbing, switching system, to keep the prices in line, and to render it possible for huge lines of stock to be dealt with at close prices. The moral is, that without the switchers, there would be no secondary market. Without a secondary market, institutions would no longer be attracted to gilt-edged stocks. They would seek their fixed interest investments elsewhere. The corollary of this would be that direction of investment would come into gilt-edged stocks at the wrong rate of interest as happens elsewhere in the world.

It is interesting to remember that all this switching activity has been developed by actuaries and actuarial students over the last 40 years. I can remember, as far back as 1937, actuarial students being avidly sought by stockbrokers to build up their switching departments. Over the years these actuarial people have gradually developed the science of gilt-edged
switching which reached its peak until now in the paper read by G. T. Pepper in 1964. It is interesting to see from paragraph 12.8 that Clarkson maintains that his method and that of Pepper produced identical results under the easier conditions of 1965 with lower rates of interest and with lower coupons on government stocks. There is no doubt that with rising rates of interest and rising coupons on new government stocks, further development of gilt-edged theory is needed. Clarkson's technique appears to be a breakthrough into new levels of attainment in this direction and it is very pleasing to see that the lead of the actuarial profession in this specialised field is being maintained.

Clarkson's mathematics is of course formidable. —Here and there he says "it is evident that" and then goes on to develop his argument. At this stage, this means that if you read the paragraph many times and dig into a certain amount of algebra you might be able to understand it. Here and there he says "it can be shown" and this refers to mathematics where you really need to put a towel round your head or consult, as I do, a professional mathematician. I must admit that my own formal mathematical studies ceased nearly 50 years ago, but fortunately I am able to refer to the next generation and to call on the assistance of Dr. Roger Plymen of the Mathematics department of Manchester University to translate the difficult passages.

Much work has been done already in graduating equities to assess their cheapness or dearness. In this connection I would refer to the work of Weaver and Hall and to the Bank of New York equity analysts. In 1972 Prevett and I produced a paper to the Faculty where we tried to graduate investment trust discounts by a mathematical formula. In all these cases a reasonable fit was obtained but unfortunately it was found that the coefficients changed rapidly and somewhat inconsistently so that the cheap or dear assessment by these graduation methods shows very little predictability and very little forecasting ability. The fact is the anomalies were caused by fluctuating coefficients which rectified themselves and were not due to specific cheapness or dearness. Clarkson clearly appreciates this situation and in Table 3 he shows figures once a week for 9 months of the most critical factors. This Table shows for the two critical C factors quite good short-term stability, particularly over the period 24th December 1976 to 29th July 1977, despite a fall of 2% in the gilt-edged yield in the meantime. Consequently it seems likely that Clarkson's coefficients are sufficiently stable for his assessments of cheapness or dearness to be reasonably valid, at any rate for a long enough period for the identification and exploitation of short-term gilt-edged movements. Altogether his method presents a fascinating picture of a professional method of monitoring the gilt-edged market. Presumably one has a daily run from the computer, watches all the time for $e_{40}$ turning to an increasing negative level and one watches the goodness of fit deteriorating and the M.A.D. chart indicators throwing up a number of anomalous situations. If all these symptoms appear at the same time, it is rather like the Delphic Oracle, Clarkson is telling us that something dramatic is going to happen to the gilt-edged market but we do not know quite what it is going to be. Probably this warning, read in conjunction with other factors such as money supply developments and so on, might very well give a clue to a major move in the gilt-edged market and might prove very valuable as a policy guide.

Coming to the detailed formula for paragraph 6.11, there are seven parameters. The three a factors, if studied closely in Table 2 are clearly correlated with the rate of interest. Would it be possible that if the I factor was incorporated with these a coefficients, a better fit might be obtained?
Clarkson has concentrated on the long-term stocks and only refers in passing to the application of the method to the shorter stocks. I imagine he has had little time to explore the short-term situation. Application to the shorts is of great importance, bearing in mind that existing methods have so far failed. The short-term stocks, where the rate of change of term is rapid, and where the rate of interest moves quickly, have proved particularly difficult to assess in terms of traditional statistical techniques. I have one or two points to make as regards the application to the shorts. It seems somewhat curious that for stocks longer than 15 years, a negative polynomial should be used for the $a$ and $n$ factors and that when it comes to less than 15 years stocks, we suddenly have to change over to a positive polynomial. This must surely make for considerable discontinuity around the 15-year line. Secondly, for the shorts, the underlying interest rate would perhaps be better based on the deposit rate rather than the undated rate, which seems rather remote from the point of view of the short-term stocks. Finally, if we find it necessary for the short-term stocks to use what I call a positive polynomial for the $h(n)$ function, it then becomes very similar to the $f(n)$ function and possibly admits of a simplification of the overall formula by combining the $h(n)$ and the $f(n)$ factors.

Mr. A. D. Wilkie, closing the discussion, said:—Much has been made of the difficult mathematics of this paper. But I seem to remember learning about both partial differentiation and Taylor's Theorem at school over 25 years ago. These are the only difficult concepts in the paper, although the algebra that is derived is complicated, and I for one find formula 6.1 very difficult to visualise.

Mr. Pepper has mentioned that investment fashions go in phases. It was interesting to trace through the references at the end of Mr. Clarkson’s paper, and to see the development of ideas in them according to date. First, Marshall’s paper in 1952 appears to have introduced to actuaries the idea of a yield curve. At least, he didn’t give any references to show where the concept came from, though both Irving Fisher and Keynes in the 1930’s were well aware of the differences in redemption yields between short-dated and long-dated securities. Marshall did not require any calculations more than could be done in conventional ways with a hand calculator. He referred only in passing to stocks of different coupons at the same term. At the time the difference in redemption yield between such stocks was quite small, and most stocks lay on a single yield curve. The range of different coupons was in any case also small.

Then in the early 1960’s Pepper and Brew introduced more complex ideas but based on the facilities available with the early use of computers. This mainly involved minimising a linear function, which can be done by simple techniques of matrix inversion (which may well not be in school mathematics, but if I say many simultaneous equations you will know what I mean). The concept also of reinvestment yield was introduced about this time, i.e. that single yield at which the payments from one stock can be reinvested to give the same ultimate proceeds as a second stock of later redemption date.

Yet another series of papers has been produced in the mid-1970’s, those by Pepper and Salkin, several by Burman, the new F.T.-Actuaries Fixed Interest Indices, and also Clarkson’s paper. These have taken advantage of advances in numerical analysis, which allow one to minimise efficiently a non-linear function of several variables with a computer. Thus the approach to the gilt-edged market has depended on the mathematical and
computing tools available, and we are indebted to those who have developed the numerical analysis techniques that have made this possible.

Other concepts have also been introduced: Feldman and Schaeffer (in a paper not published so far as I know) introduced the idea of zero-coupon yields. This involves discounting each payment in the future at a yield appropriate to its date of receipt, but these yields are not necessarily all the same for any one stock. This gets round the difficulty of the redemption yield which apparently discounts payments made at the same time at different rates depending on whether they come from one stock or from another.

Now Clarkson has introduced a concept of price as simply a function of coupon and term, and tried to ignore compound interest concepts. This seems a valuable idea but it has its limitations.

Clarkson's constraints apply only to stocks of one term. But in the market there are only two stocks with identical redemption dates. There were at 15th February 1978 11 pairs of stocks with the same number of coupon payments to be made, three groups of three stocks each and three groups of four stocks each, all with the same number of future payments. A comparison of these groups shows that their prices usually behave sensibly, i.e. in accordance with Clarkson's constraints. However, the following is possible:

One could sell £100 nominal of 9¾% Treasury 1981 at a price of £103·311 (including accrued interest) and give up an income of £9·75 a year for three and a half years, and a redemption amount of £100. One could buy 9½% Exchequer 1981 at a price of £99·687 and get an income of £9·85 for the same term, and a redemption amount of £103·63. Your income is increased by 0·97% and your redemption amount by 3·63%. However, your increased income is received 4 months later on 4th August instead of 1st April. How do you tell whether the extra payment is worth the delay?

Alternatively:

You could sell 12¾% Exchequer 1981 at £110·969, giving up an income of £12·75 for 4 years and a redemption amount of 100. You could buy £1·46 nominal of 8½% Treasury 1980/82 at £96·995, and £98·54 nominal of 14% Treasury 1982 at £111·176. This gives you the same £100 on redemption, but an income of £13·92, an increase of 9·17%. However, again the payments are made later on 15th January and 16th March instead of on 23rd November. Is the delay worth it?

It therefore seems necessary to bring in something connecting stocks of a different term. A simple constraint that could be used is that if you get the same quantity of income but get it earlier it is worth more, or at least no less. Formally: if there is a stock with a coupon due for a term of \( n \) at a price of \( P(g, n) \) the total income to be received is \( gn \). Another stock with a slightly shorter term of \( n - \varepsilon \) with an income of \( g(n/(n-\varepsilon)) \) gives the same total income, but sooner. Expanding by Taylor's theorem, for two variables, and letting \( \varepsilon \) tend to zero gives:

\[
\frac{\partial P}{\partial g} \bigg|_{n} - n \frac{\partial P}{\partial n} \geq 0.
\]

This is saying no more than that reinvestment rates do not fall below zero, since you can always put money on deposit even at zero interest. It may be possible to introduce second order constraints showing that you cannot switch from one stock to two or more other stocks of shorter date and get the same income sooner. I leave that to a further researcher to do.
Let us then go back to Figure 2 of Clarkson’s paper showing indifference curves between capital and income for different investors. One investor chooses the solid lines, and another investor chooses dotted lines. But if we are investors how do we choose where our indifference curves ought to be? Surely we need to compare a given capital and a given amount of income for an appropriate term on some compound interest basis and this takes us back to redemption yields, or something very like them.

There is a sort of two-way process in a market like this. Investors’ preferences, taken to maturity, and expressed through their indifference surfaces (since they are really surfaces over three variables, capital income and term) determine the prices of stock in the market according to the available supply at any one time. The Government can of course affect the market too by changing the supply. Those who write papers, construct indices or whatever, describe the market as they see that it happens to be. But by describing it they set a standard for what it ought to look like. So investors who take account of anomaly switching will look, not at their own preferences, but at what they imagine the market’s preferences to be according to the model. The model begins to determine the market, and people choose their investments with a shorter time horizon, like 1 year, rather than imagining holding them to maturity. Fortunately there are many different competing models so we might think that the market is determined by the interaction of competing models, rather than the interaction of investors’ real preferences. However, I hope that real preferences do not get wholly forgotten.

It seems to me that the last word on the subject has not yet been spoken. Clarkson’s ideas could be further developed, and so could the zero-coupon yield curve concept, and Grimes’ multiple regression technique. It is an advantage to start by analysing the statistic that is most nearly constant for all stocks, and that ends up as something like the redemption yield.

Finally may I make some observations relating to the F.T.-Actuaries Indices. The irredeemables’ yield has been criticised. It was criticised also at the Institute meeting on the subject, and the Committee has taken this into account and decided to change to simply the weighted average of all irredeemable stocks. A note appeared in Saturday’s Financial Times, and the first quotation of the new index will appear in fact in tomorrow’s paper.

Mr Burman agrees with Mr. Clarkson in finding that the root mean square error increases when there is a sudden jump in the market. This suggested to Mr. Knox that jobbers marked their prices without taking into account the underlying pattern that ought to exist. I would have expected this to have occurred, but in the F.T.-Actuaries Government Securities Indices we have not found that this increase in root mean square error happens. Perhaps that has something to do with our curves. But perhaps it is the case that jobbers react correctly to the pressures of the market even when prices change by quite a lot.

Mr. Clarkson, replying to the discussion, said:—I must say at the outset that I am most gratified by the discussion. The mathematical content has been given a clean bill of health, and the points of disagreement that have arisen are more to do with detailed methods of development than with fundamental points of principle. I shall endeavour to comment tonight on most of the points of detail, leaving to a written reply consideration of matters of principle such as the alternative approaches suggested by Dr. Feldman and Mr. Grimes.

There were two areas that attracted repeated critical comment, namely,
the discussion in 9.2 of causes of non-linearity and the number of parameters contained in the model. The demonstration in 9.2 of a possible cause of non-linearity was included purely as an illustration in a simplified special case. The validity of the general model does not depend on this result, but the example is instructive in showing how the usual bond theory assumptions of linearity can break down. A very much fuller discussion of this point is set out in (9), where it is shown how the assumptions in the original Burman and White model are relaxed to give net investors a greater weight in the determination of the prices of low-coupon stocks and gross investors a greater weight in the determination of the prices of high-coupon stocks. On the question of the number of parameters, I do not accept that seven is an unnecessarily high number. Expressed in its simplest terms, the fitting of the price model is equivalent to specifying, in the capital-term diagram, the contours formed by stocks of equal running yield. I have shown in detail how the four arbitrary functions in the general price model represent four quite distinct features in the capital-term diagram, and I stand by my justification for the number of parameters in each function. Because of the immediate graphical interpretation, the five parameters \( a_0, a_1, a_2, b_0 \) and \( b_1 \) are quite harmless, in that they can have little effect on the statistical stability. Their only purpose is to provide continuity and smoothness with varying term, and I regard their fitted values as unimportant. The fitted values of \( c_{15} \) and \( c_{10} \), on the other hand, are very important, since they monitor changes in the equilibrium position between stocks of differing coupon.

I had expected Mr. Knox to ask some searching questions on the practical aspects, and in the event I was not disappointed. His criticism of one of my assumptions in Part I is not, however, justified. I do not say that only one stock can be held by any one investor for each term, only that we cannot assume that more than one stock is held, which is quite different. I enjoyed the eloquent way in which he dismissed the irredeemables as an irrelevance in present day markets. But they do exist, their yields cannot be ignored as a market factor, and the most natural choice for the limiting value of the par yield curve for large values of term is the yield on irredeemables. Chart 3 shows that this limiting condition has led to highly satisfactory shapes for par yield curves at various dates in the past. I agree with Mr. Knox that the adjustment factor for accrued interest should ideally be related to both coupon and term rather than being constant at 0.9. Mr. Ayers also referred to this point, but doubted whether such a modification was feasible. I have found that the constant value of 0.9 is a highly satisfactory approximation, and I was pleased to hear Mr. Burman confirm this. The inaccuracy involved is negligible except around the period when a stock goes ex-dividend, and in practice the income-tax rate suffered on the coupon is a much more important consideration. If greater accuracy is required, one could calculate for each stock an average value, based on closing prices during specially ex-dividend periods, of the ratio

\[
\frac{\text{price in cum-dividend form} - \text{price in ex-dividend form}}{\text{half-year's coupon}}
\]

A convenient method would be to use a geometric average of daily values of this ratio with a value of \( \alpha \) of around 0.15. This modification would, I think, marginally improve the practical usefulness of the model. On the question of weighting by market capitalisation, however, I have no hesitation in stating that I disagree with Mr. Knox. The Mean Absolute Deviation techniques are very much weakened if weighting by market capitalisation is used, and I stand by my arguments in 7.1 for equal weighting. Mr Ayers
supported these arguments, and I assume from the absence of specific reference to the point that the other contributors to the discussion have no objections to the use of equal weighting.

Dr. Feldman puts up a very strong defence of linear models, and is reluctant to accept that any adjustment for non-linearity is worth making. The significant improvement in the goodness of fit, together with the stability of the parameters $c_{15}$ and $c_{40}$ which monitor departures from linearity, is in my view convincing evidence that a non-linear model is required, and the Bank of England have been using a non-linear model since 1973. On the detailed point of the convergence of equation 6.1, I would refer Dr. Feldman to equation 7.1, which provides the most convenient expression for the calculation of graduated prices. The function $h(n)$ was chosen in such a way as to ensure that it tends to a limit as $n$ tends to infinity. Also, for all sets of data tested, the fitted parameters were such that

$$f(n) \gamma(P) \rightarrow \infty \text{ as } n \rightarrow \infty,$$

so that

$$f \rightarrow \frac{q}{1} \text{ as } n \rightarrow \infty,$$

which is the limiting condition one would expect. From Chart 3 it is clear that the par yield curves also behave in a satisfactory manner for large values of $n$.

I am grateful to Mr. Burman for explaining the quite dissimilar specifications of indifference curves that we have chosen for our respective models. My choice is of course much more arbitrary than his, but this is related to the differing applications to which the models are put. Having fitted his price model he takes a cross-section through it to give the par yield curve as the principal application. With my model, however, the par yield curve cross-section is interesting but is much less important than being able to represent the price structure in terms of statistically stable components, and it is with this latter application in mind that I have chosen an exponential curve for the whole of the length of the indifference curve. Again because of the different emphasis on areas of application, I have not experimented to any great extent with short-dated or medium-dated stocks, and I accept the comment that applying my model separately to each of the three maturity areas is somewhat inefficient in terms of numbers of parameters. Mr Burman also asked about the effect of varying $i_0$. I had checked this on several occasions, and found each time that a change of 1% either way made no significant difference to the graduated prices. As would be expected from the discussion in 6.4 of the choice of $i_0$ and the definition of $h(n)$ as a function which tends to a limit as $n$ tends to infinity, the best fit for varying $i_0$ occurs in the vicinity of the yield on irredeemables.

From their practical experience of other models, both Mr. Pepper and Mr. Plymen are well aware that stability in the values of $c_{15}$ and $c_{40}$ is crucial if the model is to provide a sufficiently reliable base against which to measure the cheapness or dearness of individual stocks. The results of Table 3 are, I think, sufficient evidence that this essential requirement is met. When the paper was in the final stages of preparation last September, extreme values of $c_{40}$ were attained and I was very relieved that a remarkably smooth progression in the daily values of $c_{15}$ and $c_{40}$ occurred over the following weeks. The detailed results over this period are listed below in the same form as in Table 3.

Mr. Plymen commented on a possible discontinuity on the changeover from one definition of $h(n)$ to the other. This does not cause any problems. Moreover, the results for long-dated stocks are for all practical purposes unchanged if $h(n)$ is altered from the long-dated form to the alternative
A Mathematical Model for

definition in 6.12. This adds weight to my comment earlier that the parameters $a_0$, $a_1$, $a_2$, $b_0$ and $b_1$ are quite harmless, in that their values do not have to be taken into account when interpreting the results of the model.

The final point I should like to comment on tonight is the suggestion by Mr. Knox that in 16.2 too strong a claim is made for the model, namely that it is sufficiently stable to allow projections of future relative price movements to be made. I think that my conclusion as stated is valid. The important components to be taken into account when making projections are the parameters $c_{15}$ and $c_{40}$ and the price error. I have already discussed the stability of $c_{15}$ and $c_{40}$ and I have found that price error graphs of the type shown in Chart 6 are very much easier to interpret than the more usual graphs showing price ratios and yield differences. There are, I agree, many factors to be taken into account when making projections, and there is of course no guarantee that the projections based on the model will be correct. As has been said on many occasions before, successful management will always ultimately depend on the investor's judgment. My experience has been that the methods described do in fact provide, in a concise form, a very useful statistical background for many of the practical decisions that are required in the management of a gilt-edged portfolio.

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Mr. Clarkson subsequently wrote:—Most of the discussion dealt with the detailed statistical application of the general model rather than with what I consider to be the most important part of the paper, namely the concept that a stable price structure can best be represented as a family of equal running yield contours in the capital-term diagram. The statistical application of this concept then reduces to specifying a family of curves.

Mr. Grimes accepts that a model is required which allows for different coupons at each term and suggests multiple regression on gross redemption yield. However, this approach is very indirect, in that gross redemption yield is itself a complicated function of price, coupon and term, and while I accept that gross redemption yield may be used by many investors to assess the attractiveness of individual stocks, it does not follow that this function provides an efficient description of the resulting equilibrium position. Dr. Feldman approaches the construction of a price model by assuming, as I do, that price is a function of the two variables term and coupon. Having found that linear variation of price with coupon gives a reasonably good fit, he uses a linear model and interprets deviations from linearity as anomalies. The choice between a linear and non-linear model depends on what evidence one requires before accepting that non-linearity is an inherent feature of the market. As I explained in my introductory remarks, I constructed my non-linear model after testing for departures from linearity and concluding that these departures from linearity existed and were of a permanent rather than a transitory nature. Because of the very sparse distribution of long-dated stocks in the capital-term diagram, it is difficult to test directly for non-linearity. However, with short-dated stocks the distribution is very much better. In particular, there are at present (April 1978) several triplets of low, medium and high coupon stocks with nearly similar term. If the prices and running yields of the low, medium and high coupon stocks are $P_1$, $P_2$, $P_3$ and $i_1$, $i_2$, and $i_3$ respectively, then, disregarding the small differences in term, a linear model implies

$$\frac{1}{P_1} - \frac{1}{P_2} = \frac{1}{P_2} - \frac{1}{P_3} = \frac{i_2 - i_1}{i_3 - i_2}$$

However, examination of current data shows that the ratio on the left hand side is always significantly smaller, which corresponds to

$$\frac{\partial^2}{\partial i^2} \frac{1}{P} < 0$$

My interpretation of this very direct test is that the use of a linear model at present for short-dated stocks would be highly unsatisfactory, in that it disregards what I would consider to be important and easily identifiable features of the price structure. My choice of a non-linear model for long-dated stocks was based on tests such as those described in 6.6, which are similar in principle but less direct.

A great deal was said in the discussion about the number of parameters in the model, and some further comment is appropriate. As described in 2.2 a balance must be struck between goodness of fit and statistical stability, but it is important to realise that this balance may vary with different applications. Thus for weekly par yield curves such as in Chart 2 and for par yield curves at widely differing dates as in Chart 3, considerable flexibility in the graduation formula is essential to pick up subtle changes in pattern over time, and for these applications stability of the parameters is of secondary importance. On the other hand, good
statistical stability is essential if the price error is to be used for the identification of anomalies and a very high degree of statistical stability is required if the parameters $c_{15}$ and $c_{40}$ are to be used to monitor changes in the equilibrium position. Although it is almost too much to hope that one single graduation formula could be used for all four of these areas of application with their conflicting requirements of balance between goodness of fit and statistical stability, equation 6.1 does in fact appear to be a satisfactory all-round graduation formula. Should it be desired to improve the statistical stability of the model by reducing the number of parameters, this fine tuning can be accomplished very easily by using definitions of $f(n)$ or $h(n)$ that involve fewer parameters. For example, $h(n)$ could be defined in terms of two rather than three variable parameters as

$$h(n) = a_0 + \frac{a_1}{n}.$$