A GENERALIZATION OF MAKEHAM'S FORMULA
FOR VALUATION OF SECURITIES

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MAKEHAM'S FORMULA

One of the most powerful devices in the theory of compound interest is Makeham's formula. The essence of the formula may be stated as follows:

Consider an admitted loan of $C$ bearing interest at rate $g$ per annum (payable $p$thly in arrear) per unit of admitted capital. The loan is to be repaid by a series of payments each of which contains a capital portion and an interest portion. Let $K$ be the value at rate $i$ per annum effective of all the capital portions. Then the value at rate $i$ of all the interest portions is

\[(g/i^{(p)})(C-K),\]

and so the value of all the total payments is

\[K+(g/i^{(p)})(C-K).\]

Formula (2) is known as Makeham's formula (Makeham, 1874).

It is particularly useful when the loan is being repaid by a large number of payments. In these circumstances, the amount of capital outstanding varies often. Hence the amount of interest in the repayments also varies often, so that direct valuation of the interest payments (or the total payments) can be tedious.

One of the disadvantages of Makeham's formula is that it requires a unique value of $g$ for its application, whereas it is quite common in practice for the coupon rate or the redemption price or both to vary over the term of a particular security. Instances of this are provided in Australia by Commonwealth Special Bonds and Inscribed Stock which are generally of this type. For example, the latest such issue, namely Series Y, issued at par on 4 February 1972, has the following nominal interest rates and redemption prices (redemption at the option of the purchaser):

<table>
<thead>
<tr>
<th>Date</th>
<th>Nominal interest rate (%)</th>
<th>Redemption price (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.2.72–31.12.72</td>
<td>5.4</td>
<td>—</td>
</tr>
<tr>
<td>1.1.73–31.12.75</td>
<td>5.4</td>
<td>100</td>
</tr>
<tr>
<td>1.1.76–31.12.79</td>
<td>5.8</td>
<td>101</td>
</tr>
<tr>
<td>1.1.80–31.12.82</td>
<td>6.0</td>
<td>103</td>
</tr>
<tr>
<td>1.1.83 (maturity)</td>
<td></td>
<td>105</td>
</tr>
</tbody>
</table>

The object of this paper is to show that, when this restriction on $g$ is relaxed, fairly simple extensions of formulae (1) and (2) can be produced.
A GENERALIZATION OF MAKEHAM'S FORMULA

In this section the problem to be considered is that of valuing (at rate \(i\) per annum effective) a loan under which both the redemption price and the nominal interest rate may vary in an unrestricted manner.

Consider the case in which such changes are always separated by time-intervals of non-zero length. Then there exists a sequence \(t_0, t_1, t_2, \ldots\), of times \((t_0 \text{ being the valuation date})\) such that, for each \(j(= 1,2,3, \ldots k)\), the redemption price is constant at \((1 + \lambda_j)\) per unit nominal capital redeemed and the interest rate (payable \(p\)thly in arrear) is constant at \(g_j\) per unit nominal loan outstanding, between times \(t_{j-1}\) and \(t_j\). For the sake of precision, it is stated here that the phrase 'between times \(t_{j-1}\) and \(t_j\)' is taken to mean 'in the semi-open time interval \([t_{j-1}, t_j)\}'. It should also be noted that the symbol \(g\) is now being used to denote interest per unit nominal rather than admitted capital. This notation is rather more convenient, but, more than this, the very notion of admitted capital is drained of any significant meaning in the situation where the redemption price is changing.

Let \(D_j\) be the amount of nominal loan outstanding immediately following any payment made at time \(t_j\).

Then \((D_{j-1} - D_j)\) represents the nominal loan repaid between times \(t_{j-1}\) and \(t_j\). Let the value of these capital payments, as at \(t_0\), be \(K_j\), using \(i\) as the valuation rate of interest.

The nominal loan outstanding at time \(t_{j-1}\), viz. \(D_{j-1}\), can be regarded as two loans, one of \(D_j\) nominal on which interest alone is paid at rate \(g_j\) per annum per unit nominal during \((t_{j-1}, t_j]\), and the other of \((D_{j-1} - D_j)\) nominal which is repaid at a premium of \(\lambda_j\) over this time interval, and with the same distribution of capital repayments as prescribed under the given loan. Therefore, using Makeham's formula for this second loan, the value at time \(t_{j-1}\) of the interest payments, made in the interval \((t_{j-1}, t_j]\)

\[
\frac{1}{i(p)} \left[ g_j D_j v^{t_j - t_{j-1}} + \frac{g_j}{i(1 + \lambda_j)} ((D_{j-1} - D_j)(1 + \lambda_j) - K_j(1 + i)^{t_j - t_{j-1}}) \right]
\]

\[
= \frac{g_j}{i(p)} \left( D_j - D_j v^{t_j - t_{j-1}} + D_{j-1} - D_j - \frac{K_j(1 + i)^{t_j - t_{j-1}}}{1 + \lambda_j} \right).
\]

The value at time \(t_0\), the issue date, of these interest payments is therefore

\[
\frac{g_j}{i(p)} \left[ D_{j-1} v^{t_j - t_{j-1}} - D_j v^{t_j - t_{j-1}} - \frac{K_j}{1 + \lambda_j} \right].
\]  \(\text{(3)}\)

Summing over the range of \(j\) from 1 to \(k\) and adding the value \(K\) of the capital repayments gives

\[
\text{Value of loan} = \frac{1}{i(p)} \sum_{j=1}^{k} \left( D_{j-1} v^{t_j - t_{j-1}} - D_j v^{t_j - t_{j-1}} - \frac{K_j}{1 + \lambda_j} \right) g_j + K
\]  \(\text{(4)}\)
where $K = \sum_{j=1}^{k} K_j$.

Formulae (3) and (4) are the counterparts of Makeham's formulae (1) and (2) respectively in the situation where the nominal interest rate and redemption price are allowed to change. In the case where no such changes in fact occur, so that $k = 1$, formulae (3) and (4) obviously reduce to (1) and (2) as they should.

All of the above working carries through if $k$ takes the value $\infty$, provided, of course, that all of the infinite sums converge.

An example of the use of formula (4) is given in the final section of the paper.

**EFFECT OF TAX ON INTEREST**

If the tax on interest in the period $(t_{j-1}, t_j]$ is assumed to be constant, it is only necessary to replace $g_j$ in formulae (3) and (4) by the rate of interest net of tax, $(1 - \tau_j)g_j$, where $\tau_j$ is the rate of tax in the time-interval in question. If tax is not assumed constant over each such interval, then it would be necessary to increase the number of intervals, since a change in the rate of tax would have the same effect on the value of the interest payments as a change in the rate of interest.

**EFFECT OF CAPITAL GAINS TAX**

If the capital gains made on redemption are to be subject to a capital gains tax of $\gamma_j$, assumed constant in the time-interval $(t_{j-1}, t_j]$, and a capital gain is defined as the excess of the actual amount redeemed over its cost to the investor, provided that this excess is non-negative, the effect of a capital gains tax so operating can be allowed for as follows.

Let $A$ be the issue price per unit nominal loan allowing for tax on interest payments and for tax at the appropriate rate on capital gains made on redemption.

Let $L_{j\ell}$ be the nominal loan redeemed at time $\ell$ after the start of the time-interval $(t_{j-1}, t_j]$, $0 < \ell \leq t_j - t_{j-1}$.

$$\sum_{\ell} L_{j\ell} = L_j = D_{j-1} - D_j.$$  

Value of the capital gains tax payments, as at time $t_0$,

$$= \sum_{j=1}^{k} \sum_{\ell} v_{t_0}^{j-1+\ell} \gamma_j (L_{j\ell}(1 + \lambda_j) - L_{j\ell}A)$$

$$= \sum_{j=1}^{k} \gamma_j \left( K_j - \frac{K_j}{1 + \lambda_j} A \right)$$

$$= \sum_{j=1}^{k} \gamma_j K_j \left( 1 - \frac{A}{1 + \lambda_j} \right),$$
the summation to include all values of $j$ from 1 to $k$ with the exception of those values for which $A > 1 + \lambda_j$.

Since this last condition is not known from the data, it is necessary to use a trial-and-error method to obtain $A$. A comparison of $i$ with the several values of $\lambda_j$ may suffice to determine whether any values of $j$ should be excluded. Otherwise it is necessary to assume that capital gains tax is payable on all redemptions and obtain an approximation for $A$ from the following equation

$$D_0 A = \frac{1}{j(p)} \sum_{j=1}^{k} \left( D_j v^{\tau_j - 1} - D_j v^{\tau_j} - \frac{K_j}{1 + \lambda_j} \right) g_j (1 - \tau_j) + K - \sum_{j=1}^{k} \gamma_j K_j \left( 1 - \frac{A}{1 + \lambda_j} \right)$$

i.e. $A = \frac{1}{j(p)} \sum_{j=1}^{k} \left( D_j v^{\tau_j - 1} - D_j v^{\tau_j} - \frac{K_j}{1 + \lambda_j} \right) g_j (1 - \tau_j) + K (1 - \gamma_j)$

An example of the use of formula (5) is given in the final section of the paper.

**Loans repayable by cumulative sinking funds**

The techniques described in the previous sections can be used to value loans repayable by cumulative sinking funds, and, once the $D_j$'s are determined, the mode of working is just as before. However, it is characteristic of many such loans that the calculation of the amounts of capital repayments presents a preliminary problem in itself. Brief consideration is now given to this matter.

With the additional definitions:

$$s_j = \text{service per annum in the interval } (t_{j-1}, t_j),$$
$$g'_j = g_j/(1 + \lambda_j),$$
$$s'_j = s_j/(1 + \lambda_j),$$

and the assumption that all of the $t_j$'s are integral, it can be shown that

$$D_j = D_{j-1} - (s'_j - D_{j-1} g'_j) s'_{t_j-1} - g'_j$$

or, if $g'_j$ is not a tabulated rate,

$$D_j = D_{j-1} (1 + g'_j)^{t_j-1} - (s_j/g_j) [(1 + g'_j)^{t_j-1} - 1].$$

i.e. $D_j = (1/g_j) [s_j - (s_j - g_j D_{j-1}) (1 + g'_j)^{t_j-1}]$.

With the use of this formula, all of the $D_j$ terms can be determined. The capital value terms $K_j$ can then be found as follows. The amount available for capital repayment at the end of the first year following the start of the $(t_{j-1}, t_j)$ period
A Generalization of Makeham's Formula for Valuation of Securities

(it is assumed that capital repayments are made annually), is equal to \( s_j - g_j D_{j-1} \) and this amount will increase each following year in G.P. the common ratio being \((1 + g'_j)\). Therefore the value of the capital repayments made in the interval \((t_{j-1}, t_j)\) as at time \(t_0\) is given by

\[
K_j = \sum_{t=1}^{t_{j-1}} \nu_{i,t-1} v_t'(1 + g'_j)^{j-1}(s_j - g_j D_{j-1})
\]

\[
= (s_j - g_j D_{j-1})(v_{i,t-1} - v_t'(1 + g'_j)^{j-1})(i - g'_j)
\]

or, if \(i = g'_j\),

\[
K_j = (s_j - g_j D_{j-1})(t_j - t_{j-1})v_{i,t-1}^{j+1}
\]

**EXAMPLES**

Examples A and B below illustrate the use of formulae (4) and (5). In fact, (5) has been used for both, this procedure being valid since (4) is the special case of (5) obtain by setting \(\tau_j = \gamma_j = 0\).

**Example A:**

A loan of \$100,000\) is being repaid over 20 years. Interest, payable half-yearly in arrear, is at 5% per annum on the nominal loan outstanding for the first 5 years, 6% per annum for the next 6 years, and 7½% per annum thereafter. Redemption is as follows:

<table>
<thead>
<tr>
<th>At the end of each of years</th>
<th>Nominal loan to be redeemed</th>
<th>Redemption at</th>
</tr>
</thead>
<tbody>
<tr>
<td>6–10 inclusive</td>
<td>$5,000 per annum</td>
<td>par</td>
</tr>
<tr>
<td>11–15 inclusive</td>
<td>$5,000 per annum</td>
<td>120</td>
</tr>
<tr>
<td>16–20 inclusive</td>
<td>$10,000 per annum</td>
<td>125</td>
</tr>
</tbody>
</table>

It is required to find the issue price to yield 7% per annum effective to an investor taking up the whole loan at inception.

**Example B:**

A loan of \$100,000\) is being repaid over 20 years by equal annual instalments of nominal capital of \$5,000 per annum. Redemption is at par for the first 5 years and at 120 thereafter. Interest, payable half-yearly in arrear, is at 5% per annum on the nominal loan outstanding for the first 10 years and at 5½% per annum thereafter. Income tax on interest payments is at 30% in the first year and 40% subsequently and tax at the rate of 40% in the first year and 50% subsequently is levied on capital gains made at redemption provided these are positive. It is required to find the issue price to yield 3½% per annum effective net of tax to an investor taking up the whole loan at inception.
A Generalization of Makeham's Formula for Valuation of Securities

Formula: \[ A = \left( \frac{k}{\sum_{j=1}^{n} \left( \frac{\sigma J(1-\tau_j) \left( \beta J - \gamma J \left[ D_j - \frac{K_j}{1+\lambda_j} \right] \right)}{(1+\lambda_j)} \right) + \gamma J K_j} \right)^{-1} \left( D_0 - \frac{k}{\sum_{j=1}^{n} \gamma J K_j} \right) \]

Example A

| \( j \) | \( t_j \) | \( \gamma_j \) | \( (1-\gamma_j) \) | \( K_j \) | \( (1+\lambda_j) \) | \( D_j \) | \( \beta_j \) | \( \left( \beta_j - \gamma_j \left[ D_j - \frac{K_j}{1+\lambda_j} \right] \right) \) | \( 12 \) | \( 13 \) | \( 14 \) | \( 15 \) | \( 16 \) |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1   | 5   | 0.50 | 1   | 0   | 100,000 | -712999 | 71,2999 | 28,701 | 28,701 | 1,435.1 | -   | -   |
| 2   | 10  | 0.50 | 1   | 0   | 14,617  | 14,617  | 75,000  | 38,126 | 33,173 | 18,556 | 1,113.4 | -   | -   |
| 3   | 11  | 0.60 | 1   | 0   | 2,251   | 2,251   | 70,000  | 33,256 | 4,870  | 2,494  | 149.6  | -   | -   |
| 4   | 15  | 0.75 | 1   | 0   | 6,555   | 8,046   | 50,000  | 18,123 | 15,133 | 7,087  | 531.5  | -   | -   |
| 5   | 20  | 0.75 | 1   | 0   | 18,576  | 14,861  |        | -2,5842 | 18,123 | 3,262  | 244.6  | -   | -   |

\[ \div i(2) = 50,485 \]

\[ \therefore \text{Price per unit nominal loan} = \frac{45,699 - 50,485}{100,000} = .96184 \]

Example B

| \( j \) | \( t_j \) | \( \gamma_j \) | \( (1-\gamma_j) \) | \( K_j \) | \( (1+\lambda_j) \) | \( D_j \) | \( \beta_j \) | \( \left( \beta_j - \gamma_j \left[ D_j - \frac{K_j}{1+\lambda_j} \right] \right) \) | \( 12 \) | \( 13 \) | \( 14 \) | \( 15 \) | \( 16 \) |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1   | 1   | 0.50 | -7  | -4  | 1281  | 4,831  | 95,000 | -96618 | 91,787 | 28,701 | -   | -   |
| 2   | 5   | 0.50 | -6  | -5  | 1281  | 17,745 | 75,000 | -84197 | 63,148 | 28,701 | -   | -   |
| 3   | 10  | 0.50 | -6  | -5  | 1281  | 22,309 | 50,000 | -70892 | 35,446 | 27,702 | -   | -   |
| 4   | 20  | 0.50 | -6  | -5  | 1281  | 35,735 | 29,479 | -50257 | 35,446 | 5,967  | -   | -   |

\[ \div i(2) = 26,024 \]

\[ \therefore \text{Price per unit nominal loan} = \frac{80,760 - 26,024 - 39,379}{100,000 - 35,048} = 66,887 \]

\[ \text{Adjustment} = \frac{80,760 - 26,024 - 39,379}{64,952} = 1.02979 \]

\[ -4 -5 \]

\[ \therefore \text{Price per unit nominal loan} = \frac{66,887 - 10,805}{64,952 - 10,805} = 77,692 \]

\[ \text{Adjustment} = \frac{64,952 + 10,805}{75,757} = 1.02554 \]
ACKNOWLEDGEMENT

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REFERENCE