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THE WHITTAKER-HENDERSON METHOD OF GRADUATION

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Mr Elphinstone, in his recent paper(1) to the Faculty of Actuaries, has reawakened interest in the method of graduation devised by Sir Edmund T. Whittaker(2, 3) some 32 years ago. Except for Professor A. C. Aitken who, in a paper(4) in the *Proceedings of the Royal Society of Edinburgh* and a note(5) in the *Transactions of the Faculty of Actuaries*, gave a brilliant alternative solution of the equations on which Whittaker's method depends, English and Scottish writers have paid little attention to the method. Robert Henderson's paper(6) published in 1924 stimulated American actuaries, and their investigations have continued up to the present time. The book by Kingsland Camp (reviewed *J.I.A.* LXXVII, 327) gives numerous practical developments of Robert Henderson's work. Students should not overlook a most valuable paper(7) by Charles A. Spoerl which deals comprehensively with almost every aspect of the subject.

The method is allied to graduation by summation formulae, but has the advantages that the run of the coefficients when the formula is extended in linear compound form is theoretically very good and, unlike summation formulae, special methods are not needed to deal with the ends of the series to be graduated. The present note, which does not pretend to be original, gives an account of the method, links up the work of Aitken and Henderson and leads to the formulae which have been found to be of most use in practice. It is thought that the tables in Appendices I and II are new.

It is desired to graduate a sequence of values $u''_x\,(a \leqslant x \leqslant b)$. The graduated values are $u_x\,(a \leqslant x \leqslant b)$. If, as is usual, third differences are regarded as giving a criterion of smoothness the expression $\sum_{x=a}^{b-3} (\Delta^3 u_x)^2$ may be taken as a measure of the roughness of the graduated values. $\sum_{x=a}^{b} (u_x - u''_x)^2$ is a measure of the distortion caused by the graduation. Whittaker's method is to minimize the expression

$$\sum_{x=a}^{b-3} (\Delta^3 u_x)^2 + \epsilon \sum_{x=a}^{b} (u_x - u''_x)^2,$$

where ϵ is a constant chosen to give a balance between roughness and distortion. The equations so derived take the form

$$u_x - u''_x = \frac{1}{\epsilon}\Delta^6 u_{x-3} \quad (a+3 \leqslant x \leqslant b-3),$$

and six other equations, three at each end.

The six equations at the ends can, however, also be brought into the form $u_x - u''_x = \frac{1}{\epsilon}\Delta^6 u_{x-3}$ by the device of introducing three additional values of u_x at each end (i.e. values for $x = a-3,\ a-2,\ a-1,\ b+1,\ b+2,\ b+3$) chosen such that $\Delta^3 u_x = 0$ for $x = a-3, a-2, a-1, b-2, b-1, b$. The equations to be solved are therefore

$$u_x - u''_x = \frac{1}{\epsilon}\Delta^6 u_{x-3} \quad (a \leqslant x \leqslant b),$$

and the six terminal equations $\Delta^3 u_x = 0$.

Whittaker (2) and Lidstone (8) showed that these equations implied that the sum and the first two moments of the graduated and ungraduated values were equal. In symbols

$$\sum_{x=a}^{b} x^r u_x = \sum_{x=a}^{b} x^r u''_x \quad (r = 0, 1, 2).$$

Whittaker expanded the equations in powers of ϵ, which he assumed would be small, and solved the equations, powers of ϵ above the first being ignored. This solution had its advantages because it was easy to derive graduated values corresponding to different values of ϵ and thus to choose the value of ϵ which seemed most suitable.

Aitken (4) gave an exact solution of Whittaker's equations. He showed that the data u''_x could be extended at either end by quantities which lay on two parabolas, one for each end. The graduated values u_x could then be expressed in terms of the original data and the additional data by an infinite series

$$u_x = k_0 u''_x + k_1(u''_{x+1} + u''_{x-1}) + k_2(u''_{x+2} + u''_{x-2}) + \ldots,$$

where $k_0, k_1, \ldots$ were the coefficients of the expansion of $\dfrac{-\epsilon E^3}{(E-1)^6 - \epsilon E^3}$ in the form

$$k_0 + k_1(E + E^{-1}) + k_2(E^2 + E^{-2}) + \ldots.$$

Aitken tabulated the values of k for various values of ϵ. He found that they diminished rapidly and could be ignored after a certain number of terms depending on ϵ.

The six roots of $(E-1)^6 - \epsilon E^3 = 0$ may be associated in pairs whose product is unity and may be denoted by $\alpha, 1/\alpha, \beta, 1/\beta, \gamma, 1/\gamma$, where, to fix the values, α, β, γ are those roots whose absolute magnitude is less than unity.

The additional data to be added at either end may be obtained by the infinite series

$$u''_x = j_1 u''_{x-1} + j_2 u''_{x-2} + \ldots \quad (x > b)$$

$$u''_x = j_1 u''_{x+1} + j_2 u''_{x+2} + \ldots \quad (x < a),$$

where $j_1, j_2, \ldots$ are the coefficients of the expansion of

$$1 - \frac{(1-E)^3}{(1-\alpha E)(1-\beta E)(1-\gamma E)}$$

in the form $j_1 E + j_2 E^2 + \ldots$.

Aitken tabulated the values of $j_1, j_2, \ldots$, which diminish rapidly in the same way as the values of $k_1, k_2, \ldots$.

It will be found that the values of u_x outside the range $a \leqslant x \leqslant b$ are equal to the values u''_x which have been added at each end.

Aitken's original papers should be consulted for a full description of his method. It will be seen, however, that it is a very practical method. First the values $u''_{b+1}, u''_{b+2}, u''_{b+3}, u''_{b+4}$ are calculated by the formula involving the j's. Since they lie on a parabola second differences are constant, which is a check on the calculations. Further values u''_{b+5}, etc., which also lie on the parabola, may easily be obtained by finite differences. A similar process is repeated at the other end. u_a, u_{a+1}, etc., are then calculated by the formula involving the

k's. The method has the advantage that the terminal conditions are automatically satisfied. There are two objections to the method.

(1) If there are few terms in the data to be graduated, the j expansion for computing u''_{b+1}, u''_{b+2}, etc., will stretch beyond the data and involve the values u''_{a-1}, u''_{a-2}, etc., which have yet to be computed.

(2) The k formula is somewhat confusing to use.

Henderson (6) gave an ingenious way of solving Whittaker's equations. In effect, as was pointed out by Joffe (9), Henderson discovered that $1-\frac{1}{\epsilon}\Delta^6 E^{-3}$ can be factorized into

$$\left[1+n\Delta E^{-1}+\frac{n(n+1)}{2}\Delta^2 E^{-2}+\frac{n(n+1)^2(n+2)}{4(2n+3)}\Delta^3 E^{-3}\right]$$
$$\left[1-n\Delta+\frac{n(n+1)}{2}\Delta^2-\frac{n(n+1)^2(n+2)}{4(2n+3)}\Delta^3\right],$$

where

$$\frac{1}{\epsilon}=\frac{n(n+1)^3(n+2)^3(n+3)}{16(2n+3)^2}.$$

These factors are really

$$\{(1-\alpha E^{-1})(1-\beta E^{-1})(1-\gamma E^{-1})\}\{(1-\alpha E)(1-\beta E)(1-\gamma E)\}$$

multiplied by an appropriate constant, thus

$$1-n\Delta+\frac{n(n+1)}{2}\Delta^2-\frac{n(n+1)^2(n+2)}{4(2n+3)}\Delta^3$$
$$=\frac{(n+1)(n+2)^2(n+3)}{4(2n+3)}(1-\alpha E)(1-\beta E)(1-\gamma E).$$

If

$$v_x=\left[1-n\Delta+\frac{n(n+1)}{2}\Delta^2-\frac{n(n+1)^2(n+2)}{4(2n+3)}\Delta^3\right]u_x, \qquad (1)$$

the equation $\left(1-\frac{1}{\epsilon}\Delta^6 E^{-3}\right)u_x=u''_x$ becomes

$$u''_x=\left[1+n\Delta E^{-1}+\frac{n(n+1)}{2}\Delta^2 E^{-2}+\frac{n(n+1)^2(n+2)}{4(2n+3)}\Delta^3 E^{-3}\right]v_x \qquad (2)$$

Formula (2) may be written

$$v_x=\frac{4(2n+3)}{(n+1)(n+2)^2(n+3)}u''_x+\frac{n(3n+5)}{(n+1)(n+2)}v_{x-1}$$
$$-\frac{n(3n+4)}{(n+2)^2}v_{x-2}+\frac{n(n+1)}{(n+2)(n+3)}v_{x-3}, \qquad (3)$$

and enables $v_a, v_{a+1}, \ldots, v_b$ to be calculated in succession provided three starting values $v_{a-3}, v_{a-2}, v_{a-1}$ are given.

Formula (1) may be written

$$u_x=\frac{4(2n+3)}{(n+1)(n+2)^2(n+3)}v_x+\frac{n(3n+5)}{(n+1)(n+2)}u_{x+1}$$
$$-\frac{n(3n+4)}{(n+2)^2}u_{x+2}+\frac{n(n+1)}{(n+2)(n+3)}u_{x+3}, \qquad (4)$$

and enables

$$u_b, u_{b-1}, u_{b-2}, \ldots, u_a, u_{a-1}, u_{a-2}, u_{a-3}$$

to be calculated in succession provided three starting values u_{b+3}, u_{b+2}, u_{b+1} are given.

The solution would be complete if the six starting values were known. Three of them may be obtained quite easily. For if

$$\left.\begin{aligned} u_{b+1} &= v_{b-2} + (n+3)\Delta v_{b-2} + \frac{(n+2)(n+3)}{2}\Delta^2 v_{b-2}, \\ u_{b+2} &= v_{b-2} + (n+4)\Delta v_{b-2} + \frac{(n+3)(n+4)}{2}\Delta^2 v_{b-2}, \\ u_{b+3} &= v_{b-2} + (n+5)\Delta v_{b-2} + \frac{(n+4)(n+5)}{2}\Delta^2 v_{b-2}, \end{aligned}\right\} \quad (5)$$

then the operation (4) produces

$$\left.\begin{aligned} u_{b-2} &= v_{b-2} + n\Delta v_{b-2} + \frac{(n-1)n}{2}\Delta^2 v_{b-2}, \\ u_{b-1} &= v_{b-2} + (n+1)\Delta v_{b-2} + \frac{n(n+1)}{2}\Delta^2 v_{b-2}, \\ u_{b} &= v_{b-2} + (n+2)\Delta v_{b-2} + \frac{(n+1)(n+2)}{2}\Delta^2 v_{b-2}, \end{aligned}\right\} \quad (6)$$

and it is clear that the terminal conditions $\Delta^3 u_{b-2} = \Delta^3 u_{b-1} = \Delta^3 u_b = 0$ are satisfied. It is by no means as easy to ensure that the other terminal conditions $\Delta^3 u_{a-3} = \Delta^3 u_{a-2} = \Delta^3 u_{a-1} = 0$ are satisfied. There are three ways to proceed.

(A) Use Aitken's method to obtain starting values u''_{a-3}, u''_{a-2}, u''_{a-1}. As has been pointed out, these values are also u_{a-3}, u_{a-2}, u_{a-1}. Then v_{a-3}, v_{a-2}, v_{a-1} can be computed from formula (1) combined with

$$\Delta^3 u_{a-3} = \Delta^3 u_{a-2} = \Delta^3 u_{a-1} = 0.$$

(B) It is evident from Aitken's tables that the j-coefficients are so small at a sufficient distance from the end of the table that the contribution of u''_x to a starting value u_{a-1} may be neglected if $x-a$ exceeds a certain minimum value. It is possible to work backwards from arbitrary figures (which may be zeros) at this distance, using suitable recurrence relations, to correct starting values at $a-1$, $a-2$, $a-3$. The following analysis shows how this is done.

Since, for $x < a$,

$$u_x = u''_x \quad \text{and} \quad u''_x = \left[1 - \frac{(1-E)^3}{(1-\alpha E)(1-\beta E)(1-\gamma E)}\right] u''_x,$$

the proper recurrence relation is

$$\{(1-\alpha E)(1-\beta E)(1-\gamma E)\} u_x = \{(1-\alpha E)(1-\beta E)(1-\gamma E)\} u''_x + \Delta^3 u''_x. \quad (7)$$

The recurrence relation for u_x $(a \leqslant x \leqslant b)$ is different from (7), being

$$\left(1 - \frac{1}{\epsilon}\Delta^6 E^{-3}\right) u_x = u''_x.$$

The values u_a, u_{a+1}, u_{a+2} obtained from (7) will therefore not be correct, but once u_{a-1} is reached correct values of u_x are obtained.

Since (7) is not in itself a convenient relation to use recourse may be had to a sequence u'''_x formed from the simpler relation

$$u'''_x = \frac{4(2n+3)}{(n+1)(n+2)^2(n+3)} u''_x + \frac{n(3n+5)}{(n+1)(n+2)} u'''_{x+1} - \frac{n(3n+4)}{(n+2)^2} u'''_{x+2} + \frac{n(n+1)}{(n+2)(n+3)} u'''_{x+3}, \tag{8}$$

which is similar to formula (4).

This is the same as

$$\left[1 - n\Delta + \frac{n(n+1)}{2}\Delta^2 - \frac{n(n+1)^2(n+2)}{4(2n+3)}\Delta^3\right] u'''_x = u''_x$$

or

$$(1-\alpha E)(1-\beta E)(1-\gamma E)\, u'''_x = \frac{4(2n+3)}{(n+1)(n+2)^2(n+3)} u''_x. \tag{9}$$

Then for $x<a$

$$u''_x = \left[1 + \frac{\Delta^3}{(1-\alpha E)(1-\beta E)(1-\gamma E)}\right] u''_x$$

$$= \frac{(n+1)(n+2)^2(n+3)}{4(2n+3)} \{(1-\alpha E)(1-\beta E)(1-\gamma E) + \Delta^3\} u'''_x.$$

Also from (9) for $x<a$

$$u''_x = \frac{(n+1)(n+2)^2(n+3)}{4(2n+3)} \{(1-\alpha E)(1-\beta E)(1-\gamma E)] u'''_x.$$

Hence for $x<a$ $\qquad \Delta^3 u'''_x = 0.$

Hence $\qquad u_x = \left[1 - n\Delta + \frac{n(n+1)}{2}\Delta^2\right] u'''_x \quad \text{for } x<a.$

From (1)

$$v_x = \left[1 - n\Delta + \frac{n(n+1)}{2}\Delta^2\right] u_x \quad \text{for } x<a$$

Hence v_{a-1} may be expressed in terms of u_{a-1}, u_a and u_{a+1}, and, by the use of

$$\Delta^3 u_{a-3} = \Delta^3 u_{a-2} = 0,$$

in terms of u_{a-3}, u_{a-2} and u_{a-1}. Substitution of u_x in terms of u'''_x gives v_{a-1} in terms of u'''_{a-3}, u'''_{a-2}, u'''_{a-1}, u'''_a and u'''_{a+1}. By means of

$$\Delta^3 u'''_{a-3} = \Delta^3 u'''_{a-2} = \Delta^3 u'''_{a-1} = 0,$$

v_{a-1} may then be expressed in terms of u'''_a, u'''_{a+1} and u'''_{a+2}. A similar procedure may be followed for v_{a-2} and v_{a-3}, and when the algebra is carried through the relations obtained are

$$v_{a-1} = \left[1 - (2n+1)\Delta + \frac{(2n+1)(2n+2)}{2}\Delta^2\right] u'''_a,$$

$$v_{a-2} = \left[1 - (2n+2)\Delta + \frac{(2n+2)(2n+3)}{2}\Delta^2\right] u'''_a,$$

$$v_{a-3} = \left[1 - (2n+3)\Delta + \frac{(2n+3)(2n+4)}{2}\Delta^2\right] u'''_a.$$

The method, therefore, is to use formula (8) to obtain u'''_a, u'''_{a+1}, u'''_{a+2}, starting with three zero values of u'''_x sufficiently far away from u'''_a and then to obtain v_{a-1}, v_{a-2}, v_{a-3} from the relationship just given.

(C) If any three arbitrary values (usually guessed approximations to the true values) of v_{a-3}, v_{a-2}, v_{a-1}, are taken as starting values, a set of graduated values will be obtained, but the set will differ from the correct values all the way up the table and the values $\Delta^3 u_{a-3}$, $\Delta^3 u_{a-2}$, $\Delta^3 u_{a-1}$ will not, as they should, be zero. u_x may, however, be corrected by subtracting from it a sequence u'_x obtained by graduating by the Whittaker-Henderson process data consisting of $b-a+1$ zeros with the condition that

$$\Delta^3 u'_{a-3} = \Delta^3 u_{a-3}, \quad \Delta^3 u'_{a-2} = \Delta^3 u_{a-2}, \quad \Delta^3 u'_{a-1} = \Delta^3 u_{a-1}.$$

In theory there is no difficulty in finding the sequence u'_x. With any three starting values u'_{b-2}, u'_{b-1}, u'_b take

$$v'_{b-2} = u'_{b-2} - n\Delta u'_{b-2} + \frac{n(n+1)}{2}\Delta^2 u'_{b-2},$$

$$v'_{b-1} = u'_{b-2} - (n-1)\Delta u'_{b-2} + \frac{(n-1)n}{2}\Delta^2 u'_{b-2},$$

$$v'_b \;\; = u'_{b-2} - (n-2)\Delta u'_{b-2} + \frac{(n-2)(n-1)}{2}\Delta^2 u'_{b-2}.$$

These are the counterparts of the relations (6) and will ensure that the terminal conditions are satisfied at the b end.

The column v'_x may now be calculated as far as v'_{a-3} by the formula

$$v'_{x-3} = \frac{(n+3)(3n+4)}{(n+1)(n+2)} v'_{x-2} - \frac{(n+3)(3n+5)}{(n+1)^2} v'_{x-1} + \frac{(n+2)(n+3)}{n(n+1)} v'_x,$$

which is another way of expressing formula (3), where u''_x is zero and v'_x takes the place of v_x.

The column u'_x may be calculated as far as u'_{a-3} by means of the formula (4) (dashed letters taking the place of undashed), and $\Delta^3 u'_{a-3}$, $\Delta^3 u'_{a-2}$, $\Delta^3 u'_{a-1}$ obtained.

The same process may be carried out starting from three values u'_{b-2}, u'_{b-1}, u'_b linearly independent from the first choice of these quantities, and again a third time starting from the three values u'_{b-2}, u'_{b-1}, u'_b linearly independent from the first two choices of these quantities.

The resulting three sets of values $\Delta^3 u'_{a-3}$, $\Delta^3 u'_{a-2}$, $\Delta^3 u'_{a-1}$ may be combined linearly to produce $\Delta^3 u_{a-3}$, $\Delta^3 u_{a-2}$, $\Delta^3 u_{a-1}$. The same linear combination of the u' sequences produces the corrective sequence u' which is being sought.

It would be a convenience if those three linearly independent columns of u'_x could be replaced by one column. This, in fact, can be done by the device of taking u'_{b-2}, u'_{b-1}, u_b' as 1, 0, 0. For then it will be found that

$$v'_{b-4}, \; v'_{b-3}, \; v'_{b-2}, \; v'_{b-1}, \; v'_b$$

become

$$\frac{(n+3)(n+4)}{2}, \quad \frac{(n+2)(n+3)}{2}, \quad \frac{(n+1)(n+2)}{2}, \quad \frac{n(n+1)}{2}, \quad \frac{(n-1)n}{2}$$

respectively, and

$$u'_{b-4}, \quad u'_{b-3}, \quad u'_{b-2}, \quad u'_{b-1}, \quad u'_b, \quad u'_{b+1}, \quad u'_{b+2}, \quad u'_{b+3}$$

become 6, 3, 1, 0, 0, 1, 3, 6 respectively. These values, of course, solve the equation $\epsilon u'_x - \Delta^6 u'_{x-3} = 0$ and the terminal equations

$$\Delta^3 u'_{b-2} = \Delta^3 u'_{b-1} = \Delta^3 u'_b = 0.$$

But they also satisfy $\Delta^3 u'_{b-3} = \Delta^3 u'_{b-4} = 0$. Hence by cutting off the last term the series could be considered as the u'_x series, where

$$u'_{b-2} = 3, \quad u'_{b-1} = 1, \quad u'_b = 0,$$

and by cutting off the last two terms as the u'_x series, where

$$u'_{b-2} = 6, \; u'_{b-1} = 3, \; u'_b = 1.$$

If, to change the notation, we write $w_1 = 0$, $w_2 = 0$, $w_3 = 1$, $w_4 = 3$, $w_5 = 6$, etc., the general term to solve the equation $\epsilon w_r - \Delta^6 w_{r-3} = 0$, as can easily be verified, is $w_r = (r-1)_{(2)} + (r+2)_{(8)}\epsilon + (r+5)_{(14)}\epsilon^2 + (r+8)_{(20)}\epsilon^3 + \dots$ This formula is not, however, very convenient to use because the coefficients of the powers of ϵ rapidly become large.

Before showing how method (C) may be applied in practice we will consider what values of n are suitable to choose for the Whittaker-Henderson process. The essentials are that the coefficients of the recurrence formula (3) should be simple and that ϵ should be sufficiently small for the formula to have a good smoothing coefficient. Aitken gives the smoothing coefficient as $\frac{1}{270}$ for $\epsilon = {\cdot}01$, $\frac{1}{180}$ for $\epsilon = {\cdot}02$, $\frac{1}{105}$ for $\epsilon = {\cdot}05$. The only satisfactory value of n is 3 giving $\epsilon = {\cdot}009$; formula (3) then becomes

$$v_x = {\cdot}06 u''_x + 2{\cdot}1 v_{x-1} - 1{\cdot}56 v_{x-2} + {\cdot}4 v_{x-3}.$$

A further advantage of this choice of n is that $u_{b-2}, u_{b-1}, u_b, u_{b+1}, u_{b+2}, u_{b+3}$ are the immediate successors of v_{b-2}, v_{b-1}, v_b by constant second differences. In the remainder of this note ϵ will be taken as ·009.

In Appendix I the values of w_r are given from $r = 1$ to $r = 45$ for $\epsilon = {\cdot}009$. In order to illustrate the use of this table the case where $b - a + 1 = 20$ will be taken.

If $u'_{a+x} = w_{22-x} \, (0 \leqslant x \leqslant 19)$

$$\Delta^3 u'_{a-3} = -1526{\cdot}5185, \quad \Delta^3 u'_{a-2} = -966{\cdot}6765, \quad \Delta^3 u'_{a-1} = -613{\cdot}7618.$$

If $u'_{a+x} = w_{21-x} \, (0 \leqslant x \leqslant 19)$

$$\Delta^3 u'_{a-3} = -966{\cdot}6765, \quad \Delta^3 u'_{a-2} = -613{\cdot}7618, \quad \Delta^3 u'_{a-1} = -391{\cdot}4920.$$

If $u'_{a+x} = w_{20-x} \, (0 \leqslant x \leqslant 19)$

$$\Delta^3 u'_{a-3} = -613{\cdot}7618, \quad \Delta^3 u'_{a-2} = -391{\cdot}4920, \quad \Delta^3 u'_{a-1} = -251{\cdot}3352.$$

Suppose the values of w_r from 22 to 3, from 21 to 2 and from 20 to 1 are multiplied respectively by A_1, B_1, C_1, and added, so as to give

$$\Delta^3 u'_{a-3} = 1, \quad \Delta^3 u'_{a-2} = 0, \quad \Delta^3 u'_{a-1} = 0;$$

by A_2, B_2, C_2, and added, to give

$$\Delta^3 u'_{a-3} = 0, \quad \Delta^3 u'_{a-2} = 1, \quad \Delta^3 u'_{a-1} = 0;$$

by A_3, B_3, C_3, and added, to give

$$\Delta^3 u'_{a-3} = 0, \quad \Delta^3 u'_{a-2} = 0, \quad \Delta^3 u'_{a-1} = 1;$$

then in matrix notation

$$\begin{bmatrix} -1526{\cdot}5185 & -966{\cdot}6765 & -613{\cdot}7618 \\ -966{\cdot}6765 & -613{\cdot}7618 & -391{\cdot}4920 \\ -613{\cdot}7618 & -391{\cdot}4920 & -251{\cdot}3352 \end{bmatrix} \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

yielding

$$\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} = \begin{bmatrix} 1{\cdot}023419 & -2{\cdot}756341 & 1{\cdot}794218 \\ -2{\cdot}756341 & 7{\cdot}170701 & -4{\cdot}438437 \\ 1{\cdot}794218 & -4{\cdot}438437 & 2{\cdot}528060 \end{bmatrix}.$$

Table 1. Application of Whittaker-Henderson process to $10^5 q_x$ assured lives 1927–29, durations 3 and over, all classes combined

x	u''_x	v_x	Δv_x	$\Delta^2 v_x$	u_x	Δu_x	$\Delta^2 u_x$	$\Delta^3 u_x$
42½	—	350	—	—	483·3	35·6	−2·4	1·7
43½	—	400	—	—	518·9	33·2	− ·7	2·2
44½	—	450	—	—	552·1	32·5	1·5	2·0
45½	526	492·6	—	—	584·6	34·0	3·5	—
46½	624	529·9	—	—	618·6	37·5	—	—
47½	595	560·0	—	—	656·1	—	—	—
48½	650	585·4	—	—	698·8	—	—	—
49½	803	615·9	—	—	747·7	—	—	—
50½	870	656·4	—	—	803·5	—	—	—
51½	862	703·5	—	—	867·3	—	—	—
52½	954	757·0	—	—	940·9	—	—	—
53½	1,020	816·0	—	—	1,026·5	—	—	—
54½	1,099	880·0	—	—	1,126·4	—	—	—
55½	1,159	947·4	—	—	1,242·6	—	—	—
56½	1,399	1,027·1	—	—	1,376·3	—	—	—
57½	1,627	1,128·6	—	—	1,527·8	—	—	—
58½	1,675	1,247·2	—	—	1,697·3	—	—	—
59½	1,915	1,384·2	—	—	1,885·3	—	—	—
60½	1,925	1,528·1	—	—	2,092·0	—	—	—
61½	2,366	1,690·5	—	—	2,317·1	—	—	—
62½	2,601	1,876·0	212·6	15·2	2,559·4	258·2	15·2	—
63½	2,916	2,088·6	227·8	15·2	2,817·6	273·4	—	—
64½	3,011	2,316·4	243·0	15·2	3,091·0	—	—	—
—	28,597	2,559·4	258·2	15·2	—	—	—	—
—	—	2,817·6	273·4	—	—	—	—	—
—	—	3,091·0	—	—	—	—	—	—

The twenty values of u''_x in Table 1 are ungraduated values of $10^5 q_x$ for the combined years 1927, 1928, 1929 of the British Offices' mortality investigation, durations 3 and over, all classes combined. They are part of the figures graduated by Spencer's summation formula in *Actuarial Statistics*, H. Tetley, 1, 210–11. Table 1 shows the working of the Whittaker-Henderson process.

The starting values 350, 400, 450 for v_{a-3}, v_{a-2}, v_{a-1} are rough guesses at the true values.

Since an error in computation is carried on to each subsequent value it is important to correct errors immediately. A convenient way to do this is to check by the formula

$$v_{x-3} = -{\cdot}15u''_x + 3{\cdot}9v_{x-2} - 5{\cdot}25v_{x-1} + 2{\cdot}5v_x,$$

immediately after a fresh value v_x has been calculated by

$$v_x = 2{\cdot}1v_{x-1} - 1{\cdot}56v_{x-2} + {\cdot}4v_{x-3} + {\cdot}06u''_x.$$

A similar check should, of course, be imposed as the u_x column is formed.

The multipliers A, B, C to give the values

$$\Delta^3 u'_{a-3} = 1{\cdot}7, \quad \Delta^3 u'_{a-2} = 2{\cdot}2, \quad \Delta^3 u'_{a-1} = 2{\cdot}0$$

are

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} \begin{bmatrix} 1{\cdot}7 \\ 2{\cdot}2 \\ 2{\cdot}0 \end{bmatrix} = \begin{bmatrix} 1{\cdot}023419 & -2{\cdot}756341 & 1{\cdot}794218 \\ -2{\cdot}756341 & 7{\cdot}170701 & -4{\cdot}438437 \\ 1{\cdot}794218 & -4{\cdot}438437 & 2{\cdot}528060 \end{bmatrix} \begin{bmatrix} 1{\cdot}7 \\ 2{\cdot}2 \\ 2{\cdot}0 \end{bmatrix}$$

$$= \begin{bmatrix} -{\cdot}7357019 \\ 2{\cdot}2128885 \\ -1{\cdot}6582708 \end{bmatrix}.$$

The calculation of the corrections to u_x of Table 1 is shown in Table 2.

Table 2. Calculation of corrections to u_x

x (1)	$-{\cdot}7357019$ $\times(w_{22}$ to $w_3)$ (2)	$2{\cdot}2128885$ $\times(w_{21}$ to $w_2)$ (3)	$-1{\cdot}6582708$ $\times(w_{20}$ to $w_1)$ (4)	Col (2)+ Col (3)+ Col (4) (5)	u_x (6)	Graduated Value Col (6)− Col (5) (7)
45½	−6,235·7	11,932·8	−5,658·4	38·7	584·6	546
46½	−3,967·2	7,550·9	−3,555·2	28·5	618·6	590
47½	−2,510·4	4,744·3	−2,215·6	18·3	656·1	638
48½	−1,577·3	2,956·6	−1,370·0	9·3	698·8	689
49½	−983·0	1,828·2	−842·8	2·4	747·7	745
50½	−607·8	1,124·7	−518·9	−2·0	803·5	805
51½	−373·9	692·5	−323·0	−4·4	867·3	872
52½	−230·2	431·0	−205·9	−5·1	940·9	946
53½	−143·3	274·8	−136·1	−4·6	1,026·5	1,031
54½	−91·3	181·6	−93·8	−3·5	1,126·4	1,130
55½	−60·4	125·2	−67·1	−2·3	1,242·6	1,245
56½	−41·6	89·5	−48·9	−1·0	1,376·3	1,377
57½	−29·8	65·2	−35·5	− ·1	1,527·8	1,528
58½	−21·7	47·4	−25·0	·7	1,697·3	1,697
59½	−15·7	33·4	−16·6	1·1	1,885·3	1,884
60½	−11·1	22·1	−9·9	1·1	2,092·0	2,091
61½	−7·4	13·3	−5·0	·9	2,317·1	2,316
62½	−4·4	6·6	−1·7	·9	2,559·4	2,558
63½	−2·2	2·2	0	0	2,817·6	2,818
64½	−·7	0	0	−·7	3,091·0	3·092
						28,598

The values of A_1, B_1, C_1, A_2, B_2, C_2, A_3, B_3, C_3 for all values of r from 10 to 40 inclusive are given in Appendix II. From this table the values of A, B, C for a particular case may easily be calculated in the manner shown above.

Three final checks may be made. The sum and the first two moments of the graduated and ungraduated values should be equal to one another. With $55\frac{1}{2}$ as origin these moments are

	Sum	First moment	Second moment
u''	28,597	70,990	462,593
u	28,598	70,979	462,686

An alternative way of forming the corrected graduated sequence is to correct the starting values v_{a-3}, v_{a-2}, v_{a-1} by means of A, B, C and the columns of v'_n also given in Appendix I.

The correction v' to be made is given by

$$\begin{bmatrix} v'_{a-3} \\ v'_{a-2} \\ v'_{a-1} \end{bmatrix} = \begin{bmatrix} 102{,}977\cdot75 & 65{,}760\cdot896 & 42{,}072\cdot450 \\ 65{,}760\cdot896 & 42{,}072\cdot450 & 26{,}956\cdot247 \\ 42{,}072\cdot450 & 26{,}956\cdot247 & 17{,}279\cdot455 \end{bmatrix} \begin{bmatrix} -\cdot7357019 \\ 2\cdot2128885 \\ -1\cdot6582708 \end{bmatrix} = \begin{bmatrix} -6\cdot9 \\ 20\cdot5 \\ 44\cdot4 \end{bmatrix}$$

The starting values v_{a-3}, v_{a-2}, v_{a-1} should therefore be taken as 356·9, 379·5, 405·6 respectively. It is thought, however, that the method of making corrections to u_x given earlier is simpler than repeating the whole Whittaker-Henderson process with the corrected starting values.

It is more usual, but not quite as accurate, to obtain the corrective sequence u' by remembering that w_r may be expressed as

$$A\alpha^r + B\beta^r + C\gamma^r + D\alpha^{-r} + E\beta^{-r} + F\gamma^{-r}.$$

Since $|\alpha|$, $|\beta|$, $|\gamma| < 1$ the terms $A\alpha^r + B\beta^r + C\gamma^r$ become progressively less important as r grows and therefore for large r

$$(1-\alpha E)(1-\beta E)(1-\gamma E)\, w_r \doteqdot 0,$$

i.e.

$$w_r - 2\cdot1 w_{r+1} + 1\cdot56 w_{r+2} - \cdot4 w_{r+3} \doteqdot 0, \quad \text{when } \epsilon = \cdot009.$$

From the table of w_r in Appendix I it is easy to confirm that this relation holds.

u'_x, which is compounded linearly from w_r, where, however, increasing x corresponds to decreasing r, satisfies the relation

$$u'_x - 2\cdot1 u'_{x-1} + 1\cdot56 u'_{x-2} - \cdot4 u'_{x-3} \doteqdot 0, \qquad (10)$$

and the relation will be more accurate at the a end of the sequence than at the b end. Knowing $\Delta^3 u'_{a-3}$, $\Delta^3 u'_{a-2}$, $\Delta^3 u'_{a-1}$, we can calculate the initial values u'_{a-3}, u'_{a-2}, u'_{a-1} very easily.

Thus in our example $\Delta^3 u'_{a-3} = 1\cdot7$, $\Delta^3 u'_{a-2} = 2\cdot2$, $\Delta^3 u'_{a-1} = 2\cdot0$, so that $\Delta^4 u'_{a-2} = -\cdot2$ and $\Delta^5 u'_{a-3} = -\cdot7$.

Writing (10) in the form

$$u'_x + 3\Delta u'_{x-1} + 6\Delta^2 u'_{x-2} + \tfrac{20}{3}\Delta^3 u'_{x-3} \doteqdot 0,$$

we have

$$u'_{a+2} + 3\Delta u'_{a+1} + 6\Delta^2 u'_a + \tfrac{20}{3}\Delta^3 u'_{a-1} \doteqdot 0,$$

$$\Delta u'_{a+1} + 3\Delta^2 u'_a + 6\Delta^3 u'_{a-1} + \tfrac{20}{3}\Delta^4 u'_{a-2} \doteqdot 0,$$

$$\Delta^2 u'_a + 3\Delta^3 u'_{a-1} + 6\Delta^4 u'_{a-2} + \tfrac{20}{3}\Delta^5 u'_{a-3} \doteqdot 0.$$

Therefore

$$\Delta^2 u'_a \doteq -3(2\cdot 0)-6(-\cdot 2)-\tfrac{20}{3}(-\cdot 7)=-6\cdot 0+1\cdot 2+4\cdot\dot{6}=-\cdot 1\dot{3},$$

$$\Delta u'_{a+1} \doteq -3(-\cdot 1\dot{3})-6(2\cdot 0)-\tfrac{20}{3}(-\cdot 2)=\cdot 4-12\cdot 0+1\cdot\dot{3}=-10\cdot 2\dot{6},$$

$$u'_{a+2} \doteq -3(-10\cdot 2\dot{6})-6(-\cdot 1\dot{3})-\tfrac{20}{3}(2\cdot 0)=30\cdot 8+\cdot 8-13\cdot\dot{3}=18\cdot 2\dot{6},$$

$$u'_{a+1} \doteq 18\cdot 2\dot{6}+10\cdot 2\dot{6}=28\cdot 5\dot{3},$$

$$u'_a \doteq 28\cdot 5\dot{3}+10\cdot 2\dot{6}-\cdot 1\dot{3}=38\cdot\dot{6},$$

and subsequent values may be obtained by application of (10).

Except for the last few values, which are in any case small, there is good agreement between the values found by this method and the earlier method of p. 107.

It is of interest to note that

$$\begin{bmatrix} v'_{a-3} \\ v'_{a-2} \\ v'_{a-1} \end{bmatrix} = \begin{bmatrix} 102{,}977\cdot 75 & 65{,}760\cdot 896 & 42{,}072\cdot 450 \\ 65{,}760\cdot 896 & 42{,}072\cdot 450 & 26{,}956\cdot 247 \\ 42{,}072\cdot 450 & 26{,}956\cdot 247 & 17{,}279\cdot 455 \end{bmatrix}$$

$$\begin{bmatrix} 1\cdot 023419 & -2\cdot 756341 & 1\cdot 794218 \\ -2\cdot 756341 & 7\cdot 170701 & -4\cdot 438437 \\ 1\cdot 794218 & -4\cdot 438437 & 2\cdot 528060 \end{bmatrix} \begin{bmatrix} 1\cdot 7 \\ 2\cdot 2 \\ 2\cdot 0 \end{bmatrix}$$

$$= \begin{bmatrix} -382\cdot 921 & 974\cdot 010 & -749\cdot 383 \\ -299\cdot 685 & 785\cdot 901 & -599\cdot 526 \\ -239\cdot 755 & 635\cdot 396 & -472\cdot 958 \end{bmatrix} \begin{bmatrix} 1\cdot 7 \\ 2\cdot 2 \\ 2\cdot 0 \end{bmatrix}.$$

C. A. Spoerl on p. 412 of his paper(7) indicates that unless the number of terms to be graduated (i.e. $b-a+1$) is small the matrix to calculate v'_{a-3}, v'_{a-2}, v'_{a-1} is approximately

$$\begin{bmatrix} -383\tfrac{1}{3} & 975 & -750 \\ -300 & 786\tfrac{2}{3} & -600 \\ -240 & 636 & -473\tfrac{1}{3} \end{bmatrix}.$$

With Spoerl's matrix $[v'_{a-3}\ \ v'_{a-2}\ \ v'_{a-1}]=[-6\cdot 7\ \ 20\cdot 7\ \ 44\cdot 5]$ instead of $[-6\cdot 9\ \ 20\cdot 5\ \ 44\cdot 4]$.

In conclusion it may be said that the Whittaker-Henderson method is simple to apply, gives good results and overcomes difficulties about graduating the ends. If the sequence to be graduated has 40 terms or fewer method (C) should be used. If the sequence to be graduated has more than 40 terms method (A) or (B) should be used. As a supplement to the tables given by Aitken the values of j and k have been computed for $\epsilon=\cdot 009$ and are shown in Appendix III. The values of j have been calculated by the recurrence formula $j_x = 2{\cdot}1 j_{x-1} - 1{\cdot}56 j_{x-2} + {\cdot}4 j_{x-3}$, starting from $j_1 = {\cdot}9$, $j_2 = {\cdot}45$, $j_3 = {\cdot}141$. Since

$$\frac{-{\cdot}009E^3}{(E-1)^6 - {\cdot}009E^3}$$

$$\equiv \frac{3}{506}\left(\frac{13 - 2{\cdot}7E - 10{\cdot}44E^2 + 5{\cdot}2E^3}{1 - 2{\cdot}1E + 1{\cdot}56E^2 - {\cdot}4E^3} + \frac{13 - 2{\cdot}7E^{-1} - 10{\cdot}44E^{-2} + 5{\cdot}2E^{-3}}{1 - 2{\cdot}1E^{-1} + 1{\cdot}56E^{-2} - {\cdot}4E^{-3}}\right),$$

the values of k have been calculated by the same recurrence formula, viz.

$$k_x = 2{\cdot}1 k_{x-1} - 1{\cdot}56 k_{x-2} + {\cdot}4 k_{x-3}.$$

The starting values are

$$k_0 = {\cdot}1541502, \quad k_1 = {\cdot}1458498, \quad k_2 = {\cdot}1241502, \quad k_3 = {\cdot}0948498.$$

These values are also given to 5 places of decimals by C. A. Spoerl ((7), p. 461).

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APPENDIX I

Values of v'_r, w_r, $-\Delta w_{r-1}$, $-\Delta^3 w_{r-3}$ (see p. 105) $\epsilon = \cdot 009$

r	v'_r	w_r	$-\Delta w_{r-1}$	$-\Delta^3 w_{r-3}$
45	874,794,380	272,830,760	−99,247,090	−13,137,119
44	556,538,820	173,583,670	−63,140,170	−8,357,383
43	354,072,160	110,443,500	−40,170,369	−5,316,453
42	225,268,730	70,273,131	−25,557,951	−3,381,894
41	143,327,290	44,715,180	−16,261,986	−2,151,260
40	91,196,953	28,453,194	−10,347,915	−1,368,472
39	58,030,520	18,105,279	−6,585,104	−870,584·8
38	36,927,760	11,520,175	−4,190,765·0	−553,913·7
37	23,499,466	7,329,410·0	−2,667,010·8	−352,497·1
36	14,953,781	4,662,399·2	−1,697,170·3	−224,371·7
35	9,514,877·0	2,965,228·9	−1,079,826·9	−142,850·61
34	6,053,129·6	1,885,402·0	−686,855·2	−90,965·22
33	3,849,876·6	1,198,546·8	−436,734·11	−57,928·81
32	2,447,809·5	761,812·69	−277,578·24	−36,884·95
31	1,555,844·3	484,234·45	−176,351·18	−23,475·54
30	988,640·85	307,883·27	−112,009·07	−14,929·642
29	628,152·48	195,874·20	−71,142·50	−9,484·389
28	399,178·20	124,731·70	−45,205·572	−6,017·182
27	253,812·69	79,526·128	−28,753·033	−3,812·299
26	161,549·65	50,773·095	−18,317·676	−2,412·776
25	102,977·75	32,455·419	−11,694·618	−1,526·5185
24	65,760·896	20,760·801	−7,484·336	−966·6765
23	42,072·450	13,276·465	−4,800·5725	−613·7618
22	26,956·247	8,475·8925	−3,083·4855	−391·4920
21	17,279·455	5,392·4070	−1,980·1603	−251·3352
20	11,064·090	3,412·2467	−1,268·3271	−162·58112
19	7,061·1379	2,143·9196	−807·8291	−105·93441
18	4,480·1913	1,336·0905	−509·91222	−69·37035
17	2,818·6626	826·17828	−317·92975	−45·45334
16	1,754·5482	508·24853	−195·31763	−29·60911
15	1,079·5141	312·93090	−118·15885	−19·02129
14	657·34946	194·77205	−70·60918	−11·93694
13	398·21115	124·16287	−42·08080	−7·23858
12	242·58391	82·08207	−25·48936	−4·18749
11	151·15660	56·59271	−16·13650	−2·27432
10	98·28242	40·45621	−10·97113	—
9	67·62235	29·48508	−8·08008	—
8	49·149	21·405	−6·324	—
7	37·035	15·081	−5·072	—
6	28·15	10·009	−4·009	—
5	21	6	−3	—
4	15	3	−2	—
3	10	1	−1	—
2	6	0	0	—
1	3	0	—	—

APPENDIX II

Values of A_1 A_2 A_3 for $r=10$ to $r=40$ (see p. 107) $\epsilon=·009$.
B_1 B_2 B_3
C_1 C_2 C_3

$r=10$	−11·6956	34·5017	−26·3009
	34·5017	−99·6586	73·6818
	−26·3009	73·6818	−52·3941
$r=11$	−9·93412	27·83043	−19·78984
	27·83043	−76·23024	52·43918
	−19·78984	52·43918	−34·47291
$r=12$	−7·74108	20·51234	−13·48458
	20·51234	−52·84668	33·24638
	−13·48458	33·24638	−19·52950
$r=13$	−5·34571	13·17992	−7·74211
	13·17992	−31·06859	16·81497
	−7·74211	16·81497	−7·67411
$r=14$	−3·08502	6·70030	−3·05792
	6·70030	−12·98263	4·19810
	−3·05792	4·19810	·71976
$r=15$	−1·21969	1·67446	·28709
	1·67446	·16620	−4·17819
	·28709	−4·17819	5·70762
$r=16$	·11451	−1·66658	2·27663
	−1·66658	8·17843	−8·59765
	2·27663	−8·59765	7·79367
$r=17$	·90872	−3·43177	3·11086
	−3·43177	11·72587	−9·86347
	3·11086	−9·86347	7·75712
$r=18$	1·24107	−3·93501	3·09468
	−3·93501	11·92604	−8·96728
	3·09468	−8·96728	6·41064
$r=19$	1·23726	−3·58514	2·56299
	−3·58514	10·04960	−6·90281
	2·56299	−6·90281	4·49333
$r=20$	1·023419	−2·756341	1·794218
	−2·756341	7·170701	−4·438437
	1·794218	−4·438437	2·528060
$r=21$	·7190502	−1·7787468	1·0131444
	−1·7787468	4·1755023	−2·1540443
	1·0131444	−2·1540443	·8727821
$r=22$	·4054314	−·8619869	·3492624
	−·8619869	1·5826931	−·3488539
	·3492624	−·3488539	−·3208520
$r=23$	·1394961	−·1393329	−·1281489
	−·1393329	−·3178598	·8497134
	−·1281489	·8497134	−1·0229985
$r=24$	−·0510534	·3385185	−·4075537
	·3385185	−1·4589837	1·4606225
	−·4075537	1·4606225	−1·2914604

APPENDIX II (*continued*)

$r=25$	−·1648840	·5909239	−·5224861
	·5909239	−1·9622497	1·6267511
	−·5224861	1·6267511	−1·2677380
$r=26$	−·2047829	·6375879	−·4968765
	·6375879	−1·9075580	1·4245976
	−·4968765	1·4245976	−1·0126417
$r=27$	−·2031963	·5825853	−·4141171
	·5825853	−1·6198802	1·1077916
	−·4141171	1·1077916	−·7187928
$r=28$	−·1643643	·4396859	−·2852911
	·4396859	−1·1371688	·7017488
	−·2852911	·7017488	−·3986019
$r=29$	−·11380194	·27992593	−·15900137
	·27992593	−·65316340	·33546077
	−·15900137	·33546077	−·13472292
$r=30$	−·06311900	·13316834	−·05348115
	·13316834	−·24113143	·05025818
	−·05348115	·05025818	·05296274
$r=31$	−·021586037	·020285184	·021376799
	·020285184	·054790388	−·136076854
	·021376799	−·136076854	·160966058
$r=32$	·008775914	−·055864247	·066082125
	−·055864247	·238487654	−·236736093
	·066082125	−·236736093	·208771400
$r=33$	·025441938	−·091144543	·080378001
	−·091144543	·302096073	−·249593772
	·080378001	−·249593772	·193690223
$r=34$	·034924270	−·108448583	·084158471
	−·108448583	·324618969	−·242263572
	·084158471	−·242263572	·172841211
$r=35$	·025068468	−·072163581	·051484591
	−·072163581	·199820866	−·135774162
	·051484591	−·135774162	·086200612
$r=36$	·035398934	−·093353381	·059268410
	−·093353381	·240136215	−·146789023
	·059268410	−·146789023	·084282322
$r=37$	·0152907306	−·0378702825	·0217441014
	−·0378702825	·0881111953	−·0449236878
	·0217441014	−·0449236878	·0168847347
$r=38$	·0098706395	−·0203929110	·0076647513
	−·0203929110	·0358527353	−·0059649682
	·0076647513	−·0059649682	−·0095648131
$r=39$	·00320034417	−·00249061586	−·00399369696
	−·00249061586	−·01077296469	·02309031302
	−·00399369696	·02309031302	−·02642948268
$r=40$	−·00182768320	·01056709550	−·01209523956
	·01056709550	−·04508287969	·04475798080
	−·01209523956	·04475798080	−·04047115369

APPENDIX III

Coefficients j_r and k_r for $\epsilon = \cdot 009$

r	j_r	k_r
0	—	·1542
1	·9	·1458
2	·45	·1242
3	·141	·0948
4	−·0459	·0638
5	−·1364	·0358
6	−·1583	·0135
7	−·1381	−·0020
8	−·0977	−·0109
9	−·0529	−·0144
10	−·0140	−·0140
11	·0141	−·0113
12	·0302	−·0077
13	·0359	−·0041
14	·0339	−·0011
15	·0272	·0010
16	·0187	·0021
17	·0103	·0025
18	·0034	·0023
19	−·0014	·0019
20	−·0042	·0012
21	−·0053	·0007
22	−·0051	·0002
23	−·0041	−·0001
24	−·0028	−·0003
25	−·0015	−·0004
26	−·0005	−·0004
27	·0003	−·0003
28	·0007	−·0002
29	·0008	−·0001
30	·0008	·0000
31	·0006	·0000
32	·0004	·0001
33	·0002	·0001
34	·0001	·0001
35	−·0001	—
36	−·0001	—
37	−·0001	—
38	−·0001	—
39	−·0001	—
40	−·0001	—