MATHEMATICAL MODELS IN PORTFOLIO SELECTION

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SYNOPSIS

A great deal of theoretical and empirical work has been carried out in recent years, particularly in the United States, on the use of mathematical models as an aid to the selection of investment portfolios containing equities. The present paper attempts to survey some of the more important advances that have been made, and to assess them in relation to the problems that managers face in the running of a portfolio.

After a discussion of the background to portfolio selection models and the use of computers, the paper discusses work done on the manner in which share prices move over time. Next the paper examines the principles behind various portfolio selection models, discussing in some detail the work of Markowitz and the more recent developments by Sharpe, together with some related models suggested by other workers. In the final part, various computer-based simulation experiments made with the Sharpe model are described and discussed.

It is stressed that the paper is in the nature of a progress report, outlining the situation as it exists today. More work, both of a theoretical and empirical nature, is necessary before these methods—or developments of them—will be in a form that can achieve universal acceptance.

I. INTRODUCTION

In recent years a great deal of theoretical and empirical work has been published on the movements of equity share prices, reflecting the considerable interest currently shown in the use of mathematical models for the selection and performance of investment portfolios containing equities. This paper attempts to survey some of the more important advances that have been made in this area, and to assess them in relation to the problems that managers face in the running of a portfolio.

Controlling a portfolio is essentially a matter of making decisions, and planning is an aid to this process in that it provides an element of continuity and a framework for the process. Equally, model building is an aid to planning. By constructing a model, different sets of data can be tested and experiments made with different strategies, before coming to any formal decision. Much of the discussion in this paper is centred around statistical and mathematical models that aim to improve portfolio selection. Such models will never by themselves
solve all the problems that arise with the selection and maintenance of a portfolio and the primary emphasis in this paper on models in no way implies that judgment does not have a part to play in portfolio management. However, models can point the way to further developments and also delineate more precisely those areas where management must of necessity provide considered judgments, as against those areas where a detailed technical examination can, in itself, extract all that is there to be gained. In the long run, such an approach to portfolio management will help to discriminate between those areas of economic activity where real wealth can be created and those where it is not being created. To this extent, effective portfolio management should serve to distinguish more sharply between good and bad portfolios and, over a period, help to improve the general efficiency of the economy.

Whilst there have been a number of U.K. papers in the past that have dealt with portfolios of fixed interest stocks, see for example Grant(1) and Pepper(2), there have not been many papers discussing equity portfolios. Furthermore, work on the latter has generally concerned individual equities and has not considered portfolios as such. Thus Weaver and Hall(3) described a multiple regression method for evaluating individual equity shares. The dependent variable used was dividend yield, whilst the so-called independent variables were payout ratios, forecast short-term earnings growth rate, forecast long-term dividend growth rate, historical earnings variability and historical earnings growth rate. A twelve-month forecast period was used. Shares which looked cheap on the dividend yield criterion were then judged attractive for purchase and vice versa. Whilst this approach may well change the objective/subjective balance in assessing individual shares, it does not attempt to look at a portfolio as a whole, where the objectives can differ from those sought from a single share.

Any portfolio selection procedure must consider the various methods which have been suggested for the effective diversification of portfolios. This involves a discussion of the respective roles played by anticipated return and associated risk. Usually, but not invariably, the application of these techniques entails the use of a computer and, to put them in better perspective, some general aspects of the possible use of computer-based methods in portfolio management are first discussed. Any model devised for portfolio selection must take account of the way in which stock prices behave, and a synthesis of the work that has been done on examining the movements in prices is accordingly attempted next. The paper then examines the principles behind various portfolio selection models, discussing in some detail the work of Markowitz and the more recent developments by Sharpe together with some related models suggested by other workers. In the final sections, some computer-based simulation experiments made with the Sharpe model are described and discussed. It must be stressed that this paper is in the nature of a progress report, outlining the situation as it exists early in 1972. More work, both of a theoretical and empirical nature, is necessary before these methods—or developments of them—will be in a form that can achieve universal acceptance.
Managing a portfolio is essentially the management of a continuous process rather than a single 'once and for all' decision problem. Currently, the stage of development is very much akin to the situation a few years back in control engineering for large-scale continuous processes. A great deal of basic methodology had then been developed, but the consequential methods seemed to many plant operators insufficiently precise to replace, or even assist, in operating what was in many instances regarded as a craft activity. A great deal of refinement was therefore still required, in the light of experience, before workable and useful systems could be derived and adopted.

2. BACKGROUND

Much of the modern theory of portfolio selection originates from the U.S.A. and stems from the pioneer work of Markowitz\(^{(4)}\) who, in 1952, first provided a reasonable analytical framework for an investor to choose between a small number of efficiently diversified portfolios. The return anticipated from a portfolio is not the only criterion for selection, as the uncertainty of achieving this return must also be considered. Markowitz took the investor's beliefs about individual shares and suggested the use of the mathematical technique of quadratic programming to compute those portfolios having the highest possible return for each different level of uncertainty. The investor's task was thereby reduced to one of choosing between a relatively small number of these efficiently diversified portfolios.

The actual method put forward by Markowitz was not, however, a very practical one. It required a large amount of information on the investor's beliefs about the relationships between the different shares analysed and, unless there are only a few of these, the calculations required are extensive. The amount of computation involved in applying the Markowitz approach was later vastly reduced by the simplified single index models of Sharpe\(^{(5)(6)}\) published in 1963 and 1967, where the difficulties are largely overcome by relating the price changes of each share to an index of the market rather than to all other shares. The simplifications made the approach relatively cheap and easy to implement, and with them a new era in portfolio selection seemed to have arrived.

In practice, however, and for reasons which will be examined later, portfolio managers have been slow to use the new methods. Portfolios which make real use of these techniques for their management are still the exception rather than the rule, and the importance of the Markowitz theory has chiefly lain in explaining why and how people try to diversify, rather than in helping them to carry this out efficiently.

To investigate why this situation has arisen, imagine that a portfolio manager is handed a highly detailed blueprint of what his ideal portfolio ought to look like. This blueprint has been prepared at great expense using a complicated method, which the manager doesn't properly understand and possibly involving extensive computer work. He is quite likely to give it no more than a passing glance before shrugging...
his shoulders and filing it somewhere, possibly in the wastepaper basket. Indeed, what use could any competent manager make of this sort of document? For a computer-based system of this kind to be of any use at all, it must be designed to provide the manager making the decisions with information that is relevant to his particular needs at that moment in time and thus help him to solve problems which he has already identified. It should provide this information in a readily intelligible form. The manager would then be able to use the system as a tool which will provide answers at once to the variety of questions which are of immediate concern to him, within the time span for which the answers are still of use and interest. It is important, too, that the manager should understand the assumptions implicitly involved in producing these answers, for otherwise he is not in a good position to appreciate fully either their value or their limitations.

These requirements may seem to be asking a great deal of both the manager and the computer model. It is nevertheless encouraging to note that the recent advances in computing facilities, and in the mathematical models available for portfolio analysis, do mean that such systems are becoming more readily available, and the range of questions that can be answered in this way is rapidly increasing. On-line computer terminals enable many calculations to be carried out very quickly and at comparatively little cost. Much of the recent theoretical work has resulted in simplifications of earlier work, producing models for portfolio analysis which are both easier to understand and cheaper to implement than earlier models. For the first time, those involved in managing portfolios can demand the computer aids relevant to their specialized requirements.

3. COMPUTER SYSTEMS IN PORTFOLIO MANAGEMENT

As mentioned earlier, it is hardly likely that with so many factors to take into account, the computer will replace either the security analyst or the portfolio manager, but it is equally true that both can benefit considerably from its intelligent use. It is worth saying at this stage that there is a great deal of misunderstanding in business generally concerning computers and their role in the planning of operations. Opinions vary from the enthusiasts who believe that the complete planning operation can be automated and put on a computer, with all sorts of elaborate choices available, to those who argue that the computer can play no part whatsoever in planning. The latter by implication exclude, of course, those tools and techniques which are only feasible with the aid of a computer, which is a severe limitation. There is an analogy here with complex allocation problems which need, for their solution along optimum lines, the use of linear programming techniques and, in turn, a computer. To ignore the computer is effectively tantamount to accepting second best. As mentioned earlier, however, this does not rule out the need to use judgment—indeed the spotlight is turned on to those areas where judgment is really required.

It is sometimes argued that it is not worthwhile using a sophisticated model or analytical procedure in some particular circumstances because the input
information is of poor quality. This can be a specious argument. If decisions have to be made on poor data, it must surely be true that the best possible analysis should be carried out, only taking care that ‘best’ incorporates the knowledge that the data are poor. A simple example will illustrate the point. Suppose that you have to decide immediately between two possible processes, A and B, for the manufacture of 100 units of article X that are wanted in a hurry. You are told that A takes 18 minutes whilst B takes 20 minutes per unit. Under these circumstances you would no doubt choose process A. But if you were also told that the estimate of 18 minutes per unit for A was an average based on very widely differing times observed for just two units, whilst that for B was based on 2,000 units all of whose times for processing fell between 19 and 21 with an average of 20 minutes, you might take a rather different view and want to delve more deeply into such matters as the effects of costs of delays in manufacture. In this instance the judgment needed is relatively clear cut. If, however, there were numerous alternatives and the products required half a dozen separate processes, it would be difficult to steer through this maze without using an appropriate analytical technique, possibly allied to a computer.

In turning to the use of computers in portfolio management, the particular problems involved in the management of portfolios must be outlined. These are conventionally split into two stages. The first stage is usually the concern of the security analyst, whose job it is to evaluate the future prospects of individual equity shares and to discover those which are currently underpriced, drawing on whichever of the possible sources of information seems relevant.

The second stage is the responsibility of the portfolio manager who has the task of combining these individual evaluations into predictions for aggregate portfolios of shares, on the basis of which he must decide what changes to make in his portfolio. A multitude of factors have to be considered. What sort of balance of holdings is most consistent with the aims of the portfolio? How best should the holdings be spread across the shares from different market sectors, or from those with different risk characteristics? How active a trading policy should he pursue? How should he balance long-term and short-term requirements, and how much liquidity is desirable? He may, for example, consider it desirable to hold a certain percentage of blue chip equities and there are likely also to be institutional and legal constraints restricting his actions. Within these broad matters of trading policy, individual decisions must be made as to when, and how much of, particular shares should be bought and sold. The way he thinks the market will move, the way he views the economic situation and the international position, as well as the effect of having to pay taxes and transaction costs of various kinds, including the jobber’s turn, are all relevant to his decisions.

The initial stage in setting up any kind of computer system for use in portfolio management is usually to establish an appropriate bank of data that can be used efficiently for assessment purposes. Information must be recorded on each share in which the portfolio may be interested and, since the storing of such data can
be an expensive business, a great deal of thought must go into deciding the quantity of data needed. For example, how far back should the data go, and should prices be stored on a monthly, weekly, daily or some other basis? How should information be recorded on bonus issues, new issues and dividends, and how should the prices be adjusted to take account of these? How much additional data about the companies themselves should be stored and what aggregate indices may be usefully recorded? The answer to these questions will obviously be partially dependent on the costs involved and in turn will affect the best way of handling the data. Consultation between the future users of the system and those responsible for its design and implementation is, therefore, essential before finally settling these details.

With a sensible bank of data to draw on, it is possible to mechanize many existing procedures and to introduce new ones. It is arguable that the mechanization process should not be carried too far, for there is a danger that if too much is taken out of the hands of the analyst, he will lose some of his insight into the workings of the market. The computer can be used on a routine basis to screen prices for unusual changes which might otherwise escape unnoticed in the sheer volume of share information. Many other forms of screening are also possible, although there are problems involved in deciding what factors are relevant. Calculations to evaluate stocks through making analytical projections of the probable levels of future earnings, to which appropriate discount factors are applied, can also be mechanized to some extent, whilst the computer provides a valuable tool for assessing the performance of the portfolio itself and for making comparisons between this and other portfolios or stock market indices.

Various forms of analysis can be used to investigate factors influencing share prices. Insight into share valuation can be gained by means of cross-sectional regression models (see, for example, Whitbeck and Kisor(7), which attempts to explain P/E ratios in terms of various measurable company and industry characteristics). Time-series analysis of price changes can be used to investigate such things as the importance of business cycles and macro-economic indicators in general (Sprinkel(8)) and the way in which stock prices tend to move together and in groups (King(9)).

4. SHARE PRICES AND THEIR MOVEMENTS

Over the years, a large amount of empirical work has been done to investigate the levels and changes of stock prices. Although some very interesting results have emerged, it is noticeable that very few definite conclusions or firm theories have been reached.

The major conclusion that can be drawn from the various statistical analyses which have been performed is basically that there is no simple and foolproof way to make a fortune through investment in ordinary shares. It is true that the level of share prices is correlated with the general level of economic activity but,
even if the latter can be forecast precisely, it does not provide a way of forecasting precise changes for individual shares. It is usually impossible to detect a reliable cyclical effect which can be expected to persist and there are no strong correlations between successive price changes.

When it comes to the consideration of risk, it has not even been possible to produce a definition which can be universally accepted. Everybody agrees that risk is something that is to be minimized where possible, and that this can be achieved by appropriate diversification, but few have a clear idea as to what they mean by risk, let alone how to measure it. The theoreticians try to beg the question and approach the handling of risk indirectly. However, since risk is presumably related in some way to the uncertainty of achieving a desired level of return, it seems sensible to work directly with expected returns and the uncertainty of actually obtaining them.

In principle the uncertainty of a return could be measured subjectively by asking the security analyst, who is forecasting the expected return from a share, to guess at different confidence limits about this forecast value, so that the probability of very high or very low returns can be estimated. So far as is known, very little attention has been paid to this approach, as it is hard enough to get an analyst to commit himself to an expected value, let alone a confidence limit. It seems unlikely that such estimates of uncertainty will be very reliable, particularly as they would depend upon assumptions made about the general state of the market. For these reasons the objective approach of measuring the variation of past returns is usually taken as a basis for estimating the uncertainty for the future and this approach seems to work fairly well (see, for example, Works\textsuperscript{(10)}).

A great deal of research has been done on the variation of share prices, as well as for the total return from shares over time. The price change of a share from between one point in time, $t$, and one time period later, $t+1$, is usually adjusted for net dividend payments accruing during the corresponding time period to give the return as:

\[ 1 + R_t = \frac{P_{t+1} + D_{t+1}}{P_t} \]

where $P$ denotes price, $D$ denotes dividends received in the time period and $R$ denotes the proportional return. It is often necessary to make adjustments for bonus and rights issues, and in many contexts it makes better sense to work with the logarithm of (1). Much attention has been paid to the correlations between the values of these single period price changes and also to the shape of the corresponding distributions of the $R_t$ values.

The random walk school of thought, which is linked to the distribution of the $R_t$ over successive intervals of time, has assumed such importance that it is worth more than a passing mention at this point. Let $X(t)$ denote a random variable that is equal to the price of a particular equity investment at time $t$. Then $\Theta = X(t+T) - X(t)$ is the change in price of the investment from time $t$ to
time $t+T$ and is also a random variable. The first form of random walk hypothesis assumes that the values of $\Theta$ for non-overlapping intervals are mutually independent. A second and further form assumes further that $\Theta$ has a normal distribution with mean zero and variance proportional to $T$. The quantity $\Theta T^{-\frac{1}{2}}$ would then have a normal distribution whose parameters do not depend upon $T$.

Kendall\textsuperscript{(11)} in 1953 published an analysis of sample serial correlation coefficients for weekly changes in a group of nineteen share indices and for weekly and monthly price changes for several commodity prices. Sample serial correlation coefficients near zero would tend to support the random walk model. Kendall did obtain many results close to zero and, from the discussion of his paper, this was obviously felt to be surprising. Many later analyses (Cootner\textsuperscript{(12)} and Fama\textsuperscript{(13)} have also given similar results, although it is fair to point out that, given the number of tests carried out, the results are not incompatible with the hypothesis that there is some small correlation present, particularly bearing in mind the apparent non-normality of the data. The open question is whether any such dependencies are of investment importance. That is, after taking into consideration trading costs, do they produce considerably greater yields than do simpler trading rules that use only the sequences of past price changes?

Investigations have also shown that the distribution of single-period price changes depends upon the length of the time period for which the individual changes are defined but, irrespective of this, it turns out to have a slightly unusual shape. Compared with the familiar bell-shaped normal distribution, the distribution of price changes is more peaked, slightly skew and has longer tails. It has been suggested that the underlying probability distributions may either be of a log-normal form (see, for example, Osborne\textsuperscript{(14)}), or else belong to a class of distributions known as stable-Paretian (see, for example, Fama\textsuperscript{(15)}). If either of these is the case, then the standard deviation is not a good measure of spread to use to describe the distribution, and some more robust statistic such as the inter-quartile range would be more appropriate. The difference between these distributions and the normal distribution is fairly slight, and studies on the effect of diversification on the spread of possible returns show that, from this point of view at least, no harm is done by regarding the distributions as normal. There is no conclusive evidence for the distribution to have infinite variance and, although the tails are certainly fat, they may well be truncated in some way.

Diversification can reduce the variability in the return achieved by a portfolio of equities, but only to a limited extent because of the way share prices tend to move together under common influences. Some studies have been done on this co-movement of prices and, although the major published results refer to American stocks (see King\textsuperscript{(9)}), there is every reason to suppose that very similar results would hold for British equities. King’s results suggest that the variation of a stock price can be thought of as due to a combination of four factors. The most important factor acts solely on the individual stock itself, mainly in response to information about the company and various unaccountable pressures of
buying and selling. It would typically give rise to about 40% of the variance of the price of the stock. The market influence is another major factor and the movement of the market as a whole may explain another 30% of the variation. Of the remaining 30%, perhaps 10% will be due to a common industry influence and 20% to the other factors, which combine in different ways for different stocks. These percentages are intended only as a rough guide, for they vary enormously from stock to stock and also change over time.

By efficiently diversifying across a number of shares, the variation due to market influence becomes the biggest source of risk for the performance of the portfolio. For this reason, much of the recent empirical research has centred on regression analysis to explain share price changes in terms of changes in an overall market index. Least-squares regression is usually employed, though it has been argued that since fat-tailed distributions are involved, more robust estimation methods such as minimum absolute deviations (MAD) should be used (Hodges (16)). The average rate at which a particular share responds to a change in the market as a whole is termed its 'volatility' and is often denoted by \( \beta \). It can be interpreted as the market’s assessment of the stock’s sensitivity to the overall level of well-being. The \( \beta \)s of the different quoted shares are scattered around the value 1. If \( R_t \) denotes the return on a particular share over the time period \( t \), and if \( I_t \) is the corresponding return over the same time period from the market index then the equation:

\[
R_t = \alpha + \beta I_t + u_t
\]  

(2)

is found to provide an effective way in which to estimate the volatility effect. In this equation \( \alpha \) would represent the notional average level of return from the share in a situation where the share’s response to a market change is zero. The final term \( u_t \) is a random disturbance term reflecting the other factors mentioned above. Regression methods can be used to estimate historical values for both \( \alpha \) and \( \beta \).

It has been empirically established that, although the possibility of error involved in estimating the volatility (\( \beta \)) of any share is not negligible, the underlying values seem to be reasonably stable over time (see Blume(17)). This is also true, though perhaps to a slightly lesser extent, of the estimated variance for \( u_t \). Evidence suggests, however, that the values of \( \alpha \) are not so stable as those of \( \beta \) or the variance of \( u_t \). All these values are obviously dependent on many factors including the operating characteristics and financing of the company, and the industry within which it operates. Much research therefore remains to be done on forecasting and explaining the volatility measures.

5. SIMULATION

The technique of simulation has been used both in examining the behaviour of individual equity shares, and for examining the behaviour of portfolios. Information about individual shares and the movements of their prices under
various assumptions forms a valuable input to the models of Markowitz and later writers, quite apart from any intrinsic merit that simulation may have as a direct approach to the management of portfolios. Such examinations normally require the use of a computer because the amount of data to be handled is voluminous and the number of computations to be made is extensive.

Levy\(^{(18)(19)}\) has described the results of some extensive simulation tests that he carried out. The tests were based on a weekly price file of two hundred U.S. securities over the five-year period 1960–65. The data were used to look at the widely held theory that stocks which have performed well relative to the market over some past period will continue their superior price action in the immediate future. If, in fact, relative strength does continue for a significant period of time, then a strategy of portfolio upgrading should prove to be lucrative. Portfolio upgrading (or letting profits run and cutting losses short) involves the sale of securities when they become relatively weak and the replacement of these securities in the portfolio with securities that are relatively strong. Such a strategy can, of course, be compared with the opposite axiom that you cannot fail by taking a profit. The simulations performed by Levy on this particular problem led him to draw two main technical conclusions. First, that portfolio upgrading was a far superior strategy to profit taking. But the risk, as measured by the monthly variation in the rate of return, was then far greater than that for a random selection of stocks. Secondly, this risk could be reduced by introducing a variable plan whereby fixed interest stocks are brought into the simulation, with the proportion of such stocks included rising as the market declined. This plan, combined with portfolio upgrading, outperformed the market as a whole and, more importantly, produced less variation in overall return than would have been produced by random selection of ordinary shares.

In more general terms, Levy draws three broad, but tentative, conclusions from his experiments. First, that technical analysis, which implies an unequal distribution of critical information about stocks and shares in the market place, can be justified. Secondly, that the random walk hypothesis (which was discussed in the previous section) cannot be upheld. Thirdly, he stresses again that computers are invaluable in making investment decisions, in particular because they allow additional research to be carried out through simulation techniques.

Jensen and Bennington\(^{(20)}\) are, however, critical of Levy’s conclusions, particularly with regard to the random walk hypothesis, since the alternative rules were both derived and tested on the same set of data. They point out that, given enough computer time and effort, it is always possible to devise a mechanical trading rule which ’works’ even on a table of random numbers, provided of course that the rule is tested on the same table of random numbers which was used to discover the rule. In such instances the rule would, almost certainly, be valueless on any other table of random numbers. Hence Jensen and Bennington have repeated Levy’s analyses using seven non-overlapping five-year time periods and found that the results are not inconsistent with a random walk hypothesis.

The direct management approach of simulation compares the results achieved
where the computer is programmed to follow a defined investment strategy with those that would have been achieved through other strategies (e.g. a random form of selection). One example of this approach is provided by Clarkson\textsuperscript{(21)}, who succeeded in defining a number of rules of thumb employed by portfolio managers for certain trust funds and then formulated a computer-based model whose actions would parallel closely those of an actual fund. This sort of simulation approach can be used to examine the possible value of such rules of thumb for controlling diversification, and of other similar devices such as filter rules for trying to pick stocks which will rise in price.

6. THE PORTFOLIO SELECTION PROBLEM

Portfolio selection is concerned with the problem of how to generate a portfolio to give the maximum possible return whilst, at the same time, minimizing the risk of poor performance. The return from either a single equity share or from a whole portfolio will always be uncertain and, subject to obtaining a satisfactory expected return, this uncertainty needs to be made as small as possible. Only in exceptional circumstances does it make sense to prefer an increased amount of uncertainty. Unfortunately, there is no straightforward way of assessing directly the future uncertainty of the return from a holding, but empirical evidence suggests that there is enough stability in the measures of historical variation (such as volatility, discussed earlier) for these to be used as reasonable indicators of uncertainty. This uncertainty of return is usually equated with an estimate of the variation of future returns, measured as a standard deviation (SD).

Proper diversification enables this standard deviation to be varied within limits for any given level of expected return, and a portfolio with the lowest possible standard deviation for its expected return is said to be efficient. In Figure 1 a portfolio is represented by its expected return (R) and its standard deviation of return (SD). All possible portfolios will be found to lie in some irregularly shaped figure such as that shown shaded. The upper boundary cannot be defined very precisely. The lower boundary of this figure, denoted by AA'B, contains these portfolios for which the standard deviation is a minimum, given the expected return. Such portfolios are said to form the set of 'efficient portfolios'.

The problem still remains as to which of the many possible efficient portfolios should be selected. On what rational basis can a trade-off between the expected return and the uncertainty of obtaining this return be made? The question is simplified by considering the possibilities of borrowing or lending at risk-free rates. This is illustrated in Figure 2.

Through lending at the rate $P_1$ by including risk-free fixed interest loans in the portfolio, the standard deviation of the efficient portfolio at X can be reduced and points along XW reached which are preferable to those of AX, in that the standard deviation is reduced for a given expected return. Similarly, by
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FIG. 1. The efficient set

All portfolios lying along the line $AA'B$ are efficient

Region containing possible portfolios

FIG. 2. The borrowing and lending portfolios
borrowing at the rate $P_2$ and investing in portfolio $Y$, any portfolio lying on this line $YZ$, which is below $YB$, can be held since again the standard deviation is reduced for any given expected return. This implies that the portion of any portfolio which consists of common equities should fall somewhere along the line $XY$ since, for such portfolios, it is not possible to improve upon the position by either borrowing or lending.

This kind of reasoning also gives rise to another important theoretical idea, the Capital Asset Pricing model of Sharpe\(^{(22)}\). Sharpe argues that in a properly diversified portfolio the uncertainty due to movement with the market as a whole is the only form of risk that need be considered. It must be possible for every individual equity share to belong to an efficient portfolio, for otherwise price adjustments would take place until this was the case. As already discussed, all efficiently diversified portfolios must lie between $X$ and $Y$ in Figure 2, which is re-drawn as Figure 3 in terms of the volatility $\beta$ instead of the standard deviation, when the curved line $XY$ becomes a straight line. (For simplicity the borrowing and lending rates are assumed to be equal with the value $P$.)

\[ \begin{align*}
\text{Volatility} & \quad (\beta) \\
\text{Expected return} & \\
R & \quad P
\end{align*} \]

**FIG. 3.** The capital asset pricing model

Because the expected return value ($R$) and volatility measure ($\beta$) of a portfolio are just the weighted average of those figures for its constituent shares, it follows that, if each individual share is in some efficient portfolio, the share itself must also lie on $XY$. The market's expected return from a share is therefore a linear function of its volatility. Since a portfolio with $\beta = 0$ must yield a return of $P$, whilst one with $\beta = 1$ should have an expected return equal to the expected
return on the market as a whole, denoted by $E(I)$, a feasible form of relationship is

$$ E(R) = P + \{E(I) - P\} \beta $$

(3)

The earlier equation (2), taking expectations, gives

$$ E(R) = \alpha + \beta E(I). $$

Equating (2) with (3) and eliminating $E(R)$ then gives

$$ \alpha + \beta E(I) = P + \{E(I) - P\} \beta $$

or

$$ \alpha = P(1 - \beta). $$

Hence the expected value for $\alpha$, namely the expected return from the share in an unchanged market, should be $P(1 - \beta)$. This does not mean that the calculated values of $\alpha$ will equal this, for the market can be wrong, whilst there are also difficulties caused by estimation errors. The expression, furthermore, cannot be legitimately extrapolated beyond the range of values of $I$, and $R$, for which it was initially established.

7. THE MARKOWITZ MODEL

The different kinds of portfolio selection models discussed in this paper take the investor’s beliefs, concerning the expected return and riskiness of different possible investments, and generate the specifications of the portfolio having the trade-off between expected return and standard deviation that is most appropriate to him. The models assume that some fixed horizon period is being considered. This is no real restriction, for an optimal long-term policy consists of a sequence of optimal short-term policies. The returns expected from the shares under consideration must be provided as inputs, but the models vary in the way the standard deviation is handled.

The earliest kind of portfolio selection model was due to Markowitz(4). Markowitz suggests that the analysis should proceed in two stages. First, as described in section 6, the set of efficient portfolios is formed. Secondly, from amongst the efficient set, the portfolio that provides the investor with the most suitable combination of risk and return is selected.

Suppose that a portfolio is to be selected from $n$ stocks $S_1, S_2, \ldots S_n$. Let $r_i$ denote the return expected from the $i$th stock and $c_{ij}$ the covariance of the return with that from the $j$th stock where both $i$ and $j$ can take the values 1, 2, \ldots $n$. Then, if $x_i$ denotes the proportion of the portfolio invested in the $i$th stock, the return expected from the complete portfolio is

$$ r_p = \sum_{i=1}^{n} r_i x_i, $$

where $\sum_{i=1}^{n} x_i = 1$ and $x_i \geq 0$ for all $i$.

(If short selling is permitted, then $x_i < 0$ may be allowed.)
The variance of the return $r_p$ is given by

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}x_i x_j$$

The efficient portfolios which provide the maximum expected return for each level of variance can be obtained by finding the values of $x_i$ to maximize the expression

$$\sum_{i=1}^{n} r_i x_i - \lambda \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_i x_j$$

subject to

$$\sum_{i=1}^{n} x_i = 1 \text{ and } x_i \geq 0 \text{ (i = 1, 2, \ldots n)}$$

for appropriate non-negative values of $\lambda$. The latter represents the trade-off being made between expected return and variance (or standard deviation).

In matrix notation the formulation can be written as

$$\text{MAXIMIZE } r'x - \lambda x' Cx$$

subject to $l'x = 1 \text{ and } x \geq 0$

(4)

This is a problem of parametric quadratic programming and Markowitz gave a method, called the critical line method, by which the whole of the frontier, or set, of efficient portfolios could be calculated. If $\lambda$ is put equal to zero, the programme will select the portfolio with the highest possible expected return. If $\lambda$ is put equal to infinity, the programme will select the portfolio with the smallest variance. Intermediate values of $\lambda$ will result in portfolios that have lower than maximum expected return and higher than minimum variance. The objective function permits the selection of efficient portfolios only. The complete set of efficient portfolios is obtained, therefore, by solving the quadratic programme for all values of $\lambda$ between zero and infinity.

The selection of $\lambda$ by the portfolio manager is equivalent to his specifying his attitude towards risk or, more generally, his utility function. If the manager's utility function can be so specified, then the best course of action is to select that portfolio for which his expected utility is the highest. (For a discussion of utility principles see, for example, Lindley(23) chapter 5.)

To carry out the Markowitz approach requires a complete table of the estimated covariances between every pair of stocks. Not only does this set an enormous and perhaps impossible task of data collection and assessment (see the comments of J. Plymen in the discussion of Hemsted(24)), but reliable, easily used and inexpensive programs to solve general quadratic programming problems of this kind are difficult to obtain, even though Wolfe(25) in 1959 gave a more efficient and more general computational procedure. Computational difficulties clearly provide one reason why Markowitz's methods have not been adopted for day-to-day portfolio selection.
8. THE SHARPE DIAGONAL MODEL

Markowitz himself was well aware of the impracticability of trying to collect data directly on all the \( n(n+1)/2 \) distinct entries in the covariance matrix concerned and suggested that an index could be used to generate this data. Sharpe noticed that, if an index is used for this purpose, then it becomes unnecessary to multiply out all the entries of the covariance matrix. The investment can be viewed as being made partly in an index, and partly in items uncorrelated either with each other or with the index. These latter items then contribute nothing to the overall variance of the portfolio. By introducing one additional constraint the covariance matrix becomes diagonal and Sharpe discovered that the Markowitz critical line method for the solution of (4) was thereby considerably simplified. His computer program enabled portfolios now to be selected from as many as 2,000 securities and at as little as 2% of the cost associated with standard quadratic programming codes. He also found empirically that the use of this simple form of the covariance matrix led to very nearly the same results as the use of a more complicated form. This is in agreement with the analyses in King\(^{(9)}\) which showed that the market and individual company factors tended to be more important than industry and other common factors.

The usual formulation of the Sharpe diagonal model is as follows:

Let the return from the \( i^{th} \) stock be of the form

\[
 r_i = \alpha_i + \beta_i I + u_i \tag{5}
\]

where \( \alpha_i, \beta_i \) are constants, \( I \) is the return from a market index and \( u_i \) is a random variable with zero mean and zero correlation with \( I \) and with the other \( u_i \)s.

Then, if \( E_I \) is the expected return on \( I \), and \( \sigma_I^2, \sigma_i^2 \) denote the variances of \( I \) and \( u_i \) respectively, the portfolio selection problem reduces to

\[
\text{MAXIMIZE} \quad \sum_{i=1}^{n} \alpha_i x_i + \beta_i E_I - \lambda \left\{ \beta_i^2 \sigma_I^2 + \sum_{i=1}^{n} x_i^2 \sigma_i^2 \right\}
\]

subject to \( \sum_{i=1}^{n} \beta_i x_i = \beta_p \)

\( \sum_{i=1}^{n} x_i = 1 \)

and \( x_i \geq 0 \) \( i = 1, 2, \ldots, n \)

The important point about this formulation is that the only quadratic terms which appear are the squared ones. The covariance matrix has been reduced to a diagonal form, and this makes the solution of the quadratic programming problem especially simple.

The data quantities required are \( \alpha_i, \beta_i \) and \( \sigma_i^2 \) for each stock and \( E_I, \sigma_I^2 \) for the index. Values for the volatility \( \beta_i \) of each stock to the index can be obtained empirically by regression analysis and empirical estimates can also be made of
\[ \sigma_i^2 \text{ and } \sigma_i^2. \] The evidence available, for example Blume(17), seems to indicate that these values are reasonably stable over time. Subjective estimates would normally be used for \( \mu_i \) and the \( \alpha_i \) values.

9. MULTI-INDEX MODELS

Since the introduction of a single index into the model can produce such computational simplifications, it is reasonable to consider the introduction of several indices to improve the realism of the model and yet still maintain simplicity. In principle there is no difficulty in doing this, and a multi-index model can be formulated as follows:

Suppose the share returns are of the form:

\[ r = by + u \]

where \( y \) is a vector of indices with covariance matrix \( C \), and \( u \) is a vector of zero mean disturbance terms with diagonal covariance matrix \( D \) and uncorrelated with the indices. The expected return from a portfolio of holdings \( x \) is given by

\[ r_p = x' B E(y) \]

and its variance by \( \sigma_p^2 = x' B C B' x + x' D x. \)

The portfolio selection problem therefore simplifies to:

\[
\begin{align*}
\text{MAXIMIZE } & Z' E(y) - \lambda \{ Z' C Z + x' D x \} \\
\text{subject to } & B' x = Z \\
& l' x = 1 \\
& x \geq 0
\end{align*}
\]

The quadratic terms again have a very simple form enabling efficient computational methods to be used.

Some empirical work has been done by Cohen and Pogue(26) to investigate the usefulness of this type of model, incorporating different indices for the various industrial sectors. They concluded that the performance of the index models is not dominated by the Markowitz formulation. Furthermore, for portfolios selected entirely from equity shares, the performance of the multi-index models was not markedly superior to that of the Sharpe diagonal model formulation. The main reason for this seems to lie in the problems of estimation that occur as soon as more than one index in employed. These conclusions are broadly in agreement with those of King(9) which have been described above.

10. SHARPE MUTUAL FUND MODEL

The last model to be described is the Mutual Fund model of Sharpe.(6) This model is a simplification of the diagonal model described above and makes the assumption, particularly appropriate to American mutual funds, that the placing of limits on the percentage of the portfolio which may be invested in any one stock or share will enforce diversification.
Mathematical Models in Portfolio Selection

Since the portfolio's variance is given by

$$\sigma_p^2 = \beta_p^2 \sigma_1^2 + \sum_{i=1}^{n} x_i^2 \sigma_i^2$$

a restriction that, say \(x_i \leq 0.05\), implies a total weighting on the \(\sigma_i^2\) terms of the order of one-twentieth of that on \(\sigma_1^2\). (It is assumed that \(n \geq 20\).) The risk from the independent variation of the different stocks has essentially been diversified away, leading the index as the only real source of risk. Data on the \(\sigma_i^2\) is therefore not required and the formula

$$\sigma_p^2 = \beta_p^2 \sigma_1^2$$

can be used for the variance of the portfolio. Working in terms of the standard deviation instead of the variance, it is necessary to maximize only

$$E_p - \lambda \beta_p$$

for all positive \(\lambda\).

This is carried out straightforwardly by selecting stocks and shares with the highest values of \((\alpha_i + \beta_i E_i) - \lambda \beta_i\). The procedure can even be carried out graphically by hand.

Sharpe's paper indicated that empirical studies gave results, using the mutual fund model, which hardly differed from those which would have been obtained from a diagonal model incorporating the same constraints. The computation is, of course, far simpler.

11. Extensions to the Diagonal Model

Several useful extensions can be made to the diagonal model. First, whilst the original description made no mention of imposing limits to the \(x_i\) values, both upper and lower bounds can be imposed on the individual \(x_i\) values. Upper bounds prevent the purchase of holdings which would be too large, perhaps because of institutional or legal constraints, or for reasons of marketability. Lower bounds can serve a similar purpose: if the model is being used to recommend changes to an existing portfolio, then lower bounds on holdings can be used to prevent the model suggesting the sale of very large blocks of stocks or individual securities.

A second extension concerns the incorporation of dealing costs. The return from securities already held in the portfolio is higher than that from those which have to be purchased by the amount of the dealing costs. These costs may be of two kinds, namely the brokerage and marketability costs associated with the volume of an individual security that is to be bought or sold. Two variables are used, \(x_i\) to denote the amount retained of the existing holding of security \(i\), and \(y_i\) to denote how much extra is bought. This introduces a few off-diagonal terms in the covariance matrix, for the returns on \(x_i\) and \(y_i\) are perfectly correlated. It seems easier to cope with these elements in the algorithm than to eliminate them by means of extra constraints. Sharpe has suggested that, to reduce the return
(and the variance) from the $y_i$ investment, the constraint on the proportions held in the portfolio should be modified, rather than changing any of the data values of the returns and variances, from

$$\Sigma x_i + \Sigma y_i = 1$$

to

$$\Sigma x_i + \Sigma (1 + T_c) y_i = 1$$

where $T_c$ is the level of dealing costs expressed as a fraction.

Two papers by Pogue\(^{(27)}\)(\(^{(28)}\)) also deal with the problem of incorporating dealing costs in portfolio selection models. His method, which is slightly different from that described above, allows for the possibility of the level of dealing costs varying with the quantity purchased. This is an attractive feature, but it does introduce many new variables into the model and so seems rather a luxury, unless the degree of any such variation is extremely large.

There are usually a variety of factors other than expected return and variance that portfolio managers are concerned about. These can often be taken into account by means of the inclusion of some additional linear constraints. Sharpe\(^{(29)}\) writes:

Rightly or wrongly, it is felt that a number of constraints must be invoked to capture the essence of a 'real world' situation. For example, a lower bound on current yield, an upper bound on realized capital gain, an upper bound on the proportion of the present portfolio to be sold in order to buy (supposedly) better securities, etc. If portfolio analysis methods are to be used in practice, it must be possible to cope easily with such requirements.

The more general problem described above, including the extension of bounds, transaction costs and extra constraints can be formulated in general terms as

$$\text{MAXIMIZE } e'x - \frac{1}{2} \lambda x' D x$$

subject to $A x = h$

and $x_{\text{min}} \leq x \leq x_{\text{max}}$

where $e'$ represents the vector of expected returns and $D$ the matrix of covariances.

The Kuhn-Tucker conditions for a solution to this problem are given by

$$e - \lambda D x - A' Z = w$$

$$A x = b$$

and $x_{\text{min}} \leq x \leq x_{\text{max}}$, where

if (i) $w_i < 0$, then $x_i = x_{\text{min}}$, or
if (ii) $w_i = 0$, then $x_{\text{min}} \leq x_i \leq x_{\text{max}}$, or
if (iii) $w_i > 0$, then $x_i = x_{\text{max}}$.

The efficient frontier can be generated by finding an initial solution for $\lambda = 0$, and then following the Kuhn-Tucker conditions as $\lambda$ is increased.

When $\lambda = 0$, implying that the expected return is to be maximized without
regard to the variance, it is only necessary to solve the linear programming problem

\[
\text{MAXIMIZE } e'x \text{ subject to } \\
Ax = b \\
x \text{ min} \leq x \leq x \text{ max}
\]

There are many standard routines available which will accomplish this. One case is worthy of special mention, namely that a trivial solution exists when there are no extra constraints. Suppose the problem is

\[
\text{MAXIMIZE } \sum_{i=1}^{n} e_i x_i \text{ subject to } \\
\sum_{i=1}^{n} b_i x_i = x_{n+1} \\
\text{ and } \sum_{i=1}^{n} a_i x_i = 1 \text{ (with the } a_i > 0) \\
x \text{ min}_i \leq x_i \leq x \text{ max}_i (i = 1, 2, \ldots n)
\]

The solution consists of setting \( x_i = x \text{ min}_i \) for \( i = 1, 2, \ldots n \) and then increasing those with the highest ratio \( e_i/a_i \) to their upper limits, \( x \text{ max}_i \), until the capacity constraint \( \sum_{i=1}^{n} a_i x_i = 1 \) is satisfied.

In general, the first solution will have a number of \( x \) values at their lower limits, a number at their upper limits and some in between. For \( \lambda = 0 \), values of \( Z \) and \( w \) can be found so that the Kuhn–Tucker equations are satisfied.

12. EXPERIMENTS WITH THE SHARPE MODEL

Hodges and Brealey\(^{(30)(31)}\) have described various experiments that they have performed with the Sharpe model. The first paper\(^{(30)}\) described a number of experiments conducted in conjunction with a group of participants in an investment seminar. An appropriate Fortran computer program was written for use with an on-line time-sharing computer system which made the operation of the model very simple and, at the same time, very flexible. The model differed from that described by Sharpe in that it was adaptive, that is instead of being purely a model to compute an ideal portfolio to hold, it computed the best portfolios into which to change, allowing for transaction costs. Further, it allowed upper limits to be imposed on the amounts held of individual securities. This enabled the model to avoid recommending the purchase of a security in quantities which would be regarded as unmarketable.

Instead of selecting portfolios of shares the analyses were carried out in terms of 22 of the F.T.-Actuaries share indices. This was done to avoid the problem of choosing a small set of shares familiar to all the seminar participants.
and in no way affects the principle of the method. The main difference it produces in the data is that the indices are more closely correlated with the market index (taken as the F.T.-Actuaries all-share index) than would be the case with individual shares. Consequently the variances of the residual (u) terms are relatively small and, since it is these that primarily cause the model to diversify, the portfolios may consist of fewer holdings than would similar analyses of individual shares.

Regression analyses of the monthly value of the 22 indices against the all-share index over a 47-month period were used to estimate the volatility coefficients (β) and the standard deviations of both I and u. The α values had to be obtained from the seminar participants in some way. These represent the returns expected from each industry index if the all-share index remains unchanged or, roughly speaking, the difference between the market’s and the individual’s expectation for that sector. The method adopted was to ask participants to rank the 22 indices in order of their supposed α values and to estimate the range (inter-quartile distance or IQD) covered by the middle 12 indices. A computer program then calculated the estimated individual values by assuming a normal distribution. This method of estimation proved a difficult concept for participants to grasp and alternative methods clearly need to be explored.

The experiments were designed to assess the importance of changing various assumptions, by inspecting the effect of making systematic changes to each of a number of inputs to the program. A few of the tests made will be described here. The first assumption tested related to the impact of differing degrees of variability of the α (see equation 5). The opening portfolio corresponded to an IQD of 3%, when it was diversified across thirteen sectors, with a maximum of 44% of the investment in a single sector. When the IQD was changed to 1%, the benefits from changing the portfolio were insufficient to cover the transaction costs and the optimal portfolio was still the original one. When the IQD was raised to 8%, only four sectors were to be held, with 71% of the investment concentrated in one sector. However, those participants who believed that they could distinguish very different prospects between sectors, baulked at the suggestion that they should hold such a concentrated portfolio.

The second assumption tested was that of transaction costs. Uncertainty in these costs arose in two ways; the magnitude of the jobber’s turn is uncertain and, in addition, the appropriate costs to incorporate depends upon the length of time over which the investor aims to write them off. As one might expect, the larger the costs, the smaller the opportunities to profit from trading. Again the structure of the optimal portfolio was found to be extremely sensitive to the input data. If transaction costs were ignored completely, a very high degree of concentration was indicated. If the costs were taken at 4%, then few changes in the portfolio were made. The lesson seems to be that if such costs are ignored—as is the case in many models—the supposedly optimal portfolio may well prove to be considerably less desirable than the investor’s original holdings.

Another assumption tested was that of changing the price of one security,
keeping all others fixed. In a typical example an increase of 1% in the price of a share was sufficient to make the investor reduce his buying programme sharply. The equivalent return from the changed portfolio showed the cost of acting as if the price of these shares had not changed. This was again substantial. In practice, portfolio managers do not drastically revise their buying programmes for every minor change in price. It is possible that they underestimate the danger of raising their buying price, but the principal explanation is probably that the individual manager continually revises his assessment of the outlook to neutralize the effect of such day-to-day changes. His buying and selling programme is characterized by gradual shifts, not because the prospective returns are large, but because the price changes reflect adjustments, with which he largely agrees, in the market's assessment of the shares.

The broad conclusions reached in the paper suggest that the results of any portfolio selection method are very sensitive to the quality of the input data. When such data are incorrectly estimated, the user will obtain portfolios which are not truly optimal and, if he is mistakenly led into making large changes to his initial portfolio, he may well succeed in worsening his position. Of course, these difficulties apply even more strongly to the portfolio manager who relies entirely on his own judgment and intuition.

The second paper describes the results of some larger scale simulation exercises carried out with the same model and a slightly revised computer program. For the exercise a 'population' of 40 securities was used and transaction costs were assumed to be fixed at 5%. A number of problems were analysed. The first concerned the relationship of portfolio performance to forecasting ability. The simulation assumed that there was some defined level of correlation between the forecast and actual return for a single year for each individual security and for the market as a whole. As the correlation between forecast and out-turn is reduced, the frontier of the efficient set shifts significantly to the left (see the unbroken lines in the schematic Figure 4 where \( r_1 < r_2 < r_3 \) and \( r \) is the coefficient of correlation) and becomes steeper. Yet, even with mediocre forecasts and despite high initial transaction costs, simulation suggests that the investor should still be able to outperform the market for an equivalent level of risk. A reduction in forecasting ability requires important changes in the way in which the portfolio is managed. Again, the simulation study suggests that the number of holdings in the lending portfolio is increased as forecasting ability drops, and the rate of portfolio turnover declines. In practice, of course, funds turn over their portfolios at least as rapidly as this without making such large gains. A plausible explanation of this divergence between theory and practice is that portfolio managers employ relatively inefficient selection procedures.

Secondly, the usual form of analysis assumes that any bias can be effectively eliminated from the expected returns and their variances. In practice it seems that most investors tend to make very exaggerated claims as to their forecasting abilities and find it difficult to provide expected values. Users of portfolio models have seldom made any serious attempt to overcome this problem, but have
simply relied upon forecasts provided by the analyst. Although there may be no systematic bias, there is bias of another kind: thus, whenever a high forecast is made it is more likely than not that the forecasting error involved is a positive one and vice versa. In other words, the forecasts produced tend to exaggerate the likely outcome. Experiments show that in each case, the disadvantages to using crude forecasts are most marked at the right-hand end (as seen in Figure 1) of the efficient set. This stems from the fact that the effect of the limited forecasting ability is to require much larger adjustments to the expected returns than to the variances. The situation is illustrated by the dotted lines in Figure 4, where
a fixed level of bias is assumed in combination with the varying levels of correlation between forecast and actual returns that were assumed earlier.

The penalties of employing biassed forecasts are also highest when the investor’s forecasting powers are weakest. In these circumstances, the investor is induced to make a large number of transactions, the expense of which can far outweigh the prospective gain. If the investor cannot recognize the limitations of his forecasts, the experiments suggest that the correlation between prediction and out-turn must be at least 0.1 before the transactions are justified. No such requirement is involved when the investor acts on the basis of biassed forecasts. One interesting aspect of these results is the wide range of situations in which crude forecasts produce portfolios which are only marginally inferior to those produced from the adjusted forecasts. This is not because the two methods lead to similar courses of action, for the portfolios differ greatly both in terms of turnover and in the number and identity of the holdings. Indeed a major disadvantage to using crude forecasts seems to lie in the appearance of the portfolio, rather than in its performance. It is such results, rather than the portfolio’s subsequent performance, that has probably led to the lack of acceptance of selection models by the investment community. If the inputs are corrected for bias, the efficient portfolios conform much more closely with accepted investment practice.

Some further experiments concerned the performance and character of the efficient portfolios when the number of securities analysed was changed. It might be expected, a priori, that the effect would be greatest when the investor’s forecasting ability is low, for in these circumstances a large set of available securities may be necessary before he encounters a security which offers a sufficiently high expected gain to overcome the costs of purchase. However, towards the bottom of the efficient set, the smallness of a set of securities seems to limit severely the extent to which even the good forecaster can take advantage of his abilities. Results seem to show that, regardless of the investor’s forecasting ability, the higher incidence of extreme changes justifies a significantly higher rate of portfolio turnover. Although residual risks tend to become more important, the portfolios remain broadly diversified. The effect on portfolio performance seems to be striking, particularly when the forecasts are only mediocre. In practice, a portfolio manager is unlikely to be able to widen his set of available securities from which to choose without impairing the quality of his forecasts of the residual returns, and his choice is between analysing a few securities well or many securities not so well. Findings so far suggest that the wider coverage is justified, even if accompanied by a substantial decline in forecasting accuracy.

13. DEVELOPMENT AND IMPLEMENTATION OF THE MODELS

In using these portfolio selection models, it is essential to be aware of their dependence on the quality of the data fed into them, and hence attempt some
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assessment of the significance of this quality. Thus the Sharpe Mutual Fund model amounts in essence to looking for securities which are priced below what would be expected from the Capital Asset Pricing model. The knowledge that there can be a large amount of estimation error in the $\beta$s and that expected returns cannot be forecast very accurately, is therefore rather disquieting.

There are two main ways in which any uncertainties in the data can be taken into account. Probably the most satisfactory way is to solve the model with the data available and then to try the effect of introducing changes in the basic data. It is usually possible to obtain sensitivity limits on many of the variables concerned, such that within these limits the portfolio recommended by the model will not change. It is then necessary only to try the effect of changing a variable when it is uncertain whether or not a variable will lie between these sensitivity limits. This kind of analysis can be carried out very effectively with an on-line computer terminal and does not require any technical knowledge of computing.

Another possible way round the data problem is to attempt to adjust the data before feeding them into the model. It is known that the $\beta$s cluster around unity, and any values a long way from unity may be rather misleading. It seems plausible to adjust the $\beta$ values towards unity by an amount dependent on their probable errors of estimation. The expected returns can also be adjusted towards that implied by the Capital Asset Pricing model. This approach will remove some of the problems caused by estimation error, and iron out some of the apparent statistically significant differences between securities. The procedure is, however, rather arbitrary and there is some doubt as to its overall effect.

The way in which computer-based models should be built around the requirements of the manager who is to use them as an aid to his decision-making has been described. The importance of ensuring that the system will produce the right kind of information, at the time when it is needed, cannot be stressed too strongly. In using a portfolio selection model, the manager will sometimes find the recommendations surprising and will want to examine the effect of changing some of his assumptions. The system must therefore be sufficiently flexible for this to be done easily. Used in this way in conjunction with an on-line computer terminal, the models can be very effective tools, but it is as well to appreciate those tasks which they are, and are not, properly equipped to perform.

They can be valuable in the analysis of a portfolio in the light of the manager’s beliefs about the probable returns from individual shares. The suggestions for changes to the portfolio which would be consistent with its risk characteristics are also of definite interest, and useful comparisons can be made with other portfolios. In performing such analyses the rate of turnover of the portfolio must be borne in mind as an important variable which effectively determines the appropriate level of transaction costs. Price limits can be obtained to show levels below which shares should be bought and above which they should be sold. Care must be taken in interpreting such limits, however, for most price changes include an information content which has the effect of moving the previous limits produced by the model in the same direction as the price.
Amongst the problems which confront the portfolio manager there are, in particular, two very basic problems that as yet are not very well answered by the models. First, they are unable to tell him very clearly how he should efficiently diversify or, secondly, how much of any particular security he should buy. The models do indicate, however, that these problems are probably much less critical than was previously thought. This point of view is reinforced by some recent pieces of empirical research carried out in America by Evans and Archer(32) on the reduction of variability by diversification, and by Scholes(33) on the effect of disposing of large holdings of stock. Moreover, the model builders have very seldom made any serious attempt to build their models around the forecasts which the portfolio manager (or his analysts) must provide as inputs to a model. The fact that these forecasts are themselves uncertain is significant, and makes it inappropriate to use such data as raw inputs for selection models. This point has been very largely overlooked and accounts for many strange results obtained by practitioners attempting to use the models.

The models have also tended to be of the 'once-and-for-all' type, specifying the ideal portfolio to be holding at a particular moment of time and not at all suited to the continuous management of a portfolio through time, as a cyclical evaluation/action/reaction/re-evaluation process. The manner in which a model can be used to manage a portfolio (with stated objectives) on a continuing basis requires further research by the model builders, before such models can be commonly applied in this manner.

Interest in portfolio selection models is steadily increasing and some American funds are now experimentally managing small proportions of their accounts by means of such models. In the U.K. there is rather less evidence of the development and use of these models than in the U.S.A., although it is clear that interest in the application of them is rising, as is the volume of research being carried out. At least one firm of stockbrokers is currently offering a service which provides estimates of the volatility coefficients $\beta$, using them to assist in the analysis of clients' portfolios. Although no attempt is made at formal portfolio selection, recommendations are made on the basis of volatility and it is to be hoped and expected that it will not be too long before some experimental portfolios will be analysed and managed along the lines of the models outlined here. Many of the technical problems concerning these models have now been overcome and the major difficulties in applying them are, first, the organizational problem of how best to make their use a meaningful part of the total management process and, secondly, the availability of suitable and relevant data. As emphasized earlier, no model will in itself completely replace judgment in portfolio management. What it will help to do, however, is to make more explicit those areas where judgment is required and thus help to focus the exercise of judgment more clearly than would otherwise be the case.

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REFERENCES


ABSTRACT OF THE DISCUSSION

The author, in presenting the paper, described the skill traditionally required in the successful management of a portfolio as an amalgam of experience, judgment and intuition, aided by some basic data for possible stocks and shares such as historical trends, price/earnings ratios and dividend yields. The use of mathematics in such an exercise had commonly been limited to a number of straightforward calculations, although all portfolio managers inevitably had some implicit or explicit model behind their reasoning and actions. Over the previous 20 years, mathematical techniques had been considerably refined for use in many business problems, whilst a number of older and well established mathematical techniques had been refurbished and looked at afresh through the advent of high speed, low cost calculating provided by modern electronic computers. Portfolio investment procedures were no exception to the general trend, and the use of mathematics, allied to computers, had helped to focus attention more powerfully and critically upon those areas where a manager's judgment and intuition were really required, and those areas where it was less necessary and the appropriate work could be both delegated and covered more quickly and effectively.

Much of the impetus and background development work had come from the United States, and the bulk of the references in the paper were American. He had, however, inadvertently omitted an interesting paper by Mr J. Plymen, 'Use of Computers for Investment Purposes' (Trans. of the Eighteenth International Congress of Actuaries, 1968, III, 987), where linear programming had been used to improve the quality of a portfolio allowing for certain constraints regarding the number of holdings.

British interest in portfolio analysis had grown in the previous five years. The two landmarks were, of course, the pioneer work of H. M. Markowitz and of W. F. Sharpe. Markowitz's approach imposed very daunting demands upon the analyst, needing as it did a great deal of input that could not readily be quantified. Sharpe's models, and the various modifications of them, had benefited from the lessons of Markowitz and had attempted to simplify the approach and the input data, accepting at the same time a different trade-off between practicability and efficacy of the model. Sharpe's models were also interesting in that their development tried to encapsulate the basic reasoning that portfolio managers appeared intuitively to attempt before acting—not thinking of some abstract theory and then fitting a model to it. The aim of a model had to be to incorporate as many as possible of those elements that the decision-maker currently included in his thinking as well as those items he would like to capture if he could. There was no question of ignoring a manager's judgment and intuition. The model provided a means to build upon, and exploit, that judgment to the fullest possible advantage.

The paper did not imply that the current models were in any sense fully adequate, or would necessarily be suitable for every type of portfolio but, if they were critically examined for the purpose in hand, alternatives and further development could be made, thus enabling the models to be refined. He stressed that any decision-making model was designed for action: it was not an esoteric means to postpone decisions by making it more complicated to come to a decision. He concluded by quoting from the Forsyte Saga: '... that means doing so long a sum every time, that I can't think how you ever get to acting' in order to point out that models were only of use if, over time, they enabled improved results to be obtained.

Mr T. Grimes, in opening the discussion, began by considering the random walk theory. The history of the random walk hypothesis was longer than that of the Markowitz theory and he believed there were few people who had studied the problem in depth who would disagree with its general applicability. The random walk theory was really not a theory; it was a mathematical model which seemed to provide the best explanation for the movements in price not only of individual stocks and shares but also of a large number of freely-traded commodities. The model was extremely simple: it was that changes in price, or log of price, were independently distributed random variables. The model appeared to apply to price changes from hour to hour, from day to day, from week to week, from month to month, and even from year to year. It had been found to apply to markets in the U.S.A., Great Britain, Australia, and various other
countries. It seemed to apply if the total return on an investment was taken into account, i.e. if dividends were included.

The random walk model was not an unexpected model for the market and he felt it would be wrong if any other model were found to be better. The higgledy-piggledy growth of equity earnings had been demonstrated, and it could be shown that a large number of economic series, including that of gross domestic product, also followed random walks. Those series showed no signs of being better explained by any other reasonably simple mathematical model and, although each seemed to follow a random walk, that was no reason why they should not be related to one another in some way. It seemed to him that it should be accepted that share prices moved according to the random walk, and the various economic random walks should be examined to see whether there were any relationships between them and share prices. Some sort of relationship might be found between gross domestic product, for example, and share prices which would enable share prices to be forecast.

The work of Markowitz had been first published in 1952 and it had been worked on, altered, simplified and very widely publicized in the years since. Even so, it had not been widely accepted by the people who had a practical interest in investment. That might be because those concerned with practical investment were not capable of understanding the techniques. They were certainly not the sort of thing that an investor was used to. He visualized an investment manager versed in the techniques speaking to a stockbroker who told him that there was a rumour that Dynamic Investment Trust was going to bid for Dilapidated Engineering. The manager's reaction would be to ask what effect that news would have on the covariance between the returns on the two stocks. If the broker went on to mention that there was a line of 200,000 Enormous Chemicals Ltd in the market, and he might be able to buy some at 10p under the midmarket price, the manager might ask himself what effect that news was likely to have on his beta coefficient. It was not that investors could not understand all the algebra but, while looking elsewhere for the reasons for not using Markowitz, it was worth remembering that those reasons might not have been thought out in detail. An investor had plenty to think about and was used to rejecting (and accepting) arguments without detailed consideration.

Markowitz, in his first paper, had presented a method which enabled a portfolio of shares to be selected given certain very detailed information about their future returns and about the variances and covariances of those returns. In order to use the techniques it was necessary to know the covariance of the return between any pair of stocks that one would consider investing in. In other words, one had to state what one considered would be the return from investing in XYZ Ltd and ABC Ltd, and what would be the variance of their two returns and, furthermore, what would be the covariance of those returns. The question might have to be put to a non-statistically minded investment analyst in terms of 'If XYZ goes up 10%, do you think ABC is likely to move quite independently or to go up, or to go down, and if it moves, by how much?' It was not surprising that the technique was not widely used. Logically, of course, a technique like that should be applied not simply to equity investment; it should also be applied to investment in gilt-edged securities and even to the amount of cash remaining on hand in the fund. The covariance of returns between ABC Ltd and War Loan would then be needed and he suggested that that would be rather difficult. In view of further developments, he mentioned that over the preceding 20 years most equities had increased in value, while long-term gilts had decreased. Whether those past data should be used in predicting the future, he left to the audience to decide.

The models developed by Sharpe and others were designed to avoid estimating all the covariances needed to use the basic Markowitz model. Rather than using the variances and covariances of each stock, they used an index. The variance of that index, its expected return, and how the return on any individual stock would be related to that index had to be estimated. A very simple model for that relationship, $y = \alpha + \beta x$ was normally used and it was then common to argue, as the author had, that the parameters in the model could be estimated from historical data, and those historical parameters used along with estimates of the return and variance of the index or indexes. That provided the basic data to obtain the usual Marko-
As the author remarked, the parameters in those relationships did not remain constant over time. Even the β coefficients which were most stable were shown in Blume's paper (ref. (17)) to regress towards a mean. There had been various other objections, mostly technical, raised to the Markowitz technique, for example, that it did not allow for tax, nor did it take into account the cost of switching investments if it should be used in a case where switching was envisaged. As the author had said, those problems could be overcome and papers had been written on the subject.

An actuarial audience might be surprised, and even shocked, to learn that all the Markowitz methods purported to advise on the investments of the assets of a fund without considering its liabilities. All of the techniques were based on the concept that there was a return which could be ascribed to each share and that return should have a variance and covariances with all the other shares. It was assumed that those variances and covariances measured the risks and co-risks of the shares, and that they could be combined to measure the risk of a fund, which was the combination of those shares. Having estimated the returns etc., one proceeded to calculate, on the assumption of a particular level of risk being acceptable, the best return portfolio. It ignored the liabilities of the fund and it gave wrong answers. For example, if one had a fund and wanted to invest it in such a way that risk was minimized—to ask, in fact, for a zero risk portfolio—it was perfectly possible. The Markowitz technique gave cash or bank deposits or short-term gilts, or something similar as an investment which had zero risk. It had zero risk because the variance of an estimate of its return at the end of the period under consideration was zero; the return had no variance. Therefore, if one performed a Markowitz analysis for the fund's minimum risk portfolio, one would be told to invest all the assets in bank deposits. That might be perfectly reasonable as an answer for a unit trust, possibly even for an investment trust, but it was not sensible for a life assurance company, not in the sense that a life assurance company should never invest in cash (obviously that might at some stages maximize its return) but that a cash portfolio for a life assurance company was never a zero risk portfolio. It might be possible to get round that objection by some further advance, by more mathematics, but the solution would not necessarily be easy. The obvious method was to force the fund to include some negative long-term assets as representatives of the liabilities and to include them in fixed amounts in any portfolio. In practice, it would not be easy to decide whether a with-profits policy should be regarded as a negative fixed asset or a negative equity-type asset, or whether a shareholder had a negative irredeemable fixed interest annuity or an equity-linked annuity. The assumptions made on those points would affect the result much more than a large error in an estimate of the covariance between two shares. Would a with-profits policy have a positive or negative correlation with War Loan?

Markowitz-type techniques were not used even for funds, such as unit trusts, that had a form of liabilities which was suitable for its use. He had recently been taking part in the Integrated Insurance Game which was being organized by the Institute of Actuaries Students' Society and it had occurred to him that the very simplified investment part of the game provided a good place to try the Markowitz techniques. In the game, four types of investment were available and their histories were given. Each investment had random characteristics and it was up to each investor to profit as he could. It was possible in the game to borrow in order to finance investment. Having decided to use Markowitz, he had proceeded to estimate the returns, variances and covariances, and then he came to the critical question: how much risk did he want to accept? The Markowitz approach required the investor to specify the highest acceptable standard deviation for his portfolio and he had tried to say what he wanted, but he could not. He had a perfectly good criterion, but it was not the Markowitz one—to avoid going broke. The criterion for having, say, a one in a thousand chance of going broke involved both return and risk. If, for example, only two investments were available, one of which was known to have a return of 10%, with a standard deviation of 1%, and the other, which was independently distributed, to have a 100% return, with standard deviation 10%, and by choosing a suitable combination of those two investments one wished to have a low risk portfolio, what would be the lowest risk portfolio that one could find, including those two investments only? The
Markowitz technique suggested that it was the portfolio which had the lowest standard deviation, i.e. one should invest all of the money in the investment which gave a 10% return with a 1% standard deviation. However, it would be an extremely conservative fund which would choose anything other than complete investment in the second. The chances of the second investment giving a smaller return than the first one he thought would be negligible, but it was not what the Markowitz method gave. It seemed to him that some hard thinking about the possible results of the Markowitz idea that one should consider variability of return as a measure of risk ought to be considered very closely. He appreciated that his suggestion that one should take the probability of going broke as a criterion was one that applied only to the Students' Society Insurance Game, but it was a more reasonable criterion than the standard deviation of return even if in practice the probability of ruin was so small as to be irrelevant.

He asked whether work would be done on the relationship between the return and the risk but hoped that, before they considered it, the academics would more seriously consider another objection to the techniques. Studies of historical data on stock market prices, using Markowitz assumptions, had shown that most of the risk in any portfolio could be removed by investing in as few as 20 reasonably spread shares. In other words, of all the portfolios with a given rate of return there was little to choose, as far as risk minimization was concerned, between all the portfolios which had 20 or more widely spread stock in them. He suggested there were very few institutional investors who could invest in as few as 20 stocks, and even fewer who would dare to if they could. He wondered what was to be gained by using Markowitz's theory. Any useful analysis must take into account that one simply could not invest in fewer than 20 stocks and it would amount to a choice of a portfolio with a minimum risk from a set of portfolios whose risks were all similar. A lot of computer time could be spent working on expensively gathered data to arrive at a result which would not be significantly better than the one arrived at without all that work. One could not in any practical situation hope to reach an absolute minimum; only to get reasonably close to it, so that a technique which was based on absolute minimization of anything, even risk, was unlikely to be very practical. In any case, the data, however difficult to gather, consisted only of opinions as to what returns and risks would be. The Markowitz 'minimum risk' portfolio was only a portfolio which had a minimum risk if the data were precisely correct, which was very unlikely. Among all the portfolios that a fund might have in practice, there was one that actually had minimum risk, and there was another, probably very different, which inaccurate estimates and the Markowitz theory would imply had minimum risk, but all of those practical portfolios had risks so close to the minimum that there was no reason to choose between them. In portfolio selection, common sense in spreading investments was all that was required to achieve the amount of efficiency in avoiding risk.

He considered the Markowitz type of exercise somewhat futile. It was all too easy to produce a theory, to describe it as interesting but requiring more work, and then to proceed to work on it. If such work were undirected by practical considerations, it could be interesting but useless. An example of that occurred in the random walk theory: having discovered the random walk, the reaction of some investigators had been to study the distribution of price changes rather than to consider other methods of forecasting them. He doubted whether anyone would ever make very much money in the market by using the knowledge that the distribution of price changes was stable paretian. It was made even less likely by the fact that pareto distributions were a whole class of distributions, not just a single one. The normal distribution was one of them, but so was the Cauchy distribution. Indeed, the normal distribution was the only pareto distribution which had a finite variance, and the normal distribution was known not to fit the data adequately. He did not expect to make very much money in the market by using any stable paretian distribution, and he did not expect that they would ever see Markowitz, or any similar or successor technique, widely accepted or even marginally useful in making practical investment decisions.

Mr M. G. Hall welcomed the paper and shared the author's disappointment that little had been published by the Institute relating to equity portfolios. The author had been kind enough
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to mention the paper by Weaver and himself (reference (3)) which had set up a mathematical model to try and evaluate ordinary shares, and he went on to say how the work was proceeding, noting with regret that many of the model's U.S. competitors had appeared to have come to an untimely end. The performance of the Weaver-Hall model was being continually monitored by selecting portfolios every month containing the top 20% of shares ranked and valuing them on a regular basis in comparison with the Financial Times-Actuaries All Share Index. The figures presented in the paper had shown that at that time the Buy/Hold portfolios had beaten the Index by 3.7% per annum, or 6.1% per annum if dear shares (bottom 20%) were sold and replaced by cheap shares. Latest figures covering the last five years showed that the 59 Buy/Hold portfolios chosen had beaten the Index by 4% per annum and the Buy/Switch portfolios had gained 5.2% per annum.

The computer program that tested the share selections had the facility for varying the level at which shares were sold and if, instead of selling the bottom 20% shares, they sold the bottom 50%, the performance on switching fell to +2.2%. That performance loss was due to the greatly increased level of switching which rose from an annual rate of 14.8% of shares held to 66.2%, and calculations showed that before expenses the performance was virtually the same. That particular program had interesting potential, therefore, for investigating the optimum level of switching.

The performance of the portfolios varied from year to year, of course, depending upon the accuracy of the analysts' forecasts, so that, for example, portfolios started in 1967 had generally performed slightly worse than the Index but, in contrast, those started in the most recent four years had all shown very good results. Apart from the performance of the share selections, there had also been interesting variations in the regression weights in the equation and, for example, the weight that the model placed on dividend yield gradually decreased following the introduction of corporation tax until finally it had no effect in the model. In addition, the weights placed upon the past and future growth rates had all varied somewhat, but nevertheless the accuracy of the forecasts had been clearly the overriding factor in the performance of the model.

The author suggested that whilst the method was interesting in that it looked at the objective/subjective balance in assessing the shares, it did not attempt to look at the portfolio as a whole. However, the speaker thought that, if one were considering only the equity portion of the fund, to have predictions of that nature was vitally important in trying to achieve a better performance than an unmanaged randomly selected portfolio typified by the All Share Index. If, as a next step, one could quantify risk, then so much the better.

In addition to that work, he had been developing mathematical models of the U.K. stock market as a whole relating changes in the price/earnings ratio to changes in company profits, money supply and other economic factors. Again, in the two or three years since the model had been introduced, its results had been encouraging, and he felt that both the mathematical models were steps in the right direction towards a more objective approach to managing a life assurance or pension fund. In addition, both models have what he regarded as an important characteristic in that they attempted to use the forecasts coming from a research department of trends in company profits etc., so keeping one's feet firmly on the ground. He believed there was a danger in using a computer to develop rules based purely on past experience that might not apply to the future. That was particularly relevant to the leading assumption in the author's work, namely the use of the beta coefficients. The author stated that the beta coefficient tended to be stable over time and, whilst that might be true in the U.S.A., he was not convinced that it was true in the U.K. market. As a very rough test, he had examined the beta coefficients for five of the Index subsectors in the preceding 10 years, divided into four 29-month periods. It was to be expected that, since the sector indices represented averages of groups of shares, there would be a greater degree of stability of the beta coefficients than in the case for individual shares, but the results had not confirmed it. Motors showed the best results, with a range of 1.12 to 1.29 for beta, but for Electricals, Breweries, Stores and Banks the ranges were respectively 0.89 to 1.29, 0.58 to 1.12, 0.72 to 1.04 and 0.55 to 1.18. In addition, there was
some evidence that the beta coefficient depended upon the period of measurement of the price change. Against that background it was worth looking more closely at what the coefficient really meant. A beta coefficient measured the slope of the regression line between the change over a given interval of time of a share price and the corresponding change in the price of the Index, and the theory suggested that, for example, in a strongly rising market one should buy shares with a high beta coefficient.

An alternative way of looking at the relation with the market was by considering the change in status of a share, and that might be done by looking at the trend in the relative price/earnings ratio, a familiar technique to most investment managers. Taking I.C.I. as an example, it would be found that the cycle in the price/earnings relative ratio was related to the cycle in pre-tax profits, and the volatility of the latter found reflection in the volatility of the share price. I.C.I. had not followed its beta coefficient very faithfully in the previous year or so (approximately 1.0 over six years) and, if one related the price/earnings ratio relative in 1971 to the pre-tax profits trend, the reason became clear. Thus, in contrast to industry in general in the U.K. in 1971, I.C.I.'s profits were relatively depressed, reflecting the difficult conditions in the international chemical industry and I.C.I.'s important interests in Europe. It was, of course, always easy to choose a particular example to disprove a theory, but nevertheless the dangers surrounding the indiscriminate use of beta coefficients were obvious, and the relative price/earnings approach in conjunction with the economic background had clear advantages. Nevertheless, he was naturally concerned at constructing a complicated portfolio theory on what could be shaky foundations.

He felt that mathematical models had an important part to play in investment in using in an objective way all the forecasts that came from investment research departments. Late in section 13 of the paper, the author suggested that the fact that the forecasts were uncertain made them inappropriate for input to selection models but the speaker felt that the beta coefficients were even more uncertain. He suggested that the next step in model building might well be concerned with developing particular models for companies to enable better profits forecasts to be made and he believed that Mr Plymen would be presenting a paper to the Faculty of Actuaries on that subject. In addition, the European Federation of Analysts were to discuss the use of Financial Planning models in industry, hopefully pointing out their uses to investment analysts. In that field he believed the actuary could well make a significant contribution.

Mr G. T. Pepper noted that of the 33 references in the paper only four were from actuarial journals. All the papers were about financial statistics applied to the field of portfolio management, in which actuaries had a particular interest. The survey had only covered the more important papers. If the scope had been extended to cover modern statistical work about rates of interest, the number of references would have almost doubled. It was almost impossible to keep in touch with the explosion of research and he was very grateful to the author, not only for the survey, but for putting the subject-matter into perspective. The problem of keeping in touch was both serious and continuing. Some of the new techniques being developed—particularly in the econometric time series work—were as different from actuarial statistics as actuarial statistics was from the arithmetic of accountancy. The best summary of modern interest rate papers he had seen was The Function and Analysis of Capital Market Rates, by James Van Horne, published by Prentice Hall.

The main part of the paper was about the Markowitz technique but various other techniques were mentioned. The one that appeared most promising was that of Sprinkel (reference (8)). Sprinkel had produced a second book called Money and Markets, published in May 1971 and acceptance of Sprinkel's approach had led to a change in the pattern of capital markets in America. The various time lags that Sprinkel had mentioned had changed as a result of the application of his techniques.

The underlying principle of the Markowitz technique was to select a portfolio so that the risk of the constituents tended to offset. In his own paper (reference (2)), he had suggested
that the larger the proportion of ordinary shares in the portfolio the greater should be the percentage of very long dated gilt-edged stocks in the remainder because low interest rates were likely in a slump. That was an elementary example of the Markowitz technique.

He was doubtful, however, whether the more formal models would prove fruitful. The main trouble was obtaining the quantitative estimates of expected yield for the array of stocks under consideration. The statistical evidence was, of course, that the past return of a stock was of no help in assessing the future expected return, but the whole Markowitz approach stood or fell on whether the past variance of a stock was any guide to its future variance. Certain work suggested that the variance of a stock was reasonably stable over time, and it was on that stability that the whole practical case for the Markowitz approach rested. As an example, the return of Poseidon during 1969 had borne no relation at all to the return in 1970, but the volatility of the share price during the upswing was probably roughly the same as during the downswing. If that stability of volatility was sufficient, then there was some hope for the Markowitz technique, which had prompted him to do a statistical investigation into the gilt-edged market a few years previously.

Over a 10-year period, and taking weekly data and 6-monthly observation periods, he had calculated the mean returns and variances on gilt-edged stocks, allowing for the passage of time. The mean returns had been compared with the yields according to the ‘normal yield curve’ and the variances with the usual compound interest volatilities. A preliminary examination had suggested nothing unexpected, which had been useful but negative information. He had hoped to establish a new concept of statistical variance to set alongside compound interest volatility, but then there had been a change in the tactics of the Bank of England in the gilt-edged market and he had abandoned the investigation when the volatility of the market changed.

There was a novel use of the Markowitz technique by W. R. White of the Bank of England in a paper presented to a monetary seminar the previous week, which represented several steps forward in the academic study of the yield curve. White, in effect, ran Markowitz backwards. If the market were assumed to be efficient in the Markowitz sense, and subject to the usual assumption that past variance was a good guide to future variance, the Markowitz technique could be run backwards to generate the expected yields of an array of stocks. When applied to the gilt-edged market, White generated the expected yield curve at the end of a time horizon. The speaker thought it might be worth trying to apply the same type of approach to equities. The trouble with Markowitz had always been estimating the expected yields not just for the stocks in which one was particularly interested, but for all possible stocks. If Markowitz could be run backwards, the value judgment could be taken at the weakest point of the analysis. By examining the expected yield thrown up by the model one could assess whether one thought anything unexpected was happening. A more practical use might be obtained through that approach than from the general Markowitz approach and he wondered whether anyone had actually tried the idea in practice.

Mr S. J. Green, referring to the need to establish an appropriate bank of data as the initial stage in setting up any kind of computer system for use in portfolio management, said he could think of only three computerized data banks in the United Kingdom, those of a London stockbroker, an Edinburgh stockbroker, and the City University. He had read that the London Business School was proposing to devote a substantial sum of money to producing a fourth. That was due, perhaps, to the fact that there was no single satisfactory solution as to what information to store and what to leave out. Unfortunately, something was always left out, be it a qualification in the auditors’ report, a loss taken to profit and loss, or some equally vital but subtle piece of information. That was why a data bank could never replace the report and accounts as a prime source for investment analysis, and any model which ignored the investment analyst had serious drawbacks.

On the question of risk, a working party of the Society of Investment Analysts had said: ‘The concept of risk in relation to return achieved by a portfolio has been receiving considerable
attention, particularly in the United States, although as yet no completely satisfactory method for measuring it has been put forward.' He suggested that the author had supplied an answer when he had said that when it came to the consideration of risk it had not even been possible to produce a definition which could be universally accepted.

He appreciated the author's reference to the lack of attention paid to a subjective measure of risk. He had always thought it was a pity because any measure of risk could not be truly objective until it was post facto. There was a fundamental difference between a priori and post facto risks. Unfortunately, many of the academics who had published on that subject did not seem to mention it. Further, there was a difference in a priori risk from analyst to analyst and from investment manager to investment manager. As an illustration, if he knew for a fact that on 30 June the Dynamic Investment Trust would pay £4 cash per ordinary share of the Dilapidated Engineering Company, whose price was now standing at £2, and instructed his broker to buy Dynamic and to store it, the broker, being a trusting soul would probably buy for himself and would not be able to resist telling his leading private client. That client, who had done well by his broker over the years might invest in Dilapidated, and the following week might tell his friends with the result that one of them might be sufficiently impressed with the story to have a little flutter. He, in turn, might tell an acquaintance on the train going to work, and the next day he might also buy a few shares. So there would be five individuals investing on the same piece of information and yet, he suggested, each one of those five was investing with a different a priori risk. It was an extreme example perhaps but, as any investment analyst knew, there were always clues, certain pieces of information which, although available to the public as a whole, were still not equally appreciated. Yet, if the information were not equally appreciated, why should the risk be equal? Post facto it might be, but not a priori.

There was a point he wanted to ask Mr Hall on the Weaver-Hall model: could similar results have been obtained by using raw input without processing it through his mathematical model? In other words, how much of the return achieved in their portfolios was due to the model, and how much was due to the raw input? There was another point in relation to beta coefficients. Surely, one of the main interests an investor might have in beta coefficients was to try and spot when they were about to change, because that was really when one was going to reduce or increase one's investment in that particular sector of the market.

On portfolio selection models Hodges and Brealey had said: 'When one considers the rarity with which transaction costs are incorporated in portfolio selection models it is no wonder that they have met with such little acceptance in the investment community.' Sharpe's model had been a great improvement on Markowitz and, in their turn, Hodges and Brealey had been a great improvement on Sharpe's model as they had allowed for transaction costs in their model. They had originally taken the cost of a switch as 2 1 2% and then as 4% to allow for the jobber's turn. Unfortunately, the jobber's turn was not so easily quantified. For example, there had been a recent occasion when fleetingly the jobber's turn in I.C.I. ordinary shares had been more than 4%. Even more recently, there had been occasions when jobbers had restricted the quotation in a leading stock to £10,000, and moved the price by more than 1% for £25,000. There was also the whole question of dealing. A skilled dealer could make a significant change to the price of a package of shares, but usually it took time and the market, as everyone knew, was changing instantaneously. In the prices they allowed in their model and the expenses they were prepared to pay they must allow also for instantaneous change. Markowitz himself had recognized that but he had written: 'For the actual choice of portfolio, however, dynamic programming techniques cannot be used; they require both too much from man and machine.' He had recognized that it was a dynamic problem requiring dynamic programming techniques and he had dismissed it because, in his view, it could not be done. Perhaps 20 years later it was possible, but so far nobody had attempted it.

What Markowitz did say was that dynamic techniques could be used if the portfolio consisted only of a limited number of sectors, such as fixed interest, ordinary and property, with no subdivision for underlying securities. That could be a useful avenue but for some reason it had not been explored. They had a dynamic problem and so far had only static models or,
to be fair, if the portfolio selection model were up-dated regularly, they had a series of static models. It was rather as if the designers of Concorde had built a scale model, complete in every detail, but had omitted to produce a wind tunnel in which to test it. The model would show what the aircraft looked like but it would not show if it worked.

He suggested those were some of the reasons why portfolio managers had not taken up the new methods so readily, particularly the portfolio selection models as apposed to econometric models, which were more widely used, and share selection models. If only the portfolio selection model builders would peddle their wares for what they were—educational toys rather than vital management tools—the portfolio managers would be more prepared to give them a try.

Mr J. Plymen agreed thoroughly with the last paragraph of section 7 of the paper. The author had admitted that the Markowitz technique, although so elegant theoretically, was just impossible of application. He had had an admission of that from Markowitz himself. Some years earlier, when making an enquiry on behalf of the Society of Investment Analysts into the progress of the computer projects, he had communicated with Markowitz, who had replied more or less that he had given up trying to develop it any more, that it did not have any real practical application, and that he had retired.

The author assumed that given distinctly limited and inadequate data a better result was likely by a complicated mathematical computer technique than with a traditional method. He did not think that was axiomatic at all. Given limited resources, if one was going to try somehow or other to apply, for example, the Sharpe method, one was going to devote a vast amount of time to assessing all sorts of esoteric factors for a large number of shares, and one might not be able to give enough time to studying the one that was going to go wrong. That brought him to the question of the alpha and beta functions. He was not at all satisfied with the statement that an American study had shown that in practice the alpha and beta functions on average had a degree of persistency. He was quite prepared to admit that in many cases that was right. There were many companies where, due to the special circumstances of the company, the beta function was more favourable than the average. In a study of some shoe manufacturing companies some years previously, that had been most interesting from the point of view of the sort of cycles that appeared. About every three years there was an adverse cycle and at such times two or three companies showed violently reduced profits. They were much more vulnerable than the others and the reason was perfectly clear: they were in a more vulnerable section of the trade, making ladies' fashion shoes rather than children's shoes. Trends like that persisted but the trouble was that one might get a marginal improvement from an elaborate technique for most of the portfolio but one of the shares selected by the portfolio could be a 'rogue', fall apart and give a very poor result. In such circumstances, an investment manager would not be in a very happy position facing a critical board or client and saying: 'Well, I chose the Sharpe portfolio, but I got my beta factors wrong.'

In the shoe trade study, whereas there were the very vulnerable companies, there was one particular company that had displayed a most remarkable and absolutely exemplary profit record for years. Its profits had grown 10 times in 10 years, and it showed every possible favourable factor, yet two years later something went wrong with the management and their marketing methods, and the whole thing had fallen apart. The shares price fell to 20% of what they had been a few years earlier. He suspected that a Sharpe technique, using historical beta functions, would have made for an investment in a company like that and would have produced a bad result. Practical analysts could think of lots of cases like that.

In the paper there was a reference to estimating the return from the index. It was dismissed in a couple of lines as if it were quite simple but, in practice, it was almost impossible. If what the average return on the index was going to be were known, there would be no need to worry greatly about selecting a portfolio. More or less anything could be bought or sold. He thought that if 50 people were asked what the average return from the index would be over the year, they would probably all say much the same but they could very well all be wrong, and it was
an estimate which could not really be checked because there was only one observation each time.

The Sharpe system was maintained as being of general application and one could think of sectors in the market where it might be of satisfactory application. Mr Pepper had referred to the gilt-edged market where clearly there was more chance of getting the variances and so on on a reliable basis. Mr Hall had mentioned the question of econometric models; an interesting point. There were a limited number of industries where a sort of computerized model of the industry could be made up, all the likely assumptions as to production, profit margins, costs and so on fed in, and out would come the earnings on a variety of assumptions. The variances might be calculated from the model. Unfortunately, there were comparatively few classes that worked like that. It might, perhaps, work with American utilities, and it might work with industries where one had a group of companies all doing the same thing and all subject to the same influences and where one had a lot of regular statistics. However, he did not think one could possibly contemplate the system being practicable on a wider front. The Sharpe technique really started from the wrong end, in that an elaborate mathematical structure was built up not on shaky foundations but on practically no foundations at all. It was really like planning a military campaign without having any troops. A better way to do it was to start off by saying: 'What can the analysts do?' Not, perhaps, what they were doing at present but what they could do in a few years time with developments in computer techniques, better accountability, and so on. At the moment he thought that all the analysts could do was to make a somewhat wobbly estimate of next year's earnings. In practice that could be done, perhaps, with a reasonable degree of accuracy on average but with very considerable variation. It might be possible, by using econometric models in special circumstances, to get closer estimates, but anything involving the risk element, other than using the beta function as a sort of 'trimmings' was going too far.

A much more profitable approach was that of the Lionel D. Edic Organisation, the American statistical economic consultant organization (and a subsidiary of Merrill Lynch), which had taken the constituents of the Dow Jones Index, put them all together (all the balance sheet data, earnings data and earnings estimates) to make a sort of conglomerate, and had said: 'This conglomerate has got certain assets and figures for next year's earnings, and so on. Let us get the computer to do a linear programming project to get the same assets, or the same next year's earnings, or growth of earnings next year, but at a lower cost.' The computer had churned away and had produced an alternative portfolio subject to those constraints, and that alternative portfolio had appeared to perform very much better than the average. A project of that sort which started off with input material that was practicable was likely to be more profitable.

Mr P. D. Praetz thought that the question of models was immensely important. The kind of model that the optimization techniques presented in the paper gave rise to was a kind of static model of risk and return, i.e. an optimum at a given point in time but, because things changed fairly rapidly in the market, not optimum at a later date. He thought that part of the solution to that lay in evolving theories. No real working theory on a single share, or even shares in aggregate, had been evolved by the theoreticians and thoroughly tested. Such variables that might contribute, like dividends, earnings and interest rates, had only been spelled out verbally and never tested. One needed something which took account of the time element, whether it be one year or 20 years.

He did not think the remarks of the opener about the Pareto distribution being widely accepted were true. It had been propagated by a few people, mainly in Chicago, who had been very prolific in their publications, but he thought the converse was true. The random walk analogy was also false. The upward drift it exhibited was mainly due to the growth in the share price over time.

Mr J. D. U. Harsant found the paper of great interest as he had been generally wondering how
to approach the problem which had been summarized so well by the author. He felt that as
competence in the use of computers within the profession grew, so the use of computers for
investment purposes would also grow. The sentence in the paper which summarized the invest-
ment problem for him was: ‘Much research remains to be done on forecasting and explaining
the volatility measures.’ That statement was made at the end of a section which postulated the
mathematical calculation of volatility measures from past experience. He agreed with other
contributors that that line was not sufficient and that some attempt should be made to check or
calculate volatility measures by other means. He suggested that a system should be devised
whereby points were awarded to each share in accordance with the way it was considered
various factors would influence the share and the volatility factor, alpha and beta, then al-
located to that share according to the rating thus derived. Examples of the headings were the
past rate of growth of profits, recent trends in the sector of the market to which the company
belonged, and the accuracy of past forecasts of that company. Difficulties with the approach
were very considerable, and he did not want to play them down. They included the need to make
sure that one had taken all the relative factors into account and that one was consistent in the
allocation of points. Nevertheless, it had the advantage that it was another check on the
historical record and also it corresponded more with the approach of some investment managers
in their traditional way.

Something not immediately relevant to investment which intrigued him a great deal was the
reference to ‘confidence limits’ associated with the return from an investment. Actuaries as a
body were generally shy of giving confidence limits although there were many instances where
they could be of great value; for example, in the calculation of rates of contribution to pension
funds.

Mr B. H. Fison spoke as an actual operator of an on-line real time computerized portfolio
management system, possibly the only one at the time. It was in the gilt-edged market but
other contributors had referred to gilts and, under the title of Mathematical Models in Port-
folio Selection, it was not inappropriate. The system was absolutely straightforward linear
programming. One had to estimate a number of possible future market outcomes expressed as
gilt-edged yield curves, the limit to the number being practical rather than theoretical. It was
then a matter of arithmetic to compute the return for each stock from the current price to
achieving each of the given yields on the date selected as a horizon, e.g. one month, three
months, or the next Budget. One finished up with a matrix of a number of scenarios across
the top—usually three to five—and 47 odd stocks downwards, and the solution of that matrix
for the optimum selection was a straightforward technique. One also fed in as many probability
sets as one liked and the end product was a series of portfolios for each set.

There were snags, however. His experience, which was not very long, was that the portfolios
tended to consist of either one stock, two stocks, or possibly a small number of stocks, although
there was a facility to cut back the turnover that might involve. There was a marketability
problem: if a fair number of gilt-edged portfolios were being managed and they were all to
finish up in one stock, it could not be done. The other snag was sheer speed. Given a new piece
of information in the market, new yield expectations etc. had to be fed in and that was very
time-consuming. The ‘seat of the pants’ man would have already dealt in the market. How-
ever, although the end product in terms of portfolio selection took a lot of time, the spin-off
in terms of the output useful to fund managers was important. Expected mean returns could
be produced very quickly and other ordinary information could be produced very fast indeed,
so that there was certainly a use for the system. He was a great believer in mathematical models,
but he had retracted from the belief that they could be used to do the investment manager’s
job. They were one of many tools which management could use in the investment field.

Mr S. Benjamin mentioned that the Institute’s Research Committee had received a request to
try to collect data on property earnings and values corresponding to the data produced for
shares on the Stock Exchange. He had come across the problem himself while carrying out an
actuarial valuation of a mortgage guarantee fund and had been quite surprised to find how limited the data were on property.

He was a 'random walk' man and, when the members of the Institute had been discussing maturity guarantees a short while before, he had been so sure that everybody else was a 'random walk' man too that he had been very surprised at the reaction of people to the very elementary statistical proofs he had presented. Had he anticipated quite such a strong reaction he might have asked for some help and got down to it a little more seriously. The point the opener had made was important. If they were going to measure risk in the actuarial context then gilt-edged redeemables were certainly important, and they were not mentioned in the paper at all.

He would like to suggest another mathematical model which was to try to find out whether people could actually forecast. One could start by getting people to write down why they recommended a particular purchase or sale. There was one snag. They would have to put down some time limit, some early time limit and some late time limit, such as: 'I think this particular share will do better than this particular section of the market as a whole at some point in time between three and nine months from now.' Or 'I think it is going to do better some time between one week from now and 100 years from now.' From the data, one could then count and see how well the forecasts did. If one particular analyst insisted on using a time interval of between tomorrow and next century, it might be possible to go back to him after a year and say: 'On your forecasts we are unable to check up, but another analyst has used an interval of about three months and he is right very regularly, so by and large we reckon he is better than you are.' He suggested as a sophistication, and perhaps a simplification of the whole thing, if the methods were going to depend on forecasts, it might be easier to persuade the chairmen of the various companies to make their own forecasts and publish them. Those could be used not merely for determining which shares would do better but also for determining how well the chairman of the board was running the company.

The random walk theory was not totally negative. It could be used to take some very important decisions. For instance, if an insurance company was thinking of joining in the game, or expanding the game, of unit-linked or equity-linked products, clearly a unit was wanted that was going to do very well. He suggested that several units could be set up and then, at any point in time, at least one of them should be doing pretty well just on the random walk theory.

Mr Meyer Melnikoff, F.S.A. (a visitor) said it was a rare privilege to be present at the Institute as a guest and to have the opportunity to comment on the paper. It was most appropriate that actuaries should take an interest in its subject but, unfortunately, in the United States his colleagues had exhibited relatively little interest in those matters. He thought that they would say that it was a subject in which probabilities could very well be applied. Wall Street was often referred to as the Las Vegas of the East.

The concept of variability was not necessarily synonymous with risk and it would be most appropriate to distinguish between uncertainty, which was perhaps more directly related to volatility and variability, and risk, which might be better defined as the chance or probability of missing a target and by how much. Furthermore, much of the mathematics in the area had been devoted to what he called the concept of a single-time purchase rather than a continuous series of purchases, even of a single security. The mathematics might well turn out to be quite different in the latter case.

The subject might very well find its greatest value not in portfolio selection as such but in other aspects of the management of portfolios. In the United States there were some organizations that managed a series of different investment funds and, in recent analyses of the performance of those funds by the Securities and Exchange Commission with some of the consultants working for them, they had begun to consider the desirability of measuring the performance of a fund not merely by showing the equivalent rate of return over a period of time, but also an index of the risk or variability of the rate of return. Increasingly in the
United States some such measure might have to be provided for each form of investment medium in order to more clearly characterize its nature. Furthermore, it seemed to him that within the operation of certain kinds of organizations, such as insurance companies that provided for bonus distributions or dividends and included in the portfolio a significant proportion of variable assets such as common stocks, the field might have great importance in determining what portion of any appreciation in the value of the stocks at any time might safely be distributed to the appropriate individuals.

Mr J. Brew was disappointed that nobody had mentioned what he thought was going to be the most likely practical application of what they were discussing, namely, the assessment of portfolio performance. He had been puzzled, as the opener had been, by the author’s reference to Blume’s paper in which he mentioned the persistence of betas. In his limited reading, American results showed that the betas for portfolios were very much more persistent than the betas for individual shares. He admitted that Mr Russell from the City University had results which showed that in the United Kingdom even the portfolio betas were not consistent over time.

He thought that methods of measuring portfolio performance were improving but the difficulty was to know what to make of the results. He hoped he spoke for Mr Hall’s Society of Investment Analysts’ working party on it when he said that they would have included quite an extensive treatment of risk if the problem had been really solved. The problem was much nearer solution and practical application because of the fact that the betas were much more persistent for portfolios than for anything else. It must be agreed that a lot of the excellent performances put up by the ‘go-go’ funds in America in the last rising market had occurred just because the portfolios had very high beta coefficients.

Mr L. G. Hall, in closing the discussion, said that effective investment was educated judgment. The author had undoubtedly furthered their education. He had not improved their judgment except in so far as, by increasing the precision of those factors which were statistically amenable, he had improved the background against which their judgment was made. Clearly, as the author’s methods achieved greater precision and wider acceptance, the fluctuations in the market ought to become less. If effective investment were, in fact, education without judgment, and if it were science without art, the progress of education would carry with it the seeds of destruction of the market and they would enter a strange dead financial world with no more competition between investment managers for performance and with investment of growing funds merely by formula. That could never happen. They would go on, refining their analytical methods, scientifically planning their portfolios, and each analyst and each portfolio manager would go on trying to keep one step ahead of the game. More computers and more effective planning, by clearing the background against which decisions were made, simply ensured that true judgment became paramount. Investment managers who had true judgment would prosper, but true judgment could not be taught. It could only be encouraged to develop by broad background training, by wide experience and, above all, by willingness to learn and to go on learning from that experience. It was Mr Pepper who had said that the author’s paper had no doubt opened the eyes of many to the scope of the work being done outside the actuarial profession. For investment work, an actuarial training was first-class, but only if it were combined with a ready willingness to go on learning.

He would advise the actuary, and especially the young actuary in investment work, to beware of developing a professional superiority complex. He should learn, and go on learning, from the other disciplines, from the accountants, the economists and the bankers, and at the same time should teach them too. He would be taking important steps to develop his own true judgment. More than that, he would be helping the profession to give the lie to observers such as Anthony Sampson who, in his new ‘Anatomy of Britain’, described the actuarial training as narrow and bleak, and suggested that the actuaries’ confidence was apt to depend on the exclusion of human, social and psychological factors. No actuary would be successful in investment, and not many in other fields, if that were true.
The author had said that the major conclusion that could be drawn from the various statistical analyses which had been performed was basically that there was no simple and foolproof way to make a fortune through investment in ordinary shares. Nevertheless, he had followed a fascinating road. He had brought together in one paper, to use his own words, a great deal of theoretical and empirical work which had been carried out in recent years on the use of mathematical models as an aid to the selection of investment portfolios containing equities. The work was likely to have a clear intellectual appeal to the actuary who combined a statistical upbringing with training and practice in investment.

The part of the paper he had found most interesting was that dealing with experiments with the Sharpe model, and one of those experiments in particular. A test had been made of the effect of varying the level of transaction costs with all the other data held at the standard values. If costs were ignored, as had been the case in much theoretical model-building work, the models indicated a very high degree of concentration of the portfolio in a small number of stocks. It was Mr Green who had spoken of the rarity of inclusion of transaction costs, of how hard it was to quantify the jobber's turn, and how important was the power of a skilled dealer. If costs were remembered, the theoretically optimal portfolio might in practice prove to be considerably less desirable than the investor's original holdings. Any success depended in the real world, of course, on the accuracy of the analyst's subjective judgments of the relative merits of the various sectors of the market, which formed part of the computer input. Even when transaction costs had been taken into account by Hodges and Brealey in the basic data input for the Sharpe model, they had been taken at 4%, to include stamp duty on the purchase, commission on both purchase and sale, and the jobber's turn. That was not all: there was usually capital gains tax to pay. Equity prices had risen so much that the impact of capital gains tax on the dealing of a net fund was devastation. He sometimes wondered if the managers of gross funds realized how lucky they were. He doubted whether the governments of both parties who had introduced and supported the tax realized what a millstone it was around the necks of those who had to manage continuing investment portfolios. He was speaking of switching, not of the individual who realized a profit merely to use it as spending money. Capital gains tax on switching threw an immensely distorting factor into investment decision-making. If, for instance, one wanted to switch from A to B, one might find that 15% of one's proceeds had to be paid from the sale of A in capital gains tax, and one's choice was not between £1,000 worth of stock A, or £1,000 worth of stock B; it was between £1,000 worth of stock A and £850 worth of stock B. He asked whether anyone dare claim that his judgment of the market was so good that he could incur costs like that and still really improve the return from his portfolio, especially in the face of the tests made with the Sharpe model.

Sometimes it was said that an investment manager must not allow himself to be mesmerized by capital gains tax but he must buy and sell stocks on their intrinsic merits and cheerfully pay tax on his realized profits; but a growing fund such as a life fund need not make sales at all. It was Mr Grimes who had made the very valid point that cash for a life company was never a zero-risk portfolio. The investment manager of a life fund would inevitably find that he was either just not making sales, or he was looking not at stocks which he thought were really too expensive but at stocks which he thought were rather expensive but on which he did not have much taxable gain. Stocks which were too high tended to stay too high. The investment community seemed to sit down under the capital gains tax and never protest against it. There were protests in full measure against the required sacrifice of 25% of the premium in the sale of a dollar investment, which had exactly the same distorting effect, but hardly any, against capital gains tax and he wondered why.

The main impact on practical day-to-day investment management of the research work under discussion was a greater appreciation of the need to consider risk as well as reward. They had heard a great deal about the beta factor and the fact that it was obtained statistically when, as the company changed its character, so it should be changing its beta. Mr Pepper had spoken very interestingly about volatility in the gilt-edged market and had made the point that the beta coefficients there changed quite substantially when the Bank of England methods
The ideal in portfolio management was clear enough if one took a very simplified view: a high alpha all the time; a high beta in a market expected to rise; a low beta in a market expected to fall; while not forgetting the degree of risk to which the fund might properly be exposed. The ability to get the beta right was much more rewarding to the manager of a gross fund than to a manager of a net fund; for the latter it was so much more expensive to switch from high beta to low beta when the market was expected to fall and, because it was more expensive, it was very much more difficult to do profitably.

Another problem which had been referred to was the need for understanding by those to whom an investment manager reported. An aggressive investment manager who had stayed too long in high beta stocks might find himself with problems when he attempted to explain that the 30% drop in his high volatility fund during a 15% market down-turn was to be expected, and the fund would rise twice as much as the index when the market turned round. He might even find himself being told, at the bottom of the market, to be more careful and to stick to low beta stocks, and the next time he reported after that he would really be in trouble. As always, the market had the last word. If everyone bought high beta stocks when the market was expected to rise, they would soon become too expensive and no longer interesting. Mr Fison had said of the gilt-edged market that all his clients could not finish up in one stock, and really it was the same point: the market always had the last word.

Mr Benjamin had talked about the possibility of mathematical models to see whether people could forecast. He wondered how long it would take to establish statistically within reasonable confidence limits whether people could forecast. He suspected that it might be so long that the person concerned would be getting somewhere near his retirement before reliable evidence would be available.

They had heard a lot about how the techniques and concepts were not useful. He thought that perhaps one or two speakers had been a little too damning in this. The opener had spoken of the two different worlds of the portfolio manager, the theoretical and the practical. Surely the truth was that the concepts must go into the subconscious if they were to influence practical behaviour, and he thought a great deal of the work that had been done in the field was going into the subconscious of investment managers and was going to gain its reward in their later activity.

He ended as he had begun, by saying that effective investment was educated judgment. The first need was for proper understanding of the liabilities of the fund which was being invested. The second was for the economic, political and market judgments which could distinguish between the merits, at any time, not just of one equity against another, nor even of one equity portfolio design against another, but of equities against gilt-edged and other fixed-interest investments and, indeed, against property investments too. Judgment was helped by education. The author had carried their education forward in a most stimulating and interesting way, and for that they owed him their very warm thanks.

The President (Mr R. S. Skerman) then proposed a vote of thanks to the author. The subject of investment was important to many actuaries in their daily work and, although, as the author himself had said, there was no substitute in making practical investment decisions for sound judgment, there were techniques which might help judgment and limit the area over which it was exercised. The author had made several valuable contributions to actuarial thinking by showing how statistical methods could be applied in fields of interest to the profession and the paper was another contribution of that kind. It had been described as a progress report, indicating where they were in the field at the moment and how they had got there. He was sure that actuaries should maintain an interest in the subject because clearly it was a developing subject, and their concern with investment was such that they ought to keep in touch with developments in the field and, indeed, contribute to them.

Mr A. W. Joseph later submitted the following written contribution:

This is not the first time that in a serious discussion on investments I have called attention
to the antics of Minoru, St Amant, Game Chick, Miss McGiggle and Googoo. These are horses in a racing game, about which I wrote nearly forty years ago (J.I.A., 1933, 64, 172). The odds given in this game by the bookie were wrong, which gave the punter the opportunity of winning if he bet in the right manner. The punter would hope for as little divergence as possible from his theoretical expected gain. The ratio of expected gain divided by standard deviation was, therefore, maximized subject to the condition that no stake on a horse was negative. The solution was oddly reminiscent of that for solving problems in linear programming, although 'reminiscent' is hardly the right word since in 1933 there was no such thing as linear programming. The reason put forward for making the ratio of expected gain divided by standard deviation as great as possible was that the risk was measured by the standard deviation; if the standard deviation of a bet on horse A was half of that for horse B then doubling the stake on A would make the risks equal but would double the expected gain on A.

There is a very clear analogy between the expected gain and standard deviation of the racing game 'Minoru' and the expected return and risk of a portfolio. But whereas in a parlour game it is easy enough to say that the gain may be increased by increasing the stake, it is not realistic to say the same thing about a portfolio of investments. There is not an unlimited amount of money to play with and furthermore it is not cost-free. It may need to be borrowed, or as the opener pointed out, may be needed to cover the liabilities of a life assurance fund. For that reason I am unattracted by the portfolio concept. I would go all the way for the highest expected return but I cannot see that anything is gained by bringing in the standard deviation in a highly sophisticated mathematical way.

The author replied in writing as follows:

First, I must thank all those who took part in the discussion for their comments. Progress in any area must give rise to some extent to controversy and one welcomes constructive expressions of doubts and difficulties as well as ideas for further progress.

I should first like to clear up a major point of misunderstanding that occurred in the opener's remarks and seemed to underlie much of the discussion. Neither in the Markowitz nor in some of the other models, such as the Sharpe model described in the paper, is there any question of the manager being forced to choose the portfolio with minimum overall risk as measured by the standard deviation-only a suggestion that for a given expected return it would be logical to choose the portfolio with minimum risk. From amongst this set, of what are referred to as efficient portfolios (see section 6), the manager can now decide on the trade-off that he is interested in as between expected return and risk. Mr Joseph would have us ignore the risk element and maximize expected return alone, but Mr Grimes assumes that we select the portfolio with minimum risk, whatever its expected return. Thus in looking at the example quoted by Mr Grimes where there are only two possible investments, we note first that for any particular expected return, which can range from 10% to 100%, there is only one possible portfolio, whose standard deviation is fixed. The following figures illustrate the range of possibilities on the efficient set, the mix being defined in terms of the proportion of the 100% investment included.

<table>
<thead>
<tr>
<th>Mix of investment</th>
<th>0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expected return (%)</td>
<td>10</td>
<td>19</td>
<td>28</td>
<td>37</td>
<td>46</td>
<td>55</td>
<td>64</td>
<td>73</td>
<td>82</td>
<td>91</td>
<td>100</td>
</tr>
<tr>
<td>Standard deviation of return (%)</td>
<td>1</td>
<td>1.35</td>
<td>2.15</td>
<td>3.08</td>
<td>4.04</td>
<td>5.02</td>
<td>6.01</td>
<td>7.01</td>
<td>8.00</td>
<td>9.00</td>
<td>10</td>
</tr>
</tbody>
</table>

The manager would now have to decide on his desired trade-off between expected return, or to choose his \( \lambda \) in the notation used in section 7 of the paper. In this instance the gain in expected value as we go from left to right in the table is so high, compared with the rise in standard deviation, that the value of \( \lambda \) would have to be infinity if the manager is to do anything other than to select the portfolio consisting entirely of the 100% investment. Any manager is
very unlikely indeed to choose a $\lambda$ of infinity and, in this particular example, it is even more certain that he would pick the higher expected return investment since it is effectively also the less risky investment. We can see this if we imagine, not improbably, that risk-free investment has a return of 6%. Then the 10% investment gives an extra expected investment return of 4% with a standard deviation of 1% whilst the 100% investment gives an extra expected investment return of 94% with a standard deviation of 10%. Hence no reasonable utility function would lead one, under the Markowitz formulation, to choose other than complete investment in the 100% return investment—the opposite conclusion to that reached by Mr Grimes in his remarks.

Secondly, I would like to take up a point made by Mr Plymen, and repeated in slightly different form by one or two other speakers. Mr Plymen stated that 'Professor Moore assumed that given distinctly limited and inadequate data one was likely to get a better result by a complicated mathematical computer technique than one got with a traditional method.' Although I am not sure what is meant by a 'traditional' method this is, I feel, putting words into my mouth. I have never made such an assumption, and indeed am fairly well known to be a great believer in simplicity of approach in mathematical and statistical problems. What I am adamant about, however, are two things. First that opinions or subjective information should be expressed on an understood and coherent scale of measurement and, secondly, that data are worth the best analysis that they can be given—allowing for any imperfections that may exist in them (see section 3 of the paper). You do not reject data merely because they are poor; you recognise that fact in their analysis. If a simple (traditional?) method can squeeze out all the information that is available in a given mass of facts and opinions, then there is no need to go further. But there are many illustrations of situations where, precisely because the data were inadequate and sketchy, rather more effort was required to analyse it properly than would otherwise be the case. What would be wrong, however, would be to analyse the data assuming that they are perfect when they are, in reality, rather imperfect. Mr S. Benjamin put his finger on an important point when he suggested—perhaps with his tongue in his cheek—that one could calibrate people's forecasting abilities by keeping records of results achieved and use these as an input. I feel that one major benefit of trying to approach portfolio management through a mathematical model is that, properly handled, it helps to define more closely where data are deficient and the benefits that could accrue from better data. Simplified methods of analysis cannot, in general, do that.

One other misapprehension arose which seems to need correction. I never meant to imply, nor did I say anywhere, that there was an either/or situation in the sense that if one had a portfolio model then the whole selection process was completely automated and, for example, all data had to be obtained in some automated way and fitted in. I tried to stress at various places in the paper that the manager was quite able to, and indeed should, exercise his experience and judgment at different points; for example, at the point of input if he wished to correct or change some of the basic information that went in, or, indeed, at the point of output where he was clearly required to exercise his judgment in defining the particular objective that he required from that portfolio.

Finally, it seems to me that the discussion as a whole was basically calling into question the role and relevance of model building when much of the input data is of the nature of forecasts, and may thus contain a greater or lesser degree of subjectivity. I am unrepentant in repeating my view that one should be objective in the use of data and information, whether the latter are subjective or objective. Only in this way can one extract the best out of information and seek out the areas where more (or better) information is required. I have the sneaking feeling that much of the antagonism shown is due to a defence mechanism which I would not be surprised to hear from managers as a class—and indeed commonly hear it from managers when talking about actuarial calculations in general—but find more surprising to hear from actuaries themselves. After all, the actuarial profession began its existence by attempting to put on a more scientific and mathematical basis calculations about the future which involved a blend of objective and subjective information—an attempt that the profession is trying to repeat again
at the present time in the field of non-life assurance. Actuaries have thus been in the forefront of efforts to put on to a more objective basis what have commonly been thought of as being completely subjective areas of business. Hence I cannot comprehend the lack of will to try expressed by many of the speakers. I believe that even today, contrary to the opener's final remark, there are instances where some of these techniques have proved themselves more than 'marginally useful in making practical investment decisions'.