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INTERNATIONAL ACTUARIAL NOTATION

THE existing international actuarial notation was founded on the 'Key to the Notation' given in the *Institute of Actuaries Text-Book*, Part II, *Life Contingencies*, by George King, and is embodied in an explanatory statement adopted unanimously by the Second International Actuarial Congress held in London in May 1898 and printed on pp. 618–40 of the *Transactions* of that Congress. At the Third International Congress held in Paris in June 1900 a further statement by Dr Sprague was submitted which rearranged the symbols in different orders and grouped them on different principles but did not introduce any changes in the symbols themselves beyond two slight additions. This statement is printed on pp. 622–51 of the *Transactions* of that Congress. At the Eleventh International Congress held in Paris in June 1937 an International Committee was appointed to consider the question of notation and given the duty of submitting definite proposals to the next Congress. This Committee met in Brussels in July 1938 and again in July 1939, and certain changes would have been proposed for adoption at Lucerne in 1940. Owing to the war this could not be done, but in order not to lose the benefit of these changes for an indefinite period the Institute of Actuaries and the Faculty of Actuaries have decided to adopt those recommendations which had received unanimous or substantial support at the meetings of the Committee.

The change is to take effect with Vol. LXXVI of the *Journal* and with the May 1950 examinations of the Institute of Actuaries.

The general principles on which the system is based are as follows:

To each fundamental symbolic letter are attached signs and letters each having its own signification.

The lower space to the left is reserved for signs indicating the conditions relative to the duration of the operations and to their position with regard to time.

The lower space to the right is reserved for signs indicating the conditions relative to ages and the order of succession of the events.

The upper space to the right is reserved for signs indicating the periodicity of the events.

The upper space to the left is free, and in it can be placed signs corresponding to other notions.

In what follows these two conventions are used:

A letter enclosed in brackets, thus (x), denotes 'a person aged x'. A letter or number enclosed in a right angle, thus $\overline{n}|$ or $\overline{15}|$, denotes a term-certain of years.

FUNDAMENTAL SYMBOLIC LETTERS

Interest

i = the effective rate of interest, namely, the total interest earned on 1 in a year on the assumption that the actual interest (if receivable otherwise than yearly) is invested forthwith as it becomes due on the same terms as the original principal.

$v = (1+i)^{-1}$ = the present value of 1 due a year hence.
$d = 1 - v$ = the discount on 1 due a year hence.
$\delta = \log_e(1+i) = -\log_e(1-d)$ = the force of interest or the force of discount.

Mortality Tables

l = number living.
d = number dying.
p = probability of living.
q = probability of dying.
μ = force of mortality.
m = central death rate.
a = present value of an annuity.
s = amount of an annuity.
e = expectation of life.
A = present value of an assurance.
E = present value of an endowment.
P, π = premium per annum. P generally refers to net premiums, π to special premiums.
V = policy value.
W = paid-up policy.

The methods of using the foregoing principal letters and their precise meaning when added to by suffixes, etc., follow.

Interest

$i^{(m)} = m\{(1+i)^{1/m} - 1\}$ = the nominal rate of interest, convertible m times a year.
$a_{\overline{n}|} = v + v^2 + \ldots + v^n$ = the value of an annuity-certain of 1 per annum for n years, the payments being made at the end of each year.
$\ddot{a}_{\overline{n}|} = 1 + v + v^2 + \ldots + v^{n-1}$ = the value of a similar annuity, the payments being made at the beginning of each year.
$s_{\overline{n}|} = 1 + (1+i) + (1+i)^2 + \ldots + (1+i)^{n-1}$ = the amount of an annuity-certain of 1 per annum for n years, the payments being made at the end of each year.
$\ddot{s}_{\overline{n}|} = (1+i) + (1+i)^2 + \ldots + (1+i)^n$ = the amount of a similar annuity, the payments being made at the beginning of each year.

The diaeresis or trema above the letters a and s is used as a symbol of acceleration of payments.

Mortality Tables

The ages of the lives involved are denoted by letters placed as suffixes in the lower space to the right. Thus:

l_x = the number of persons who attain age x according to the mortality table.
$d_x = l_x - l_{x+1}$ = the number of persons who die between ages x and $x+1$ according to the mortality table.
p_x = the probability that (x) will live 1 year.
q_x = the probability that (x) will die within 1 year.

$\mu_x = -\frac{1}{l_x}\frac{dl_x}{dx}$ = the force of mortality at age x.

m_x = the central death-rate for the year of age x to $x+1 = d_x \Big/ \int_0^1 l_{x+t}\,dt$.

e_x = the curtate 'expectation of life' (or average after-lifetime) of (x).

If $m \to \infty$ then instead of writing (∞) a bar is placed over the principal symbol. Thus:

$\bar{a}$ = a continuous or momently annuity.

$\bar{\mathrm{A}}$ = an assurance payable at the moment of death.

A small circle placed over the principal symbol shows that the benefit is to be complete. Thus:

$\mathring{a}$ = a complete annuity.

$\mathring{e}$ = the complete expectation of life.

Note. Some consider that $\bar{e}$ would be as appropriate as $\mathring{e}$. As $e_x = a_x$ at rate of interest $i = 0$, so also the complete expectation of life $= \bar{a}_x$ at rate of interest $i = 0$.

When more than one life is involved the following rules are observed:

If there are two or more letters or numbers in a suffix without any distinguishing mark, joint lives are intended. Thus:

$l_{xy} = l_x \times l_y, \quad d_{xy} = l_{xy} - l_{x+1:y+1}$.

Note. When, for the sake of distinctness, it is desired to separate the letters or numbers in a suffix, a colon is placed between them. A colon is used instead of a point or comma to avoid confusion with decimals when numbers are involved.

a_{xyz} = an annuity, first payment at the end of a year, to continue during the joint lives of (x), (y) and (z).

A_{xyz} = an assurance payable at the end of the year of the failure of the joint lives (x), (y) and (z).

In place of a life a term-certain may be involved. Thus:

$a_{x:\overline{n}|}$ = an annuity to continue during the joint duration of the life of (x) and a term of n years certain; that is, a temporary annuity for n years on the life of (x).

$\mathrm{A}_{x:\overline{n}|}$ = an assurance payable at the end of the year of death of (x) if he die within n years, or at the end of n years if (x) be then alive; that is, an endowment assurance for n years.

If a perpendicular bar separates the letters in the suffix, then the status after the bar is to follow the status before the bar. Thus:

$a_{y|x}$ = a reversionary annuity, that is, an annuity on the life of (x) after the death of (y).

$\mathrm{A}_{z|xy}$ = an assurance payable on the failure of the joint lives (x) and (y) provided both these lives survive (z).

If a horizontal bar appears above the suffix then survivors of the lives, and not joint lives, are intended. The number of survivors can be denoted by a letter or number over the right end of the bar. If that letter, say r, is not distinguished by any mark, then the meaning is *at least* r survivors; but if it is enclosed in square brackets, $[r]$, then the meaning is *exactly* r survivors. If no letter or

number appears over the bar, then unity is supposed and the meaning is *at least one* survivor. Thus:

$a_{\overline{xyz}}$ = an annuity payable so long as at least one of the three lives (x), (y) and (z) is alive.

$a_{\overline{xyz}}^{2}$ = an annuity payable so long as at least two of the three lives (x), (y) and (z) are alive.

$p_{\overline{xyz}}^{[2]}$ = probability that exactly two of the three lives (x), (y) and (z) will survive a year.

${}_nq_{\overline{xy}}$ = probability that the survivor of the two lives (x) and (y) will die within n years $= {}_nq_x \times {}_nq_y$.

${}_n\mathrm{A}_{\overline{xy}}$ = an assurance payable at the end of the year of death of the survivor of the lives (x) and (y) provided the death occurs within n years.

When numerals are placed above or below the letters of the suffix, they designate the order in which the lives are to fail. The numeral placed *over* the suffix points out the life whose failure will finally determine the event; and the numerals placed *under* the suffix indicate the order in which the other lives involved are to fail. Thus:

A_{xy}^{1} = an assurance payable at the end of the year of death of (x) if he dies first of the two lives (x) and (y).

A_{xyz}^{2} = an assurance payable at the end of the year of death of (x) if he dies second of the three lives (x), (y) and (z).

$\mathrm{A}_{\substack{xyz \\ 1}}^{2}$ = an assurance payable at the end of the year of death of (x) if he dies second of the three lives, (y) having died first.

$\mathrm{A}_{\substack{\overline{xy}:z \\ 3}}$ = an assurance payable at the end of the year of death of the survivor of (x) and (y) if he dies before (z).

$\mathrm{A}_{x:\overline{n}|}^{1}$ = an assurance payable at the end of the year of death of (x) if he dies within a term of n years.

$\left.\begin{matrix} a_{\overline{yz}|x} \\ {}^{1} \\ \text{or} \\ {}^{2}a_{\overline{yz}|x} \end{matrix}\right\}$ = an annuity to (x) after the failure of the survivor of (y) and (z), provided (z) fails before (y).

Note. Sometimes to make quite clear that a joint-life status is involved a symbol ⌐¬ is placed above the lives included. Thus $\mathrm{A}_{\overline{xy}:\overline{n}|}^{1}$ = a joint-life temporary assurance on (x) and (y).

In the case of reversionary annuities, distinction has sometimes to be made between those where the times of year at which payments are to take place are determined at the outset and those where the times depend on the failure of the preceding status. Thus:

$a_{y|x}$ = annuity to (x), first payment at the end of the year of the death of (y) or, on the average, about 6 months after his death.

$\hat{a}_{y|x}$ = annuity to (x), first payment 1 year after the death of (y).

$\mathring{a}_{y|x}$ = complete annuity to (x), first payment 1 year after the death of (y).

ANNUAL PREMIUMS

The symbol P with the appropriate suffix or suffixes is used in simple cases, where no misunderstanding can occur, to denote the annual premium for a benefit. Thus:

P_x = the annual premium for an assurance payable at the end of the year of death of (x).

$P_{x:\overline{n}|}$ = the annual premium for an endowment assurance on (x) payable after n years or at the end of the year of death of (x) if he die within n years.

P^{1}_{xy} = the annual premium for a contingent assurance payable at the end of the year of death of (x) if he die before (y).

In all these cases it is optional to use the symbol P in conjunction with the principal symbol denoting the benefit. Thus instead of $P_{x:\overline{n}|}$ we may write $P(A_{x:\overline{n}|})$. In the more complicated cases it is necessary to use the two symbols in this way. Suffixes, etc., showing the conditions of the benefit are to be attached to the principal letter, and those showing the conditions of payment of the premium are to be attached to the subsidiary symbol P. Thus:

${}_nP(\bar{A}_x)$ = the annual premium payable for n years only for an assurance payable at the moment of the death of (x).

$P_{xy}(A_x)$ = the annual premium payable during the joint lives of (x) and (y) for an assurance payable at the end of the year of death of (x).

${}_nP({}_n|a_x)$ = the annual premium payable for n years only for an annuity on (x) deferred n years.

${}_tP^{(m)}(A_{x:\overline{n}|})$ = the annual premium payable for t years only, by m instalments throughout the year, for an endowment assurance for n years on (x) (see below as to $P^{(m)}$).

Notes. (1) As a general rule the symbol P could be used without the principal symbol in the case of assurances where the sum assured is payable at the end of the year of death, but if it is payable at other times, or if the benefit is an annuity, then the principal symbol should be used.

(2) $P^{(m)}_x$. A point which was not brought out when the international system was adopted is that there are two kinds of premiums payable m times a year, viz. those which cease on payment of the instalment immediately preceding death and those which continue to be payable to the end of the year of death. To distinguish the latter the m is sometimes enclosed in square brackets, thus $P^{[m]}$.

Note. As a general rule V or W can be used as the main symbol if the sum assured is payable at the end of the year of death and the premium is payable periodically throughout the duration of the assurance. If the premium is payable for a limited number of years, say n, the policy value after t years could be written ${}_tV[{}_nP(A)]$, or, if desired ${}^n_tV(A)$.

In investigations where modified premiums and policy values are in question such modification may be denoted by adding accents to the symbols. Thus, when a premium other than the net premium (a valuation premium) is used in a valuation it may be denoted by P′ and the corresponding policy value by V′. Similarly, the office (or commercial) premium may be denoted by P″ and the corresponding paid-up policy by W″.

COMPOUND SYMBOLS

(Ia) = an annuity
(IA) = an assurance
} commencing at 1 and increasing 1 per annum.

If the whole benefit is to be temporary the symbol of limitation is placed outside the brackets. Thus:

$(Ia)_{x:\overline{n}|}$ = a temporary increasing annuity.

$(IA)^1_{x:\overline{n}|}$ = a temporary increasing assurance.

If only the increase is to be temporary but the benefit is to continue thereafter, then the symbol of limitation is placed immediately after the symbol I. Thus:

$(I_{\overline{n}|}a)_x$ = a whole-life annuity
$(I_{\overline{n}|}A)_x$ = a whole-life assurance
} increasing for n years and thereafter stationary.

If the benefit is a decreasing one, the corresponding symbol is D. From the nature of the case this decrease must have a limit, as otherwise negative values might be implied. Thus:

$(D_{\overline{n}|}A)^1_{x:\overline{n}|}$ = a temporary assurance commencing at n and decreasing by 1 in each successive year.

If the benefit is a varying one, the corresponding symbol is v. Thus:
(va) = a varying annuity.

COMMUTATION COLUMNS

Single lives

$$D_x = v^x l_x,$$

$$N_x = D_x + D_{x+1} + D_{x+2} + \text{etc.},$$

$$S_x = N_x + N_{x+1} + N_{x+2} + \text{etc.},$$

$$C_x = v^{x+1} d_x,$$

$$M_x = C_x + C_{x+1} + C_{x+2} + \text{etc.},$$

$$R_x = M_x + M_{x+1} + M_{x+2} + \text{etc.}$$

When it is desired to construct the assurance columns so as to give directly assurances payable at the moment of death the symbols are distinguished by a bar placed over them. Thus:

$$\overline{C}_x = v^{x+\frac{1}{2}} d_x, \text{ which is an approximation to } \int_0^1 v^{x+t} \mu_{x+t} l_{x+t}\, dt,$$

$$\overline{M}_x = \overline{C}_x + \overline{C}_{x+1} + \overline{C}_{x+2} + \text{etc.},$$

$$\overline{R}_x = \overline{M}_x + \overline{M}_{x+1} + \overline{M}_{x+2} + \text{etc.}$$

Joint lives

$$D_{xy} = v^{\frac{1}{2}(x+y)} l_{xy},$$

$$N_{xy} = D_{xy} + D_{x+1:y+1} + D_{x+2:y+2} + \text{etc.},$$

$$C_{xy} = v^{\frac{1}{2}(x+y)+1} d_{xy},$$

$$M_{xy} = C_{xy} + C_{x+1:y+1} + C_{x+2:y+2} + \text{etc.},$$

$$C^1_{xy} = v^{\frac{1}{2}(x+y)+1} d_x l_{y+\frac{1}{2}},$$

$$M^1_{xy} = C^1_{xy} + C^1_{x+1:y+1} + C^1_{x+2:y+2} + \text{etc.}$$

SELECTION

If the suffix to a symbol which denotes the age is enclosed in square brackets, it indicates the age at which the life was selected. To this may be added, outside the brackets, the number of years which have elapsed since selection, so that the total suffix denotes the present age. Thus:

$l_{[x]+t}$ = the number in the select life table who were selected at age x and have attained age $x+t$.

$d_{[x]+t} = l_{[x]+t} - l_{[x]+t+1}$.

$a_{[x]}$ = value of an annuity on a life now aged x and now select.

$a_{[x-n]+n}$ = value of an annuity on a life now aged x and select n years ago at age $x-n$.

$N_{[x]} = D_{[x]} + D_{[x]+1} + D_{[x]+2} + \ldots.$

$\ddot{a}_{[x]} = N_{[x]} \div D_{[x]} = 1 + a_{[x]}$,

and similarly for other functions.

When Dr Sprague presented his statement, he mentioned that an objection had been raised that the notation in some cases offers the choice of two symbols for the same benefit. For instance, a temporary annuity may be denoted either by ${}_n a_x$ or by $a_{x:\overline{n}|}$. This is, he says, a necessary consequence of the principles underlying the system, and neither of the alternative forms could have been suppressed without injury to the symmetry of the system.

The following are the changes which have now been introduced:

(1) *Interest.* In place of the symbol $j_{(m)}$ = the *nominal* rate of interest convertible m times in a year when the effective rate is i, the symbol $i^{(m)}$ has been substituted; the former definitions of $i^{(m)}$ and $\bar{\imath}$ as the effective rate of interest when interest is convertible m times a year and momently respectively have been abolished.

(2) *Annuities-due.* In place of the symbol a for an annuity under which the first payment is to be made at once the symbol $\ddot{a}$ has been adopted. This adoption of the trema .. as a symbol of acceleration has permitted the introduction of $\ddot{s}$ for the accumulated amount of an annuity-due at the end of the term, for which no symbol previously existed.

(3) *Mortality.* The use of Q for the probability of dying within a longer term than 1 year has been omitted.

(4) *Compound symbols.* The use of D in a compound symbol to indicate decreasing benefits has been introduced.

(5) *Commutation columns.* The old definition of $N_x = D_{x+1} + D_{x+2} + D_{x+3} +$ etc. has been abolished in favour of $N_x = D_x + D_{x+1} + D_{x+2} +$ etc. This renders the use of $\mathbb{N}_x$ and $\mathbf{N}_x$ for the latter series unnecessary.

Though the use of $j_{(m)}$ to represent the nominal rate of interest is superseded by $i^{(m)}$ as an international symbol, it was recognized by the Committee that its use locally, where it may be a matter of convenience and cannot be misunderstood, is not prevented. A suggestion that U should be introduced into the official international notation as a symbol for the surrender value of a policy was not adopted, though it was agreed that, if a symbol were considered necessary, U should be chosen.