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REPORT ON THE WILKIE STOCHASTIC INVESTMENT MODEL

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ABSTRACT

A FIMAG Working Party was set up in 1989 to consider the stochastic investment model proposed by A. D. Wilkie, which had been used by a number of actuaries for various purposes, but had not itself been discussed at the Institute. This is the Report of that Working Party. First, the Wilkie model is described. Then the model is reviewed, and alternative types of model are discussed. Possible applications of the model are considered, and the important question of 'actuarial judgement' is introduced. Finally the Report looks at possible future developments. In appendices, Clarkson describes a specific alternative model for inflation, and Wilkie describes some experiments with ARCH models. In further appendices possible applications of stochastic investment models to pension funds, to life assurance and to investment management are discussed.

KEYWORDS

Stochastic Investment Models; Wilkie Model

1. INTRODUCTION

1.1 This is the Report of a Working Party, set up in 1989 under the auspices of the Financial Management Group (FIMAG), and under the chairmanship of the first-named author. The origin of the Working Party was a remark by Professor S. Benjamin in the discussion at the Institute of the paper by Purchase *et al.* (1989); the authors of the paper then being presented had made use of the stochastic investment model proposed by Wilkie (1986a), and Benjamin was concerned that this model was being used without having had the scrutiny of the Institute (though it had been discussed by the Faculty in 1984).

1.2 The terms of reference of the Working Party were as follows:

A stochastic investment model has been developed by David Wilkie for actuarial use. The brief of the Working Party is:

to critique the model from a statistical viewpoint;
to indicate the practical circumstances in which the model can be used and the interpretation of any results;
to consider the types of investment model which exist, indicating their potential use in practice;

to enumerate the practical situations where an investment model would make a substantial contribution to the resolution of an actuarial problem;
to suggest the areas where further development of a model would reap the highest rewards, or failing that, whether research is better concentrated by assuming the current model is sufficiently useful.

1.3 Generally, therefore, the Working Party was asked to review the stochastic investment model proposed by Wilkie and to comment on those areas where the use of such a model might be appropriate. Professor Wilkie assisted the Working Party by attending its meetings, giving evidence as desired by it, and preparing Appendix B of the Report.

1.4 Each member, or group of members, was allocated a task in accordance with the terms of reference. The Working Party also agreed that an external authority on statistics should be invited to comment on the Wilkie model and we are grateful to Professor Andrew Harvey of the London School of Economics for his advice in this area. His report is made use of in Section 3 of the Report.

1.5 We now have pleasure in presenting our Report. Since different authors have been responsible for different parts of the Report and the appendices, a considerable variety of opinions have been expressed in a variety of styles. This is as it should be, since among actuaries there are different opinions about the use of stochastic investment models, about the use of random simulation methods in general, and about Wilkie's model in particular. There are many areas which require further research, and we hope that our Report will stimulate further discussion and lead to a greater awareness of the uses and limitations of existing stochastic techniques.

1.6 In Section 2 the model proposed by Wilkie is outlined. In Section 3 considerations with respect to the estimation of the Wilkie model are discussed. In Section 4 alternative models are considered. In Section 5 implications relevant to any subsequent application of the model, as it stands, are presented. Throughout the discussions of the Working Party one theme remained central—the extent to which 'actuarial judgement' might comfortably over-ride purely theoretical and statistical considerations. A short Section 6 is therefore devoted to this topic. Finally, in Section 7, we present some thoughts on future developments and implications for the actuarial profession.

1.7 Additional material is given in the appendices. In Appendix A, Clarkson's alternative model for inflation is described. In Appendix B, contributed by Wilkie, some experiments with ARCH models are described. In Appendices C, D and E the implications of stochastic investment models in relation to pension funds, life assurance and investment management respectively are considered from different viewpoints.

2. DESCRIPTION OF THE WILKIE MODEL

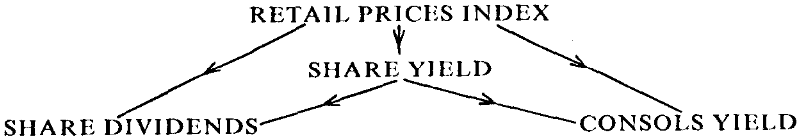
2.1 Detailed descriptions of the Wilkie model are set out in a number of papers (Wilkie, 1984c, 1986a, 1986b and 1987a) and it was discussed at the Faculty of

Actuaries on 19 November 1984 and at the Institute of Actuaries Students' Society on 29 January 1985. The reader is referred to these papers for complete details of the Wilkie model and its statistical justification. It may, however, be helpful to set out below a brief description of the model.

2.2 The variables used in the model are:

$Q(t)$ a retail prices index,
$D(t)$ an index of share dividends,
$Y(t)$ the dividend yield on these share indices,
$C(t)$ the yield on 2·5% Consols, which is taken as a measure of the general level of fixed-interest yields in the market.

2.3 The interdependence is shown diagrammatically below, where the arrows indicate the direction of influence.



2.4 The model used for $Q(t)$ is:

$$\nabla \ln Q(t) = QMU + QA\,(\nabla \ln Q(t-1) - QMU) + QSD \cdot QZ(t)$$

where the backwards difference operator ∇ is defined by $\nabla X(t) = X(t) - X(t-1)$, and $QZ(t)$ is a sequence of independent identically distributed unit normal variates.

The values proposed by Wilkie for the parameters in his 'standard basis' are:

$$QMU = 0{\cdot}05, \quad QA = 0{\cdot}6, \quad QSD = 0{\cdot}05.$$

2.5 The model for $Y(t)$ is:

$$\ln Y(t) = YW \cdot \nabla \ln Q(t) + YN(t),$$

where:
$$YN(t) = \ln YMU + YA \cdot (YN(t-1) - \ln YMU) + YE(t),$$
$$YE(t) = YSD \cdot YZ(t),$$

and $YZ(t)$ is a sequence of independent identically distributed unit normal variates.

The values proposed by Wilkie for the parameters are:

$$YMU = 0{\cdot}04, \quad YA = 0{\cdot}6, \quad YW = 1{\cdot}35, \quad YSD = 0{\cdot}175.$$

2.6 The model for $D(t)$ is:

$$\nabla \ln D(t) = DW \cdot \left(\frac{DD}{1-(1-DD)B}\right) \cdot \nabla \ln Q(t) + DX \cdot \nabla \ln Q(t)$$
$$+ DMU + DY \cdot YE(t-1) + DE(t) + DB \cdot DE(t-1),$$

where the backwards step operator B is defined by $BX(t)=X(t-1)$, $DE(t)=DSD\cdot DZ(t)$, and $DZ(t)$ is a sequence of independent identically distributed unit normal variates. The term in parentheses above involving DD represents an infinite series of lag effects, with exponentially declining coefficients:

$$DD,\quad DD\cdot(1-DD),\quad DD\cdot(1-DD)^2,\quad \text{etc.}$$

The sum of these coefficients is unity, so that part of the formula represents the lagged effect of inflation with unit gain. This means that if retail prices rise by 1% this term will also eventually rise by 1%. It can alternatively be described as the carried forward effect of inflation, $DM(t)$, where:

$$DM(t)=DD\cdot\nabla\ln Q(t)+(1-DD)\cdot DM(t-1),$$

from which it can be seen that the amount that enters the dividend model each year is DD times the current inflation rate, plus $(1-DD)$ times the amount brought forward from the previous year, and that this total is then carried forward to the next year.

The values proposed by Wilkie for the parameters are:

$$DW=0{\cdot}8,\quad DD=0{\cdot}2,\quad DX=0{\cdot}2,\quad DMU=0{\cdot}0,\quad DY=-0{\cdot}2,$$
$$DB=0{\cdot}375,\quad DSD=0{\cdot}075.$$

2.7 The model for $C(t)$ is:

$$C(t)=CW\cdot\left(\frac{CD}{1-(1-CD)B}\right)\cdot\nabla\ln Q(t)+CN(t),$$

where:

$$\begin{aligned}\ln CN(t) = {} & \ln CMU \\ & + (CA1\cdot B+CA2\cdot B^2+CA3\cdot B^3)\,(\ln CN(t)-\ln CMU) \\ & + CY\cdot YE(t)+CSD\cdot CZ(t),\end{aligned}$$

and $CZ(t)$ is a sequence of independent identically distributed unit normal variates. The term in parentheses in CD has a similar form to the DD term in the dividend model, though the parameter value is different. It represents the current value of expected future inflation as an exponentially weighted moving average of past rates of inflation.

The values proposed by Wilkie for the parameters are:

$$CW=1{\cdot}0,\quad CD=0{\cdot}045,\quad CMU=0{\cdot}035,\quad CA1=1{\cdot}20,$$
$$CA2=-0{\cdot}48,\quad CA3=0{\cdot}20,\quad CY=0{\cdot}06,\quad CSD=0{\cdot}14.$$

3. REVIEW OF THE MODEL

3.1 *Introduction*

In this section considerations with respect to the estimation of the model proposed by Wilkie are discussed. Statistical properties of the model are examined under the following headings: description of the nature, structure and details of the model in a multivariate time series context; period of data; adequacy of the model.

3.2 *Description of the Nature, Structure and Details of the Model in a Multivariate Time Series Context*

3.2.1 This section draws heavily on the contents of the report prepared by Professor Harvey, which examined the model proposed by Wilkie within the wider context of time series models as they could be applied in actuarial work.

3.2.2 Time series modelling, just like actuarial subjects, has its own terminology; we shall try to minimise its use and explain as many of the terms as we can without sacrificing continuity.

3.2.3 Multivariate time series models (i.e. models with several time-dependent variables) can be classified into two broad categories:

(i) statistical (economically atheoretical) models; and
(ii) econometric models.

3.2.4 In the statistical models, the future behaviour of each of the variables is explained in terms of prior values (often termed lagged or time-lagged values) of itself and/or other variables in the set under investigation. These models are referred to in the time-series terminology as 'vector auto-regression models—VARs'.

3.2.5 In the econometric models, one adds further ingredients, such as economic theory, connecting some of the variables, and possibly other, so called exogenous, variables which are used to explain the future behaviour of the variables in the set under investigation (so-called endogenous variables).

3.2.6 VAR models are the most widely used in economics. However, they are subject to a number of criticisms, such as their lack of regard of the underlying economic theory and limitations imposed by their linearity. These criticisms must be weighed against the fact that the parameters of the VAR models are straightforward to estimate, using Ordinary Least Squares (OLS) estimates. However, if one imposes restrictions on the parameters, either because of some theoretical reasons or judgement, OLS can give biased estimates and more sophisticated techniques must be used. These techniques are known as Generalised Least Squares (GLS) procedures.

3.2.7 Alternative formulation of VAR models can be derived by the introduction of an ordering of the variables in such a way that each variable depends only on lagged values of itself and variables of lower order. This has the consequence that the residuals (error terms) are mutually uncorrelated, which is often convenient. Unless parameter restrictions are imposed, the ordering is arbitrary and the OLS procedures can be applied to each equation separately.

3.2.8 Econometric models, on the other hand, require restrictions to be imposed in order to make them viable. Parameter restrictions are imposed on the basis of economic theory, and further restrictions, particularly on the lags, are often made on the basis of estimates. Econometricians have developed a vast battery of tests and diagnostic checks for arriving at a final model specification. Nevertheless, specification of econometric models remains difficult because of uncertainty over prior theory, and this leads many economists to prefer the use of VAR models. In this respect, the virtues of VAR models are set out in an influential article by Sims (1980).

3.2.9 Turning, now, to a consideration of the model proposed by Wilkie, the model incorporates series of values of retail price inflation, share dividend yields, and dividend values (and hence share prices), and the yields on British Government 2·5% Consols. The model is structured such that variables are ordered, and depend only on lagged values of themselves, and variables of a lower order. Thus, the composite model is essentially driven by the retail price inflation component, and, broadly, can be viewed within the framework of the alternative formulation of VAR models described above, although the model for the yield on 2·5% Consols contains a minor non-linearity.

3.2.10 Parameter restrictions are imposed, partly on the basis of prior assumptions, and partly on the basis of testing a more general model (although these tests are not reported in published papers). To quote directly from a Working Party communication from Wilkie: "In some cases parameters that were just significant at a 5% level were omitted, because there seemed to be no economic rationale for them. In other cases, the parameters were pushed in a certain direction precisely because there was an economic rationale for this." Thus, the model is not economically atheoretical, nor is it a fully fledged econometric model. There is nothing wrong with such an approach, though the importance of testing restrictions is noted.

3.2.11 To quote directly from the report prepared by Harvey: "The methodology is essentially as advocated by Box & Jenkins (1976). This methodology is less popular than it was, and it can be particularly difficult to apply in a multivariate context. It makes the whole modelling procedure more complex than it need be, and it can lead to formulations of lag structures which are not easy to handle." More recent developments in multivariate time-series modelling suggest that standard Box-Jenkins methods are gradually being replaced by more comprehensive and sophisticated techniques, such as the direct formulation of structural time series models, with lag dependent variables; for example, see Harvey (1989).

3.2.12 As mentioned, the retail price inflation component of the model is the most important, in that it essentially drives the composite model. The model provides that the expected value of inflation will lie between the previous value and a mean value; in time series terminology, the model is auto-regressive, a straightforward ARIMA (1,1,0) model. Thus, the interpretation of the model is that retail price inflation has a normal distribution, with constant mean, and standard deviation increasing at a decreasing rate over time towards an upper limit. The mean of the distribution of retail price inflation at any time is equal to the mean parameter of the model, which will be the mean rate of inflation for the period over which the model is estimated. This interpretation will hold if simulations are generated using the model, provided that initial values are neutral.

3.3 *Period of Data*

3.3.1 The Working Party concluded that a model should be estimated using post-1945 data, including as recent data as are available. This was because fundamental changes had affected the processes generating the data, before and after the Second World War. Nevertheless, pre-1945 data were of interest, for the same reason, in order to assess the impact of such fundamental changes.

3.4 *Adequacy of the Model*

3.4.1 The Working Party discussed, primarily, the adequacy of the published model. In particular, the model for retail price inflation, estimated from data for the period 1919–1982, was examined, because this essentially drives the composite model. However, more recently Wilkie has estimated a similar model from data for the period 1945–1989; in a Working Party communication he concludes that no higher order linear model would give a significantly better fit.

3.4.2 Deliberations were very much a compromise between, on the one hand, considerations of the fit of the model, and the associated advantages and disadvantages of more sophisticated models; and, on the other hand, considerations of parsimony, and the associated attractions of a more simple model—for example, a model in which the yield on 2·5% Consols might be derived directly from a model for retail price inflation.

3.4.3 The Working Party agreed that there was little evidence to suggest that a better fitting parsimonious model could be estimated using standard Box–Jenkins methodology.

3.4.4 Nevertheless, there were concerns over the fit of the model to the data. Such concerns result from the analysis of residuals, testing for independence, and normality. While Wilkie concludes from his tests that there is little significant evidence to suggest that the residuals are not independent, or non-normal, other tests are less conclusive. For example, Kitts (1990) concludes that there is some evidence to suggest that the residuals are not independent, so that the frequency of occurrence of sustained periods of extreme inflation and deflation are under-estimated; and that there is very strong evidence to suggest that the residuals cannot be assumed to have a normal distribution.

3.4.5 More generally, the sorts of features which a model of retail price inflation ought to reflect were discussed, as a result of which several members of the Working Party expressed concerns that the model for retail price inflation might be 'too tame'. These discussions can be classified into three areas:

(1) the existence of bursts of inflation, indicating that once an upward trend in inflation is established, there is a tendency for it to continue; this problem can be approached through the incorporation of time-dependent (i.e. varying) variances of the residuals;

(2) the existence of large, irregular shocks, such as those of the mid-1970s, which are difficult to reconcile with most standard models; this problem can be approached through the incorporation of a non-linear, random-shock element, for example, using mixture models; and
(3) the possible non-normality of residuals, through asymmetry, etc.; this problem can be approached through the use of alternative distributions of residuals.

4. ALTERNATIVE MODELS

4.1 *Introduction*

4.1.1 In this section alternative models are considered, under the headings of non-constant variance of residuals, random shocks effects, the non-normal distribution of residuals and finally stationarity.

4.2 *Non-Constant Variance of Residuals*

4.2.1 Engle (1982) uses auto-regressive conditional heteroscedastic (ARCH) models to generalise the implausible assumption of a constant variance. ARCH models have non-constant variances conditional on the past, but constant unconditional variances. Essentially, the variance of the current stochastic disturbance is assumed to depend on the magnitudes of previous residuals; this can be extended so that the variance depends on the actual value of the series in the previous period, thereby capturing the notion that there is more variability at higher levels of inflation. Engle finds that the ARCH effect is significant for a series of quarterly U.K. inflation data (1958 quarter II through 1977 quarter II), and that the estimated variances increase substantially during the chaotic 1970s. ARCH models require different estimation procedures (see Engle, 1982); but these are nevertheless relatively straightforward.

4.2.2 Also, these findings are consistent with those of Taylor (1986), who studies 40 daily financial time series, and, finding residual correlation over a wide period of lags, concludes that linear models are inadequate because of changes in either the unconditional or conditional variance of returns.

4.2.3 While a model incorporating ARCH effects may well appear more realistic than a linear model in the short term, in the medium and long term the advantages are less. Broadly, in the medium term, ARCH effects will make little difference to predicted values of the series, or their mean square errors, but will still affect extreme values. In the long term ARCH effects will make very little difference at all. Therefore, it might be concluded that considerations of the incorporation of ARCH effects are of more concern for short-term models, and perhaps medium-term models used for extreme values.

4.2.4 In Appendix B Wilkie demonstrates the effect on his existing model of incorporating ARCH effects.

4.3 *Random Shocks Effects: Mixture Models*

4.3.1 The estimation of the model required restrictions to be imposed on the parameters, because of sharp changes in the series in the mid-1970s.

4.3.2 One approach to this problem, that has received widespread attention in econometric and statistical literature, is an assumption of unobserved heterogeneity, involving the use of a mixture of distributions. Here, the stochastic disturbances are assumed to correspond to two or more distributions. For example, a large proportion might be assumed to have one normal distribution, while the remaining proportion might be assumed to have a different normal distribution. An interesting, broad introduction to the subject of unobserved heterogeneity can be found in Vaupel & Yashin (1985).

4.3.3 However, the inclusion of a random shock element would be difficult in practice. There are problems in the estimation of mixture models, such as identification. For example, it would be difficult to estimate the appropriate period between shocks, and their appropriate magnitude, from the sparse data available. Techniques exist, but remain the subject of controversial debate.

4.3.4 Also, while a model incorporating random shocks may well appear more realistic than a linear model in the short term, in the medium and long term the effect will be limited to a slight difference in the mean square errors of predicted values; and perhaps in the predicted values themselves, if the random shock distribution is asymmetrical. Therefore, again, it might be concluded that considerations of the incorporation of random shock effects are of more concern for short-term models, and perhaps medium-term models used for extreme values.

4.4 *Non-Normal Distribution of Residuals*

4.4.1 In Wilkie's published papers he considers the residuals to be approximately normal, although somewhat negatively skewed and definitely fat-tailed. To overcome this problem, he suggests the use of the same model, but with a larger standard deviation for the residuals. This gives the opportunity for more extreme values to arise, at the expense of reducing the number of small residuals. However, Wilkie notes that this is an arbitrary adjustment, and cannot be supported explicitly by the data.

4.4.2 Apart from this, the model provides for negative and positive movements in inflation with equal probability and it provides for a significant probability of negative inflation. The Working Party concluded that, in practice, this might be unlikely. In this respect, it was noted that some practitioners who have applied the model have made adjustments to reduce these effects.

4.4.3 One straightforward approach to this problem is to use the empirical distribution of actual residuals from the fitted model. However, it is difficult to alter the distribution in any way for sensitivity analysis, and, broadly, in the medium and long term, the incorporation of the actual residuals, rather than the associated normal distribution, will make very little difference.

4.4.4 A second, less straightforward, approach would be to use a different distribution for the residuals, such as the Pearson Type IV, *t*-distribution or Stable Paretian. Strictly, this requires refitting the model, since maximum likelihood estimators of the parameters using a different distribution for residuals might be different from those using a normal distribution for residuals (which are often the same as the least squares estimates). It is not known how different the results would be; but, in the medium and long term, the results will be very similar, because of the Central Limit Theorem, unless a distribution with non-finite moments, such as a Stable Paretian distribution, is used.

4.4.5 One author, R. S. Clarkson, developed ideas that he had put forward at the Faculty discussion on Wilkie's paper into the outline of a non-linear stochastic model for inflation, where the expected value of the residual varies with the recent level of inflation and where specific upward trend and irregular shock components are incorporated. This model, which relies very much on actuarial judgement rather than being formulated within a conventional statistical framework, provides an improved fit for post-war U.K. inflation data and results in significantly fewer negative values of inflation than is the case with the Wilkie inflation model. The Working Party asked Clarkson to set out, in summary form, his thinking on a non-linear model and this is presented in Appendix A. Clarkson has also presented a paper on this model at the Brighton AFIR Colloquium (Clarkson, 1991).

4.5 *Stationarity*

4.5.1 It is, perhaps, worth introducing this section by emphasising that stationarity is often the most important limitation of economic and financial time-series models, in that they are unable to anticipate future changes in the process which generated historic data. In this context, stationarity must not be confused with the separate issue of comparability; for example, the Maturity Guarantees Working Party (1980) suggested that confidence of stationarity could be gained from the fact that some estimated parameter values were reasonably similar for all the indices studied, both U.K. and U.S.A.

4.5.2 Considering the model under review, there is an implicit assumption that the parameters of the model remain constant over time. However, there is considerable evidence to suggest that the process generating the data is changing over time. Wilkie drew attention to this in a Working Party communication: "The evidence, however, about inflation in the past [pre-1919] contradicts the present model, and indicates that appropriate models can indeed change."

4.5.3 However, it is difficult to see how one can make any forecast of unpredictable changes, nor is it obvious how the process might change over time in any predictable way. One remedy for this dilemma is to represent future uncertainty by using a higher variance for the random residuals, but this introduces other difficulties already noted.

5. APPLICATION OF THE MODEL

5.1 *Introduction*

5.1.1 The Working Party considered the practical application of the model in the traditional areas of actuarial work: pension funds, life assurance and investment management; and these are presented in Appendices C, D and E respectively.

5.1.2 A description of the practical application of the model in the field of general insurance was given by Daykin & Hey (1990). The reader is referred to the bibliography for a list of further papers on stochastic modelling.

5.1.3 However, the Working Party considered it extremely important to record that the use of the model, or any alternative, without a full understanding of its limitations is dangerous and is not to be recommended. It was also considered helpful to set out some general comments on any applications. These comments fall under the following headings: general considerations; period of simulation; number of simulations; and sensitivity.

5.2 *General Considerations*

5.2.1 To quote directly from the report prepared by Harvey: ". . . the aim of stochastic investment modelling is to investigate the medium and long-run behaviour of [the relevant economic and financial variables]. The idea is not simply to predict [these variables]. In fact, in the long run, the predictions . . . will, if the model is stable (in technical terms 'stationary') converge to the underlying mean level implied by the model. The additional aim is to determine the spread of possible values of [the variables] in the medium and long run. In other words, what is required is the covariance matrix of future values of [the variables]. This can be worked out analytically. Alternatively, the covariance matrix . . . may be found by simulation. Indeed, if the model has been amended so that certain variables enter in a non-linear way, an analytic solution becomes difficult and simulation is the preferred approach."

5.3 *Period of Simulation*

5.3.1 On the one hand, considering the long term, some statisticians quote a general rule of thumb, which dictates that a model estimated from a period of data of length n should be used only for forecasts up to $n/2$ in the future. The rationale for this rule lies in the fact that the sensitivity of the model to the effects of non-stationarity increases with the length of the forecasting period. Thus, the rule of thumb would suggest that, if the model is estimated from data for the period 1919–1982, simulations should be restricted to a period of about 30 years; however, if the period of data is restricted to 1945–1989, then simulations should be restricted to a period of about 20 years. These periods fall somewhat short of the 50 or more years ahead, for which Wilkie wanted the model to be usable for actuarial purposes.

5.3.2 On the other hand, considering the short term, the model is not, and, indeed, was not, intended to be appropriate for short-term forecasting or simulation. This is because of the issues discussed in Section 4. To quote directly from a Working Party communication from Wilkie: "I [did] not put forward the model to be of practical use in tactical investment management, but rather for strategic".

5.3.3 These constraints to the period of simulation used for the model can be summarised in the comments of one of the authors, S. J. Green: "I am not convinced that the Wilkie model is an accurate forecasting tool in the short term or the long term, although perhaps satisfactory for the medium term. I suspect that is because I subscribe to chaos theory (without understanding the mathematics); hence, no linear model will satisfy me."

5.3.4 It is worth noting that concern about the use of the model in the short term might be of particular importance in the area of contemporary life assurance profit testing, for example, because of the relative sensitivity of these techniques to the projections of early cash flows.

5.4 *Number of Simulations*

5.4.1 The number of simulations required will depend on the distributional statistic that is to be estimated, and on the required confidence in that estimate.

5.4.2 The median can usually be estimated from a single simulation, by using the equivalent deterministic model; i.e., effectively setting the standard deviations of the stochastic disturbances to zero. It is perhaps worth noting that this estimate is usually not the mean, because of the normal to log-normal transformation.

5.4.3 In order to estimate the means and standard deviations of distributions, larger numbers of simulations are required. Wilkie, in a Working Party communication, suggests that "one can get quite good estimates of means and variances with only 100 simulations".

5.4.4 Estimates of more extreme values would require many more simulations; however, because of the problems set out in Section 3, the validity of estimates of extreme values is open to question. Again, to quote from a Working Party communication from Wilkie: "It is . . . perhaps not appropriate to try to be too precise about the extremes anyway." It is worth noting, therefore, that the model may not be considered appropriate for the estimation of extreme values, such as probabilities of ruin, etc.

5.4.5 The reader is usefully referred to Section E1.3. and Table E1.3. of the Report of the Maturity Guarantees Working Party (1980) in which the use of simulation methods and consequent confidence limits are discussed.

5.5 *Sensitivity*

5.5.1 The sensitivity of the model to violations in the underlying assumptions has been discussed as and when appropriate. However, it is worth noting that to test the sensitivity of the model to non-quantifiable effects is not possible; and for other effects, such as non-stationarity, effective tests are often difficult to devise and/or interpret.

5.5.2 Considering non-stationarity, in particular, it has been suggested that it would be possible to allow for changes in the parameters of the model, either deterministically or stochastically. However, it must be stressed that such formulations would not be based on theoretical or statistical grounds. This question is considered by Wilkie in a number of his papers.

6. ACTUARIAL JUDGEMENT

6.1 The training and education of the actuary is, or should be, directed towards the correct application of professional judgement by the practitioner, relying on numerical analysis, perhaps, for guidance. If the use of actuarial judgement were to be replaced by the automatic application of numerical answers, then, indeed, the future of the profession would be seriously at risk.

6.2 Throughout the discussions of the Working Party, this theme remained central—the extent to which 'actuarial judgement' might comfortably over-ride purely theoretical and statistical considerations. Therefore, it is appropriate to consider this topic explicitly, since the model relies on a number of empirical judgements.

6.3 Considering the estimation of the model, the period of data chosen will have affected the estimated parameters of the model, and the structure itself; parameter restrictions were also imposed.

6.4 In applying the model, as it stands, there is the question of whether the parameters of the model might be adjusted to reflect current initial conditions, or even to incorporate the economic and financial view of the user.

6.5 Wilkie's model requires eleven initial values, which could be taken to be either neutral (typically, reflecting long-term estimates) or actually relating to current conditions. It is noted that the effects of variations in these initial conditions can take some time to wear off, which may be of concern for short-term applications.

6.6 In particular, with respect to the parameter representing the mean rate of inflation in the long term, Wilkie, in a Working Party communication, suggests that: "actuaries should treat [the mean inflation] parameter just as they do at present, using their own judgement and past experience to decide on an appropriate . . . value". It must be stressed that such an approach would not be based on theoretical or statistical grounds.

7. FUTURE DEVELOPMENTS

7.1 The increasing computer power available to the actuary makes the use of stochastic techniques relatively simple and their increasing use is highly likely. The increasing financial sophistication of clients and institutions which actuaries serve will increase pressure on the profession not to rely so much on a deterministic approach for future planning.

7.2 If this is the case, then clearly the awareness of stochastic approaches will need to be revised significantly, so that the actuary can choose the most appropriate model, if any, for his needs, and fully understand the implications of doing so. This increased awareness must start with the education of the student, by incorporation of the subject into the appropriate part of the syllabus. At this point, it is worth quoting from Harvey's conclusion:

"A stochastic investment model is undoubtedly useful in giving actuaries some idea of the kind of movements to be expected in key financial areas. The Wilkie model represents an important first step in this direction. However, it is not definitive, and it is important to recognise that the whole approach can be developed and improved in many ways. Wilkie built his model using Box-Jenkins methods; and approaches based on other kinds of econometric and time series modelling procedures may be better and simpler.

"If the actuarial profession is to take stochastic models seriously, and to use them in an intelligent and critical way, it is necessary for those in the profession to understand the way such models are built, and the nature of their strengths and weaknesses. This leads to the conclusion that econometrics and time series analysis should figure to a much larger extent in actuarial training. Thus, if stochastic investment models are to appear on the syllabus for the Institute of Actuaries examinations, they should do so within the wider context of a course on econometrics and time series. The Wilkie model, or a modified version of it, would just be one illustration of a much wider set of techniques which are of general interest to actuaries."

7.3 Considering these comments and its own deliberations, the Working Party decided that it should not be seen to make judgements suggesting that the model proposed by Wilkie, or any other single particular model, is that which is most appropriate for widespread actuarial use. Further in this respect, several members of the Working Party expressed concerns that practitioners (both actuaries and non-actuaries) without the appropriate knowledge and understanding of econometric and time series models might enthrone a single particular model, and scenarios generated by it, with a degree of spurious accuracy. Thus, while this report has drawn attention to the relative advantages and disadvantages of alternative model formulations, the Working Party agreed that considerably more research is required in this area. The proper application of this research cannot take place until a wider number of actuaries obtains the appropriate level of knowledge either through the examination system or by other suitable means.

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APPENDIX A

AN OUTLINE OF A NON-LINEAR INFLATION MODEL

By R. S. Clarkson

A.1 This appendix shows how a general stochastic model for inflation can be derived from the general characteristics of inflation. The model has three quite separate non-linear components, and the Wilkie model for inflation is the special case of this model with all three non-linear components suppressed. A fuller discussion of the model appears in Clarkson (1991).

A.2 On the basis that there are three main mechanisms which affect the year-on-year inflation rate:

(i) a tendency for the rate to return to some 'intrinsic' value,
(ii) a random error component which operates every year, and
(iii) a random shock component which operates infrequently,

I believe that an appropriate model is:

$$i_{t+1} = i_t - A_t(i_t - B) + D_t + E_t,$$

where i_t is the rate of inflation in year t (equivalent to Wilkie's $\nabla \ln Q(t)$),

$A_t > 0$,
B is the 'intrinsic' mean rate of inflation,
D_t is the random error term, and
E_t is the random shock term.

A.3 Some very general properties of these auxiliary functions A_t, D_t and E_t can be deduced from certain characteristics of inflation. Inflation is essentially a symptom of instability in an economic system. Because of various linkages with wages and prices, negative inflation rates are uncommon, but instability as a result of domestic circumstances (e.g. lax monetary policy) or external effects (e.g. the first and second 'oil shocks') can result in a marked rise in the inflation rate above the intrinsic rate. When the rate of inflation is low and has been relatively stable for a number of years, the year-on-year variability is significantly lower than when recent rates have been high.

A.4 A constant value of A_t would imply a constant rate of return to the intrinsic rate. To provide a mechanism for the rate of inflation rising well above the intrinsic rate as a result of domestic conditions, it is desirable to use a lower value of A_t when there is an upward trend in inflation. This can be achieved by replacing the term $-A_t(i_t - B)$ by $-A(i_t - B) + C \cdot \text{Trend}^+\{i_t\}$, where A and C are positive constants and $\text{Trend}^+\{i_t\}$ is the recent trend rate of change of inflation when this is positive and zero otherwise.

A.5 To allow for the higher variability of inflation when recent rates have been high, the random error term can be expressed as $D(t)\cdot Z(t)$ where $D(t)$ is positive and increases with the recent average rate of inflation and $Z(t)$ is a random unit normal variate.

A.6 Since the centralising and random error terms provide the mechanism for a large negative impulse when inflation is high, but cannot provide the mechanism for a large positive impulse when inflation is low, the random shock term can be regarded as 'upward only' and defined as $p(t)\cdot E(t)$ where $p(t)$ reflects the probability that the random shock operates at time t and $E(t)$ is the value of the shock if it occurs at time t.

A.7 This model for inflation, namely:

$$i_{t+1} = i_t - \mathrm{A}(i_t - B) + C\cdot\mathrm{Trend}^+\{i_t\} + D(t)\cdot Z(t) + p(t)\cdot E(t),$$

contains one linear and three non-linear components. Because of feedback from one non-linear component to another, the patterns of inflation that can be generated by this apparently simple formulation are many and varied. In particular, the model can operate for a period in what may be called its 'quiescent phase', with relatively low and stable rates of inflation. Then any one of the non-linear components can trigger an 'active phase' in which inflation rates are higher and more variable. The linear centralising component provides the mechanism, which in due course takes the model back to another quiescent phase.

A.8 It is interesting to note that the Wilkie model for inflation is the special case of this model with all three non-linear components suppressed (i.e. C and $E(t)$ both zero and $D(t)$ constant).

A.9 Although this model appears to be very much more complicated than the Wilkie model, it is relatively straightforward to derive suitable auxiliary functions and parameter values. I believe that a highly satisfactory representation of U.K. inflation over the past 30 years is obtained using the following auxiliary functions and parameter values:

$A = 0{\cdot}5$	(slightly higher than the corresponding Wilkie parameter of 0·4),
$B = 0{\cdot}04$	(i.e. an intrinsic rate of just over 4%),
$C = 1$	(the most 'natural' value in that this is linear extrapolation when no other mechanisms apply),
$\mathrm{Trend}^+\{i_t\}$	calculated as the geometrically weighted least squares straight line fit with parameter 0·5,
$D(t)$	50% of the geometrically weighted average force of inflation with parameter 0·5, subject to a minimum of 0·015,
$E(t) = 0{\cdot}1$	(subject to the probabilities defined by $p(t)$),
$p(t) = 0$	if any of $p(t-1)$, $p(t-2)$, $p(t-3)$ or $p(t-4)$ had value 1, otherwise
$= 0$	with probability 0·94, and
$= 1$	with probability 0·06.

APPENDIX B

INCORPORATION OF ARCH EFFECTS INTO THE WILKIE MODEL

By A. D. Wilkie

B.1 I have carried out some further investigations on my inflation model, with a view to testing whether an ARCH model might be appropriate. However, I have used the approach of Engle, rather than that of Clarkson.

B.2 First, I have refitted my model to the data series from 1919 to 1989, and get results: $QA=0{\cdot}5909$, $QMU=0{\cdot}0375$, $QSD=0{\cdot}0521$.

B.3 I have then considered the residuals in three forms: untransformed, E; the absolute values, $|E|$; and the squares E^2. The first order autocorrelation coefficients of these are as follows:

for	E	0·0250
	$\|E\|$	0·4408
	E^2	0·4714.

B.4 The first of these is not significant, which is as expected considering the method of fitting. The second two of these are significantly different from zero, which lends support to the hypothesis of heteroscedasticity.

B.5 I then compared E in each of its three forms with the force of inflation, QD, in three corresponding forms: QD, $|QD|$ and QD^2. I calculated the correlation coefficients between $E(t)$ and $QD(t-1)$, taking each of them in each of their three forms, making nine regressions in all. The results are as shown below.

y	x	Correlation coefficient	Regression parameters $y=a$	$+$ $b \cdot x$	
E	QD	0·0	0·0	0·0	as expected
E	$\|QD\|$	−0·07	0·0041	−0·0721	n.s.
E	QD^2	−0·06	0·0018	−0·3054	n.s.
$\|E\|$	QD	0·13	0·0325	0·0779	n.s.
$\|E\|$	$\|QD\|$	0·36	0·0207	0·2644	
$\|E\|$	QD^2	0·28	0·0294	1·0378	
E^2	QD	0·10	0·0023	0·0094	n.s.
E^2	$\|QD\|$	0·30	0·0006	0·0368	
E^2	QD^2	0·24	0·0019	0·1441	

B.6 The lines marked 'n.s.' show regressions that are not significantly different from zero. Four of the lines, however, do show significant results. The bottom line is in the form suggested by Engle for an ARCH model.

B.7 I have repeated the calculations for the period from 1945 to 1989. The results are generally rather more significant. The bottom line for this period is as shown below:

$$E^2 \quad QD^2 \qquad 0{\cdot}33 \qquad 0{\cdot}0011 \qquad 0{\cdot}0821$$

B.8 I have not carried out the proper system for fitting an ARCH model. However, it is not difficult to simulate using an ARCH model. What I have done is to put:

$$QSD(t)^2 = A + B \cdot QD(t-1)^2.$$

B.9 I have then run 100 simulations, using the parameters from my standard basis, except for QSD, where I use the following parameters:

	A	B
Standard basis	0·0025	0·0
First ARCH model	0·002	0·2
Second ARCH model	0·0015	0·4
Third ARCH model	0·001	0·6

B.10 In each case if QD has its median value of 0·05, then the corresponding value of QSD is also 0·05, the same as in my standard basis. In fact the first line, in which the value of B is zero, is precisely my standard basis.

B.11 In each simulation the average rate of inflation (% p.a.) was calculated, and the means and standard deviations of these rates are as shown in Table B1. It can be seen how the ARCH model makes no difference over one year, but puts up the standard deviations of the results over longer periods.

B.12 The results indicate that ARCH models are worth investigating further. However, the process of fitting an ARCH model to the data, in order to estimate parameters from the past, is not a trivial exercise.

Table B1. *Means and standard deviations from 100 simulations of average rate of inflation (% p.a.) over the given term on the basis shown*

	Basis							
	Standard		First ARCH		Second ARCH		Third ARCH	
Term	Mean	s.d.	Mean	s.d.	Mean	s.d.	Mean	s.d.
1	4·93	5·23	4·93	5·23	4·93	5·23	4·93	5·23
5	5·30	4·33	5·32	4·72	5·34	5·16	5·36	5·66
10	5·01	3·42	5·01	3·89	5·01	4·47	4·99	5·09
20	5·01	2·60	4·94	2·80	4·77	2·96	4·53	3·62
30	5·09	2·21	5·02	2·44	4·88	2·69	4·68	3·41
40	5·11	2·05	5·09	2·24	4·99	2·44	4·83	2·92
50	5·03	1·80	5·07	2·09	5·03	2·47	4·95	3·49

APPENDIX C

PRACTICAL APPLICATIONS OF SIMULATION PROCESSES: PENSION FUND ASPECTS

C.1 *General Principles of Simulation Processes*

C.1.1 Simulation is a method used to study the dynamic characteristics of a system. Often the purpose of a study is to gain an understanding of how the various factors that determine the system interact with one another and evolve with time. If a system can be modelled by analytical means, this is generally superior to a simulation approach. In practical situations, if the model is at all realistic the analytical solution is usually not possible.

C.1.2 Basic methods of simulation are easily understood and relatively simple to apply. However, it can be difficult to draw accurate conclusions from simulation studies and the practitioners must be sure that they have adequate experience in the practical details of carrying out simulation procedures.

C.1.3 In the case of a pension fund, the 'deterministic' approach, using expected values, i.e. means for the various factors, withdrawal rates, etc., is well established and is used in the classical actuarial valuation. In this approach, we can investigate how the system changes if we change the means, but it does not give any indication of the likely variability of the results, or the likelihood of any particular outcome. The pension fund system is too complex to solve analytically except when we make sweeping simplifications; it is for this reason that a simulation approach is the preferred method to study the variability of pension fund aspects, e.g. solvency ratios, contribution rates, etc.

C.1.4 A key factor in the simulation is the time horizon over which one carries out the investigation. Many of the factors that enter the simulation process are clearly not 'random' in the short term to the extent that they can be 'steered' to some degree by the government, and salary inflation can be influenced by the unions and the employers. It is in the longer term that the 'stochastic' nature of those factors asserts itself. It is well known that the Wilkie model described below is normally only suitable for long-term projections, but, providing the parameters for the model are chosen with care, it may be used for shorter periods as well.

C.1.5 Another problem area in the simulation of pension fund financing is the assumption regarding early leavers (withdrawals and early retirements) and new entrants. Variation in these factors can totally swamp the variation due to the random nature of salary inflation and rates of return, and any conclusions one draws from a simulation must be qualified.

C.2 *Description of the Basic Wilkie Model (Assets and Salary Inflation)*

C.2.1 There are four basic series (all time dependent):

$Q(t)$ a retail price index,
$D(t)$ an index of share dividends received,
$Y(t)$ the dividend yield on the shares, related to the share price index by $P(t)=D(t)/Y(t)$, where $P(t)$ is the price index (e.g. *Financial Times–*Actuaries All Share), and
$C(t)$ the yield on 2·5% Consols (irredeemable), a proxy for the general level of long-term fixed-interest yields.

C.2.2 The equations appear more meaningful if we introduce a series for the annual force of inflation, $i(t)$ say, defined as:

$$i(t)=\ln\,(Q(t)/Q(t-1))\ \text{(Wilkie's } \nabla\ln Q(t)\text{)}.$$

C.2.3 For use in pension fund work, it is essential to allow for salary inflation. In the studies published a very simple relationship was used, typically a constant gap. We propose to use for the formula for $j(t)$, the force of salary inflation:

$$j(t)=JMU+JA\cdot(i(t)-QMU)+JE(t).$$

C.2.4 A reasonable fit can be obtained using:

JMU $=0{\cdot}07$ (i.e. 2% gap between RPI and national average earnings (NAE)).
QMU $=0{\cdot}05$ (or as in the definition of $Q(t)$), and
JA $=0{\cdot}8$ (a value of 1 would imply $j(t)=0{\cdot}02+i(t)+JE(t)$, i.e. gap 2% or RPI plus the random component, a value of 0 would imply $j(t)=0{\cdot}07+JE(t)$, i.e. random fluctuation around the long-term mean).

C.2.5 We have fitted the parameters for the j series using linear regression, over the 45-year period 1946 to 1990, and also over three non-overlapping 15-year periods. The model parameters are not stable: for the three periods the JA were 0·59, 0·86 and 0·83 respectively, with the 45-year value being 0·84. A value of 0·8 seems a reasonable compromise.

C.2.6 As in the case of the estimation of the parameters for the Wilkie model, one has to exercise judgement as to what period of past history (if indeed any) is the best predictor for the future, and fit the model accordingly. We should mention that, in recent studies using the Wilkie model, some authors have used different parameters for the RPI series formula than originally proposed to get a better fit to the observed values of RPI in the light of recent experience.

C.2.7 We could add an additional term to represent an autoregression of j, say $JB\cdot(j(t-1)-JMU)$, but we have not investigated this any further.

C.3 *Development of the Pension Fund Formulae*

C.3.1 The current and future financial viability of the pension fund is largely a function of the emerging cashflows (both income and outgo).

C.3.2 In a particular investigation, the practitioner will have to augment the model by specific formulae that describe the process being simulated. For example, investigation into the variability of the pension expense under FAS 87 will have more difficult detailed calculations than if the discontinuance solvency ratio is being looked at. However, certain 'basic' processes will be common, and this appendix addresses this area.

C.4 *Accumulating Fund*

C.4.1 The basic recurrence relationship is:

$$A(t) = (1+G(t))\cdot A(t-1) + CM(t) + CC(t) + I(t) - B(t) - E(t),$$

with terms as defined below.

C.4.2 $(1+G(t))$ is a factor representing capital growth; it will be stochastic and calculated from the Wilkie model—e.g. for an all-share portfolio:

$$1 + G(t) = \frac{P(t)}{P(t-1)} = \frac{D(t)\cdot Y(t-1)}{Y(t)\cdot D(t-1)}.$$

C.4.3 This formulation assumes that the asset distribution remains constant throughout the projection period. If this is not the case, and some other investment strategy is pursued, one has to estimate the growth and income of each asset class separately, and also implement a strategy for the investment of new money or disinvestment. In some recent studies developments of the model allowed explicitly for a number of asset classes, including property and overseas equities, using the same basic formulae as for the two U.K. equities series, but with different parameters.

C.4.4 $CM(t)$, $CC(t)$ are the member and company contributions respectively. One would need to reflect assumptions about their level and demographic assumptions, for example, contribution holidays or changes in percentage contributions planned should be allowed for explicitly. There will be a stochastic element through the salary inflation $j(t)$ series. On the assumption of a stable population, the recurrence for 'stable' CC and CM can be written as:

$$CM(t) = CM(t-1)\cdot(1+j(t-1)), \text{ and}$$
$$CC(t) = CC(t-1)\cdot(1+j(t-1)).$$

C.4.5 For cases where the contribution rate(s) vary with time, $CMR(t)$ and $CCR(t)$ say, the best way to describe this is through the use of salary roll, $SAL(t)$, so that:

$$SAL(t) = SAL(t-1)\cdot(1+j(t-1)),$$
$$CM(t) = SAL(t)\cdot CMR(t), \text{ and}$$
$$CC(t) = SAL(t)\cdot CCR(t).$$

C.4.6 CMR and CCR may be input as data series or derived from other values calculated in the model, e.g. to allow for amortisation of surplus/deficiency.

C.4.7 $I(t)$ is the investment income of the portfolio; it will be stochastic and generated from the Wilkie model through $Y(t)$ and $C(t)$. A suitable formula needs to be decided on for redeemable securities, through the use of a yield curve as one possibility.

C.4.8 $B(t)$ is the benefit outgo; it should be stochastic and will depend on $i(t)$ and $j(t)$, but the precise form would depend on the definitions of benefits. A computer system capable of generating cashflow could provide this information, but practical issues (such as speed of the computer's processor) will probably prevent direct use to simulate the necessary thousands of final projections, and an appropriate model of the liabilities will need to be built for each case. It is possible that a suitable 'parameterised' model can be developed which would make the use of the stochastic model more cost-effective.

C.4.9 $E(t)$ is an expense outgo; in normal circumstances it will be sufficient to use some formula using percentage of salary roll, the accumulated fund, etc.

C.5 *Projecting Past Service Reserves*

C.5.1 If we adopt the definition of Past Service Reserve to be that applicable to SSAP 24 and FAS 87 (i.e. projected unit credit liability—*PBO*, say) we have the recurrence relationship (assuming all cash flows take place at the end of a year):

$$PBO(t) = PBO(t-1) + i \cdot PBO(t-1) + NC(t-1) - B(t) + \text{ASSUMPTIONS},$$

where i is the valuation rate of interest, NC is the 'normal cost' on the PUC basis and $B(t)$ is the benefit outgo as in C.4.8 (all items calculated on the valuation basis). There should be no problem with these items assuming stable populations, but the ASSUMPTIONS item, representing the stochastic element will be difficult to calculate accurately, due to the stochastic nature of $i(t)$ and $j(t)$ (the RPI and NAE series) and the fact that benefits will in general depend on NAE and RPI.

C.5.2 As for $B(t)$ in C.4.8, in theory a computer could calculate the $PBO(t)$s directly for each scenario, but will be defeated by the volume of calculations. Further investigations would be necessary to determine how to estimate (model) the $PBO(t)$ and/or the ASSUMPTIONS through a parameterised model of the pension fund liabilities.

C.6 *Calculation Procedures*

C.6.1 The full process can be summarised as detailed below.

C.6.2 Decide on the time horizon for the exercise, e.g. 40 years.

C.6.3 Choose the appropriate eleven starting values (the salary inflation series is fully specified through $Q(t)$:

one for the inflation series $Q(t)$,
one for the dividend yield $Y(t)$,
five for the dividend index $D(t)$, and
four for the Consols yield $C(t)$.

C.6.4 Generate the random vectors for $i(t)$ (derived from $Q(t)$), $Y(t)$, $D(t)$,

$C(t)$ and $j(t)$, $t = 1$ to 40; this would be essentially a 'black box' operation, with different parameters for special situations.

C.6.5 Generate the benefit outgo, expense, *CM* and *CC* vectors.

C.6.6 Project the *PBO*.

C.6.7 Project the assets; this is again a 'black box' procedure requiring only the cashflow vectors and the parameters describing the investment process and starting asset distribution, together with an investment 'strategy', e.g. keeping the asset distribution fixed.

C.6.8 Calculate any auxiliary results, e.g. ratio of $PBO(40)$ to $A(40)$, vector of ratios PBO/A, expected contributions assuming amortisation of surplus over n years, etc. Store these for future processing. We may also store results for any of the intermediate years 1 to 40.

C.6.9 Repeat steps C.6.4 to C.6.8 as many times as practical; minimum 100 times, but preferably 1,000 times.

C.6.10 Present the results stored in C.5.8; typically one would sort the results to obtain medians, deciles, etc., but other formats e.g. probability distribution curves, may be preferred.

APPENDIX D

APPLICATIONS OF THE WILKIE INVESTMENT MODEL IN LIFE ASSURANCE OFFICES

D.1 *Introduction*

D.1.1 This appendix first lists the obstacles that need to be overcome for the model to be used for life office work. This is followed by an analysis of the types of actuarial work in a life office and a discussion of how a stochastic model would assist with these types of work.

D.1.2 Reference has already been made in Section 5 to the shortcomings of the model in estimating extreme values, probabilities of ruin, short-term values, etc., and this should be borne in mind when considering the questions raised in this appendix.

D.2 *Obstacles to be Overcome*

D.2.1 There are very few actuaries who have both an interest and expertise in both of the very different areas of business management and advanced statistics. This is bound to hinder the implementation and use of the model.

D.2.2 There are computer costs to consider. Looking at the model in isolation, it is certainly true that for a large life office the software costs would be relatively trivial.

D.2.3 A model to value maturity guarantees under unit-linked policies is a relatively simple and self-contained system. The Maturity Guarantees Working Party (1980) estimated 19 man-weeks of development time. One of the members of the present Working Party recalls this as a gross over-estimate.

D.2.4 However, in considering systems costs, we must first remember that they would be *in addition to* many other development projects which are being externally imposed on life offices. Second, in order to be of the most practical use, the model would need to be integrated with all the office's existing actuarial systems and those to be developed in future.

D.2.5 In the experience of at least two members of the Working Party, an embryonic asset model in a life office rapidly metamorphoses into a liability model: in attempting to apply the asset model, the actuary needs to relate the assets to the nature and term of the liabilities. It is a time-consuming exercise to construct and maintain a complete liability model (including correct allowance for taxation) in a suitable form and environment for integration with the asset model. A standard software package is unlikely to achieve this integration with the office's existing systems.

D.2.6 In view of this, the life office will incur non-trivial costs both for the development of the software and for the use of the hardware. Even though computer costs have fallen dramatically, large numbers of simulations of a model of a large office will still incur heavy run time costs.

D.2.7 Offices may be reluctant to commit themselves to these costs at a time when they are attempting to minimise their expenses, unless there is a very visible benefit.

D.2.8 As a result, offices may decide not to carry out full-scale model projections, but to confine themselves to some form of sensitivity testing.

D.2.9 The Working Party considered that the model should be regarded as a research tool. If so, the results of the model will be difficult to interpret and apply to the life office data, and it may be difficult to draw reliable conclusions from them.

D.2.10 In devoting his attention to the detail of the model, there is a danger that the actuary may lose sight of the wood for the trees.

D.2.11 More importantly, he will need to convince *management* that the model is valid. The abstruse and highly theoretical nature of the model means that he is on a hiding to nothing in this respect.

D.2.12 It is clear that there are very few actuaries for whom these techniques are transparent, or even comprehensible: that is why the Institute looked to a specialist Working Party for actuarial certification of the techniques, together with guidelines as to how they can be applied by the 'layman'.

D.2.13 It may be difficult to get the message across that traditional methods developed by the profession for estimating long-term investment performance are far too crude—in fact virtually non-existent.

D.2.14 A new technique will have to be sold in simple language, with as few formulae as possible.

D.2.15 Further, it may not enhance the image of the profession to use what one might call "horrendously incomprehensible techniques to prove something blindingly obvious". ("An actuary is a man who uses highly precise methods to go from unwarranted assumptions to foregone conclusions.") There is a danger that this will appear to happen in some instances. The image of the actuary as a pure technician is one that is all too common and must be avoided at all costs.

D.2.16 It is clear that we need to demonstrate that there are real benefits from using the model in order to overcome these objections.

D.3 *Areas of Life Office Work*

D.3.1 All types of actuarial work in a life office incorporate as major assumptions estimates of investment returns and inflation in future years. So, in principle, an investment model can be applied to all types of life office actuarial work. (This appendix does not consider those types of work usually covered under pension funds, e.g. determination of funding rates.)

D.3.2 For most types of life assurance policy, investment performance is a most critical parameter in any actuarial calculation—so that it is important to try to incorporate as accurate an estimate of it as possible into any calculation. Traditional methods of estimating long-term investment performance are far too crude—what they boil down to is: there is an expanding funnel of doubt, so let us take an arbitrary amount off the current investment yield available. Any further insight we can get into the position, by using a stochastic model, must be worthwhile.

D.3.3 Let us consider how the model could be applied to various types of actuarial work in a life office, beginning with a very simplistic picture of life office actuarial work and then introducing more complications.

D.3.4 First, look at premium rating and product design. The simplest type of premium calculation uses life contingencies techniques to derive a premium rate for, say, a without-profit endowment assurance.

D.3.5 Estimated mortality rates are applied to the product terms to derive a projected series of cash flows under the contract. An interest rate is then chosen—suppose for the moment that it is intended to be a best estimate of the interest to be earned in future. This interest rate is used to discount the values of benefits and premiums, and to derive a premium rate which equates the two.

D.3.6 Supposing that the choice of investment medium is predetermined, the model can be used to project investment returns for each individual year over the term of the contract.

D.3.7 The parameters input into the model will, of course, have to allow for some of the practicalities of life office investment, such as dealing costs, investment management costs, and the consequences of active trading in investments by the managers. It will usually be impracticable to include directly within the model the effect of tax: this will interact in a too complicated manner with the other financial characteristics of the liabilities of the office. Tax will have to be dealt with elsewhere in the actuarial calculations (see D.4.4).

D.3.8 The simplest approach would be to use the model to derive a best estimate of the investment yield over the whole term of the contract, without any attempt at weighting, and then apply that yield using traditional actuarial methods. This is very cheap to do, and gives the actuary much more confidence in his calculations, provided he believes the underlying assumptions. However, at a more detailed level of analysis, the results of such a calculation are somewhat difficult to interpret.

D.3.9 A more complicated approach would be to incorporate the projected benefit and premium cash flows into the model simulations, and derive a statistically valid best estimate of the actuarial values of benefits and premiums.

D.3.10 This approach would give the actuary even more confidence in his results, but is usually going to be far too cumbersome to be practical: the full version, with thousands of simulations, would be vastly more expensive and time-consuming—and remember that we are later on going to generalise the task considered to include virtually all life office actuarial work.

D.3.11 There are obviously major practical difficulties in running thousands of simulations each time we want to do an actuarial calculation of any kind (not least of which are the difficulties of reproducing and checking the results).

D.3.12 It has been suggested that the cost of carrying out, say, 1,000 simulations of a full model office with, perhaps, 5,000 model points would be enormous, and would keep a mainframe in constant use for several weeks. We have not investigated this, but if it were so it would hardly be likely to appeal either to the DP manager or to the financial director who wants the results by yesterday.

D.3.13 So, the aim of the next variant must be to cut down the volume of calculation. This can be done either by reducing the number of simulations or by reducing the volume of data to which the simulations must be applied.

D.3.14 Simply reducing the number of simulations would lead to unacceptable random errors. A better approach would be to develop a technique which condenses the results of a large number of simulations into a relatively small number (say 10) of 'representative' scenarios, each of which would be assigned a weighting factor. After *once* performing a full set of simulations, the scenarios and weighting factors would be chosen so that, as nearly as possible, they are equivalent to the full set of simulations, in that they give the same expected value of any quantity that is reasonably likely to be required.

D.3.15 The number of scenarios chosen would depend on the application, the importance attached to 'accurate' investment assumptions, and the shape (skewness, curtosis) of the function being estimated. The actuary would get a feel from experience of the likely degree of error introduced by cutting down the number of scenarios. For some applications, a single scenario might be enough. Even though that may look like a deterministic approach, the model will have enabled us to improve our estimates of the parameters used.

D.3.16 We shall often be considering adverse economic scenarios and risks of ruin, so some improbable adverse scenarios will need to be included, with suitably low weighting factors.

D.3.17 There is clearly some scope for research in developing these techniques, if this has not already been done.

D.3.18 Thus, for each case to be tested, the calculations will reduce to anything between 1 and, say, 20 profit tests, in each of which the economic assumptions are derived from one of the standard scenarios and will vary from one projection period (e.g. year of policy duration) to the next.

D.3.19 As for reducing the volume of data to which the simulations are applied, it is already quite common to use a complicated method to derive values for sample policies or pivotal ages or terms or durations in force, and then use these to help derive values for other cases—perhaps by linear or geometric interpolation of some slowly varying quantity such as the ratio or difference between the value as calculated by the full sophisticated method and the value as calculated by a simple method. There may be a need to cover such techniques in the examination syllabus.

D.3.20 If full use were made of these techniques to reduce the volume of calculation, then the details of the model could be partly separated from the other parts of the actuarial calculation, and the extra work involved would thus be kept within reasonable bounds—but it would still be substantial.

D.3.21 The office would be using a profit-testing technique in any case, for some applications.

D.3.22 The model itself would not be incorporated directly into the profit-testing program, and although still highly desirable, it would not be essential for each life office actuary to be intimately familiar with the structure of the model. Consistency of assumptions throughout the office could be ensured by the use of office standard economic scenarios.

D.4 *Elements of the Actuarial Basis*

D.4.1 Having accepted that a profit-testing technique is to be used, the principle can be extended to bring more elements into the actuarial basis.

D.4.2 Commission and expenses: expense inflation would be an additional economic variable derived from the model and included in the standard scenarios.

D.4.3 Profit loading: a profit target could be included, and might well be related to some of the economic variables, e.g. a required rate of return of $x\%$ above price inflation or above the return on equities.

D.4.4 Taxation, in the context of the whole office: the model would supply an analysis of e.g. different types of investment income and gains which may possibly be taxed differently. In the more extreme economic scenarios, it would be reasonable to treat the rate of tax as an economic variable.

D.4.5 Withdrawals and other liability options would be incorporated, perhaps also as economic variables.

D.4.6 Similar methods would be used for unit-linked business, bringing in the various charging methods in the unit-linked contracts, and bringing in the unit-linked fund performance as a function of various economic parameters.

D.4.7 Other items will be brought into the profit test—in particular, the published valuation reserve (see D.6.1).

D.4.8 The main point to note is that the model can be envisaged as co-existing with established pricing techniques.

D.5 *Published Valuations*

D.5.1 A solvency valuation has to incorporate suitably pessimistic estimates of a number of items, including future investment performance. If the actuary has an idea of an acceptable level of the risk of ruin, he can use the model to derive investment scenarios which correspond to this risk of ruin. This is perhaps a more scientific way to establish theoretically appropriate valuation bases.

D.5.2 In principle, there is no reason why techniques similar to profit testing should not be used to perform the detailed valuation calculations—the only real differences being in the choice of conservative assumptions, and in the starting point for the calculations; the current duration of the contract rather than the contract start date.

D.5.3 The conservative assumptions will influence the choice of standard scenarios, as stated in D.3.16.

D.5.4 However, it is an unfortunate fact that the actuary may instead be more concerned to demonstrate that his published valuation basis is comparable with those of other offices, and the model does not readily lend itself to this, in the absence of the relevant information on other offices.

D.5.5 The office has to demonstrate that its valuation is at least as strong as the minimum statutory valuation basis—which is based on traditional methods and concepts such as the net premium valuation method—and this tends to encourage the use of traditional techniques as opposed to cashflow analyses. (For unit-linked business a cashflow analysis is encouraged to test the adequacy of expense reserves.)

D.5.6 The regulations also specify that policies should be valued individually, and this appears to rule out the use of sampling techniques as described above—although in practice offices do carry out tests on a sample basis in order to satisfy themselves of the adequacy of a valuation basis—and sampling is especially important here in view of the large number of policies to be valued, their diversity, and the requirement that the actuary should be satisfied of the office's solvency at all times, which may require frequent monitoring.

D.5.7 So it is likely that the model results will be translated into a form to which traditional techniques can be applied, i.e. a low fixed rate of interest together, perhaps, with a minimum asset margin—e.g. investment reserve as a proportion of total assets.

D.5.8 The position might change in future if the valuation regulations were modified to reflect more modern actuarial techniques—which seems unlikely in the European context.

D.6 *Valuation Reserves*

D.6.1 Returning to product pricing, the profit test will naturally need to incorporate the cost of financing the valuation reserves in each year of the policy term. (The required solvency margin will also be brought in.) In theory, a full cashflow method using simulation techniques would (unlike the deterministic case) require at least as much work as $t^2/2$ separate valuations, where t is the number of projection periods in the policy term, and there is no obvious short cut. Nevertheless some short cut must be developed.

D.6.2 All valuation reserves calculated must, of course, be tested against the minimum basis prescribed by the valuation regulations, together with any informal guidelines published by the authorities, and it is only reasonable to allow for the possibility of the regulations changing to adapt to extreme economic circumstances.

D.6.3 In assessing the cost of financing valuation reserves, it will, as usual, be important to distinguish between the case where the capital resource is plentiful and that where it is not.

D.7 *Corporate Financial Modelling*

D.7.1 Another task for life office actuaries is to project into the future the business and in particular the accounts of the office. The results of these projections help the office to project its capital requirements, the emergence of distributable surplus, its manpower, systems and accommodation requirements, and its tax position. The results may cause the office to attempt to alter its new business volume and/or mix, in order to manipulate those quantities, which are likely to be subject to some degree of external constraint. The results may also enable a notional market value to be placed on the office, which may be of some relevance e.g. in the circumstances of a potential takeover or merger.

D.7.2 In principle, the model may assist with these projections. There are three key differences:

(1) The results are more likely to be required in the form of a range of different scenarios, rather than a single best estimate or weighted average or conservative estimate, so simulation results are of less direct application;
(2) The ultimate recipients of the information are more likely to be external to the company and less likely to be actuaries; they are less likely to understand and accept the model, and they may well want to see illustrated economic scenarios of their own choice; and
(3) Finely tuned economic assumptions may be of lesser importance here, as they may well be swamped by assumptions as to new business levels.

D.7.3 For these reasons, the model is unlikely to be helpful in practice here.

D.8 *Investment Decisions*

D.8.1 So far, it has been assumed that investment policy is predetermined. Another area of life office actuarial work is to assist in the determination of investment policy. This function is, of course, not unique to life offices, and is covered in Appendix E. A special feature of life office investment policy is the need to relate assets and liabilities. The types of calculation described above can be applied to both assets and liabilities on various assumptions as to investment policy. Provided that we believe in the parameters input into the model, this will provide as fair a comparison as can be made between different investment policies, and should provide a valuable insight into the merits of alternative investment philosophies.

D.8.2 For this to be constructive, it is necessary that the investment manager also accepts the model. This may be difficult to achieve. It has also been pointed out that the model is intended as a long-term economic model and not a short-term one. Investment management is all about a continuing series of short-term decisions—even though these may result in some assets being held over a long period. The model would therefore be much more useful in this area if it were amended to be suitable also in the short term.

D.9 *With-Profits Business and Bonus Philosophy*

D.9.1 So far we have not considered participating business. This business requires the actuary to consider equity between the various categories of existing and new policyholders. A bonus policy and investment strategy must be devised which are consistent with policyholders' expectations, which do not result in a need to hold back the distribution of surplus unduly, or constrain investment policy unduly in order to demonstrate solvency, and which is flexible enough to ensure that equity will be broadly maintained in a range of possible future economic scenarios. The model will be of particular use here to decide on a likely range of future economic scenarios to be considered, and to optimise the investment strategy once the bonus philosophy has been decided.

D.9.2 However, it must be remembered that in practice such decisions will also be heavily influenced by statutory solvency requirements, the actions of competitors and crude published measures of relative strength, such as Form 9 ratios, and that these factors defy modelling. Some of the decision makers may not have the time, inclination, or technical expertise to absorb the results of comparatively simple actuarial investigations, let alone complex statistical models.

D.10 *Discretionary Surrender Values and Policy Alterations*

D.10.1 Similar considerations apply to these as to premium rating, as the objective of the calculation is much the same: to compare the position of the office with and without the block of business under consideration, bearing in mind equity to policyholders according to circumstances.

D.11 *When is a Stochastic Model Appropriate?*

D.11.1 We have utilised the concepts of a best estimate and a conservative estimate. Both these concepts depend on the idea of a statistical distribution. We have assumed that a stochastic model improves the accuracy of these estimates. It is worth looking at a few simple cases to see when this is true.

D.11.2 A best estimate usually means the expected value of a dependent variable such as the product profitability (P, say), which is a function of a series of independent variables x_1, x_2, etc. In principle, the distribution of these independent variables needs to be input into the actuarial calculation in some way, and the model is a convenient way of doing so.

D.11.3 However, if P depends linearly on x_1, then the expected value of P depends only on the mean, or expected value of x_1, so we need only consider that single case, and a deterministic approach will be adequate.

D.11.4 Even then, it is still important for management to understand the risks that they are running, and the stochastic approach can often provide a valuable insight into ruin probabilities, and distributions of likely outturns.

D.11.5 A conservative estimate is usually defined as some percentile of the distribution, i.e. in terms of a risk of ruin. This is difficult to calculate by deterministic methods.

D.11.6 Of course, another possibility is to use 'actuarial judgement' to adjust the results of a deterministic calculation. This may well produce as accurate an answer as can be obtained from a stochastic model based on arbitrary parameters, and is a lot less distracting. However, if one can obtain confidence in the parameters of the model, one will have much more confidence in the model's results than in any method involving guess-work.

D.11.7 Now let us look at one or two examples of profit tests on different types of life assurance contract.

D.11.8 A short-term non-profit assurance. Here, P will be approximately a linear function of the interest rate earned over the term of the contract. So a deterministic method would be quite satisfactory.

D.11.9 An immediate annuity which it is intended to match with precisely matching assets. Clearly a deterministic method is adequate. However, it might even in this case be instructive to use the model to demonstrate whether or not that is in fact the most efficient investment policy.

D.11.10 A long-term non-profit assurance. A deterministic model is clearly inadequate, because P is the net result of the effects of several different investment parameters taking effect over many years, and all the 'independent' variables will be inter-related. The current guess-work techniques are not satisfactory.

D.11.11 A policy with an investment guarantee—an extreme case would be a unit-linked policy with a maturity guarantee, but there are plenty of other possible examples—P expressed as a function of x_1, etc., has a kink in it, so again a deterministic method cannot be used, and it is even more difficult to attempt to apply judgement to the problem.

APPENDIX E

APPLICATIONS OF STOCHASTIC MODELS TO INVESTMENT MANAGEMENT

E.1 *Introduction*

E.1.1 The discussion below is not limited to applications of the Wilkie model exclusively, but of stochastic investment models in general—which we shall often refer to as 'time series forecasting models'. Some of the applications referred to are taken from a rather wider range of securities operations than pure investment management. It is worth noting that financial economists use a great variety of stochastic investment models, in particular those representing interest rates and term structure. See, for example, some of the papers listed in the bibliography. In addition, many organisations offering asset allocation services use some kind of stochastic model.

E.2 *Trading (of Stocks, Bonds, Currencies, Commodities or of Indices based on them, or of Index Futures)*

E.2.1 Trading decisions are taken on a very short-term basis which, depending on what is being attempted, may be measured in seconds, minutes, days or occasionally a few weeks. Clearly, a model designed for long-term actuarial projections will not be suitable for trading.

E.2.2 However, time-series forecasting models might have some trading applications if based on data which have been collected on a frequent (say daily) basis. The success of these forecasting tools depends on the inefficiency of the market to which they can be applied. This can be measured by the autocorrelations of successive lagged returns.

E.2.3 If significant autocorrelation is observed for lags 1 to n, this suggests that new information is absorbed into prices over a number, n, of periods (days). Trading rules can then be devised to exploit these inefficiencies.

E.2.4 These methods have achieved a measure of success for currencies and some commodities (Taylor, 1986), but for many stock and bond markets, although a one-period lag is usually seen, this is rarely sufficient for the creation of a successful trading rule—since dealing expenses outweigh the small amount that can be earned by exploiting the inefficiency.

E.2.5 It must be pointed out that many successful forecasting models are non-linear and often employ techniques which deal with conditional heteroscedasticity—changes in the variance of returns which depend on its recent history.

E.2.6 The major applications of stochastic models to investment (as opposed to actuarial) problems, will probably be for derivative instruments, which are now such an important addition to the range of investments on many of the major exchanges.

E.3 *Traded Options*

E.3.1 Short-term (i.e. with lives measured in days, weeks or months) put and call options can be bought or sold ('written') on many of the instruments discussed in the previous section. So long as the underlying price model is a random walk with constant variance and constant interest rates, the ubiquitous Black & Scholes option pricing formula cannot be improved upon for 'European' options (which can only be exercised on a single date). Cox, Ross & Rubinstein (1979) were the first to propose a numerical (binomial) procedure for valuing 'American' options (which can be exercised at any time) and which can incorporate individual or complex special provisions, and this idea has been widely developed.

E.3.2 Taylor (1986) describes modifications to Black & Scholes when the variance exhibits conditional heteroscedasticity.

E.3.3 Time series forecasting models do clearly have some application in the pricing of even short-term traded options.

E.4 *Warrants*

E.4.1 Longer-term options are often referred to as warrants. When companies issue warrants (which might originally have been attached to equities or bonds), on exercise new shares are created and the exercise cost is paid to the company. In contrast, traded options do not usually involve new shares or new funds for the company—the contract is settled between the options clearing house and the market players.

E.4.2 Warrants can be issued by intermediaries unconnected with the company. These are often referred to as 'covered' warrants—implying that the issuer has already bought cover (i.e. stock or other warrants) which will enable him to meet his obligations if the option is exercised. In practice 'cover' is, initially, partial, and the issuer relies on hedging techniques to maintain adequate cover if the market level moves.

E.4.3 Securities houses have issued put and call warrants on the various international stock market indices or on individual equities or gilts (index-linked or conventional). These are of the 'covered' variety with lives typically of between one and three years. A thriving OTC (Over The Counter) market—i.e. tailored for the client as to expiry date and exercise price—exists in currencies and bonds.

E.4.4 Investment Trusts have issued 'bonds' whose cash flows are linked to the *Financial Times*-Actuaries All-share Index, and warrants linked to the FTSE Index (or to the trust's share price, which, for a well-diversified fund, probably amounts to almost the same risk).

E.4.5 Thus a FTSE warrant, exercisable at a price E on a future date, will be worth $(P-E)$ (if the index has a price $P>E$ at that date), or zero otherwise. In order to value the warrant we require the probability distribution of P, and a discounting factor. Using the lognormal distribution for index movements with a constant variance and constant interest rate leads us, inevitably, to Black & Scholes' formula.

E.4.6 Using the Wilkie model, we would impute a distribution for the FTSE index (which differs from the lognormal) at the exercise date, so that the warrant value will differ from the Black & Scholes model price. The Wilkie model will, presumably, produce lower warrant values when the equity market looks dear, and higher warrant values when the market appears cheap (when compared with the long-term equilibrium parameter values).

E.5 *Convertible Bonds*

E.5.1 Convertible bonds entitle the holder to a (usually) fixed coupon, but with the option to convert the bond into equity (or another bond) between specified dates. A number of methods are used to evaluate convertibles:

(a) income differentials—where the present values of the income streams expected from the equity and bond are compared;
(b) regression methods— of the conversion premium against the purely fixed interest part of the convertible; and
(c) a bivariate extension of the Black & Scholes method—since both the equity and bond markets are subject to (correlated) stochastic processes.

E.5.2 The stochastic model can probably not add much light to the appraisal of an individual equity. However, it would be possible to postulate a convertible bond, the 'stock' into which it converts being the 'equity market' as measured by an index. Using the three methods described above, it would be possible to establish the market price (in prevailing conditions) of such a security.

E.5.3 In addition, the stochastic investment model could be run on a standardised set of such bonds, the distribution of the values of each convertible bond at final exercise date being defined by the higher of P, the index value, and E, the fixed interest value of each bond at that date, after expressing these values per unit nominal value of each bond, the present values of the expected values of these distributions would be used to price the convertibles. Since equities pay dividends, it is probably unsound to assume that the option to convert is exercised only at the final opportunity. It is not clear how the simulation program should be constructed in the general case, although at the extremes of the equity price the decision whether or not to convert is much simpler.

E.5.4 These could be compared with the traditional methods, and any differences might be used to adjust individual company issues after making appropriate adjustments for volatility and income differences.

E.6 *Interest Rate Caps, Floors, Swaps and Swap Options*

E.6.1 If a model could be found which gave a satisfactory representation of short-term rates, this could be combined with the gilt model in order to produce an early warning of any inconsistency between short and long-term yields. In general terms, if the short model produced, say, a higher realised 10-year yield (by rolling up cash deposits) than the long model (with an acceptable variance difference), we might expect long yields to rise. An interesting test would be to the period following the rise in base rates to 15% in autumn 1989—with long yields little more than 9%.

E.6.2 These methods have obvious applications to interest rate swaps—where two parties agree to exchange a fixed-interest liability for a floating one—and vice versa. The decision whether or not to embark upon the swap could, perhaps, benefit from an illustration of the distribution of the present values of the difference between the income streams.

E.6.3 'Swap options' give the holder the right (in exchange for a premium) to initiate a swap from his floating rate liability into a predetermined fixed rate. These operations have become notorious after Hammersmith & Fulham Borough Council wrote options with an underlying capital value in excess of £6bn, the validity of which would prove extremely painful to the ratepayers. Premia are usually calculated by a series of Black & Scholes calculations (one for each roll over date). It would be interesting to investigate whether an application of the methods discussed above would have resulted in different premia being charged.

E.6.4 Many floating rate loan agreements offer a 'cap' facility, whereby the negotiated rate formula (LIBOR plus x) cannot exceed a predetermined (capped) level. A fee is charged for the provision of this ceiling, and to reduce the fee a 'floor' might be requested. If the borrower is willing to agree that the rate he is charged cannot fall below a 'floor' level, whatever happens to market rates generally, the cost of the 'cap' can be reduced.

E.6.5 Caps and floors are normally valued by a Black & Scholes method—since they represent a series of options expiring at each rate-fixing date. More precise valuation methods could be derived using stochastic models.

E.7 *Split Capital Investment Trusts*

E.7.1 The market values of investment trusts are usually considerably smaller than their net asset values—particularly when interest rates are high. This exposes the trusts to predators who can purchase the assets at a discount. During the last few years, some trusts have arranged a restructuring of their equity into a number of different classes of capital—each of which will appeal to a particular type of investor. Such investors will pay a premium for a vehicle which meets their needs so precisely—thus eliminating the discount problem.

E.7.2 Typical asset classes are as follows—in decreasing ranking order of priority on a winding-up:

(a) *Zero Dividend Preference Shares*
These pay no income but, since they are sold at a deep discount to their maturity value, offer a gross yield comparable with that of gilts. Tax payers will pay CGT only after indexation—reducing the tax to a fraction of what would be payable on a conventional bond.

(b) *Stepped Preference Shares*
The initial net income (typically 5%) and par value both grow at a constant (usually 5%) compound rate. CGT will be zero or negative if inflation

exceeds the growth rate. In some trusts (b) takes priority over (a), but in both cases the promised benefits can only be paid if the trust's assets and income are sufficient.

(c) *Income Shares*
All the income after (b) has been satisfied is paid to the income shareholders. On winding up, the capital repayment (which ranks after those of (a) and (b)) is a fixed amount. These shares will appeal to funds which pay no income tax. The CGT situation is less important as a capital loss will often be made, but a gain in excess of inflation would be unusual.

(d) *Capital Shares*
The assets remaining after all the above obligations have been satisfied are paid to the capital shareholders on a winding up. These shares do not carry any entitlement to dividends, and will generally appeal to long-term speculative funds whose effective CGT rate is lower than their income tax rate.

(e) *Warrants*
These give holders the right to subscribe for capital shares at a fixed price. They are more speculative than the capital shares and carry no entitlement to income. A number of capital shares really behave more like warrants, since they have no 'intrinsic' value—the net asset value being negative.

E.7.3 The assets of these trusts can, for simulation purposes, be assumed to comprise gilts and U.K. equities in known proportions. If we assume that the fund's assets are not altered during the life of the trust, it is quite straightforward to run any number of simulations and produce a distribution for the values of each class of capital on the winding-up date—from which the distribution of rates of return can be deduced.

E.7.4 Classes (a)–(c) have a zero floor and fixed cap—there is no additional benefit to holders if the assets perform well. They have, therefore, the same profile as a written put option. Classes (d) and (e) receive all the (geared) benefits of superior performance. They suffer a much greater chance of 100% capital loss than the zero coupon shares, due to the priority ranking of this latter class. The income paying shares could suffer a complete loss of capital in extreme scenarios, but the total return on these shares might still be acceptable, due to the initial high and growing yield.

E.7.5 There is, presumably, an obligation on the fund managers to satisfy the reasonable expectations of the holders of all classes of capital. For example, too high a fixed-interest content in the portfolio would favour the income shareholders (initially, at least) to the possible detriment of capital and warrant holders.

E.7.6 As investment conditions change, the portfolio will have to be rebalanced, so as to retain the best chance of meeting the obligations/expectations of the various classes. This will present the greatest difficulty to devising the simulation program—as indeed it might to the fund managers!

These simulation exercises would be of interest to institutional investors in the various asset classes and to the fund mangers who will be able to evaluate the consequences of rebalancing (or not) the portfolio.

E.8 *Sinking Funds*

E.8.1 Sinking funds can be set up to repay a fixed sum on a fixed date, n years hence, using a fixed annual sum. Depending on the term, it might be possible to use matching theory to select a suitable portfolio of bonds, and thereby fix the interest rate after just a few annual servicing instalments have been received.

E.8.2 For very long periods, however, matching will not be possible. The best we can do is to use the Wilkie model to estimate the distribution of the value of the fund at maturity (using a unit amount to service the fund), and the annual service can then be adjusted to produce the required maturity value with a prescribed degree of confidence.

E.8.3 The current model does not admit any term structure of interest rates, so the following assumption might be used in practice: all coupons and servicing contributions arising at time k are invested at par into a notional gilt with a life of $(n-k)$ and a coupon equal to the interest rate generated by the model at that time.

E.8.4 After many years with an inverted yield curve, it is not clear whether this rule is likely to over- or under-estimate the shorter yields. Clearly, a short-term model as discussed in Section E.6, would be sufficient to enable a crude term structure to be incorporated.

ABSTRACT OF THE DISCUSSION

Mr P. R. Lockyer (opening the discussion): Professor Wilkie's paper 'A Stochastic Investment Model for Actuarial Use' (*T.F.A.* **39**, 341) was written in 1984, and has been one of the more influential papers of the last decade, providing the basis for further research by actuaries all around the world. There have been many advances since then, not least in the power of computers. With suitable software and a powerful PC, econometric models can be analysed in a fraction of the time that it would have taken in 1984, enabling the modeller to consider many more variations and alternatives. In addition, there have been the changes in time series analysis referred to in §3.2.11.

In a deterministic valuation, the elements of the actuarial basis will vary according to the purpose of the valuation. It is unlikely that the actuarial basis used to value a life insurance company for statutory purposes would be the same as that used to assess the contribution rate to a pension fund. The same issues apply to time series analysis; a model built to analyse the development of a fund and its liabilities over 20 years will be different to one built to forecast the behaviour of the economy over the next 2 years. Furthermore, a time series model is a combination of objective and subjective analysis. It requires decisions about the weight to ascribe to various factors, in the same way that decisions are made to establish the elements of a deterministic basis.

In analysing the Wilkie model the report concentrates on the inflation series. Many actuaries will have a view as to future inflation, but may not have a view as to its volatility. The BZW Equity-Gilt study shows that there is as much room for differences of opinion about its standard deviation as there is about its mean level, even before any subjective account is taken of the possible effect of the United Kingdom's entry into the European Monetary System. The standard deviation of the inflation rate has varied from 3·0% to 5·6% in the two 25-year periods ending 1965 and 1990 respectively. The arithmetic means of the inflation rate in these two periods are 3·2% and 9·1%. Furthermore, it is important not to extrapolate our short-term experience, but to have some regard to the longer term. The BZW study shows that the last year of negative inflation was 1943, but it also discloses that there have been 12 such years of negative inflation since 1922. Modellers have to be aware of mixing the recent experience of high and volatile inflation with an actuarial judgement that the mean rate of inflation will be lower in the future. The figures imply that a lower mean inflation rate may lead to lower variability in inflation. If a model combines a low mean inflation rate with high volatility, it will result in what is considered to be an excessive number of negative values, especially in relation to our lack of experience of negative inflation in recent years.

The report also considers the shape of the residuals. In Appendix B the analysis by Professor Wilkie of ARCH effects in the inflation model shows that the autocorrelations of the absolute values of the residuals and their squares are significant. One of the problems when constructing such models is the amount of weight that should be given to any particular value. Consider a model based on quarterly inflation rates for the period from quarter 3 of 1963 to the end of 1991, consisting of 114 values. The resulting model could be written as:

$$DIr_t = A \cdot DIr_{t-1} + B \cdot DIr_{t-2} + Q + C \cdot \mathrm{N}[0,1]$$

where DIr_t is the inflation rate in period t, $A = 0{\cdot}55$, $B = 0{\cdot}21$, Q is the seasonal element, $C = 0{\cdot}01$ and N[0,1] is the normal distribution with mean 0 and standard deviation 1. The autocorrelation of the residuals E from this model is $-0{\cdot}04$, of the absolute values of the residuals $|E|$ is 0·37, and of the squares of the residuals E^2 is 0·41, and they exhibit the same characteristics as Professor Wilkie exposed. However, if the residuals are re-examined there are two which stand out: the third quarters of 1975 and 1979. The effect of these two periods can be shown by re-estimating the model using dummy variables for these. Using the dummy variable for 1975, there is only a marginal change in the parameters, but the autocorrelation of the residuals, their absolute values and their squares are significantly reduced to 0·06, 0·15 and 0·17.

Furthermore, when considering any model, it is relevant to take account of the observation in §4.2.3, which covers the effect of any complexity on the resulting output for the time period under consideration. Incorporating ARCH or shock effects may improve the fit of the model or appear more realistic in the short term. However, in the medium and long term the advantages are less apparent.

The probabilities of negative values with a mean inflation rate of 5% are 22% for the Wilkie model and 16% for the quarterly model just referred to. However, if the mean inflation rate is increased to 9%, these probabilities are reduced to 8% for the Wilkie model and 4% for the quarterly model. Since the mean inflation rate for the period 1963–90 was 9%, a probability of 4% of negative inflation is perhaps closer to the subjective assessment that one would make of the probability of negative inflation, based on the experience of the last 30 years.

It would have been interesting if the Working Party had extended its work to other aspects of the model; for example, the basic structure of equity yields driving gilt yields or the quality of the data series for the equity model. It would also have been useful if they had put forward proposals on how such models should be described in reports: the least that should be included being the means, standard deviations, correlations and auto-correlations of the variables.

In § D.3.4 it is suggested that the number of scenarios is reduced to 10, each of which is assigned a weighting factor. The implication is that these scenarios are established by analysing the results of the simulations from a stochastic model. The danger with this approach is the temptation to manipulate the scenarios and their weightings. This is in addition to the difficulty of consolidating what may be subtly different simulations into one composite scenario.

In Appendix E the authors explained how stochastic models can be used for derivative trading. However, although they discussed the merits of time series forecasting models in short-term trading, it is not clear whether this included tactical asset allocation. Professor Wilkie has stated that he did not intend his model to be of use in tactical investment management, but, by definition, the model, by postulating a relationship between gilt and equity yields, will comment on the relative value of these securities in prevailing market conditions. A number of investment managers have used this type of analysis to construct tactical asset allocation tools; for this purpose such models would include more exogenous variables to take account of the factors that influence security prices over the short term.

In § 7.1 the authors note that the increased computer power available to actuaries means that stochastic techniques are more viable. The elements of any actuarial basis combine the experience of the past with a view of the future. This also applies to the parameters used in a time-series model. Deterministic valuations provide a static view of the future development of a fund, whereas stochastic valuations show the spread of outcomes; from which the actuary and his client can test a variety of courses of action. Stochastic valuation techniques should be viewed as a useful extension of the actuaries' role. They permit actuaries to help their clients in the management of their funds.

Mr A. D. Smith: In § 4.4.2 the Working Party criticises the model because it permits deflation, implying that this should not be possible in practice. When I examined recent RPI figures I found that the index fell between October and November 1990, and again in the following month. Although we have not recently experienced deflation over a whole year, I do not think it can be ruled out in the future. I think that the fact that the Wilkie model allows such a scenario should be regarded as a strength rather than a weakness.

However, in this deflationary scenario there are two features which can arise from the model and which have not been discussed in the report. In his original paper Professor Wilkie postulated what might happen when yields fell below 2½%. He assumed that the debt would be refinanced by another issue of perpetuities. This ignores the problem that, in a low interest rate scenario, 2½% Consols would be likely to trade at a discount relative to truly irredeemable fixed-interest stock, on account of the option against the bearer. The model for C_t might then not be appropriate.

If C_t were taken as the yield on truly irredeemable stocks, we avoid the problem of this option, but another one appears. Although the real yield CN_t is always positive, it is possible in times of deflation for the model to produce negative nominal yields on perpetuities. This outcome is rather hard to interpret, but it would seem to imply a negative price. In some applications of the model a quirk like this may not matter, because of the tiny probability that it will happen. However, if the model is used for portfolio insurance or dynamic hedging, the flaw could be more serious.

The model explains current inflation in terms of past inflation only. Current dividend yields are explained in terms of past dividend yields, and also past and present inflation. Dividends are explained by past dividend levels and a lagged change in dividend yields. The current yield on perpetuities is influenced by past and present inflation, past perpetuity yields and present dividend yields. In fact, the directions of dependencies between current values of the variables are a function of the way the model is set up. We could restate the model in such a way that the current force of inflation would appear to be affected by current investment performance. This point is similar to, but different from, the point made by the Working Party in § 3.2.7.

Any such restatement would not restructure the dependence of the current variables on past values of other variables. The model does not allow current inflation to be affected by past investment performance, for example, and this is a genuine restriction. By assuming that certain possible dependencies are absent, Professor Wilkie has reduced the number of fitted parameters, making estimation easier and more reliable. However, it can be argued on economic grounds that some of the dependencies ruled out by Professor Wilkie should have been left in, particularly the dependence of inflation on the past history of other economic variables within the model. It might be better to allow more interdependence of the variables, especially if the lag structure could then be simplified. In § 3.4.3 we are told that "The Working Party agreed that there was little evidence to suggest that a better fitting parsimonious model could be estimated using standard Box–Jenkins methodology." I wonder whether they have tested a simplified lag structure with more complex correlations? A revised model of this sort would probably be more consistent with the 'efficient market' hypothesis.

When describing applications, the Working Party appear to believe that the only way to make full use of a stochastic model is via simulations. In my opinion this represents a rather narrow view. It would, for example, be possible to perform recursive convolutions estimating the joint probability distribution of the variables at each point in time. There are numerous intermediate possibilities which trade mathematical complexity for running time. Even if we are determined to use simulations, there are variance reduction techniques available which reduce the number of simulations needed to produce a given level of accuracy. With such techniques, the possibilities are much more extensive than envisaged by this report.

Mr T. A. Roff: I consider here the application of stochastic investment models in life assurance offices and particularly in with-profits offices, which have a significant proportion of assets invested in equities and properties. This is often not an ideal match for the guarantees provided or the bonus smoothing policy being adopted, as the assets behave in a more volatile manner than the liabilities. Additionally, the regulators' solvency test is particularly severe where a significant proportion of equities are held. This is because, for statutory valuation purposes, the yield on equities is taken as the current running yield with no allowance for future dividend growth. The minimum liabilities which an office must hold to satisfy the regulators are significantly lower if an office is invested heavily in fixed-interest securities. It is, therefore, important to investigate the future financial position of a with-profits fund, allowing for the volatility in equity performance and for the impact of the statutory valuation regulations. Using a stochastic investment model to generate a large range of alternative investment scenarios can help provide insight into this issue.

The stochastic investment model can be combined with a liability model which interacts with the asset data. The interactive model can then automatically adjust bonus levels and investment mix, based on rules specified by the user. One example of the use of an interactive model would be to project the ratio of assets to minimum liabilities under a large range of alternative investment scenarios. If this ratio is greater than 1, then the regulators are satisfied—if not, the office, on the regulators' basis, is insolvent. Adding up the number of scenarios where the office fails to achieve the desired ratio gives a measure of the probability of failing the regulators' solvency test. The results could be presented in an alternative form by quantifying the amount of capital which needs to be set aside now, to ensure that the ratio is maintained at the desired level in all but a specified proportion of cases. More important than the absolute probabilities of failing the regulators' solvency test are the relative probabilities—these can be determined by comparing the results on alternative bonus or investment strategies. In this way the impact on projected solvency of higher reversionary bonus or a higher proportion in equities can be quantified.

Building a stochastic investment model appears to be complicated and time consuming. Building a liability model is hard enough—without having to run it 1000 times and wait 2 weeks for the output. The interactions required between assets and liabilities at first sight look too numerous and too complicated to program. To cap it all, the methods and results are apparently so complicated that it will be impossible to explain them to senior managers. This whole process sounds a bit daunting—but it need not be:

(1) There is no need to build your own stochastic investment model. The Wilkie model is well documented and can be programmed, along with a random number generator, comfortably within a week. A mean variance model is equally straightforward. This is a model where means and variances are specified for a few asset types, correlations between asset types are also specified and returns are assumed to be log-normally distributed. A VAR model would also present no problems if the underlying formulae and error term distribution were known. There is no consensus on which of these is the best stochastic investment model—and there probably never will be—but that does not invalidate the exercise, as the relative results, if not the absolute results, are still meaningful.
(2) Stochastic investigations are of a strategic nature—to help determine strategic investment policy or strategic bonus policy— and a high level of accuracy is not required. The model need only use a relatively small number of model points, and certainly nowhere near the 5,000 suggested in Appendix D. For projecting key ratios 100 scenarios will generally be perfectly adequate, as long as reliance is not being placed on the most extreme scenarios. For determining the level of capital which needs to be set aside to satisfy the regulators, in all but a specified proportion of cases the results are more sensitive and the number of scenarios may have to be increased. The underlying principle should be that the model captures the key features of the underlying data, but can run 100 scenarios in a relatively short time—overnight at the longest.
(3) There are a large number of interactions between assets and liabilities which could be allowed for. These include the impact of bonus levels on lapses, the impact of inflation on new business growth and the impact of inflation on expenses. However, I would suggest that, at least to start with, capturing three key interactions is sufficient. The three which I would choose are bonus rates depending on the investment scenario, minimum liabilities depending on the yield on assets, and perhaps investment mix depending on the solvency level. Allowing for these interactions gives the facility to investigate the impact of difficult levels of guarantees and different investment strategies. It also gives the facility to assume that management takes some remedial action when faced with adverse investment conditions or an adverse solvency position.
(4) Our experience has been that these techniques can be successfully communicated to senior managers, and that the results can be presented very effectively in graphical form. The senior managers are not interested in the detailed techniques or in the detail of the stochastic investment model; what they are interested in is comparing the results of adopting alternative bonus and investment strategies.

Mr R. C. Urwin: I am a regular user of the Wilkie model in pension fund asset/liability modelling. The process has been used by a number of pension funds in the U.K., in each case helping to address the crucial question of strategic asset allocation—that is, how a pension fund can balance its objectives of limiting the expected future cost of a pension fund and the possible variation in that cost. The way that this particular investment problem is set up requires a stochastic model. The Wilkie model has helped in addressing that problem and providing some answers. Obviously the key question is: how good are the answers?

The question is not that easy, because most of the alternative models are not published for reasons of proprietary interest. This attitude is questionable in so far as public inspection can inspire confidence in the model. As part of its use among pension funds in the U.K., the Wilkie model has been openly inspected by a large number of investment houses, and so far I have heard no significant criticism of it. I have also studied United States practices on asset/liability modelling, where there is a rather longer track record, and the Wilkie model appears to acquit itself pretty well. This is stopping short of a full, technical comparison, but it is clearly quite powerful.

Further work needs to be done on the distribution of tails. The fat tails characteristic of most investment time series needs better definition if we are to cope with the key risk question, which is: what is the downside expectation of any particular investment policy?

The proper presentation of the relationship between assumptions and results is also important. Our long-range assumptions are inevitably fuzzy, and we need to present the results more fuzzily than perhaps we have got used to, otherwise we will have model results dictating actions to the exclusion of the sound exercise of judgement. The Working Party emphasised that there is a danger for the profession in relying on purely numerical processes. I am more concerned about the supervisors of investment funds, like trustees, who submit to asset/liability modelling processes, because they may not fully recognise the inevitable limitations. Stochastic modelling does not replace the need for judgement. It merely provides a more effective framework for such judgements to be exercised, and I find the maxim that 'models are to be used, but not believed' particularly relevant and helpful.

Professor G. T. Pepper: When I decided to join this profession I thought that I would be joining one that would continue to play a dominant role in the application of statistical techniques to financial matters. However, in § 7.2 we have Professor Harvey's report:

"The Wilkie model represents an important *first* step ['first' is my stress] . . . the whole approach can be developed and improved in many ways . . . approaches based on other kinds of econometric and time series modelling procedures may be better and simpler . . . econometrics and time series analysis should figure to a much larger extent in actuarial training."

I fear that Professor Harvey was being polite. The ARIMA technique is about 20 years old, VAR is about 15 years old, ARCH techniques are about 10 years old and co-integration is about 6 years old. More recently the statistical and econometric models have been brought together by, for example, econometricians testing for co-integration. The initial results are apparently adding to the stability of the models. I have in mind work from Brian Henry at the Bank of England, or Hall, who is now at the London Business School.

In a paper by Professor Terence Mills, entitled, 'Equity Prices, Dividends and Cointegration, Error Correction and "Confidence"' (*Scottish Journal of Political Economy*, August 1991), equity prices, dividends and gilt-edged yields were shown to be cointegrated with a single cointegrating vector. The intriguing thing about the results was that the error correction was the logarithm of the old 'confidence index'—that is the index invented by Ellinger which has been used extensively by Plymen. The very latest techniques and a very old technique have come together.

I hope that this Working Party will sit continuously with a mandate to keep in touch with all the modern time series developments, and to communicate them to the profession.

Mr G. J. Clark: The principal gain from using econometric modelling techniques in actuarial work is that they allow us to capture the dynamics of the real world. In setting a basis for a conventional valuation of a pension fund, I am happy to make the assumption that over the long term it would be possible to achieve returns on the fund's investments of between 1% and 3% in excess of earnings increases. I am equally confident that, over the long term, the nominal returns from equities will vary from year to year by more than the nominal returns on gilt-edged securities. I would like my advice to clients to reflect this, and the best way in which to do this is to make use of the type of stochastic models under discussion.

If we attempt to model one series in isolation, for example price inflation or equity returns, we can never hope to produce accurate forecasts. Anyone who is able to predict accurately the returns on the equity markets, even over the long-term time periods under consideration, would presumably seek retirement at 30 and keep his secret very close to his chest. However, we can hope to capture something of the relationships between the economic variables under consideration. Professor Harvey clearly recognises this in his report as quoted in § 5.2.1. The fact that the Working Party chose to concentrate on the inflation series and to include Appendix A, dealing with an alternative inflation series in isolation, and the assertion made in § D.3.8 that one could simply take in isolation the investment yield predicted by such a model, perhaps does not make it clear that this has been recognised by the Working Party.

I disagree with the comment in § C.1.5 that the assumptions regarding pension fund new entrants and early leavers can swamp the economic uncertainties. In my experience, this would only arise if the uncertainty in the future workforce was very great. Corporate business plans will have been formulated on certain assumptions regarding manning levels. These should be available to the scheme actuary. In §§C.4.8 and C.5.2 the authors suggest that the volume of calculation required is likely to defeat a computer. Whilst disagreeing with this, I am more concerned that the authors then proceed to suggest using a parameterised model for the liabilities. This appears to contradict many of the comments in the paper by suggesting that a crude approximation might suffice. I am also concerned by the assertion in §D.2.12, that we should be seeking ways in which the techniques can be applied by the layman. I believe that is equivalent to the statement that we should produce a set of guidelines to enable the layman to decide on an appropriate deterministic actuarial basis. More generally, I believe that the techniques provide a very powerful management tool for use within insurance companies, and I would suggest that, rather than parameterising the economic model, as seems to be suggested in §D.3.14, one should concentrate on parameterising the classes of business which the office conducts.

Mr R. J. Squires: I do not know if the model could be improved, but I am sure that it is a great improvement on what we had before. The main reason is that it reflects the stability of the investment income from a portfolio of equities in contrast to their market value. We can better spend our time learning how to apply it than trying to improve it.

The main problem in applying the model to life assurance is the complexity of the cash flow models and the resulting volume of calculation required. The authors have made some pertinent comments on this subject in Appendix D. In particular, I agree with their comments on the need to find ways to simplify the calculations and the presentation of results, and I think their suggestion of using a limited number of distinct scenarios is very useful. I now suggest a further technique for simplifying the process.

In the paper 'A Unitised Fund Approach to With-Profit Business', by Squires & O'Neill (*J.I.A.* **117,** 279), we demonstrated a technique for using a unit price as the bridge between the liability and asset parts of our model. This is an obvious device with unit-linked business, and, although matching is not usually a problem for linked business, it provides a means of assessing the effect of guarantees and choosing appropriate investment strategies. It is also useful for modelling with-profits business, in that some payments can be considered to be related to asset shares, and these can be calculated using unit-linked techniques.

In our paper we considered a very simple liability model resulting from the concept of paid-up matching. Dealing with a mixed portfolio of business with continuing premiums is much more complex. It is relatively simple with uniform investment growth assumptions, because model point projections for the various plans can be overlapped. The problem comes with variable investment returns, where each generation needs to be projected separately. That there are many overlapping generations is important in relation to many of the topics we wish to address. Dividends on the whole fund will provide a substantial proportion of the maturity proceeds for one generation, and the stability of this income is important. The solution to this problem is to find a way of calculating the effect of investment performance on the items of income and expenditure in the projected Revenue Account and the projected Valuation Balance Sheet, without having to go back to the underlying model points. Each of these items can be related to factors dependent on the investment performance and parameters determined by the portfolio of business. By making a projection on a uniform growth assumption, the values of these parameters can be derived. Revised projections can then be made for other patterns of investment performance.

The accuracy of this simplified model can be tested, by making a second projection from the underlying model points using a different uniform growth assumption. In some cases it may be necessary to make two projections initially, where the nature of the contracts means that two parameters have to be evaluated for a single item of revenue, for instance. Then a third projection would be required to give an independent check.

It may not be possible to model all the revenue items in this way with great accuracy, but that is not important, provided the investment results which are being studied are not invalidated. The development of the simplified model will involve a significant amount of work the first time round, but it should make it possible to run the full stochastic model, and provide a basis for presenting the chosen scenarios.

Mr C. D. Daykin: The paper gives illustrations of the use of stochastic modelling for life insurance companies and for pension funds, but not for general insurance companies. This was covered by the paper 'Managing Uncertainty in a General Insurance Company' by Daykin & Hey (*J.I.A.* **117**, 173). We found the Wilkie model very useful in our work. We wanted an investment model, but we did not have the resources to develop our own. We thought that the Wilkie model, although perhaps not perfect in every respect, was extremely helpful in giving some idea of the stochastic nature of inflation and investments. In our paper, we noted that the relatively high incidence of negative inflation that seemed to be produced by the model with its standard parameters seemed a little unrealistic in relation to modelling a future period, but we would not want to rule out the possibility of periods of negative inflation. We suggested a modification to the model for our purposes, using a slightly higher mean rate of inflation and a lower standard deviation.

This relatively simple modification was our attempt to use actuarial judgement in applying the model. It seemed to us that it was not appropriate simply to fit a model like this to a long historic time series and then to say that it automatically applied for projecting into the future. Inevitably there are problems in fitting past data, since only one realisation of the past is available. To obtain a good number of items of data it is attractive to use a long fitting period, as Professor Wilkie did. Much has changed over a period of 70 years, and much of the variation exhibited historically could reflect fundamental changes to the economy which will not happen again, at least not in that precise way. However, it could be argued that fitting the model to the past may help to generate some of the shifts which will, of course, occur in the future, although different from the ones we have seen in the past.

We were concerned that our simple modification could have unduly reduced the variability of the model. Periods of relatively high inflation have been a feature of the post-war years in a number of economies. Our co-researcher, Pentikäinen, suggested that the model should allow the possibility of sudden shocks to the system, moving the rate of inflation to a relatively high level. In the 1989 report of the Finnish Working Party such a facility was included, although it did not suggest parameters for this as part of their standard basis. Pentikäinen and his colleagues also considered that the residuals of the inflation model were not normally distributed, and adopted a postively skew distribution for the white noise term.

Modelling the equity dividends and yields presents even more numerous problems, as does the interaction between the various series. For practical use, there needs to be an extension of the model to cover fixed interest, dated bonds, property and other types of investment.

I agree with the authors that there is a need for actuaries to understand time-series models better, and I note their suggestion that they should be included in the syllabus for training actuaries. In my capacity as Deputy Chairman of the Education Joint Committee, I can say that they will be in the new syllabus, but, because of competing pressures on the syllabus, inevitably the level of coverage will be relatively low. I hope that Continuing Professional Development will help to address this problem further.

The Working Party are right to stress that actuaries should be aware of the limitations of these models. This is true of any models we use, including the traditional life table or a deterministic approach to compound interest. It is important for actuaries to have a better feel for the uncertainty about the future and to use stochastic methods where appropriate. After all, it is at the heart of the role of the actuary to find ways of managing that uncertainty.

I disagree with the comment in § 5.4.4 that the methodology should not be used to estimate extreme values. One of its virtues is that, in giving a distribution, there is a full range of results produced, although, to get suitable estimates towards the ends of the range, more than 100 simulations are needed, and there will be a need to address more carefully the distribution of the residuals. Nevertheless, the model was useful in our paper, at least for making comparisons between the relative impacts of different factors, as opposed to producing absolute values, for example of the risk of ruin.

A model can give helpful insights, but the results are only as good as the model, and should be interpreted critically and with a good measure of actuarial judgement. We should beware uncritical acceptance of the output from the model, but do not let us discourage actuaries from paying more attention to cash flow modelling, and from using models of this type and their successors to provide an indication of the impact of uncertainty on inflation and asset returns.

Mr H. D. White: As a practitioner in the development and use of these models since the mid-1980s, I identified a similar problem to that mentioned in § 3.4.5(1). The periods with higher variance can be better considered as periods of instability than inflationary bursts. The approach I used was to calculate two estimates for inflation: a short-term estimate based on the previous 2 years' values; and a longer-term estimate based on a weighted average of my long-term inflation assumption, and the value at time (−3). When these two estimates diverge, the variance of the observed values becomes larger. Using this device, I was able to obtain more satisfactory extreme values without using an unreasonably high variance. However, I still had not explained a sufficient part of the variance to produce entirely satisfactory periods of stability. I will investigate the alternatives of random shocks and alternative distributions, as suggested in § 3.4.5(2) and (3) with interest.

There is a danger of treating the model as a black box and not really understanding what is going on inside. The stochastic model generates some cash flows in the first place and then some market values. This allows the practitioner to identify the real risks and to take action to reduce them. Given a definition of the liabilities, the risk is normally either falling investment returns or inflation, or perhaps a combination of these. Any investment produces an income stream, and there is normally a mismatch with the liabilities by amount and currency. I consider inflation as an alternative currency to purely monetary values. Potentially, any mismatch leads to problems of reinvestment and currency mismatching. Low future rates of return are a significant element of the risk involved, and any model that allows inadequately for negative inflation may be too optimistic a tool for an investment problem, where the risk is reinvestment to match monetary commitments. Equally, it is dangerous to use a model that underestimates inflation if the liabilities are sensitive to this.

My approach is first to simulate on realistic assumptions, next to identify the nature of the risk, then to develop a suitable investment policy, and finally to check more simulations to see that the investment policy has not changed the nature of the risk. For example, a bad investment policy might give a solvency risk due to market value fluctuations. This can hide a more fundamental risk of reinvestment at low rates of return. I derive my final results on assumptions which include a proper margin for conservatism. To do this, it is necessary to have a stochastic investment model which contains long-term parameters that can be set by the user without invalidating the model, as suggested in § 6.6.

It is dangerous for the actuary who does not understand the stochastic model to take an arbitrary margin. In some cases this has nullified both the veracity of the model and the conclusions that have been derived. In other cases the margin may be inappropriate, not really reflecting the nature of the risks involved. For example, we have developed a variant of the method in § D.3.14 using a number of standard investment futures. On certain life policies we identified that the real risk is high inflation and expenses, the opposite of the pricing assumptions which have been made in the past. An actuary who is inexperienced with the model could easily reduce the inflation rate without realising that he is reducing the risk involved. The risk can be identified by a study on a realistic basis before the margin is taken.

Considering § 6.4, I have some concern about the effect of initial parameters. On the downside they may bias the initial performance between different categories of investment. On the positive side, experience has demonstrated that modelled results are mildly predictive. My concern is that the same problem may have a different solution, depending on the time we start from. Nevertheless, I would still use initial assumptions in the model. The financial view adopted should reflect the purpose of the investigation rather than the view of the practitioner.

Mr P. D. Needleman: I wish to focus on the uses of stochastic investment models in life assurance, and in particular, for with-profits business. Although the Wilkie model is the most commonly known, there are many alternative types of model which can be used, some of which are mentioned in the paper. Whilst it is important to appreciate the advantages and disadvantages of any particular type of model and to understand its limitations, we are in danger of overlooking a very important tool if we fail to make use of such models because of concerns over the possible shortcomings in the stochastic model used to derive the different investment scenarios.

Appendix D lists a large number of obstacles to be overcome which are, I believe, somewhat overstated. It then states in §D.2.13 that "traditional methods . . . for estimating long-term investment performance are far too crude", and later suggests that a stochastic model improves the accuracy of these estimates. There is no great advantage in using a stochastic investment model purely in this respect—this is precisely where actuarial judgement will continue to play a major role. Stochastic investment models can play a vital role when we consider the impact of the volatility of investment returns on our business. We all have our own particular views as to the expected level of future investment returns, but are much less certain when we consider the volatility of returns.

Why is volatility suddenly a problem when traditional actuarial techniques have served us well for so long?

(1) With-profits offices have a high proportion of their assets in equity type investments. These have volatile returns, and there is a mismatch between these assets and the nature of the liabilities.
(2) Competitive pressures are resulting in offices providing significant guarantees, both explicit and implicit.
(3) Because of a rapid expansion of business over the 1980s and significantly lower investment returns in the early 1990s, capital is now a much more scarce resource.

I now comment on three specific uses of stochastic modelling: premium rating and product design; valuation reserves; and financial modelling. In all cases the interaction between the assets and liabilities is crucial, and the stochastic input from the investment model can only be used effectively if both assets and liabilities are projected, and the model allows for the appropriate interaction between the two.

Stochastic models have been used extensively in the U.S.A. to assist in product design and pricing. The profitability of certain products, particularly those with options or guarantees, can be substantially different when measured using a stochastic model, as compared to using deterministic profit testing techniques. An example is a contract with guaranteed surrender values—the rate of withdrawal needs to be linked to the level of investment return credited to the policyholders' account and also to competitors' credited rates.

Stochastic models have already been used in the U.K., with dramatic impact, for the purpose of establishing valuation reserves for investment guarantees on unit-linked contracts. At the required level of ruin probability, the reserves are so large that virtually all companies ceased offering such guarantees. After a few years of relatively poor equity performance, there is again a high demand for some form of investment guarantee on unit-linked contracts, and there are high implicit levels of guarantees on with-profits contracts. It is very difficult to establish the appropriate levels of reserves or the appropriate amount of free capital required to write these contracts without using stochastic techniques. Again, it is essential to build in realistic rules for the behaviour of the liabilities, otherwise the results will be spurious—no matter how good the investment model which is used.

Probably the most important use of stochastic investment models is in the financial modelling of a with-profits office. The liability model does not need to be very detailed, but to be realistic it must interact with the investment scenario. In particular, the level of total payouts, the level of reversionary bonus rates and the calculation of the statutory minimum liabilities, will all depend on the investment returns achieved and the yields on the assets. The investment mix may also be altered if the statutory solvency position falls below a certain level. An interactive asset/liability model can be a very powerful tool, enabling management to examine the interaction of an office's bonus philosophy, investment strategy, new business volumes and statutory solvency position, in a way which provides much greater insight than can be achieved using the traditional deterministic approach.

Professor S. M. Schaefer (a visitor): I am from the London Business School. My own profession of financial economics has benefited substantially from greater sophistication in econometrics in recent years. This has had a powerful effect on the standards of research, and now plays a very important part in training. There are similar contributions to actuarial thinking to be gained from portfolio theory, asset pricing theory and, particularly, contingent claims theory. Both economists and actuaries could gain from more interaction between the two professions than there seems to be at the moment.

In estimating the parameters of the model, such as the mean inflation rate, and what that mean might be in 20 or 30 years' time, I think that econometrics will be practically no use at all. Most of the serial dependency in economic time series will have evaporated long before you get to 20 years. Although this will not solve the problem completely, a financial economist would look first at the information that can be obtained from market prices. Taking the inflation rate as an example, modern theories of the term structure in papers such as that by Cox, Ingersoll & Ross (*Econometrica*, **53**, 385), allow implied estimates of the expected long-term inflation rate and interest rate volatility to be extracted, using data on both the index-linked and nominal markets.

My colleagues in the financial economics profession have made relatively little progress on the liabilities side. If I were to discuss the topic of this paper with them, they would feel that financial economics had a certain amount to say about characterising the distribution of asset values over time. I suspect that they would also feel that, in not being able to say a great deal more about the risk characteristics of liabilities, and more particularly about the link between asset and liability values, an important part of the story was missing.

Mr P. J. Nowell: I have been doing some work on asset/liability modelling, and we have chosen an investment model which is more economically determined, using a covariance approach and building in Wilkie's model on inflation. Having performed very many simulations, when considering the relationship between investment policy, bonus smoothing, and investment models, probably the key element is the bonus smoothing. The portfolio of liabilities is the absolute key in determining the kind of smoothing pattern that can be achieved. The Maturity Guarantees Working Party in the 1970s initially calculated some very high reserve requirements, because it concentrated on a new office starting up. Looking at an existing portfolio of investments, there is a trade-off effect of one year against the other, and therefore a mature office could give some kind of maturity guarantee at relatively low cost.

Relatively simple models, indeed almost any model, will fit a certain number of the issues that we are facing today. The important thing is that a stochastic model is used so that variability is introduced, because deterministic models have a considerable amount of danger. One approach when considering legislative frameworks, or just explaining matters to the layman, is that the model can be reduced almost to scenario testing, looking at some reasonably plausible 'outlying' situations, and testing for those.

It would be dangerous to apply any particular model, particularly a sophisticated model, to any legislation, because there would be problems in its universal application. Maybe, if the profession, and others outside it, do more work in this area on specific applications, we might find that the models being used converge on one particular type. It is at that point that we, as actuaries, would have more confidence in all using that model for a particular application.

Mr M. H. D. Kemp: I agree with the statement at the end of § C.1.3, where it states: "The pension fund system is too complex to solve analytically except when we make sweeping simplifications", in principle, but not necessarily with the lessons derived from it. There is a paper by Dufresne, 'Movements of Pension Contributions and Fund Levels when Rates of Return are Random' (*J.I.A.* **115**, 535), which defines how to solve 'analytically' for a variety of measures based on a particular simplified model of a pension fund. It is surprising just how closely those results mirror those obtained from stochastic simulations using more complicated models. There is a danger of over-engineering our 'black boxes' when carrying out asset/liability studies. This ought to give heart to those people who try to simplify them to overcome problems such as the need for several thousand simulations and excessive run times.

Paragraph C.6.2 states: "Decide on the time horizon for the exercise, e.g. 40 years." I have never come across a client who wants a 40-year projection, and only one or two who want 20 years.

It would be useful if the Wilkie model were extended to cover the approximately one-third of assets held by U.K. pension funds that are neither U.K. equities nor conventional gilts.

Professor A. D. Wilkie: In April 1952—almost 40 years ago—Redington's paper on immunisation, 'Review of the Principles of Life Office Valuations' (*J.I.A.* **78**, 286), was discussed in this hall. In the same year, Markowitz's paper on 'Portfolio Selection' (*Journal of Finance*, **VII, No 1**, 77) was published, and since then there has been first a stream, then a torrent, and now a flood of papers, articles and books applying statistical methods in investment. Redington's work has been one additional source for this flood, which has developed enormously in America and much less so in Britain. Until this evening much of the actuarial profession in the U.K. seemed to have set its face against the proper application of statistical methods in investment.

I feel that the actuarial profession is in danger of being beguiled by the notion, perhaps put forward by those who are better at talking than at counting, that we should use no more mathematics in our work than we can comfortably explain to our clients, trustees, directors, or whoever, whom we assume are mathematical laymen. However, if the clients could do the mathematics themselves, there would be no need for them to employ actuaries.

I was disappointed that the Working Party did not find the opportunity to review the various other stochastic investment models that have been presented in the literature, even though the list of references gives many important articles in the field. I am rather sorry that my model has become known as *the* Wilkie investment model. My original paper was called 'A Stochastic Investment Model for Actuarial Use', and perhaps I would prefer it to be described as just the first Wilkie investment model. I would claim, however, that my approach is so far the only published one that seriously attempts to integrate the three important investment series: those for retail price inflation; share prices; and fixed interest yields, in a structure which is economically coherent and satisfactory for long-term projections. The details of the model and the values of the parameters are far from sacrosanct, and everybody is welcome to use them as they like. We have heard that others have prepared similar models, possibly better ones, but since the details have not usually been made publicly available and subjected to informed scrutiny, it is difficult to assess them.

The main competing model, which is used very widely in the U.S.A., is based on the work of Ibbotson and Sinquefield. Although their collection of statistics is valuable, I criticise their work on a number of counts. They use cumulative wealth ratios or indices of total return rather than decomposing share prices into dividend and dividend yield. They do not look at returns on fixed-interest securities according to changes in interest rates. They allow for simultaneous correlations in the returns, but not, so far as I can see, for lagged correlations. In effect, they are using a pure random walk model for long-term simulation. My own model is in no way inconsistent with the random walk model in the short term. It is, however, designed for a longer-term projection.

Although my model, as published, is based on observations at annual intervals, it is possible to use it for simulation by filling in the gaps at monthly, weekly or daily intervals, with what are known as 'Brownian bridges' that is random walks with given starting and finishing points.

Professor Harvey, advising the Working Party, was correct in saying that my model was produced some time ago, and that time series analysis has developed substantially since then. Further, there are several more years of data since my original model was produced, and the model could well be extended by considering also series for salaries, short-term interest rates, other parameters to describe the shape of the yield curve, yields on index-linked stocks, property rentals and yields, and all the same kinds of series in other countries besides the U.K., which would then need to be linked by a suitable model for exchange rates. I am sorry that no-one else has seriously taken up this extension of my model and published it, other than Dr Toutounchi, whose Ph.D. thesis at Heriot-Watt University extended my sort of analysis to various sub-sectors of the FT-Actuaries Indices. However, I can say that the additional years of experience from 1982 to 1991 have in no way caused me to change the basic structure of my model, although some of the parameters have changed. For example, the high real rate of dividend growth during the 1980s suggests that an appropriate value of DMU, the mean real rate of dividend growth, might be taken as a small positive number rather than as zero. My view has always been that actuaries using any stochastic investment model should use it to model fluctuations around the central estimates that they would have chosen anyway. I do not believe that the values of any of my parameters are sacrosanct for all time; they are simply values that I thought, when the model was first published, were reasonable ones to use on the basis of the evidence so far, and in the absence of any alternative views.

I agree that more advanced time series analysis models can be used. This will almost certainly make the model more complicated. Mr Clarkson has put forward his own ingenious elaboration of my model for inflation, but before making use of a model such as his, I would want to know a great deal more about its long-term statistical properties, which he does not seem to have investigated, and I would want to have a method for estimating parameters using appropriate statistical techniques, so that I can judge the significance of the different terms in the model rather than using the somewhat *ad hoc* approach put forward in Appendix A.

The one problem I recognised when my model was first presented was the question of whether the residuals from the model should be taken as normally distributed or not. I found, as most investigators of short-term share price movements have found, that the distribution of price changes is 'fatter-tailed' than the normal distribution. This can be dealt with in a number of ways, as noted in §4.4. The Stable Paretian or Lévy-stable distributions (which are alternative names for the same thing) have many theoretical attractions, but also introduce substantial theoretical complications. I should like to see more work done on these, but, alternatively, there is the possibility of using a mixture of distributions, rather along the lines of Clarkson's approach. A further approach is to assume that the variance of the residuals varies with time, possibly by using an ARCH model. In Appendix B I showed results by making arbitrary assumptions about the parameters of the ARCH model. I have subsequently carried out some investigations, and show in a written contribution that ARCH models help a little.

There are other methods of dealing with non-linear and non-stationary time series. Taylor's book is mentioned in the references, and there are also books by Priestly on *Non-Linear and Non-Stationary Time Series Analysis* and by Howell Tong on *Non-Linear Time Series: a Dynamic Systems Approach.* Fitting these more complicated time series is not easy.

Mr A. J. Wise (closing the discussion): Looking at the terms of reference, the Working Party were asked to critique the model from a statistical viewpoint. Close examination of the statistical pattern of residuals points to considering adjustments if more realistic modelling is required, especially over the short term. More complex models would be able to reflect a pattern of inflation which shows more stability when the rate is low and less stability when the rate is high. The likelihood or otherwise of negative inflation was also commented upon in the report, and also by several speakers. The report notes that a heuristic approach to modelling inflation might be more realistic; and Mr Clarkson's suggestion is outlined in Appendix A.

The potential for introducing more and more realism in modelling real world events is unending, so two questions should be asked of any model:

(1) Is the model sound? I think we may reasonably conclude that the Wilkie model is sound. It can be improved upon, but that does not render it unsound.
(2) Is the model good enough for any specific purpose or do we need greater realism? Consider, for example, its application in life assurance work. The practical difficulties of implementing stochastic modelling are exemplified by the comments on life assurance in Appendix D. Although Mr Needleman thought these drawbacks to be a little overstated, they do suggest that the Wilkie model may, indeed, be good enough at present.

The report does not give any evidence that an improved model might presently be required by life offices. Both Mr Roff and Mr Squires gave constructive views as to the practical implementation for life offices, including the extent to which numbers of model points and scenarios can be tailored to the application in hand. Simplification of the practical work was also a point made by Mr Kemp in relation to pension fund work and by Mr Smith in relation to calculation procedures.

This and other models are already in use, as indicated by Mr Needleman in relation to life assurance; and Mr Urwin confirmed that the model is already in daily use, investigating the implications of alternative investment strategies for pension funds. He agreed with the Working Party's emphasis on the importance of applying actuarial judgement in work of this type.

Appendix E refers to applications of stochastic models in general to investment management. However, most of this is concerned with time periods of less than 1 year, to which the Wilkie model, as it currently stands, does not apply. Wilkie's statistical analysis is based on inflation and investment returns measured over annual time periods, and it is intended for long-term actuarial use. His methodology could, no doubt, be adapted for investment modelling over shorter time periods, which may well require the introduction of exogenous factors. The nature of the model and its parameters would, therefore, probably look different for shorter time periods. I presume that Appendix E is arguing rather for the potential utility of the short-term model for investment management, while § D.8.2 makes the same point in relation to life assurance.

This might be a fruitful area for future research, where actuaries could increase their contribution to investment theory and practice. Having said that, I am convinced that actuaries who normally deal with long-horizon work, such as pension funds, should be wary of trespassing in the short-horizon territory of the investment manager. Increasing use of investment models by actuaries may require careful handling and perhaps professional guidance at some stage. Mr Lockyer was thinking on similar lines, I think, when he commented on information to be shown in an actuary's report.

The final item in the Working Party's brief was to suggest areas for further development of the model. The paper does not specifically address this point, but nor does it say that future research would be better concentrated by assuming that the model is sufficiently useful as it stands. Surely, the need for future development of the model is likely to depend on the area of actuarial investigation being considered, and the requirements of the actuary's employer or client within that area.

We should take heed of the Working Party's advice not to rely on new methodology if it is not well understood; but who can doubt that modelling of uncertainty in future economic conditions is here to stay as a major item in the actuarial toolkit? Professor Wilkie has fashioned a tool which works so long as it is used carefully and well. The needs for refinement of this tool will emerge with time.

I hope, therefore, that there is the time and opportunity to secure what the Working Party conclude is requred, and which Professor Pepper, Mr Daykin and Professor Schaefer clearly advocated; namely, a wider understanding among actuaries of econometric and time-series models, and of their latest developments.

The President (Mr H. H. Scurfield): It is the role of a learned profession to research and push back the boundaries of knowledge. We are fortunate that Professor Wilkie is one of those with the intellect to do just that. We are also indebted to the Financial Management Group for the useful work which it has already done, and in particular for the critique which is in front of us.

We have discussed Professor Wilkie's model, and recognised its value. We have seen the need for more work to be done on it in order to generate increasing confidence in its practical application. We have also heard from the authors of the need to develop credibility and to demonstrate that there are real benefits from using the model in order to overcome objections—and one of those objections relates to its complexity. It is important that our research does more than simply help the few actuaries whose mathematical ability enables them readily to build such models. While I agree with the conclusions of the paper, I believe that, if reliance is to be placed upon the model, this must necessarily be an area of specialisation for a few actuaries rather than trying to take actuaries, generally, deeply into this field.

In projecting the past, we need to be sure that nothing fundamental has happened or is happening which represents a significant one-off change. The economic events of the last 2 years lead me to wonder if we are not going through a major discontinuity at the present time.

Above all, it is clear that the profession values highly the work done on the subject, and that more work is a priority both within the profession and with others who are involved. Behind the actuarial profession is a motto, making certainty out of uncertainty. Modelling is something which ought to make a real contribution to that, and is therefore quite fundamental to the profession.

This has been a good discussion, following the production of a very useful paper. Thank you, Mr Geoghegan and your joint authors; thank you FIMAG; and thank you Professor Wilkie, without whom the paper could hardly have been written.

Mr T. J. Geoghegan (replying): The Working Party debated the area of actuarial judgement at length. In short, it agreed that any conclusions indicated by use of the Wilkie model, or any model for that matter, must be subject to the considered judgement of the actuary. This is particularly emphasised in the appendices, which describe the application of this model to areas of classical actuarial work: pension funds, life offices and investments. Mr Daykin was right to point out that we deliberately omitted reference to general insurance, as it was comprehensively covered in the Daykin & Hey paper (*J.I.A.* **117,** 173). Actuarial judgement can only properly be applied if the actuary's knowledge of stochastic modelling and of time series is of a satisfactory standard. Otherwise, the model could be used indiscriminately. The number of actuaries who have achieved these standards it not very great. I was delighted to hear from Mr Daykin that consideration has been given to the inclusion of stochastic modelling and time series in the examination syllabus.

Where do we go from here, and what does the future hold? I believe that the use of stochastic techniques in actuarial work will grow significantly in the future, especially with the advent of more powerful computers. I welcome this growth very much, and many thanks should be given to Professor Wilkie for his contribution in laying a foundation for its future development. Deterministic approaches to financial problems have significant drawbacks—not least in the acceptance by our clients of absolutely unchangeable assumptions stretching onwards for 40 years or more. The credibility of the actuary is sometimes tested by his or her ready acceptance of solutions based on these rigid long-term assumptions. The proper use of stochastic techniques, used to supplement a deterministic approach, can be a powerful tool.

The creation of a further working party to extend the thinking behind the present paper might fruitfully be considered. Greater interaction between actuaries and econometricians should be welcomed and encouraged, as was mentioned earlier by Professor Schaefer.

WRITTEN CONTRIBUTIONS

Mr R. M. FitzHerbert: While my data have been primarily Australian in the context of investment decision making, there are sufficient similarities between Australia and the U.K. to make some of this research relevant.

First (and most importantly), long-term movements in share price indices (or dividend indices)—have little to do with inflation, which contradicts an essential assumption of the Wilkie model. The reason for this is that, while the bulk of company assets are 'real items', such as plant and stock, these assets (with the exception of land and buildings) are treated as monetary items under historical cost accounting. Consequently the stewardship of shareholders' funds is largely accounted for in money terms and not in real terms.

In the paper 'Cycles and Trends in Australian Share Prices' (*Journal of the Securities Institute of Australia*, March 1979), I demonstrated that Australian share prices had shown much the same rate of growth in the period 1875–1925 (when there was little inflation) as in the period 1925–75 when the rate of inflation was quite large. Share price indices have generally shown an upward trend as a result of retained profits, which have been around 4% p.a. of shareholders' funds for as long as I have been able to obtain data (1950). It is for this reason that share prices have shown a roughly similar upwards trend before and after 1925. If inflation had operated as a separate factor, then the U.K., Australian and U.S. share indices would now be *many* times their current level.

I was surprised at the element of Wilkie's model which requires a higher dividend yield in times of high inflation. If, as required by this model, the dividend stream is an inflation hedge, then I would have thought that, in times of high inflation, ordinary shares would command an above-average premium over other assets—and therefore a lower dividend yield.

My second point relates to the use of dividend yields rather than underlying asset values in equity models. As well as taking differences to achieve stationarity, raw data can be transformed. When modelling a time series which represents an asset with accumulated income, the logarithm should be a simple straight line, provided the rate of retained income is constant. The underlying asset value of a share portfolio will primarily increase as a result of retained profits, and therefore a simple straight line fitted by least squares will be a reasonable first approximation to the underlying trend. An autoregressive fluctuation about such a simple straight line is not irrational.

In 1980 the Maturity Guarantees Working Party identified a trend reverting tendency of prices, which they modelled by taking the logarithm of the dividend yield as a first order autoregressive process. I do not believe there is any rational reason for the dividend yield fluctuating around 4% (or any other figure), and in this respect the general thrust of Wilkie's formula may be a theoretical improvement. However, the U.S. experience of 1929 and the Australian experience of 1987 suggest that it is a mistake to use dividend yields as an ingredient of the model, because at neither of these times was the dividend yield unacceptably low. With the benefit of hindsight, dividend payments had reached unsustainable levels and were subsequently reduced. In the *Journal of the Securities Institute of Australia* in March 1979 I suggested that this trend reverting phenomenon was an autoregressive fluctuation about 1·5 times underlying net asset value. This has some rationality because, when share prices are significantly in excess of this figure it is easy for the corporate sector to soak up liquidity by issuing shares, and conversely, when share prices are below underlying asset values the vultures abound. Thus there are observable forces which could account for a reversion to net asset value. The figure of 1·5 is an empirical estimate which accounts for goodwill, undervalued property and so on. It may also represent the difference between book values and the replacement cost of assets.

In 'normal times' dividend yield and asset value models may produce much the same sort of result. However, at times a model based on underlying asset values would have been superior. For this reason equity models based on aggregate balance sheet data may be more reliable than those based on dividend yields.

My final point concerns computers and the sophistication of stochastic models. Modern computers are essential in applying stochastic models, but they can keep our attention focused on numbers and formulae which can bamboozle everybody, including the researchers. For example, in 1980 Godolphin fitted a linear trend to the de Zoete equity index (justified by profit retentions) and a first-order autoregressive process to the residuals—which has a rational explanation provided the trend is accepted as an approximation to 1·5 times net asset value. For some reason he seemed to prefer a complicated ARIMA model for which a rational explanation is hard to comprehend. In so doing the Maturity Guarantees Working Party may have missed an important point. If the rate of inflation is unacceptably high then policies may be introduced to cause it to decline. Conversely, if the rate of inflation is so low that it is no longer an issue then cynics may argue that 'vote-buying' will lead to policies which will increase inflation. Thus, there is some rational explanation for modelling the force of inflation as a first-order autoregressive process, even if it does not exactly fit the facts. With more sophisticated models there may not be any rational explanation, even if the model is a superior statistical fit.

Section 6 is most important. If stochastic investment models are to find their way into standard actuarial practice they will need to make sense—or be capable of rational explanation in words—as well as fitting the data. This requires a basic understanding of time-series models, which can be introduced by education, but it also requires more rational explanation and fewer formulae and figures from modellers.

Mr H. D. White: It is inappropriate to develop complex liability models as suggested in §§D.2.4 to D.2.7. More can be learned from simplified models of the liabilities, thus allowing for more investment simulations as in §C.4.8. This will also assist the actuary when making appropriate correlations between the investment and claims and withdrawal experience, as in Appendix C.

Much can be learned about life business from modelling, and undoubtedly some of the conclusions will be unfashionable and unwelcome to some managements. For example, a properly balanced portfolio is less likely to be the market leader in the short to medium term, but more likely to be so in the long term. Stochastic research proves that smoothing of bonus can be expensive, and it is particularly important to make reductions in bonus rates when they are needed. Nevertheless, it is the function of the actuary to give such advice when it is appropriate. These new modern techniques need be no more confusing than our traditional actuarial practices to the layman. There is perhaps more of a problem with professional actuaries not understanding the techniques than management.

Professor A. D. Wilkie: I said in the discussion that I would quote some figures in a written contribution. I have carried out some investigations for the series of Retail Price Indices, taken at annual intervals from 1923 to 1991, and found that the standard deviation of the residuals (the *QSD* of my model), if assumed constant, has a value of about 0·04. If I use an ARCH model as shown in

Appendix B, the values of A and B are about 0·01 and 0·144 respectively, giving a similar value to the standard deviation in the long run. However, the standardised residuals for the ARCH model are no more normal than for the constant variance model, and there are still one or two large values that leave one feeling uncomfortable.

In the discussion Mr Smith made two detailed remarks that require comment. He stated that "the directions of dependencies between current values of the variables are a function of the way the model is set up". True, but I have not previously explained that the first investigation I undertook studied the series using conventional multivariate ARIMA techniques, and as a consequence of this investigation I found that it was possible to order the variables in the cascade model I then adopted. A further check of this was obtained by calculating the cross-correlation coefficients at all lags for each pair of residuals. If any significant lag in the 'wrong' direction had remained, it would have been necessary either to rearrange the structure of the cascade or to use a simultaneous multivariate model. Therefore the cascade structure is a result of the investigation, not a prior assumption.

Secondly, Mr Smith observed that the simulated 'Consols' yield can become negative. This is correct in two different senses. In the investigation of past data it is possible that the allowance for prospective inflation would be greater than the nominal yield, so that the real rate of interest would appear to be negative. In the fitting program I have therefore applied a minimum value for the real part, and chosen a figure of 0·5% for it. In fact, with the chosen parameters the minimum value does not apply. Then, in future simulation, it is possible that the allowance for prospective inflation becomes negative and sufficiently large for the simulated nominal rate to be negative. Again, it is necessary to insert a minimum value for the nominal rate, and again I have chosen 0·5%. I cannot say that such a minimum value never applies in any simulation, but it is certainly rare with the chosen parameters.

It is worth observing that the Treasury Bill rate in the U.K. was about 0·5% for most of the periods 1933–38 and 1946–51 (see *The British Economy, Key Statistics, 1900–1964*, London and Cambridge Economy Service), and was lower than that, down to 0·01% or even negative, in the United States of America from 1933 to 1946 (see *A History of Interest Rates, 2000 B.C. to the Present*, by Sidney Homer, Rutgers, New Jersey, 1963). Nominal interest rates on certain bank deposits in Switzerland have carried negative rates from time to time. All these examples are of short-term, rather than long-term rates, which in recorded history so far do not seem to have fallen below 2·5% (see Homer, *op. cit*).