SOME REMARKS RELATING TO STOODLEY’S FORMULA

by

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§1. Introduction

The formula suggested almost 50 years ago by Stoodley (cf. Reference 1) to allow for a reduction in rates of interest with the passage of time does not seem to have achieved the recognition which one might have expected, particularly in view of its easy application in practice. In his remarks summarising the discussion following his paper Stoodley conceded that the problem of choosing the parameters of the formula led to somewhat intractable equations, which could be solved only approximately by trial and error techniques. However, with modern calculation methods it is a trivial matter to solve this problem accurately and in §3 below we indicate one possible approach. On the other hand, the use of modern calculators considerably reduces the specific advantage of the original formula. Accordingly in §4 we consider the more general logistic curve (of which Stoodley’s formula is a special case), which permits somewhat greater flexibility in modelling future trends.

As originally proposed, Stoodley’s formula used a three-parameter curve to model future values of \( i_t \), the annual rate of interest at time \( t \). However the subsequent discussion turned attention towards the force of interest and it is perhaps more convenient to refer to Stoodley’s formula as expressed by the equation

\[
\delta(t) = p + \frac{\alpha}{1 + rest},
\]

(1)

where \( \delta(t) \) denotes the force of interest at time \( t \).

For completeness we recall in §2 the practical advantages of this formula.

§2. The practical advantages of Stoodley’s three-parameter formula

It is a simple exercise to verify that \( \delta(t) \), as defined by equation (1) above, is a monotonic function—increasing if \( r < 0 \) and decreasing if \( r > 0 \). Accordingly the formula is appropriate only when we wish to model a steady trend in interest rates. In particular, it may be
suitable when we wish to model rates which fall with the passage of

time.

Since equation (1) may be expressed as

$$\delta(t) = (p+s) - \frac{r_{est}t}{1+re^{st}}$$

$$= (p+s) - \frac{d}{dt} \{\log(1+re^{st})\}$$

it follows immediately that, if we let

$$v(t) = e^{-\int_0^t \delta(r)dr},$$

then

$$v(t) = e^{-(p+s)t} \cdot \frac{1+re^{st}}{1+r}$$

$$= \frac{1}{1+r} \cdot e^{-(p+s)t} + \frac{r}{1+r} \cdot e^{-pt}$$

$$= \frac{1}{1+r} \cdot (1+i_1)^{-t} + \frac{r}{1+r} (1+i_2)^{-t}$$

where $1+i_1 = e^{(p+s)}$ and $1+i_2 = e^p$.

This last equation (4) illustrates the considerable practical advan-
tage of Stoodley's formula. If we are able to model $\delta(t)$ with values
of $p$ and $s$, such that $i_1$ and $i_2$ are "tabulated" rates of interest, then
$v(t)$, the present value of 1 due at time $t$, is obtained simply as a
weighted average of the corresponding present values at these
tabulated rates. Likewise, any annuity value under the varying
force of interest defined by the model is obtained as the weighted
average of the corresponding annuity values at rates of interest $i_1$ and
$i_2$. Even if it is necessary to use values of $p$ and $s$ which do not lead
to tabulated rates of interest, the simplicity of the formula lends
itself to easy application.

The accumulation of 1 from time $t_1$ to $t_2$ is obtained as

$$A(t_1, t_2) = e^{\int_{t_1}^{t_2} \delta(r)dr}$$

$$= e^{\int_{t_1}^{t_2} \delta(r)dr}$$

$$= \frac{v(t_1)}{v(t_2)}$$

$$= \frac{(1+i_1)^{-t_1} + r \cdot (1+i_2)^{-t_1}}{(1+i_1)^{-t_2} + r \cdot (1+i_2)^{-t_2}},$$

(from equation 4 above).

Again, this last expression is easily calculated.
§3. The determination of the parameters

One obvious method of choosing suitable parameters \( p, r \) and \( s \) for the formula (1) is to specify the values of \( \delta(0), \delta(t_1) \) (where \( t_1 \) is some given future time) and \( \delta(\infty) \) (the value to which \( \delta(t) \) tends asymptotically as \( t \) becomes large). The values to be used will be matters of judgment and, when choosing \( \delta(0) \), we will be influenced by current market conditions.

Suppose then that \( t_1 > 0 \) and \( \delta_0, \delta_1, \delta_\infty \) are given. We wish to determine the values of \( p, r, s \) in equation (1) for which

\[
\begin{align*}
\delta(0) &= \delta_0 \\
\delta(t_1) &= \delta_1 \\
\delta(\infty) &= \delta_\infty.
\end{align*}
\]

(6)

Note that, as we have remarked above, the nature of the formula requires that \( \{\delta_0, \delta_1, \delta_\infty\} \) be a monotonic sequence. (In practice we may be more likely to model a decrease in interest rates, but occasionally we may wish to consider an increasing sequence. In this latter case, however, it may not prove possible to fit Stoodley's formula to the given values. This point is discussed below.)

Since

\[
p + \frac{s}{1 + re^{st}} = (p + s) + \frac{(-s)}{1 + \frac{1}{r}e^{-st}},
\]

without loss of generality we may assume that in equation (1) above \( s \) is positive. Then \( \lim_{t \to \infty} \delta(t) = p \), so that

\[
\delta_\infty = p.
\]

(7)

Putting \( t = 0 \) in equation (1), we obtain

\[
\delta_0 = p + \frac{s}{1 + r}
\]

(8)

\[
= \delta_\infty + \frac{s}{1 + r} \quad \text{ (from (7))},
\]

so that

\[
r = \frac{s}{\delta_0 - \delta_\infty} - 1.
\]

(9)

Hence, substituting the values of \( p \) and \( r \) from equations (7) and (9), we may write equation (1) in the form

\[
\delta(t) = \delta_\infty + \frac{s}{1 + \left[\frac{1}{\delta_0 - \delta_\infty} - 1\right]e^{st}}.
\]

(10)
This last equation ensures that $\delta(t)$ takes the required values, $\delta_0$ and $\delta_\infty$, when $t = 0$ and $\infty$ respectively. It remains to choose $s$ so that $\delta(t_1) = \delta_1$.

Letting $t = t_1$ in equation (10), we have

$$\delta_1 = \delta_\infty + \frac{s}{1 + \left[\frac{s}{\delta_0 - \delta_\infty} - 1\right] e^{st_1}}$$

which (by rearrangement) may be expressed as

$$\hat{g}(s) = 0,$$

where

$$g(s) = \left[ s, \frac{\delta_1 - \delta_\infty}{\delta_0 - \delta_\infty} - \delta_1 + \delta_\infty \right] e^{st_1} - s + \delta_1 - \delta_\infty.$$  \hspace{1cm} (12)

In relation to this last equation note that $g(0) = 0$. Also, since $\{\delta_0, \delta_1, \delta_\infty\}$ is a monotonic sequence,

$$0 < \frac{\delta_1 - \delta_\infty}{\delta_0 - \delta_\infty} < 1,$$

so that $\lim_{s \to \infty} g(s) = \lim_{s \to -\infty} g(s) = \infty$.

It remains to find the positive root of equation (11). (By hypothesis $s$ is positive.) First, however, we must determine whether or not such a root exists.

It follows trivially from equation (12) that

$$g'(s) = e^{st_1} \left[ s, t_1 \frac{\delta_1 - \delta_\infty}{\delta_0 - \delta_\infty} + \frac{\delta_1 - \delta_\infty}{\delta_0 - \delta_\infty} + t_1 (\delta_\infty - \delta_1) \right] - 1$$

so that $g'(s) \leq 0$ according as

$$s, t_1 \frac{\delta_1 - \delta_\infty}{\delta_0 - \delta_\infty} + \frac{\delta_1 - \delta_\infty}{\delta_0 - \delta_\infty} + t_1 (\delta_\infty - \delta_1) \leq e^{-st_1}.$$  \hspace{1cm} (15)

Since

$$t_1 \frac{\delta_1 - \delta_\infty}{\delta_0 - \delta_\infty} > 0,$$

the inequalities (15) show that there is a value of $s$, say $s^*$, such that

$$g'(s) \leq 0 \text{ according as } s \leq s^*.$$  \hspace{1cm} (16)

This means that, as $s$ increases, $g(s)$ decreases to a minimum value at $s = s^*$ and thereafter increases. Since $g(0) = 0$, there is a positive root of equation (11) if and only if $g'(0) < 0$ and, when this condition is satisfied, the positive root is unique.
Some Remarks Relating to

From equation (14) we obtain

\[ g'(0) = \frac{\delta_1 - \delta_\infty}{\delta_0 - \delta_\infty} + t_1(\delta_\infty - \delta_1) - 1 \]

\[ = (\delta_\infty - \delta_1)t_1 - \frac{\delta_0 - \delta_\infty}{\delta_0 - \delta_\infty}. \]

Thus \( g'(0) < 0 \) if and only if

\[ (\delta_\infty - \delta_1)t_1 < \frac{\delta_0 - \delta_\infty}{\delta_0 - \delta_\infty}. \]  \hspace{1cm} (17)

This last condition (17) is crucial. There is a positive root of equation (11) if and only if condition (17) is satisfied and, when such a root exists, it is unique. Thus Stoodley’s model can be fitted to the given triple \( \{\delta_0, \delta_1, \delta_\infty\} \) if and only if the condition is satisfied. Note that, when \( \{\delta_0, \delta_1, \delta_\infty\} \) is a decreasing sequence, the condition (17) always holds. (The left-hand side of the inequality is negative while the right-hand side is positive.) On the other hand for an increasing sequence \( \{\delta_0, \delta_1, \delta_\infty\} \) the condition may not hold. For an increasing sequence it is a simple matter to verify that the inequality (17) is equivalent to either of the conditions

\[ t_1 < \frac{(\delta_1 - \delta_0)}{(\delta_\infty - \delta_0)(\delta_\infty - \delta_1)} \]  \hspace{1cm} (18)

and

\[ \delta_1 > \delta_\infty - \frac{(\delta_\infty - \delta_0)}{1 + t_1(\delta_\infty - \delta_0)}. \]  \hspace{1cm} (19)

If \( \{\delta_0, \delta_1, \delta_\infty\} \) is a given increasing sequence, the inequality (18) provides an upper bound on \( t_1 \) for the Stoodley model to be fitted. Likewise, if \( \delta_0, \delta_\infty \) and \( t_1 \) are given (with \( \delta_0 < \delta_\infty \)), the inequality (19) provides a lower bound on \( \delta_1 \) for the model to be applicable.

It is perhaps worth remarking that, since in relation to long-term contracts we are more likely to wish to model a fall in interest rates, the inadequacy of the model for certain increasing sequences is not too serious an objection.

Given that the positive root of equation (11) exists and is unique, we may determine its value to any desired degree of accuracy rapidly and very simply by any one of the many methods available for the solution of non-linear equations (e.g. Newton-Raphson, Regula Falsi, etc.). Then, the appropriate value of \( s \) having been found, \( \delta(t) \) is given by equation (10).

In practice, having determined the parameters \( p, r \) and \( s \) corresponding to the given values of \( t_1, \delta_0, \delta_1 \) and \( \delta_\infty \), one might regard these as preliminary parameters forming the basis for further experiment with a view to obtaining tabulated rates \( i_1 \) and \( i_2 \) in equation (4).
In this case the values of $\delta(0)$, $\delta(t_1)$ and $\delta(\infty)$ arising from the formula finally adopted may not agree exactly with the given values $\delta_0$, $\delta_1$ and $\delta_\infty$. However, if the differences are sufficiently small, the practical advantages of having tabulated rates $i_1$ and $i_2$ in equation (4) will outweigh any disadvantages arising from the slight differences between $\delta(0)$ and $\delta_0$, etc.

§4. The more general four-parameter formula

Equation (1) is a particular case of the more general logistic formula

$$\delta(t) = p + \frac{q}{1 + re^{st}}.$$  \hspace{1cm} (20)

For this four-parameter formula, if—as before—we let

$$v(t) = e^{-t^s(t)r}dr,$$

then

$$v(t) = e^{-(p+q)}t \left[ \frac{1 + re^{st}}{1 + r} \right]^{q/s}.$$  \hspace{1cm} (21)

For Stoodley’s formula $q = s$, so that this last equation reduces to the much simpler form given by equation (4).

Although equation (21) lacks the immediate practical convenience of equation (4), with modern methods of calculation it is a trivial matter to evaluate $v(t)$ as given by the more general formula. Moreover, since this formula contains an extra parameter, it provides greater flexibility in modelling the future trend of interest rates. One would hope to be able to fit the formula to four specified values of $\delta(t)$—as opposed to the three values which define Stoodley’s original formula.

Note that, since

$$p + \frac{q}{1 + re^{st}} \equiv (p+q) + \frac{(-q)}{1 + e^{-st}}$$

without loss of generality we may again assume that $s$ is positive. Then $\delta(t)$ is a monotonic function, increasing if $qr < 0$ and decreasing if $qr > 0$.

Suppose that $0 < t_1 < t_2$ and that $\{\delta_0, \delta_1, \delta_2, \delta_\infty\}$ is a given monotonic sequence. We wish to determine (if possible) the parameters $p$, $q$, $r$ and $s$ of equation (20) such that

$$\begin{align*}
\delta(0) &= \delta_0 \\
\delta(t_1) &= \delta_1 \\
\delta(t_2) &= \delta_2 \\
\delta(\infty) &= \delta_\infty
\end{align*}$$  \hspace{1cm} (22)

and

$$\delta(\infty) = \delta_\infty.$$
Some Remarks Relating to

Since, by hypothesis, $s > 0$, it follows as before that
\[ \delta_{\infty} = p. \]

Also
\[ \delta_0 = p + \frac{q}{1 + r}, \]
so that
\[ q = (1 + r)(\delta_0 - p) = (1 + r)(\delta_0 - \delta_{\infty}). \]

Hence equation (20) may be written as
\[ \delta(t) = \delta_{\infty} + \frac{(1 + r)(\delta_0 - \delta_{\infty})}{1 + re^{st}} (s > 0). \tag{23} \]

This last equation ensures that $\delta(t)$ takes the required values when $t = 0$ and $\infty$. It remains to choose $r$ and $s$ so that the specified values, $\delta_1$ and $\delta_2$, are attained when $t = t_1$ and $t_2$ respectively.

Letting $t = t_1$ in equation (23) and rearranging the resulting equation, we obtain
\[ r = \frac{1 - \alpha_1}{\alpha_1 e^{st_1} - 1} \tag{24} \]
where
\[ \alpha_1 = \frac{\delta_1 - \delta_{\infty}}{\delta_0 - \delta_{\infty}}. \tag{25} \]

Similarly, by putting $t = t_2$ in equation (23), we obtain
\[ r = \frac{1 - \alpha_2}{\alpha_2 e^{st_2} - 1} \tag{26} \]
where
\[ \alpha_2 = \frac{\delta_2 - \delta_{\infty}}{\delta_0 - \delta_{\infty}}. \tag{27} \]

Note that, since $\{\delta_0, \delta_1, \delta_2, \delta_{\infty}\}$ is a monotonic sequence,
\[ 0 < \alpha_2 < \alpha_1 < 1. \tag{28} \]

By equating the two expressions for $r$ (from equations (24) and (26)), we find that
\[ h(s) = 0, \tag{29} \]
where
\[ h(s) = (1 - \alpha_1)\alpha_2 e^{st_2} - (1 - \alpha_2)\alpha_1 e^{st_1} + (\alpha_1 - \alpha_2). \tag{30} \]

Note that $h(0) = 0$, $\lim_{s \to -\infty} h(s) = \alpha_1 - \alpha_2 > 0$, and $\lim_{s \to \infty} h(s) = \infty$.

Since, by hypothesis $s > 0$, it is possible to fit the four-parameter formula to the given values $\delta_0$, $\delta_1$, $\delta_2$ and $\delta_{\infty}$ if and only if equation (29) has a positive root. By considering the sign of $h'(s)$, we find—as before—that $h(s)$ has one turning point at which a minimum
value is attained. Accordingly there is a positive root of \( h(s) \) if and only if \( h'(0) < 0 \) and, if this condition is satisfied, the positive root is unique.

Since
\[
h'(0) = t_0(1-\alpha_1)x_0 - t_1(1-\alpha_2)x_1.
\]
it follows that \( h'(0) < 0 \) if and only if
\[
\frac{t_1}{t_2} > \frac{(1-\alpha_1)x_2}{(1-\alpha_2)x_1}.
\]

On substitution for \( \alpha_1, (1-\alpha_1), \alpha_2 \) and \( (1-\alpha_2) \) (from equations (25) and (27)), this last condition becomes
\[
\frac{t_1}{t_2} > \frac{(\delta_0 - \delta_1)(\delta_\infty - \delta_2)}{(\delta_0 - \delta_2)(\delta_\infty - \delta_1)}.
\]

Thus, in order that we may fit the four-parameter formula to the given set of values \( \{\delta_0, \delta_1, \delta_2, \delta_\infty\} \), it is both necessary and sufficient that condition (31) be satisfied. When the condition holds, it is a simple matter to find the appropriate value of \( s \) from equations (29) and (30). Then \( r \) is given by equation (24) (or (26)). The values of \( s \) and \( r \) having been determined, \( S(t) \) is given by equation (23).

The inequality (31) is of some interest. The right-hand side lies between 0 and 1 and is simply the "cross-ratio" of the quadruple \( (\delta_0, \delta_\infty, \delta_1, \delta_2) \).

§5. An illustrative example

Consider the situation in which we wish to model a fall in the force of interest from a current level of 16% to a "long term" level of 8%, using a value of 12% after five years.

We thus have \( \delta(0) = 0.16 \), \( \delta(5) = 0.12 \) and \( \delta(\infty) = 0.08 \). As has been shown in §3, there is an essentially unique three-parameter Stoodley curve which will fit these values. The parameters of this curve are given in Table 1 below. For this curve the value of \( \delta(10) \) is 0.0971.

Suppose now that we wish to refine our model by specifying in addition the value of \( \delta(10) \) and using a four-parameter logistic curve. Let the value of \( \delta(10) \) be \( \delta^* \). In order that we may fit the logistic curve to the four specified values of \( \delta(t) \), it follows from condition (31) that
\[
\frac{5}{10} > \frac{(0.16 - 0.12)(\delta^* - 0.08)}{(0.16 - \delta^*)(0.12 - 0.08)}
\]
which (since \( \delta^* \) must be less than 0.16) is equivalent to
\[
\delta^* < 0.1067.
\]

Hence, since \( \delta(10) \) is necessarily greater than \( \delta(\infty) \), we must have
\[
0.08 < \delta^* < 0.1067.
\]
Some Remarks Relating to

As illustrations of the four-parameter formula we consider two cases, which correspond to extreme situations for which the inequality (32) is satisfied. For our first example we assume that \( \delta(10) = 0.106 \) and for the second that \( \delta(10) = 0.081 \). Effectively, in the second example, the long term value of \( \delta \) has been attained after ten years.

The values of the parameters \( p, q, r \) and \( s \) for these two cases are given in Table 1 below. Graphs of the two four-parameter curves and of the Stoodley curve are also given. The shape of the curves and the manner in which they intersect is of some interest.

**Table 1**

Parameters of logistic formula for force of interest

\[
[\delta(0) = 0.16; \delta(5) = 0.12; \delta(\infty) = 0.08]
\]

<table>
<thead>
<tr>
<th>Formula</th>
<th>Parameter</th>
<th>( p )</th>
<th>( q )</th>
<th>( r )</th>
<th>( s )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stoodley</td>
<td>( (q = s) )</td>
<td>0.08</td>
<td>-0.00666670</td>
<td>1.46636735</td>
<td>0.19730938</td>
</tr>
<tr>
<td>( \delta(10) = 0.106 )</td>
<td>0.08</td>
<td>0.08105263</td>
<td>-1.08333375</td>
<td>0.08105263</td>
<td>0.01482166</td>
</tr>
<tr>
<td>( \delta(10) = 0.081 )</td>
<td>0.08</td>
<td>0.08105263</td>
<td>0.01315790</td>
<td>0.87134173</td>
<td></td>
</tr>
</tbody>
</table>

**Y=DELT A(T)**

\( \text{DELT A(0)} = 0.16; \text{DELT A(5)} = 0.12; \text{DELT A(\infty)} = 0.08 \)

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THREE PARAMETER (STOODLEY)

FOUR PARAMETER (\( \text{DELT A(10)} = 0.106 \))

FOUR PARAMETER (\( \text{DELT A(10)} = 0.081 \))

Finally, as an indication of the financial consequences of each of these three formulae, in Table 2 we give the net single and annual (payable in advance) premiums on the appropriate model for the
Stoodley's Formula

future force of interest for a capital redemption policy with sum assured £1,000 with a term of 5, 15, 25 or 35 years.

**Table 2**

Net Premiums for capital redemption policies with sum assured £1,000

\[ \delta(0) = 0.16; \delta(5) = 0.12; \delta(\infty) = 0.08 \]

<table>
<thead>
<tr>
<th>Premiums</th>
<th>Model for ( \delta(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Stoodley ( \delta(10) = 0.0971 )</td>
</tr>
<tr>
<td><strong>Single premiums</strong></td>
<td>£</td>
</tr>
<tr>
<td>Term 5 years</td>
<td>499.87</td>
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<tr>
<td>Term 15 years</td>
<td>185.40</td>
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<tr>
<td>Term 25 years</td>
<td>80.86</td>
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<tr>
<td>Term 35 years</td>
<td>36.18</td>
</tr>
<tr>
<td><strong>Annual premiums</strong></td>
<td>£</td>
</tr>
<tr>
<td>Term 5 years</td>
<td>131.37</td>
</tr>
<tr>
<td>Term 15 years</td>
<td>26.29</td>
</tr>
<tr>
<td>Term 25 years</td>
<td>9.67</td>
</tr>
<tr>
<td>Term 35 years</td>
<td>4.05</td>
</tr>
</tbody>
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**REFERENCE**