SIMPLE ALGEBRAIC FORMULAE FOR ESTIMATING THE RATE OF INTEREST

BY H. KARPIN, A.S.T.C. (Sc.), A.A.S.A., A.I.A.

The problem of estimating the rate of interest in a given transaction is a common one. Various alternatives to the process of inverse interpolation have been published including the following:

A. W. Evans (J.I.A. 72, 447-54)
M. T. L. Bizley (J.I.A. 88, 95-9)
J. I. Craig and Editors (J.I.A. 60, 341, etc.)

LOANS REPAYABLE BY LEVEL INSTALMENTS

2. The contributions of Evans and Bizley dealt with this problem, Evans by means of a formula which gave a unique value to the rate of interest and Bizley by means of a series the successive terms of which, alternately positive and negative, successively improved the result and allowed for the approximate magnitude of the error to be known.

Evans's formula is effective over a wide range of loan durations and interest rates but requires the use of logarithms in evaluating a relatively complex expression.

Bizley's method requires the use of a prepared set of coefficients and the successive terms in the series calculated therefrom reduce in absolute magnitude fairly quickly so long as \( \frac{n-a_{\text{final}}}{a_{\text{final}}} \) (the ratio of total interest to amount of loan) is less than about 0.6. Beyond this value convergence is slower and beyond a value of about 1 the successive terms diverge. Consequently, although exceedingly powerful for loans of short duration combined with low rates of interest, the method has limited application.

3. The author now puts forward a formula which is relatively simple, its use requiring neither a high degree of mathematical skill nor the assistance of logarithms nor prepared coefficients. The formula is:

\[
i = \frac{2p(3+p)}{2np+3(n+1)} \quad (1)
\]

where

\[
p = \frac{n-a_{\text{final}}}{a_{\text{final}}}
\]

For a derivation of the formula, see Appendix 1.]
This formula may be compared with that of Evans which may be stated as follows:

\[ i = \frac{2}{(n+1)q} [(1+p)^q - 1], \text{ where } q = \frac{1}{5} + \frac{4}{3} \left[ \frac{1}{n+1} + \frac{1}{10} (1+p)^{\frac{1}{2}} \right] \]

and \( p \) has the same meaning as in (1) above.

Bizley's series is

\[ i = \phi - \frac{n-1}{3} \phi^2 + \frac{(n-1)(2n+1)}{9} \phi^3 - \frac{(n-1)(2n+1)(11n+7)}{135} \phi^4 + \frac{(n-1)(2n+1)^2 (13n+11)}{405} \phi^5 - \frac{(n-1)(2n+1)(3n+1)(50n^2 + 89n + 41)}{2835} \phi^6 + \ldots \]

where

\[ \phi = \frac{2p}{n+1}. \]

It might be noted that \( \phi \) is the rate of interest which would be arrived at by use of the method laid down in the First Schedule of the Money-lenders and Infants Loans Act, 1941-1961, New South Wales Act No. 67, 1941 (see § 13).

4. In using Bizley's formula one proceeds to the desired order of accuracy by introducing, in order, succeeding terms in the series. As the coefficients of the powers of \( p \) are functions of \( n \) alone, a table of coefficients may be prepared which can be multiplied by relevant powers of \( p \) (or if the formula is in the above form, by powers of \( \phi/2 \)), and this facilitates calculation. A table of coefficients was appended to Bizley's paper.

5. The results (K) obtained by the author's formula (1) are compared in Table 1 with those obtained by the methods of Evans (E) and Bizley (B). The comparative results, being shown to six decimal places only, do not of course allow for a comparison of relative accuracy but do give an indication of the relative range of application under the circumstances noted below.

Evans has noted that some improvement can be made in the results obtained by his formula by adding .001 to \( q \) for every complete 20 in the value of \( n \). This adjustment was not made in the table of comparative results shown.

The results shown as derived from Bizley's series take account of only the first six coefficients of his series, this number of terms being given in his table of published values. Where the use of six terms of the series does not give a satisfactory result a horizontal stroke appears.
Table 1. A comparison of the values of \( i \) found by using the formulae of Evans (E), Karpin (K), and Bizley (B) for various values of \( n \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>E</th>
<th>K</th>
<th>B</th>
<th>E</th>
<th>K</th>
<th>B</th>
<th>E</th>
<th>K</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
</tr>
<tr>
<td>13</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
</tr>
<tr>
<td>25</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
</tr>
<tr>
<td>49</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
</tr>
<tr>
<td>97</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
</tr>
<tr>
<td>199</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
</tr>
<tr>
<td>298</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
</tr>
<tr>
<td>397</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
</tr>
<tr>
<td>596</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
</tr>
</tbody>
</table>

It will be noted that Evans's formula gives values of positive error for low values of \( n \) and passes through the correct value to a negative error as \( n \) increases and that the author's formula gives values which move in converse fashion.

**FOURCE OF INTEREST**

6. A simple formula was found for the force of interest analogous to Formula (1) for the rate of interest but arrived at independently (see Appendix 2).

The formula

\[
\delta = \frac{2p(3+p)}{2np+3(n+1+p)}
\]  

(2)

bears a similarity to that for the rate of interest \( i \) and a comparison of the results obtained by the two formulae is given on next page.
7. Unlike Formula (1), Formula (2) always gives a positive error which increases fairly rapidly. However the similarity between the two formulae and the fact that they were arrived at independently suggests that some useful relationships may be obtained from them.

Thus we have

\[ \frac{1}{\delta} - \frac{1}{i} = \frac{3}{2(3+p)} \]

which is a peculiar result when it is considered that \( p \) varies with \( n \) as well as \( i \).

As the approximate formulae for \( i \) and \( \delta \) give their best results when \( p \) is small, the substitution of zero for \( p \) is suggested and this gives

\[ \frac{1}{\delta} - \frac{1}{i} = .5. \]

This may be compared with the value obtained for \( \frac{1}{\delta} - \frac{1}{i} \) when we expand

\[ \frac{1}{\log(1+i)} \left( = \frac{1}{\delta} \right) \]

as

\[ \frac{1}{\delta} - \frac{1}{i} = 1 + \frac{i}{2} + \frac{i^2}{12} + \frac{i^3}{24} + \ldots \]

It was further found that if \( \frac{1}{2} i \) were substituted for \( p \) the relationship

\[ \frac{1}{\delta} - \frac{1}{i} = \frac{3}{2(3+\frac{1}{2}i)} \]

gave much more accurate results.*

* It will be noted that if terms in \( i^2 \) or higher powers of \( i \) in the expansion of \( \frac{1}{\log(1+i)} \)

are neglected and we take \( \frac{1}{\delta} - \frac{1}{i} = \frac{1}{2} - \frac{i}{12} \), we may put \( \frac{3}{2(3+p)} = \frac{1}{2} - \frac{i}{12} \), from which

\[ p = \frac{3i}{6-i} = \frac{1}{2} i. \]
From this we obtain

$$\delta = \frac{i (6 + i)}{6 + 4i}$$

(3)

a useful approximation to the force of interest.

8. Values of $\delta$ calculated from Formula (3) are compared below with the true values:

<table>
<thead>
<tr>
<th>$i$</th>
<th>Value of $\delta$ from Formula (3)</th>
<th>True value of $\delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>.005</td>
<td>.00498754</td>
<td>.00498754</td>
</tr>
<tr>
<td>.01</td>
<td>.00995033</td>
<td>.00995033</td>
</tr>
<tr>
<td>.015</td>
<td>.01488861</td>
<td>.01488861</td>
</tr>
<tr>
<td>.03</td>
<td>.02955882</td>
<td>.02955880</td>
</tr>
<tr>
<td>.05</td>
<td>.04879032</td>
<td>.04879016</td>
</tr>
<tr>
<td>.085</td>
<td>.08158123</td>
<td>.08157999</td>
</tr>
<tr>
<td>.125</td>
<td>.11778846</td>
<td>.11778304</td>
</tr>
</tbody>
</table>

It may be open to argument whether it is worthwhile to use this formula in preference to the usual method from the expansion

$$\delta = 1 - \frac{i^2}{2} + \frac{i^3}{3} - \ldots$$

but in the examples shown the answer is correct to within a small fraction of 1d. per cent, or the same as would be obtained by taking the usual formula as far as the term in $i^4$.

**Loan repayable by $n$ equal instalments and a lump sum on the due date of the final instalment**

9. It is sometimes required to estimate the rate of interest when only a portion of the loan is repayable by level instalments, the balance being payable on the same date as the final instalment. The solution to this problem consists in estimating the value of $i$ in the basic equation

$$A = \frac{I + C}{n} a_m + (A - C) v^n$$

where $A$ is the amount of the loan, $(A - C)$ the lump sum falling due at the end of $n$ equal periods and $I$ the total interest included in $n$ equal instalments.

10. In the past, several methods have been suggested of estimating the yield on debentures purchased at a premium or discount, and this
problem is a special case of the former one with the particular circumstance of \( C \) being negative when the security is bought at a discount. The following formula has been suggested in the Compound Interest and Annuities Certain Text Book for this purpose

\[
i_1 = \frac{g - \frac{k}{n}}{1 + \frac{n+1}{2n}k}
\]

where \( k \) (when positive) is the premium and \( g \) the dividend per unit of redemption value. In the terminology of the basic equation above,

\[
k = \frac{C}{A-C} \quad \text{and} \quad g = \frac{I+C}{n(A-C)}
\]

and if we put

\[
p = \frac{I}{C} \left( = \frac{ng}{k} - 1 \right) \quad \text{and} \quad r = \frac{A}{C} \left( = \frac{1}{k} + 1 \right)
\]

then we obtain

\[
i_1 = \frac{2p}{n(2r-1)+1}, \quad \text{or} \quad i_1 = \frac{2p}{a} \quad \text{where} \quad a = n(2r-1)+1
\]

This formula which gives a positive error for debentures purchased at a premium and a negative error for those purchased at a discount, does not give as good results in its application to the more general case with which we are here concerned as in the special case of a redeemable debenture. However, the results are much improved if we multiply by the factor

\[
\frac{3a^2}{3a^2 + b} \quad \text{or} \quad \frac{3a^2 + b}{3a^2 + 2b}
\]

where \( b = (n^2 - 1)p \) (see Appendix 3) giving

\[
i = \frac{2p}{a} \cdot \frac{3a^2}{3a^2 + b} \quad (4a)
\]

or

\[
i = \frac{2p}{a} \cdot \frac{3a^2 + b}{3a^2 + 2b} \quad (4b)
\]
Simple Algebraic Formulae for Estimating the Rate of Interest 303

Formula (4a) of course can be expressed as

\[ i = \frac{6ap}{3a^2 + b} \]

which makes calculation slightly easier.

The same adjustments can, of course, be used to improve the results in the special case of a redeemable debenture.

Actually the second adjustment is a refinement of the first, and usually gives somewhat better results. However, the second adjustment slightly worsens the result when used in the case of debentures purchased at a discount.

11. In *J.I.A.* 60, 341 etc., J. I. Craig developed a method of improving the results in the case of a redeemable debenture by entering in the following equation, the value of \( i_1 \) derived from the expression

\[ i_1 = \frac{g - \frac{k}{n}}{1 + \frac{n+1}{2n} \cdot k} \]

\[ i_2 = i_1 - \frac{nki_1^2}{12 \left(1 + \frac{n+1}{2n} \cdot k\right)} \left(1 - \frac{i_1}{2}\right) \]

A further method suggested by the Editors of the same *Journal* was derived from the expansion of \( \delta n \) expressed in terms of the force of interest as used in Appendix 2 to this note. The relation suggested was

\[ i_3 = i_1 - \frac{5\beta(1-\beta)}{1 + \frac{n+1}{2n} \cdot k} \left(1 - \frac{i_1}{2}\right) \frac{k}{n}, \quad \text{where} \quad \beta = \frac{n^2i_1^2}{60}. \]

From the above equations \( i_1 - i_3 = (i_1 - i_2)(1-\beta) \) so that the deductive correction in the Editors' method may be derived by multiplying the deductive correction in Craig's method by \( 1-\beta \).

Examples of the results obtained by the formulae referred to in this section are given in Table 2. For each value of \( n \) the last three cases shown may be regarded as examples of redeemable debentures, the value of \( C \) in the last two cases being negative.

Table 2 shows that formula (4b) gives better results than does (4a) while \( C \) is positive, and (4a) better than (4b) when \( C \) is negative. Formulae (4a) and (4b) both give generally better results than do the other formulae in the general case considered under the heading of this section where \( A \) is substantially greater than \( A-C \). It is recognized of course that
Craig and the Editors did not envisage use of their methods under those circumstances.

Table 2

<table>
<thead>
<tr>
<th>Amount of loan</th>
<th>Level instalment</th>
<th>Number of level instalments</th>
<th>Lump sum repayment at end of term</th>
<th>Approximation to $i = 0.05$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$I + C$</td>
<td>$n$</td>
<td>$A - C$</td>
<td>Craig</td>
</tr>
<tr>
<td>600</td>
<td>120·4874</td>
<td>5</td>
<td>100</td>
<td>-0.051220</td>
</tr>
<tr>
<td>300</td>
<td>51·19496</td>
<td>5</td>
<td>100</td>
<td>-0.050886</td>
</tr>
<tr>
<td>200</td>
<td>28·09748</td>
<td>5</td>
<td>100</td>
<td>-0.050609</td>
</tr>
<tr>
<td>150</td>
<td>16·54874</td>
<td>5</td>
<td>100</td>
<td>-0.050375</td>
</tr>
<tr>
<td>120</td>
<td>9·619496</td>
<td>5</td>
<td>100</td>
<td>-0.050174</td>
</tr>
<tr>
<td>90</td>
<td>2·690252</td>
<td>5</td>
<td>100</td>
<td>-0.049896</td>
</tr>
<tr>
<td>80</td>
<td>3·3805040</td>
<td>5</td>
<td>100</td>
<td>-0.049779</td>
</tr>
<tr>
<td>600</td>
<td>69·75229</td>
<td>10</td>
<td>100</td>
<td>-0.052673</td>
</tr>
<tr>
<td>300</td>
<td>30·90092</td>
<td>10</td>
<td>100</td>
<td>-0.051909</td>
</tr>
<tr>
<td>200</td>
<td>17·95046</td>
<td>10</td>
<td>100</td>
<td>-0.051293</td>
</tr>
<tr>
<td>150</td>
<td>11·47523</td>
<td>10</td>
<td>100</td>
<td>-0.050786</td>
</tr>
<tr>
<td>120</td>
<td>7·59009</td>
<td>10</td>
<td>100</td>
<td>-0.050361</td>
</tr>
<tr>
<td>90</td>
<td>3·70495</td>
<td>10</td>
<td>100</td>
<td>-0.049788</td>
</tr>
<tr>
<td>80</td>
<td>2·40991</td>
<td>10</td>
<td>100</td>
<td>-0.049550</td>
</tr>
<tr>
<td>600</td>
<td>45·12130</td>
<td>20</td>
<td>100</td>
<td>-0.055507</td>
</tr>
<tr>
<td>300</td>
<td>21·04852</td>
<td>20</td>
<td>100</td>
<td>-0.053895</td>
</tr>
<tr>
<td>200</td>
<td>13·02426</td>
<td>20</td>
<td>100</td>
<td>-0.052618</td>
</tr>
<tr>
<td>150</td>
<td>9·012129</td>
<td>20</td>
<td>100</td>
<td>-0.051581</td>
</tr>
<tr>
<td>120</td>
<td>6·604852</td>
<td>20</td>
<td>100</td>
<td>-0.050723</td>
</tr>
<tr>
<td>90</td>
<td>4·197574</td>
<td>20</td>
<td>100</td>
<td>-0.049579</td>
</tr>
<tr>
<td>80</td>
<td>3·395148</td>
<td>20</td>
<td>100</td>
<td>-0.049108</td>
</tr>
</tbody>
</table>

PRACTICAL APPLICATIONS

12. It is expected that various practical applications of the formulae herein will be found by those engaged in financial calculations. In Appendix 4 an approximate formula for $a_2$ in terms of $i$ and $r$ is obtained from Formula (1). As explained in the Appendix it gives its best results for low values of $ir$ and would be used only with regard to the magnitude of the error which would be acceptable.

13. Many commercial loan contracts are entered into without the true rate of interest being known. This occurs when loans are made at what are known as ‘flat’ rates of interest. The first schedule of the Money-lenders and Infants Loans Act, 1941–1961 of N.S.W., referred to in § 3, sets out a method of arriving at the rate of interest which can give quite misleading results particularly for long-term loans. A further difficulty arises when
it is required to know the principal outstanding either for the purpose of discharging the loan prior to the agreed term or for the annual accounts.

14. The solution of this latter problem requires the evaluation of \( \frac{a_{r}}{a_{m}} \), being the unexpired term of an \( n \)-period loan. \( a_{m} = \frac{n}{1 + p} \) will be known and the approximate rate of interest \( i \) may be obtained from Formula (1).

An alternative method to the approximate evaluation of \( \frac{a_{r}}{a_{m}} \) from the estimate of \( a_{r} \) as developed in Appendix 4 is that of substituting for \( r, n \) and the approximate value of \( i \) in the expansion of \( \frac{a_{r}}{a_{m}} \), as follows:

\[
\frac{a_{r}}{a_{m}} = \frac{r}{n} \left\{ \frac{1 + n - r}{2} + \frac{n - r}{12} i^2 - \frac{n - r}{24} i^3 \right\} 
\]

It has been suggested to the author that the expansion of

\[
\frac{r}{n \left( n - a_{m} \right)} 
\]

viz.

\[
\frac{r \left( r + 1 \right)}{n \left( n + 1 \right)} \left\{ 1 + \frac{n - r}{3} i + \frac{(n - r)(n - 3r - 7)}{36} i^2 \right\} 
\]

is more rapidly convergent than the above expansion and as

\[
\frac{a_{r}}{a_{m}} = \frac{r}{n} \left[ 1 + p \left\{ 1 - \frac{n - \frac{r - a_{m}}{r}}{n - a_{m}} \right\} \right] 
\]

we have

\[
\frac{a_{r}}{a_{m}} = \frac{r}{n} \left[ 1 + p \left\{ 1 - \frac{r + 1}{n + 1} \left( 1 + \frac{n - r}{3} i + \frac{(n - r)(n - 3r - 7)}{36} i^2 \right) \right\} \right] 
\]

This expansion as far as terms in \( i^2 \) usually does give better results than the former one as far as terms in \( i^3 \).

**APPENDIX 1**

Formula (1), \( i = \frac{2p(3 + p)}{2np + 3(n + 1)} \), where \( p = \frac{n - a_{m}}{a_{m}} \).
Simple Algebraic Formulae for Estimating the Rate of Interest

was originally arrived at by experimenting with the expansion

\[ p = \frac{n}{a_{ni}} - 1 = \frac{n+1}{2} i + \frac{n^2-1}{12} i^2 - \frac{n^2-1}{24} i^3 + \ldots \]

using terms as far as those containing \(i^3\).

However, it can be arrived at by neglecting powers of \(i\) beyond the second, as follows, a negative error being introduced which has the effect of offsetting the error brought about by neglecting the term in \(i^3\).

As a first approximation, ignore second and greater powers of \(i\), giving

\[ i = \frac{2p}{n+1} \]

Substitute this value of \(i\) in the expansion using the term in \(i^2\), giving

\[ p = \frac{n+1}{2} i + \frac{n^2-1}{12} \cdot \frac{2p}{n+1} i \]

which gives

\[ i = \frac{6p}{3(n+1) + (n-1)p} \]

Since \(p\) is usually small compared with \(n\), we may put \((n+1)p\) for \((n-1)p\) without great loss of accuracy and obtain

\[ i = \frac{6p}{(n+1)(3+p)} \]

and substitute this value in the expansion as follows

\[ p = \frac{n+1}{2} i + \frac{n^2-1}{12} \cdot \frac{6p}{(n+1)(3+p)} i \]

giving

\[ i = \frac{2p(3+p)}{2np+3(n+1)} \]

APPENDIX 2

Formula (2) for the force of interest \(\delta\) was developed as follows:
Making use of the rapid convergence of the series.

\[ \frac{x}{e^x - 1} = 1 - \frac{1}{2} x + \frac{1}{12} x^2 - \frac{1}{720} x^4 \ldots \]

we get

\[ \frac{n}{a_{ni}} = \frac{-n\delta}{e^{-n\delta} - 1} \cdot \frac{e^\delta - 1}{\delta} \]
Simple Algebraic Formulae for Estimating the Rate of Interest

\[ p = \frac{n}{a_m - 1} = \frac{\frac{1}{3} (n + 1) \delta + \frac{1}{3} (n^2 - 1) \delta^2}{1 - \frac{1}{3} \delta + \frac{1}{3} \delta^2}, \]

whence

\[ p \left(1 - \frac{1}{3} \delta + \frac{1}{3} \delta^2\right) = \frac{1}{3} (n + 1) \delta + \frac{1}{3} (n^2 - 1) \delta^2; \]

and

\[ p = \frac{1}{3} (n + 1 + p) \delta + \frac{1}{3} (n^2 - 1 - p) \delta^2.\]

Neglecting the term in \( \delta^2 \),

\[ \delta = \frac{2p}{n + 1 + p}, \]

and substituting for \( \delta \) in the term in \( \delta^2 \), we get

\[ p = \frac{1}{3} (n + 1 + p) \delta + \frac{(n^2 - 1 - p) p}{6(n + 1 + p)} \delta \]

and if we take as an approximation

\[ n^2 - 1 - p = (n + 1 + p)(n - 1 - p) \]

we have

\[ p = \frac{1}{3} (n + 1 + p) \delta + \frac{1}{3} (n - 1 - p) p \delta \]

\[ = \frac{\delta}{6} (n + 1 + p)(3 + p), \]

using similar methods to those in Appendix 1, and

\[ \delta = \frac{6p}{(n + 1 + p)(3 + p)}, \]

which is now substituted for \( \delta \) in the term in \( \delta^2 \).

We thus obtain

\[ p = \frac{1}{3} (n + 1 + p) \delta + \frac{\frac{1}{3} (n - 1 - p) p}{3 + p} \cdot \delta \]

from which we obtain Formula (2) in § 6, namely,

\[ \delta = \frac{2p(3 + p)}{2np + 3(n + 1 + p)}\]
APPENDIX 3

To estimate the value of \( i \) in the equation

\[
\frac{I + C}{n} a_m + (A - C) v^n = A
\]

in § 9, and to obtain Formulae (4a) and (4b) of § 10, we put

\[
p = \frac{I}{C} \quad \text{and} \quad r = \frac{A}{C}
\]

and the equation becomes

\[
1 + p - ni (r - 1) = \frac{n}{a_m}
\]

Expanding \( \frac{n}{a_m} \) we obtain

\[
p - ni (r - 1) = \frac{n+1}{2} i + \frac{n^2-1}{12} i^2 \ldots \ldots \ldots
\]

or

\[
2p = \{2n(r-1) + (n+1)\} i + \frac{n^2-1}{6} i^2 \ldots \ldots \ldots
\]

\[
= ai + \frac{n^2-1}{6} i^2,
\]

where

\[
a = n (2r-1) + 1
\]

Thus, as a first approximation

\[
i = \frac{2p}{a}
\]

Substituting in the term in \( i^2 \) we have

\[
2p = ai + \frac{n^2-1}{6} \cdot \frac{2p}{a} \cdot i
\]

from which

\[
i = \frac{6ap}{3a^2 + (n^2-1) p}
\]

\[
= \frac{6ap}{3a^2 + b'}
\]

where

\[
b = (n^2 - 1) p
\]

which is Formula (4a).
Substituting again in the term in $i^2$ we have

$$2p = ai + \frac{n^2-1}{6} \cdot \frac{6ap}{3a^2+b} i$$

$$= a \left[ 1 + \frac{b}{3a^2+b} \right] i$$

whence

$$i = \frac{2p}{a} \cdot \frac{3a^2+b}{3a^2+2b},$$

which is Formula (4b).

**APPENDIX 4**

To find the value of an annuity for $r$ periods given the rate of interest, $i$, per period, we use Formula (1) which gives

$$\frac{2p'(3+p')}{2rp' + 3(r+1)} = i, \quad \text{where} \quad p' = \frac{r}{a_r} - 1.$$ Solving for $p'$ and discarding the negative root, we have

$$p' = \frac{1}{2} \left( \sqrt{9+6i+i^2r^2} - (3-ir) \right)$$

While the square root term may be obtained quickly by use of a calculating machine, a good approximation can be obtained otherwise because the square root is close to 3.

Thus

$$\sqrt{9+6i+i^2r^2} \approx \frac{1}{2} \left( 3 + \frac{9+6i+i^2r^2}{3} \right)$$

$$\approx 3 + i + \frac{i^2r^2}{6}$$

It follows that

$$p' \approx \frac{1}{2} \left( i + ir + \frac{i^2r^2}{6} \right)$$

and that

$$\frac{1}{a_r} \approx \frac{1}{r} \left( 1 + \frac{1}{2}i \right) + \frac{1}{2}i \left( 1 + \frac{ir}{6} \right)$$

It should be noted that this formula overstates the value of $a_r$ by about $\cdot1\%$ or $\cdot2\%$ when $ir = 1$ and by about $1\%$ when $ir = 2$. When $ir = \sqrt{12}$ the derived value of $a_r$ has a maximum value for variation in $r$, the overstatement of $a_r$ at that stage being about $5\%$. 