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CRITERIA OF SMOOTHNESS

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1. INTRODUCTION AND HISTORY

1.1 The actuarial profession has for decades, indeed centuries, used age-related tables, particularly (although not exclusively) of mortality, which in preparation have been subjected to the process known as graduation. Five purposes of such graduation have recently been set out by Vallin, Pollard and Heligman[11] and these may be summarized as follows:

(1) to smooth the data, make them easier to handle, remove irregularities and inconsistencies;
(2) to make the result more precise on the reasonable assumption that the real mortality underlying the observations is a smooth curve, i.e. to remove sampling and other errors;
(3) to aid inferences from incomplete data;
(4) to facilitate comparisons of mortality;
(5) to aid forecasting.

Of these purposes, (1), (2) and (3) may all be paraphrazed as implying that the purposes are to make the table smooth, and (4) and (5) as indicating uses to which a table may better be put if smoothed. So the purposes of what the profession has come to term 'graduation' all boil down to the production of a smooth table, and for the rest of this paper until the last section I shall use the word 'smoothing' rather than 'graduation'. But what is smooth?

1.2 The main purpose of this paper is to consider what is meant by 'smooth', or rather what has been achieved in the past by the process of producing tables which have been regarded as smooth. There is a risk, which must be mentioned early, of the arguments going round in circles. In answering the question "What is smooth?" we look at what has been done to produce smoothness, we follow certain smoothing processes, we look at various orders of differences, and we come back to the original question. Perhaps a subsidiary purpose of the paper could be stated as the breaking of this vicious circle. An earlier draft has been criticized because of the cyclical nature of the development of the arguments which is, however, inevitable; some of these criticisms will be mentioned.

1.3 The paper will be particularly concerned with mortality tables, and it must be remembered that mortality observed over a number of years of experience depends upon many different factors. Mortality has been observed to change with the passing of time, but year passed through is not the only side of the environmental die. Year of birth has some bearing on the subject, apart from being the year passed through minus the age. And so do all previous years passed through by the lives being observed. The mortality of a cohort born at a certain time has rarely been traced throughout the lives of the members of the cohort as, e.g., by Giles and Wilkie,[8] and by the time it has been so traced the results have been of little more than academic interest to actuaries, although they have been of medical interest. But apart from age and time variations, mortality also depends upon changes in the different causes of death, the effects of epidemics and changes in climate, occupation, various types of selection, and (dare we say it) sex and possibly race. So we must not lose sight of the fact that mortality is a somewhat complex function which does not necessarily have to follow a simple mathematical formula. This leads to the further observation that there is likely always to be some conflict between smoothness and goodness of fit, and sometimes the one has to be sacrificed for the benefit of the other. However, the paper is not primarily concerned with fidelity to data, which has been amply covered in the different text books which have been compiled over the years, and no further mention will be made of it apart from stating that fidelity can prevent the attainment of absolute smoothness (whatever that may be).

1.4 This paper could never have been conceived had it not been for the earlier work by Bizley[4] and the discussion thereon. But the history needs to go back further, to the preparation of my earlier work[2] in which I started as a puppet, with the late Wilfred Perks and Ronald Barley pulling the strings. Their original suggestion was that a synthesis of tests was required, and my initial work concentrated entirely on tests of adherence to data. It was then that Barley wrote the letter from which I quoted in the discussion on Bizley's paper, the quotation ending ". . . there is a lot more to be said about smoothness." Until then I had given smoothness too little thought, but in due course the vague definition was produced that "a series is smooth if it displays a tendency to follow a course similar to that of a simple mathematical function", the word 'simple' being used with the intention of cutting out polynomials of a very high degree. On reflection, the definition needs amplification by stating that the simpler the mathematical function, the smoother would the series be regarded, so that a series tending to follow the course of a function with n parameters would be taken as smoother than one following a function with $(n+1)$ parameters.

1.5 Tetley's[10] excellent textbook (but now superseded many times) is understood to have been criticized for his use of the quotation *natura non agit per saltum*, but this gives as good an appreciation as any of what actuaries understand by smoothness. The quotation may have been dropped for a time, but it certainly appears in the current textbook by Benjamin and Pollard[3] who also quote Bizley's definition that "a . . . curve is smooth at those points which are such that the absolute value of the rate of change of curvature with respect to distance measured along the curve is small." But how small is small? Benjamin and Pollard accept that Bizley's definition is equivalent to a requirement that third-order differences be small, again without specific quantification, and follow the accepted practice of concentrating on the first three orders of differences without saying exactly why. I will refer to this accepted quality as 'smoothness of the third order' or 'third-order smoothness' without assuming that this is necessarily the required criterion. What seems to be missing is a more precise definition of what is understood by smoothness in age-specific, duration-specific or time-specific series, which need not necessarily be the same as the definition which other professions might give to the quality. It is clearly desirable that when a series is being smoothed the operator keeps in mind what he aims to achieve; also that a clear definition be given to actuarial students.

1.6 Bizley quoted generously from my own paper, and it may seem ungracious to repay him with criticism. He devised a system of 'symmetrical differential coefficients' (S.D.C.'s) such that the S.D.C.'s of x with regard to y are identical with the S.D.C.'s of y with regard to x. His criterion was that the third S.D.C. should be small, the acme of smoothness being a circle or a straight line. The straight line might be an acceptable acme in theory, but to follow it in practice would mean defining the object of smoothing as to reduce the series to as near a straight line (the simplest mathematical formula) as possible, or in other words to achieve smoothness of the first order; this would usually be an unattainable perfection. The circle as another object of perfection was bound to emerge from Bizley's algebra, but the process of smoothing a time series is unlikely to be satisfactory if it reduces it to a circle, so this example of second order smoothness is also unacceptable. But surely, if a circle is ideally smooth, and so is a straight line tangential to it, a curve midway between circle and tangent must also be smooth. And if first and second order smoothness are too perfect, must we stop at the third order?

1.7 The old concept used to be that a series is smooth if its third differences are small, or if they themselves are smooth. I have already asked "How small is small?" Bizley stated correctly that if q_x is smooth then 1000 q_x must also be smooth, but the third differences of the latter are 1000 times larger than those of the former, so a better criterion (if one considers third differences at all) would seem to be the ratio between the third difference of a function and the function itself, but how small should that ratio be? If $y=2^x$, successive differences are equal to each other and each line of differences repeats the previous line. If $y=3^x$, successive differences increase and each line of differences repeats the previous line multiplied by 2. Are we then to say that these curves are not smooth? And this raises another question, how smooth is smooth? The apparent requirement that third differences should be smooth if they are not small seems to take us into sixth differences, and so on, and one is reminded of the indeterminate definition, used by the now discredited German dictatorship in the years of World War II, that a man was a Jew if his grandfather was a Jew (and, presumably, so on). Indeed, if $y=f(x)$ is smooth, are $y=e^{f(x)}$ and $y=\log f(x)$ (assuming $f(x)>0$) not also smooth? Or does the transformation make them rather less smooth in some sense? The function $y=a+bf(x)$ is surely just as smooth as $y=f(x)$, since it can be derived from the latter by a change of origin and scale of the y axis on the same graph. What then about $y=a+b\,f(x)+c\,f(x)^2$? Does this have a degree less smoothness, in some sense? If not, then we could argue that any polynomial of $f(x)$ was equally smooth, which clearly conflicts with our intuition.

1.8 The only other paper which has attempted to define smoothness was that by Elphinstone[7] who saw that the smoothing process is concerned with relations between neighbouring rates, or serial correlations. In place of the traditional acceptance of the third order of differences as the necessary medium for assessing smoothness, he devised a measure of roughness of any order, by defining roughness of the order k as the product of the exposed-to-risk and the square of the kth difference of the superimposed errors inherent in the crude rates, all divided by the factorial function ${}_{2k}C_k$. The fact that he pursued this definition no further may have been because he was not wholly satisfied with it, or may have been because his paper was primarily concerned with summation methods of smoothing, about which he had plenty to say. The definition was not even given in his synopsis. Without answering "What is smoothness?" and "How smooth is smooth?" he has asked a third question "To what order should it be measured?" His was a brave try, but I find it not wholly satisfying; so, I suspect, did Elphinstone.

1.9 To conclude this section I must refer once more to Barley. Mention has already been made of a quotation from his letter which first programmed me into a particular line of thought. The same letter described the whole process of smoothing as a matter of answering the question "When is a wave not a wave?" Unfortunately the letter no longer survives. But how much richer the profession would have been if it had had the benefit of a paper by Barley on this subject.

2. A MATHEMATICAL EXCURSION

2.1 It is tempting to see whether we can draw on the ideas of mathematical analysis for a more precise definition of smoothness. But books such as Apostol[1] do not use the term. Instead we find concepts of continuity and differentiability. What can we make of these?

2.2 Let us start by restricting ourselves to functions in Cartesian co-ordinates. This eliminates the circle, even though it is a function in Polar co-ordinates. Mortality data expresses q_x or μ_x as a function of x; we can think of μ_x which is defined for all x in a suitable compact range. We can certainly say that a function that is not continuous is not smooth. A step function, discontinuous at a finite number of points, we would not consider smooth; how much less smooth is a function that is everywhere discontinuous, such as

$$f(x) = \begin{cases} 0 \text{ if } x \text{ is rational,} \\ 1 \text{ if } x \text{ is irrational,} \end{cases}$$

even though it would look like two straight lines when drawn!

2.3 Differentiability seems to be a second requirement. A function that is not differentiable at a finite number of points would feel 'angular'; how much less smooth is a function that is nowhere differentiable, such as a realization of a

Wiener process, useful though such a function is in representing many types of continuous stochastic process.

Continuous and once differentiable functions then: but how many orders of differentiability do we need? A function with a single discontinuity in the second derivative may pass as fairly smooth. Consider

$$f(x) = \begin{cases} -x^2 \text{ if } x<0 \\ x^2 \text{ if } x \geqslant 0, \end{cases}$$

which is continuous and differentiable once everywhere, but has a discontinuity in the value of $f''(x)$ at $x=0$. We might feel there was rather a conspicuous point of inflexion, but I do not think we would reject it as not being smooth.

2.4 Most of the parametric formulae used to smooth mortality rates are such as can be differentiable any number of times, except for those that involve blending functions over some discrete range, which almost always have a discontinuity in the derivatives of some order at the joins.

2.5 Cubic splines, however, have discontinuities in their third derivative at the 'knots', though they are designed so that the first and second derivatives exist everywhere. Yet cubic splines look and feel smooth in practice. Indeed, the original 'spline' was a flexible thin metal strip that, when constrained in some way, would find a position that minimized its internal strain, and so in some physical sense got itself as smooth as it could in the circumstances. The fact that in some circumstances its comfortable position was in fact a cubic spline suggests that in some way these functions are ideally smooth, even though derivatives of all orders do not exist everywhere.

2.6 Having derivatives of all orders everywhere is in any case not a sufficient condition for smoothness. Consider the simple sinusoidal function:

$$f(x) = a \sin (bx),$$

which drawn on a suitable scale and with suitable values of a and b will look like a gently rolling smooth wave. If a is held constant, and b is increased the smooth wave form becomes more and more compressed and corrugated until the curve begins to look like a rasp, and eventually becomes a jagged oscillation. If at the same time a is reduced, the curve may keep its smooth wave form, but on a smaller and smaller scale, so that it begins to resemble a straight line, until we magnify the scale of x and $f(x)$ again and see the original smooth wave we started with.

2.7 This suggests that smoothness is a matter of scale. A well-polished piece of wood feels smooth to the touch. Through a magnifying glass we can see how serrated its surface may still be, on a different scale. A well-made road surface may give a smooth ride even to an ill-sprung car, but will feel very rough to the fingers and will hardly provide a smooth surface for a child's toy car.

2.8 If smoothness is then a matter of scale, are we not forced back to consider changes in the function over some discrete step size, compatible with the size of the fingers, or the car wheels? In effect, look at differences instead of differentials, as indeed actuaries have traditionally done. The natural step size for mortality rates is a year. We know that within a calendar year the numbers of deaths may fluctuate irregularly because of chance, weather, accidents or epidemics; and we usually ignore this for actuarial purposes. Even if the same fluctuations could be shown to occur within years of age, we should still ignore them (except perhaps if we were dealing with events such as retirement, recruitment, etc. which may occur by definition at fixed ages).

2.9 I shall therefore restrict myself for the rest of this paper to considering mortality tables, where q_x is quoted at annual intervals. The only area where this is not satisfactory is just at and after birth, where we know that the incidence of deaths is not at all even over the first few days, weeks, and months of life.

3. CRUDE DATA: THE NEED FOR SMOOTHNESS

3.1 It is arguable whether a paper of this nature should first consider smoothness and then apply whatever criteria are deduced to whatever data are to be smoothed, or have been smoothed; or whether it should first consider the crude data and why these are not deemed already to be sufficiently smooth. Before we can do the latter we come back to the question posed in the last three words of § 1.1 and we risk going round in circles before we are off the ground. Inevitably the paper started with an historical section which could not avoid mentioning smoothness, so it seems appropriate now to look at the crude data thrown up by observations.

3.2 The paper will be concerned in particular with mortality data, which are the data most frequently observed by actuaries, although there is no reason why similar considerations should not apply to sickness inceptions, withdrawals, marriages, obsolescence of machinery, and so on. The crude data derived from observations will usually consist of exposed-to-risk or populations, and actual deaths (or whatever event is being observed) and from these may be derived crude values of, for example, q_x or colog p_x.

3.3 These crude rates will tend to progress with greater or less regularity from age to age but such progression may be uneven, and may not always be in the same direction. Table 1 shows over a range of 40 ages the crude mortality rates at durations 2 and over underlying the A 1967–70 ultimate table, together with columns showing the first three orders of differences of these rates, the figures having been derived from the exposed to risk and actual deaths given in *J.I.A.* **101,** 160–2. The rates themselves show a tendency to decrease through most of the twenties of age and to increase thereafter, but these progressions are somewhat irregular. Considering the "relations between neighbouring rates" (see § 1.8), an examination of the first differences shows that they change sign 9 times, while the signs of both second and third differences show more sign changes than non-changes. It is clear that however many columns of successive differences are calculated, the series will not become regular until, perhaps,

Table 1. *Assured lives 1967–70, durations 2 and over: crude values of rates of mortality at ages $20\frac{1}{2}$ to $59\frac{1}{2}$ inclusive, and their first three orders of differences*

Age x	$10^6 q_x$	$10^6 \Delta q_x$	$10^6 \Delta^2 q_x$	$10^6 \Delta^3 q_x$
$20\frac{1}{2}$	1,087	−237	155	−98
$21\frac{1}{2}$	850	−82	57	−100
$22\frac{1}{2}$	768	−25	−43	150
$23\frac{1}{2}$	743	−68	107	−282
$24\frac{1}{2}$	675	39	−175	344
$25\frac{1}{2}$	714	−136	169	−158
$26\frac{1}{2}$	578	33	11	−59
$27\frac{1}{2}$	611	44	−48	63
$28\frac{1}{2}$	655	−4	15	−12
$29\frac{1}{2}$	651	11	3	−9
$30\frac{1}{2}$	662	14	−6	43
$31\frac{1}{2}$	676	8	37	−12
$32\frac{1}{2}$	684	45	25	97
$33\frac{1}{2}$	729	70	122	−351
$34\frac{1}{2}$	799	192	−229	502
$35\frac{1}{2}$	991	−37	273	−458
$36\frac{1}{2}$	954	236	−185	50
$37\frac{1}{2}$	1,190	51	−135	702
$38\frac{1}{2}$	1,241	−84	567	−935
$39\frac{1}{2}$	1,157	483	−368	466
$40\frac{1}{2}$	1,640	115	98	−70
$41\frac{1}{2}$	1,755	213	28	−2
$42\frac{1}{2}$	1,968	241	26	75
$43\frac{1}{2}$	2,209	267	101	−155
$44\frac{1}{2}$	2,476	368	−54	46
$45\frac{1}{2}$	2,844	314	−8	488
$46\frac{1}{2}$	3,158	306	480	−948
$47\frac{1}{2}$	3,464	786	−468	762
$48\frac{1}{2}$	4,250	318	294	−450
$49\frac{1}{2}$	4,568	612	−156	496
$50\frac{1}{2}$	5,180	456	340	−480
$51\frac{1}{2}$	5,636	796	−140	−34
$52\frac{1}{2}$	6,432	656	−174	1,080
$53\frac{1}{2}$	7,088	482	906	−1,488
$54\frac{1}{2}$	7,570	1,388	−582	1,187
$55\frac{1}{2}$	8,958	806	605	−1,274
$56\frac{1}{2}$	9,764	1,411	−669	1,942
$57\frac{1}{2}$	11,175	742	1,273	
$58\frac{1}{2}$	11,917	2,015		
$59\frac{1}{2}$	13,932			

thirty-seventh differences, when the column over this range of ages will have only three entries.

3.4 So, the crude series is seen to be not very regular. Irregularities can arise (a) from errors in the data, (b) from the fact that the underlying mortality rates really are irregular, (c) from the fact that however large the observations, they only represent a sample from a hypothetical universe, or (d) from a preponderance of 'duplicates', if numbers of policies rather than numbers of persons have been counted in the exposed-to-risk or the deaths. Let us examine these in turn.

3.5 It is always hoped that errors in compiling the data will have been eliminated or reduced to insignificant proportions before the crude rates are calculated, otherwise the whole exercise is a waste of time. Where there are errors resulting from mis-statements of age, as in population censuses or national statistics derived from death certificates, their effects have usually, to some extent, been minimized by suitable grouping of the data into age-groups. It must be assumed that whenever errors of significant dimensions are present they will be corrected or, if this is not possible, the data will be discarded.

3.6 If it is believed that for some reason the underlying probabilities really do have irregularities (rather than progressing regularly with age) it is necessary to decide whether for reasons of convenience it is desirable to smooth them out, or whether some irregularities should be retained in an otherwise smoothed series. It might be found inconvenient if an uneven series of probabilities resulted in life assurance premium rates which progressed unevenly or which decreased with a rise in age, either of which might occur if the rates of mortality employed were irregular. On the other hand, if it is thought that the true rates really do decrease with age this could be a feature which it would be wrong or even dangerous to ignore, particularly in the calculation of term-assurance premium rates.

3.7 If the observed and limited data give uneven mortality rates and it is felt that mortality, being a mainly natural process, should not be expected to progress in jerks from age to age, this feeling can only be justified if it is thought that a much larger set of observations would give a more regular progression of probabilities; this gives rise to the admittedly arguable concept of an underlying smooth series which would have been observed if the data had been infinitely large, and to assess this underlying series it is necessary to try to eliminate the random errors which have arisen through the limitation of the observations. This concept has been criticized, and the critics presumably also disagree with § 11.55 of Benjamin and Pollard.

3.8 Duplicates cause wider deviations from this universe than would arise just from sampling errors, and their effects also need to be ironed out if it is desired to estimate the underlying rates.

3.9 The three preceding paragraphs indicate the reasons why it is desirable to smooth the observed data, and these may be summarized as (1) convenience and (2) the need to estimate the true underlying probabilities. To some extent these could conflict. Where rates of mortality do fall with age, as in the case of deaths in the twenties of age when greatly affected by accidents, it is necessary to balance the convenience of rates which do not decrease against the knowledge that the decrease is really there. There could be other respects in which local roughnesses inherent in the data may have to be retained in an otherwise smooth series. Whether such roughnesses are in fact retained depends upon the reliability of the data, upon the purposes to which the series is to be put, and to some extent upon the personal opinion or choice of the smoother having regard to all the relevant circumstances.

3.10 It has already been implied that this paper is concerned mainly with the smoothness of probabilities, which have by definition to lie between zero (impossibility) and unity (certainty), and §3.2 referred to other types of probability besides mortality. Reference will also be made briefly later to diverging functions. The view has been expressed to me that smoothing should be confined to probabilities or similar functions, such as q_x, colog p_x, q_x/p_x or μ_x. But as will be seen later, at times the function upon which the operation has been carried out has been l_x. And it will also be seen that the resulting smoothness can depend upon the function chosen.

4. METHODS OF SMOOTHING

4.1 Mr E. A. Johnston wrote to me, in a private letter, that the methods (of smoothing) now in use seem to be more automatic and less sophisticated than in the days of Starke and Perks. This paper is not intended to be primarily about the different methods, but a later section will examine a number of mortality tables to see what the profession has accepted in the past as being sufficiently smooth, and thus to deduce what criteria have been applied, albeit subconsciously; and it will be necessary to recall by which methods they were smoothed. Accordingly the different possible methods will now be catalogued with particular reference to their possible smoothing properties, but it will be assumed that readers either already have a working knowledge, or can have recourse to Benjamin and Pollard's textbook which covers most of them; also that, automatic or not, sophisticated or unsophisticated, the advantages and disadvantages of the various methods are understood.

4.2 The graphic method consists of drawing a graph through or near the crude values of the function to be smoothed. Smoothness has been alleged to be achieved or improved by hand-polishing the values to make the third differences as small as possible. This means that the process is, in effect, attempting to fit a series of third difference curves, i.e. to fit overall what I choose to call a poly-polynomial. This method is not usually used for standard tables, but for the purpose of comparing with the examples to be given in a later section I have used the table of smoothed l_x's produced by Lambert.[9] A description of Lambert's work was given by Daw[6] and I am grateful to him for making Lambert's figures available to me.

4.3 The summation method consists of applying a finite difference formula involving successive summations to the crude values. The accepted 'smoothing coefficient' or 'smoothing index' is related to third differences, and it follows that the more powerful the smoothing index of the formula chosen, the closer the process comes to the fitting of a poly-polynomial. My example of a table smoothed by this method will be the A 1924–29 ultimate table.

4.4 The kernel method recently described by Copas and Haberman[5] is an elegant variation of the summation method, which avoids the necessity of filling in the smoothed values at the two ends of the table. The figures published in their paper give rates of mortality to four places of decimals only, and unfortunately they have been unable to supply figures to further places for the purposes of my examples.

4.5 The method of osculatory interpolation, which aims at achieving identical third differences at points where different sections of the smoothed function (or do I mean the smoothed curve?) meet one another, is also a finite difference method which fits a poly-polynomial. Such a method, producing a series passing through (say) 22 pivotal points calculated from the crude values, and linking 7 third difference curves at 6 points of intersection, is no doubt substantially smoother in its results than a 21st difference curve passing through the same pivotal values. The English Life Tables No. 10 will provide the figures for the examples.

4.6 The spline method is a variation on the method of osculatory interpolation, but third differential coefficients take the place of third differences. Again the method is tantamount to the fitting of a poly-polynomial, and again it is an improvement on the fitting of a single formula with many parameters. Figures for the examples are taken from the English Life Tables No. 13.

4.7 The curve fitting or parametric method, or the fitting of a formula, has sometimes been said to give ideally smooth results, but whether it does must surely depend upon which formula or curve has been fitted. Various standard tables are available for examples and I have examined FA 1975–78, A 1967–70, A 1949–52, H^M, and $a(90)$, using the aggregate table for H^M and the ultimate tables for the others, as well as PA(90) and English Life Tables No. 12. These cover a wide variety of different formulae. The $a(m)$ and $a(f)$ tables have also been examined to show the effects of blending two curves.

4.8 For the sake of completeness it is necessary to mention smoothing by reference to a standard table, which usually consists of adjusting the standard table values by the application to them of a simple formula. Clearly the degree of smoothness must depend very largely upon the smoothness of the standard table used, and for the purpose of this paper no example is included.

4.9 It will be seen from the preceding paragraphs of this section that every method of smoothing is a device or an attempt, to fit or nearly to fit, a curve or series of curves, each of which follows a mathematical formula. This will no doubt be regarded as a controversial statement, particularly by actuaries who advocate the use of any of the above methods other than curve fitting. But it is submitted that it represents the facts, and helps in determining just what it is that actuaries understand by smoothness. Perhaps the statement is merely repeating, at greater length, what I tried to say in 1950.

5. CONCEPTS OF SMOOTHNESS

5.1 Traditionally, the profession has considered smoothness to be in some way related to third differences, although consideration of divergent exponential functions makes it difficult to devise a suitable measure, and no reason has been given for the particular importance of this order of differences rather than, say, fifth or sixth differences. The fact that actuaries are usually concerned with functions which do not diverge or, if they do, diverge only slowly, has no doubt had some effect on the thinking. Before starting to write this paper, I inclined to the view that in a smooth curve the second differences should not change sign more than once unless the series had an inherent inflexion which was an essential feature needing to be retained; and that third, or some other order of, differences should be small when expressed as a ratio of the function itself, just how small needing to be determined. On re-reading Bizley it will be found that this was virtually what L. V. Martin said in the discussion, apart from the fact that he was referring to differentials rather than differences, and the discussion was not considering a higher order than the third. It is perhaps a tribute to Martin that his comments could affect one subliminally. The Appendix gives an elementary note on second differences.

5.2 Let it be considered at this stage whether it is necessary to go even as far as third differences. Is it sufficient if second differences pass through zero as infrequently as possible, once for the curve itself (and, indeed, every straight line passes through zero once unless it is parallel with the axis) and twice for each inherent inflexion? The answer to this must be in the negative, as any number of sequences of second differences could be found to meet this criterion. So it has to be necessary to add that third differences should be as small as possible, and if this should prove to be not very small, then differences of some higher order should be small.

5.3 The use of higher orders is implicit in the old suggestion that third differences should be either small or smooth. If they are themselves smooth this presupposes that a higher order becomes small, and this in turn leads to the concept that smoothness is not something one can measure by size, but something which can be expressed in terms of the order of differences at which they become insignificant. It has to be remembered that a difference is the result of adding and subtracting different values of the main function, after applying various weights; thus, the third difference is made up of four values with the weights $-1, 3, -3, 1$ and the fifth difference of six values with the weights $-1, 5, -10, 10, -5, 1$. The differences can only be expressed to as many decimal places as are shown for the main function, and the error caused by the omission of the next decimal place becomes multiplied up as more differences are calculated. By the time one gets to third, fourth or fifth differences at least one decimal place has to be discarded if one is not to retain an unreliable figure.

5.4 Diverging series do not arise when probabilities are being examined, but they should be given brief consideration. Perhaps a criterion of smoothness should be applied to the reciprocal of a function when the value of the function exceeds unity, and to the function itself when less than unity; the slope of the function is of course the negative of the slope of its reciprocal at the point where the function has a value of exactly unity. Alternatively it might be said that a satisfactory degree of smoothness is obtained so long as successive differences tend to a curve no less simple than that of the function itself.

5.5 In § 3.10 it was seen that smoothness could be affected by the function upon which it is decided to operate. And it follows from § 5.3 that smoothness can also depend upon the scale used, and some smoothness may be lost if insufficient significant figures are retained in the smoothed table.

5.6 If all methods of smoothing are tantamount to fitting a curve or a series of curves, and thus ironing out as many irregularities or roughnesses as possible, the concept seems to be materializing that smoothness implies the following, or the tendency to follow, some rule or law. Indeed Bizley's concept itself seems to assume that the function becomes differentiable after being smoothed, and Elphinstone's concept of serial correlations implies some regularity or continuity.

5.7 Mention was made in § 5.2 of inherent inflexions. One of these frequently occurs in mortality tables around the age at which motor accidents suddenly reach important dimensions, i.e. age 17 in the United Kingdom, particularly in the case of male mortality; in this instance a non-natural set of occurrences causes a sudden *saltum* followed by a decrease in mortality over a number of years of age. Another, but quite different, disturbance occurs just after birth, causing the plunging semi-neckline in the mortality curve which defies differences to become small in the first few years of life. An inflexion could occur around retirement age if a table were constructed relating to employees and pensioners in a certain industry, and similar disturbances might be caused by different types of retirement, but these do not in fact apply to the PA(90) table which was based only on the mortality of lives who had retired at or after the normal retiring age, with mortality at lower ages filled in by a blend with assured lives' mortality. There could be a disturbance in assured lives' mortality tables around the more common maturity ages where the size of the data may have dropped rapidly with age. And in any female table there could be irregularities in the progressions with age as a result of childbearing deaths, or around the menopausal ages, but as each such effect would be averaged over a number of ages it would probably not appear as a *saltus*. There could possibly be roughnesses in the mortality of medically examined assured lives if these include cases where there was originally a non-medical proposal, but where as a result of some of the answers the underwriters requested an examination; the non-medical cases clearly accepted without examination would be excluded from the medical data, which would thus suffer a degree of reverse selection over and above the selection by examination, but this effect would suddenly be eliminated if non-medical proposals are only accepted up to a certain age.

5.8 To sum up,

(a) A series is smoothed to the greatest degree possible if it is fitted to the simplest possible function consistent with statistical adherence to the crude data, the word 'simplest' implying that the function has the smallest possible number of parameters. (The analogy to Occam's razor is evident.)

(b) Second differences should pass through zero no oftener than once for the curve and twice for each acceptable inherent inflexion or roughness; this automatically excludes sinusoidal curves from acceptable smoothness.

(c) There is no absolute measure of smoothness, but a series may be regarded as satisfactorily smoothed if successive differences follow a curve no less simple than the series itself.

(d) As an alternative to (c), a series may be regarded as smooth to the k'th order if k'th differences are insignificant bearing in mind the decimal place at which the figures in the series itself have been cut off.

(e) It follows from (d) that to some extent smoothness depends upon the scale used.

(f) Smoothness can also depend upon the function upon which it is decided to operate.

5.9 The next section will examine a number of standard tables of mortality to see to what extent they comply with these criteria, and thus to see whether the criteria fulfil what the profession has accepted in the past and the present. We may then begin to see how smooth is smooth, how small is small, when is a wave not a wave, and when is a blend not a blend.

6. EXAMINATION OF STANDARD TABLES

6.1 We now proceed to the examination of a number of tables, and in particular of their differences. In these examinations, it must be remembered that the last figure shown in the series to be differenced may, due to the cut-off, be anything up to ·5 out. It follows that the last figure of the calculated second differences can easily be wrong by 1, and sometimes by 2, and accordingly when looking at the sign changes in second differences any values of ± 1 should be regarded as if they were zeros. and similarly any values of ± 2 so long as they are not too frequent. When assessing the sizes of third, fourth, fifth or sixth differences it is best to discard one decimal place (assuming the main series lies between 0 and 1) and to remember that even then the last figure may be out by 1 or 2. Once a stage has been reached at which differences are insignificant there is no point in going any further as this would do no more than magnify the errors due to the cut-off, possibly making subsequent differences progressively larger. Another effect of the cut-off is that a smoothed series taken in the first instance to

Table 2. *Second differences of rates of mortality from standard tables of assured lives' mortality*

Age x	FA 1975–78 $10^6\Delta^2q_x$	A* 1967–70 $10^8\Delta^2q_x$	A 1949–52 $10^5\Delta^2q_x$	A 1924–29 $10^5\Delta^2q_x$	H^M $10^5\Delta^2q_x$	Age x	FA 1975–78 $10^6\Delta^2q_x$	A* 1967–70 $10^8\Delta^2q_x$	A 1949–52 $10^5\Delta^2q_x$	A 1924–29 $10^5\Delta^2q_x$	H^M $10^5\Delta^2q_x$
0	9	(0)			6443	55	45	10066	11	18	13
1	5	(0)			694	56	50	10984	12	19	18
2	5	(0)			354	57	54	11983	13	17	13
3	5	(1)			42	58	62	13066	14	15	18
4	2	(0)			45	59	66	14237	16	13	19
5	4	(1)			36	60	74	15506	18	15	22
6	2	(1)			37	61	83	16876	19	19	21
7	3	(0)			32	62	90	18354	21	25	24
8	1	(1)			36	63	101	19945	24	33	30
9	3	(1)			14	64	113	21658	24	41	25
10	1	(0)	0	0	16	65	124	23492	29	48	36
11	1	(3)	0	0	18	66	139	25458	30	46	32
12	2	(4)	0	0	5	67	155	27557	33	44	38
13	1	(7)	0	0	23	68	172	29792	36	41	42
14	2	(6)	0	0	5	69	195	32165	39	39	40
15	1	4538	0	−1	16	70	216	34676	42	38	52
16	1	−30395	0	−3	−3	71	244	37323	45	39	47
17	2	316	0	−3	7	72	274	40102	49	42	64
18	2	348	0	−2	−2	73	307	43005	53	47	52
19	1	380	0	−1	−10	74	349	46023	55	51	70
20	1	418	0	0	−1	75	391	49138	60	54	63
21	3	457	1	0	−10	76	445	52336	63	56	78
22	1	503	−1	0	−2	77	503	55588	67	58	83
23	3	550	0	0	−7	78	569	58868	69	58	80
24	1	605	1	0	−3	79	646	62141	73	55	101

Table 2 (*cont.*)

25	4	661	−1	0	−1	80	735	65362	75	52	86
26	2	726	1	1	2	81	832	68485	78	51	101
27	3	796	0	1	−1	82	947	71457	80	49	108
28	3	873	0	1	−1	83	1074	74211	79	47	118
29	4	957	1	2	4	84	1218	76686	82	46	127
30	3	1048	0	2	0	85	1376	78807	79	45	97
31	5	1150	1	2	2	86	1555	80494	79	45	150
32	5	1260	1	2	0	87	1746	81673	75	45	112
33	5	1382	1	2	5	88	1953	82261	73	45	180
34	6	1513	2	3	0	89	2170	82182	67	45	103
35	7	1660	1	2	2	90	2390	81363	61	45	−41
36	6	1818	3	3	2	91	2603	79736	55	45	633
37	9	1991	2	2	5	92	2797	77254	47	46	−451
38	8	2184	4	1	0	93	2951	73870	37	47	524
39	9	2390	3	1	5	94	3048	69568	28	48	−143
40	11	2620	3	1	6	95	3057	64349	18	50	−2166
41	12	2869	5	1	−2	96	2951	58229	7	52	3221
42	12	3142	5	2	9	97	2709	51262	−3	54	−1058
43	15	3440	5	3	2	98	2310	43517			16521
44	15	3768	4	4	5	99	1745	35088			5555
45	17	4124	6	5	7	100	1028	26095			
46	19	4513	6	6	6	101	189	16678			
47	21	4939	6	6	5	102	−720	6980			
48	23	5404	5	7	7	103	−1632	−2827			
49	25	5912	7	7	11	104	−2479	−12585			
50	27	6464	7	8	6	105	−3189	−22129			
51	32	7068	8	9	10	106	−3714	−31300			
52	33	7724	7	10	10	107	−4031	−39959			
53	37	8440	10	12	12						
54	41	9218	9	15	11						

* In the published A 1967–70 tables, the earliest ages were only tabulated to five decimal places, with three 0's shown for tidiness with the rest of the table. In the second differences shown above, the values up to age 14 are shown in brackets for $10^5\Delta^2 q_x$ since the additional 0's in $10^8\Delta^2 q_x$ would be meaningless.

a small number of decimal places only, is automatically treated more leniently than one taken to, say, 7 or 8 decimal places, the differences of which will take more stages to reach the point of insignificance.

6.2 Table 2 shows the second differences of the rates of mortality taken from certain published standard tables of mortality for assured lives, using the ultimate rates. The tables operated upon are the three most recent, i.e. FA 1975–78, A 1967–70 and A 1949–52, all smoothed by different parametric methods, A 1924–29 which was smoothed by a method of summation, and H^M which was smoothed by yet another parametric method. To avoid a proliferation of figures, higher order of differences than second are not shown in the tables. Table 2a shows the second differences of the national logarithms of the rates of mortality from the same tables, and comparisons of the figures in Tables 2 and 2a will be made in Section 7.

6.3 FA 1975–78 was the first standard table for female assured lives. The data were smoothed by the fitting of a 5-parameter formula, and although there were few statistics for the youngest ages the formula was used to produce values down to age 0. For this reason the plunging semi-neckline was not reproduced. It will be seen from table 2 that the second differences have only one sign change, which occurs near the top end of the table. If $10^5\Delta^3 q_x$ is calculated (dropping the sixth decimal place) it will be found that it lies between 0 and 2 for all x up to 69, while $10^5\Delta^4 q_x$ lies within the same range for all x up to 88. Differences for the whole range from 0 to over 100 become insignificant if the process goes to the fifth order, $10^5\Delta^5 q_x$ lying wholly between -3 and $+7$. The highest value, 7, occurs at ages 100–102 where it may be compared with $10^5 q_x$ of approximately 50,000. It may certainly be said that the table has fifth order smoothness, but if we go back to third differences we shall see that the highest $10^5\Delta^3 q_x$ is only 91 (disregarding the sign) which is less than 2 per 1000 of the corresponding q_x. Can we then say that the table, in effect, has third order smoothness? I believe we can.

6.4 A 1967–70, the latest standard table for male assured lives, was smoothed by the fitting of a 4-parameter formula, and tabulated to 8 decimal places after age 16. Up to age 16 the same number of digits were printed, but after the fifth decimal they were all shown as noughts so at this part of the table where a blend with national mortality was made (but again without the plunging semi-neckline) there were virtually only 5 decimal places. Second differences show three sign changes; two of these occur between ages 15 and 17 where an otherwise smooth curve has had to be affected by the sudden rise in accidental deaths at age 17, an inherent feature which was deliberately not smoothed out of the table. The resulting roughness has to be accepted, and the fact that rates of mortality decrease from age 17 to 28 does not otherwise affect the smoothness as the formula was chosen to accommodate this feature although it could not also accommodate the *saltum* at age 17. There is only one other sign change, which is satisfactory; this occurs at the top end of the table, and may be compared with the similar feature in FA 1975–78. (The reason for these single sign changes late in life is not obvious; could it perhaps be related to the need for the rate of change to be toned down to prevent the probability ever exceeding unity?) It will be found that $10^7\Delta^5 q_x$ above age 16 is in single figures as far as age 96, the highest value of 17 occurring at age 100, which may be compared with a value of $10^7 q_x$ of over 4 million. But if, as in the case of FA 1975–78, we were to cut off at $10^5\Delta^3 q_x$ we would find the highest value above age 16 to be 10 at ages 101–104 where $10^5 q_x$ lies between 43,000 and 51,000. So over the adult range this table is even smoother than FA 1975–78, which was perhaps to have been expected from the smaller number of parameters in the formula. We are beginning to conceive smoothness as a relative quality by which tables may be compared with one another, as well as an absolute quality assessing A 1967–70 as also having third degree smoothness, or fifth degree smoothness if as many decimal places as have been published are taken into account. (For ages below 17, where only 5 decimal places effectively appear in the published series, it will be found that $10^4\Delta^3 q_x$ is virtually zero up to age 14, the high values at ages 15 and 16 being due to the inherent roughness already noted as a feature of the data.)

6.5 A 1949–52 was smoothed by a 5-parameter formula and tabulated from age 10, to 5 decimal places only, with the rate of mortality arbitrarily (as a result of the formula employed) constant from age 10 to 22. This was also a male assured lives' table, and it was decided to iron out the decreasing mortality in young adult life. Ignoring two values of -1 early in the table there was only one sign change in the second differences, again at the top end of the table. $10^4\Delta^3 q_x$ lay between -1 and $+1$ throughout, so this table also had third order smoothness; the value of -1 from age 88 onwards compares with $10^4 q_x$ lying between 2,000 and 4,000. The deliberate wave-cut already noted over the younger ages could be said to be due to a preference for smoothness and a continuously rising curve (after the range of constant rates of mortality) to the retention of what is now regarded as an inherent feature of male mortality.

6.6 A 1924–29, also a table of male assured lives' mortality, was smoothed by a summation formula with strong smoothing powers. Second differences change sign once only, on passing into the twenties of age, the feature at the upper end of the table observed in the later assured lives' tables being absent. Perhaps the parametric methods used later have introduced a feature which is not really inherent in the data, and the summation formula has given a truer representation; but this is only conjecture. $10^4\Delta^3 q_x$ (which is as far as the published decimal places for A 1924–29 allow us to go) is 0 or 1 throughout the range from 10 to 96, so this table also has third order smoothness.

6.7 The H^M table, which was a much earlier assured lives' table for males, was smoothed by another parametric formula, Makeham's formula. Details are given by Woolhouse[12] from which it will be seen that the original smoothed function derived from the process was l_x from age 28 onwards. This was expressed to integers, the resulting integral values being taken as absolute, with the result that at the highest ages only one significant figure is valid. Nevertheless, second differences of the rate of mortality have been calculated and shown in table 2; the plunging semi-neckline is seen to have been reproduced at the

Table 2a. *Second differences of natural logarithms of rates of mortality from standard tables of assured lives' mortality*

Values of $10^3\Delta^2\ln q_x$ according to:

Age x	FA 1975–78	A 1967–70	A 1949–52	A 1924–29	H^M	Age x	FA 1975–78	A 1967–70	A 1949–52	A 1924–29	H^M
0	14	−6			692	55	0	−2	−2	3	3
1	7	−5			131	56	1	0	0	4	2
2	8	−9			144	57	−1	−2	−1	−1	0
3	13	13			−8	58	1	−1	−2	−2	2
4	3	−7			−1	59	0	−1	0	−3	1
5	12	16			−2	60	0	−1	0	−2	1
6	6	20			14	61	0	−1	−1	−2	2
7	10	−2			22	62	1	−1	0	1	0
8	4	24			54	63	−1	−1	−1	2	2
9	11	27			22	64	1	−1	0	3	−1
10	5	0	0	−3	35	65	0	−1	−1	4	3
11	4	78	0	−4	51	66	0	−1	0	0	−1
12	8	83	0	−3	14	67	1	0	0	0	2
13	4	100	0	−2	65	68	0	−1	−1	−2	0
14	9	23	0	−3	9	69	1	−2	0	−3	0
15	3	−20	0	−6	32	70	0	0	0	−4	0
16	4	−321	0	−15	−16	71	1	−1	−2	−2	1
17	7	0	0	−13	5	72	0	0	0	−2	1
18	6	0	0	−8	−11	73	1	−2	0	−2	−1
19	2	1	0	−5	−23	74	1	0	−2	−3	1
20	3	3	0	0	−5	75	0	−2	0	0	−1
21	7	2	9	0	−18	76	1	0	−1	−3	0
22	1	5	−9	0	−4	77	2	−1	−1	−1	1
23	7	5	0	0	−11	78	0	−2	−1	−1	−1
24	−1	8	8	0	−5	79	1	0	−2	−3	0

Table 2a (*cont.*)

25	8	8	−8	0	−1	80	1	−1	0	−2	0
26	0	12	10	4	2	81	1	−2	−3	−2	−2
27	4	11	−2	4	−2	82	2	−1	0	−2	1
28	1	14	1	5	−1	83	0	0	−3	−3	−1
29	4	14	8	7	4	84	1	−3	0	−2	0
30	0	15	0	8	0	85	1	0	−3	−2	−3
31	4	14	7	7	3	86	1	−2	−2	−3	1
32	2	13	8	7	−2	87	1	−1	−2	−1	−1
33	1	12	7	4	6	88	1	−2	−2	−1	−1
34	2	10	13	9	0	89	−1	−2	−2	−2	−1
35	3	9	4	2	1	90	0	−1	−2	−2	−7
36	0	7	16	7	1	91	0	−1	−3	−2	12
37	2	5	7	1	4	92	0	−3	−2	0	−16
38	1	5	16	−1	0	93	−2	−1	−3	−2	8
39	0	2	6	−1	3	94	−2	−3	−2	−2	−8
40	2	2	3	−2	5	95	−2	−1	−3	0	−48
41	1	2	11	−2	−4	96	−3	−2	−3	−1	64
42	0	−1	5	2	7	97	−4	−3	−2	−2	−25
43	2	1	2	1	−1	98	−5	−1			246
44	0	−1	−2	4	2	99	−5	−3			−12
45	0	0	1	5	4	100	−6	−1			
46	1	−2	0	4	2	101	−6	−3			
47	1	−1	−2	3	1	102	−8	−2			
48	0	0	−4	4	1	103	−6	−2			
49	0	−3	−1	2	5	104	−8	−3			
50	0	0	−3	2	−1	105	−7	−1			
51	1	−2	−1	3	3	106	−7	−3			
52	0	0	−4	1	2	107	−7	−1			
53	0	−3	0	4	2						
54	0	0	−2	3	0						

infantile ages, and the disturbance around age 17 shows three sign changes in the second differences; after that there is only one sign change (ignoring 1's) in the late twenties, and several in the nineties due to the cut-off to only one significant figure. Third differences are negligible up to age 86, after which they too are seriously distorted by the cut-off.

6.8 Table 3 shows the second differences of the *a*(90) ultimate tables, and of the PA(90) tables. These were forecast tables based on projections from a number of experiences, smoothed by two-parameter formulae down to age 50, blended down to age 20 with A 1967–70 for males, and with figures adjusted from the A 1967–70 table for females. Second differences of the rates of mortality for *a*(90) males, ultimate, change sign only once, at the top of the table; for *a*(90) females, ultimate, they change sign twice in the twenties of age, possibly due to the blend, but there is no sign change at the top of the table; this sign change at the high ages is also present in both PA(90) tables, but the only other sign changes are in the male table in the twenties of age, where there is probably no call for comment, and in the fifties where a sudden negative second difference in a long row of positive differences causes two adjacent sign changes. This last feature has been caused by the adjustments at age 56 downwards, described in the preface to the tables, and clearly the blend has caused a local roughness. Third differences of the rates of mortality for the *a*(90) ultimate tables are small throughout, the highest values of $10^4\Delta^3 q_x$ being 9 at ages 103–106 in the male table and 14 at 105–106 in the female table, compared with $10^4 q_x$ in each case of over 4,000; so at the third order these tables are not quite as smooth as the assured lives' tables, but they certainly have fourth order smoothness. The PA(90) female table has similar third differences, as does PA(90) male from age 58 onwards, but the male table has relatively large third differences at ages 55 and 56, as well as in the twenties, again due to the blend. Table 3a shows the second differences of the natural logarithms of the rates of mortality from the same tables.

6.9 Table 4 shows the second differences of the rates of mortality of the a(*m*) and *a*(*f*) ultimate tables, which were also forecast tables. The *a*(*m*) was smoothed by blending two Gompertz 2-parameter curves, the *a*(*f*) by blending two Makeham 3-parameter curves, in each case with arbitrary values below age 50. In both tables second differences are all positive (apart from an insignificant -1 at age 49 in the female table) and the values of $10^4\Delta^3 q_x$ are all negligible, so that the blends have achieved third order smoothness. Second differences of the logarithmic functions are shown in table 4a.

6.10 The English Life Tables are based on data in which the stated ages cannot be checked, and indeed the census returns and death certificates are known to contain some inaccuracies. From time to time different systems of grouping the data have been adopted to minimize their effects. Table 5 shows the second differences of the rates of mortality of E.L.T. No. 13 (smoothed by the spline method), E.L.T. No. 12 (smoothed by fitting a 7-parameter curve), and E.L.T. No. 10 (smoothed by osculatory interpolation). All these tables were published to 5 decimal places only. The three male tables have two sign changes each in the second differences in a run starting at age 16, due to the feature resulting from accidental deaths already mentioned in § 5.4; a similar feature appears in the female tables, but to a scarcely significant extent. Otherwise, E.L.T. No. 13 has no second difference sign changes in either the male or the female table, both the E.L.T. No. 12 have one sign change late in the age range, while both E.L.T. No. 10 have a wobble at age 86 causing two second difference sign changes, as well as a similar wobble at age 4. Apart from these wobbles, the reasons for which are not clear, the tables satisfy the second difference test; but it is interesting that it is the parametrically smoothed E.L.T. No. 12 which shows the late age sign changes already observed in the parametrically smoothed assured lives' tables. E.L.T. Nos. 12 and 13 show the semi-neckline from ages 0 to 2 while in both E.L.T. No. 10 it continues through to age 5. E.L.T. Nos. 12 and 13 have no significantly large third differences, except for E.L.T. No. 13 males at ages 15 and 16 due to the start of the accidental effect. E.L.T. No. 13, both tables, suffer from the wobble in the eighties where $10^4\Delta^3 q_x$ at ages 85 and 86 respectively has values of -13 and $+16$ in the male table and -18 and $+21$ in the female table, compared with $10^4 q_x$ of between 2,250 and 2,500 for the males and 1,800 and 2,000 for the females. If a ratio of between $\frac{1}{2}\%$ and just over 1% is acceptable then these tables all have third order smoothness, but it can be said that E.L.T. No. 10 was not smoothed so powerfully as E.L.T. Nos. 12 and 13, having suspect smoothness in the eighties of age. Table 5a shows the second differences of the logarithmic functions from these E.L.T's.

7. DIFFERENCES OF NATURAL LOGARITHMS

7.1 The function q_x usually tends toward the shape of a Gompertz curve over a considerable range of ages, and this suggests that a lower (i.e. more powerful) order of smoothness would be demonstrated by the differences of the logarithm of the function. Tables 2a, 3a, 4a and 5a show the second differences of the natural logarithm of q_x for all the standard tables so far examined. It will be seen from table 2a that after the childhood, early adult, and twenties periods of age, the second differences of $\ln q_x$ from FA 1975–78 and A 1967–70 only change by small amounts, and the third differences accordingly are all small. These two tables thus display third order smoothness in the logarithmic functions over most of the ages, and second order smoothness over a considerable range. Similar third order smoothness appears in A 1949–52 only from the middle forties upwards, and in A 1924–29 from the late thirties; while H^M shows second order smoothness between ages 50 and 89, irregularities elsewhere being partly due to the way q_x was derived from integral values of l_x.

7.2 Table 3a shows that the four annuitant and pensioner tables have third order smoothness in the logarithmic functions from the late thirties, apart from the hiccup already observed in PA(90) males around age 56, and these four tables also have second order smoothness over a considerable range. The $a(m)$ and $a(f)$ tables (see table 4a) show a number of irregularities due, no doubt, to the blends involved in their construction.

Table 3. *Second differences of rates of mortality from the latest standard tables for annuitants and pensioners*

Values of $10^6\Delta^2 q_x$ according to:

Age x	a(90) males ultimate	a(90) females ultimate	PA(90) males	PA(90) females	Age x	a(90) males ultimate	a(90) females ultimate	PA(90) males	PA(90) females
20	3	5	−16	0	65	178	127	201	152
21	5	11	−25	0	66	196	140	219	168
22	4	9	−16	0	67	210	155	236	186
23	5	5	0	0	68	228	172	255	205
24	5	−4	21	0	69	247	189	275	228
25	7	−2	15	0	70	267	208	296	251
26	6	−5	15	0	71	287	230	319	276
27	7	−4	14	0	72	310	252	344	306
28	9	4	1	0	73	333	277	367	336
29	7	7	0	0	74	358	304	394	369
30	11	7	57	31	75	383	334	422	405
31	10	7	10	8	76	412	365	448	443
32	11	9	6	3	77	438	398	479	485
33	12	10	7	6	78	468	435	508	528
34	15	10	7	5	79	496	473	537	575
35	14	12	10	6	80	528	514	567	623
36	17	12	11	7	81	558	556	598	673
37	17	12	14	7	82	586	601	625	727
38	21	13	17	9	83	619	648	655	778
39	21	13	20	9	84	645	694	680	834

Table 3 (*cont.*)

40	24	12	25	11	85	675	744	705	886
41	25	11	29	11	86	698	791	727	937
42	30	11	35	14	87	723	839	743	987
43	30	11	42	13	88	743	886	760	1032
44	34	8	49	17	89	758	929	767	1071
45	38	10	58	18	90	770	969	772	1102
46	41	8	67	21	91	778	1003	771	1127
47	44	10	78	21	92	777	1033	760	1138
48	49	10	89	25	93	772	1051	746	1137
49	54	12	100	28	94	759	1063	721	1122
50	62	14	112	31	95	737	1063	689	1092
51	83	16	123	35	96	710	1049	651	1043
52	104	18	133	37	97	673	1022	600	979
53	113	29	143	44	98	627	982	547	895
54	115	41	149	47	99	573	924	481	794
55	110	56	155	52	100	513	853	412	677
56	101	68	−336	60	101	444	764	335	547
57	91	77	52	64	102	369	663	252	401
58	80	85	114	74	103	287	548	168	250
59	76	86	123	80	104	202	420	78	91
60	77	86	135	91	105	115	287	−11	−70
61	86	85	145	99	106	23	143	−100	−230
62	103	88	158	111	107	−68	0	−188	−382
63	129	96	172	123					
64	155	110	186	136					

Table 3a. *Second differences of natural logarithms of rates of mortality from the latest standard tables for annuitants and pensioners*

Values of $10^3\Delta^2 \ln q_x$ according to:

Age x	a(90) males ultimate	a(90) females ultimate	PA(90) males	PA(90) females	Age x	a(90) males ultimate	a(90) females ultimate	PA(90) males	PA(90) females
20	1	13	−16	0	65	−1	0	0	1
21	3	35	−30	0	66	1	0	0	−1
22	1	25	−23	0	67	−1	−1	−1	0
23	5	13	−10	0	68	0	1	1	0
24	5	−13	19	0	69	0	−1	−1	−1
25	9	−7	17	0	70	−1	0	−1	0
26	7	−14	20	0	71	0	0	1	0
27	11	−10	21	0	72	0	−1	−1	0
28	13	10	1	0	73	0	0	−1	−1
29	12	16	0	0	74	−1	0	0	0
30	15	15	97	98	75	−1	0	0	0
31	15	12	10	13	76	1	−1	−1	−1
32	13	14	2	−4	77	−2	−1	−1	−1
33	13	12	2	3	78	0	1	0	0
34	14	10	1	−1	79	0	−2	0	−1
35	9	8	4	−1	80	−1	0	−2	0
36	9	5	2	1	81	−1	0	0	−2
37	7	3	5	0	82	−1	−1	−1	0
38	6	2	4	0	83	0	−1	−1	−1
39	4	−1	5	−1	84	−1	−1	−1	−1

Table 3a (*cont.*)

40	3	−2	5	1	85	1	1	−1	−1
41	2	−3	5	0	86	−1	−1	−1	−2
42	2	−4	5	0	87	−1	0	−1	−1
43	−1	−4	5	−1	88	−1	−2	−1	−1
44	1	−6	5	1	89	−2	−2	−1	−2
45	−1	−3	3	−1	90	0	0	−2	−1
46	0	−4	3	1	91	−2	−2	−1	−3
47	−2	−4	4	0	92	−2	−2	−2	−1
48	−1	−3	1	−1	93	0	−1	−1	−2
49	0	−2	0	1	94	−3	−2	−2	−3
50	−1	−2	1	0	95	0	−2	−1	−1
51	1	0	−1	−1	96	−3	−2	−2	−3
52	3	−2	−2	1	97	−1	−2	−2	−3
53	3	3	−2	−1	98	−2	−3	−2	−2
54	−1	3	−3	1	99	−2	−1	−1	−3
55	−3	6	−4	−1	100	−1	−3	−3	−3
56	−2	6	−41	0	101	−3	−3	−1	−2
57	−6	5	−3	0	102	−1	−2	−3	−4
58	−6	5	−1	1	103	−3	−3	−1	−2
59	−4	2	0	−1	104	−1	−3	−3	−4
60	−5	1	0	0	105	−3	−2	−1	−2
61	−4	−1	0	0	106	−1	−3	−3	−3
62	−3	−1	−1	−1	107	−3	−3	−1	−4
63	−1	−1	1	1					
64	−1	−1	−1	−1					

Table 4. *Second differences of rates of mortality according to the* a(m) *and* a(f) *tables*

Age x	$10^5\Delta^2 q_x$ $a(m)$ ult.	$10^5\Delta^2 q_x$ $a(f)$ ult.	Age x	$10^5\Delta^2 q_x$ a(m) ult.	$10^5\Delta^2 q_x$ $a(f)$ ult.
20	1	1	60	10	9
21	0	0	61	16	13
22	0	0	62	16	14
23	1	1	63	14	13
24	1	0	64	20	16
25	1	1	65	17	22
26	0	0	66	22	18
27	0	0	67	21	26
28	1	1	68	23	27
29	0	0	69	28	31
30	1	1	70	48	34
31	0	0	71	66	37
32	0	0	72	74	43
33	1	1	73	92	49
34	0	0	74	99	55
35	1	1	75	51	59
36	0	0	76	35	68
37	0	0	77	26	75
38	1	1	78	15	82
39	0	0	79	9	110
40	1	1	80	57	156
41	1	0	81	67	101
42	1	0	82	70	69
43	1	1	83	75	46
44	2	0	84	79	61
45	2	1	85	84	69
46	2	0	86	85	75
47	3	1	87	91	75
48	5	0	88	93	80
49	28	−1	89	96	79
50	12	3	90	102	80
51	9	1	91	100	82
52	7	8	92	105	84
53	6	1	93	104	81
54	10	6	94	105	82
55	8	6	95	106	80
56	8	7	96	103	81
57	14	6	97	98	72
58	9	8	98	96	74
59	14	11	99	89	67
			100	79	60
			101	73	57
			102	59	56

Table 4a. *Second differences of natural logarithms of rates of mortality according to the* a(m) *and* a(f) *tables*

Values of $10^3\Delta^2\ln q_x$ according to:

Age x	$a(m)$ ult.	$a(f)$ ult.	Age x	$a(m)$ ult.	$a(f)$ ult.
20	−1	0	60	−3	2
21	−2	−2	61	2	5
22	−3	−4	62	0	3
23	1	1	63	−1	3
24	0	−2	64	1	3
25	0	1	65	−1	5
26	−2	−3	66	0	1
27	−2	−3	67	0	4
28	0	2	68	0	3
29	−2	−3	69	1	2
30	0	1	70	3	3
31	−1	−2	71	6	1
32	−3	−1	72	5	2
33	1	0	73	6	2
34	−2	−2	74	4	2
35	1	0	75	−3	2
36	−2	0	76	−4	0
37	−2	−3	77	−6	1
38	1	1	78	−6	1
39	−2	−1	79	−6	3
40	1	0	80	0	5
41	0	−1	81	−1	−2
42	0	−2	82	0	−6
43	−1	1	83	0	−7
44	2	−1	84	−2	−4
45	2	0	85	1	−5
46	3	0	86	−2	−3
47	−5	−2	87	0	−3
48	7	1	88	−1	−3
49	26	−3	89	0	−2
50	6	2	90	−2	−4
51	2	1	91	0	0
52	0	7	92	−1	−4
53	−2	0	93	−1	−1
54	1	4	94	−2	−3
55	0	3	95	0	−2
56	−1	5	96	−2	−1
57	2	2	97	−1	−3
58	−2	4	98	−2	−2
59	2	5	99	−1	−1
			100	−2	−2
			101	−2	−3
			102	−1	0

Table 5. *Second differences of rates of mortality according to English Life Tables*

Age x	E.L.T. No. 13 Males $10^5\Delta^2 q_x$	E.L.T. No. 13 Females $10^5\Delta^2 q_x$	E.L.T. No. 12 Males $10^5\Delta^2 q_x$	E.L.T. No. 12 Females $10^5\Delta^2 q_x$	E.L.T. No. 10 Males $10^5\Delta^2 q_x$	E.L.T. No. 10 Females $10^5\Delta^2 q_x$	Age x	E.L.T. No. 13 Males $10^5\Delta^2 q_x$	E.L.T. No. 13 Females $10^5\Delta^2 q_x$	E.L.T. No. 12 Males $10^5\Delta^2 q_x$	E.L.T. No. 12 Females $10^5\Delta^2 q_x$	E.L.T. No. 10 Males $10^5\Delta^2 q_x$	E.L.T. No. 10 Females $10^5\Delta^2 q_x$
0	1823	1378	2234	1721	4783	3368	55	14	5	15	8	16	13
1	19	24	28	34	657	546	56	15	7	15	8	14	12
2	9	4	23	3	134	125	57	18	8	15	10	14	10
3	5	4	2	4	66	33	58	20	7	16	10	17	13
4	2	3	0	2	−67	−27	59	22	11	16	12	24	17
5	0	2	1	2	41	24	60	25	12	15	14	30	20
6	1	0	0	1	9	11	61	27	14	18	16	30	17
7	−1	0	1	0	9	11	62	26	17	19	18	30	16
8	1	1	1	1	9	10	63	30	17	19	20	31	22
9	1	0	1	2	8	8	64	31	22	21	22	35	29
10	1	0	1	1	9	8	65	31	22	24	25	35	38
11	3	2	3	0	8	5	66	28	27	26	29	40	37
12	4	2	3	0	9	6	67	26	29	30	32	42	36
13	6	1	6	2	8	3	68	23	31	34	35	45	41
14	6	2	7	1	3	3	69	22	34	37	38	47	49
15	10	−2	2	−1	2	−4	70	21	38	42	42	51	54
16	−25	0	−8	0	−7	−5	71	27	40	46	46	61	70
17	−3	−2	−8	−1	−7	−5	72	31	43	51	50	67	73
18	−4	−2	−3	0	−4	−2	73	36	45	55	55	63	66
19	−1	0	−3	−1	−5	−1	74	42	50	58	60	56	55
20	0	−1	−3	1	−4	0	75	45	51	63	64	50	45
21	0	1	−2	0	−1	−1	76	51	53	65	66	60	50
22	0	0	0	0	−5	−1	77	56	60	67	70	72	67
23	2	1	3	1	−2	0	78	60	64	67	74	67	65
24	1	1	2	0	0	2	79	64	68	66	78	60	63

Table 5 (*cont.*)

25	1	0	3	0	4	2	80	66	74	65	79	53	61
26	2	1	2	1	2	0	81	64	76	62	80	62	76
27	2	1	1	1	1	−1	82	65	79	56	79	76	86
28	0	0	1	1	1	0	83	63	81	49	77	63	75
29	2	1	0	0	4	2	84	62	78	45	74	55	60
30	0	0	1	0	3	1	85	59	79	34	70	38	40
31	1	2	1	2	5	2	86	56	76	24	64	−92	−137
32	1	0	1	1	3	2	87	53	75	16	54	69	71
33	2	2	1	1	3	1	88	49	74	4	44	72	73
34	2	1	2	0	3	1	89	47	84	−6	34	73	76
35	3	1	2	2	1	2	90	42	92	−17	22	72	78
36	3	3	1	1	1	0	91	38	99	−25	9	74	77
37	4	1	3	2	1	1	92	33	106	−36	−4	71	81
38	3	2	3	1	2	1	93	29	109	−43	−17	73	76
39	4	3	3	1	5	4	94	25	109	−51	−31	69	81
40	4	2	4	2	5	4	95	20	109	−55	−44	71	79
41	5	2	4	2	7	4	96	16	105	−60	−56	63	75
42	5	3	5	2	6	4	97	9	97	−63	−65	63	74
43	4	1	6	1	4	3	98	10	98	−66	−61	57	71
44	6	2	6	2	4	4	99	11	97	−65	−84	52	66
45	5	2	8	3	2	4	100	9	97	−65	−77	48	63
46	6	2	8	2	1	2	101	11	98	−65	−79	38	56
47	7	1	9	2	2	3	102	10	98	−62	−81	31	48
48	6	2	10	3	4	4	103	10	97		−81		40
49	7	3	11	4	7	6	104	10	98		−80		30
50	8	2	12	4	11	7	105	10	97		−75		
51	9	3	14	4	9	6	106		97		−73		
52	10	3	12	5	8	5	107				−68		
53	10	4	14	5	9	7							
54	13	5	14	6	15	11							

Table 5a. *Second differences of natural logarithms of rates of mortality according to English Life Tables*

	Values of $10^3\Delta^2 \ln q_x$ according to:												
Age x	E.L.T. No. 13 Males	E.L.T. No. 13 Females	E.L.T. No. 12 Males	E.L.T. No. 12 Females	E.L.T. No. 10 Males	E.L.T. No. 10 Females	Age x	E.L.T. No. 13 Males	E.L.T. No. 13 Females	E.L.T. No. 12 Males	E.L.T. No. 12 Females	E.L.T. No. 10 Males	E.L.T. No. 10 Females
0	2426	2206	2287	2220	702	598	55	0	0	−3	2	3	4
1	117	205	100	275	446	409	56	−1	3	−2	1	2	1
2	92	17	254	2	193	201	57	1	1	−4	3	−1	1
3	75	49	23	41	161	72	58	0	1	−3	0	2	0
4	35	63	−8	20	−232	−126	59	0	2	−3	2	2	2
5	−4	56	12	36	101	53	60	0	3	−3	2	4	4
6	20	−5	−7	19	11	23	61	1	3	−3	2	2	−1
7	−30	−6	17	−10	26	45	62	−2	4	−2	1	2	−1
8	22	38	20	29	42	60	63	0	1	−3	2	−1	1
9	29	−3	24	80	48	57	64	−1	4	−1	1	2	3
10	29	−1	26	41	63	60	65	−2	0	−2	1	−1	3
11	87	92	76	−2	54	30	66	−3	4	−1	1	0	2
12	90	82	60	−1	51	30	67	−3	0	−1	2	0	1
13	87	20	92	67	28	4	68	−3	2	1	0	−1	0
14	29	40	51	20	−5	3	69	−4	0	−2	0	0	1
15	33	−79	−41	−40	−10	−30	70	−4	0	1	0	−1	2
16	−288	−12	−114	−7	−40	−27	71	−1	−1	0	−1	0	2
17	−29	−51	−81	−29	−30	−23	72	−3	1	1	1	1	1
18	−36	−46	−26	−3	−17	−9	73	−1	−2	−1	−1	−2	−1
19	−12	0	−25	−23	−17	−4	74	0	−1	0	−1	−2	−2
20	−1	−23	−27	20	−13	−1	75	−1	−2	1	−1	−2	−5
21	−2	23	−19	−1	−3	−4	76	−1	−1	−1	−1	−3	−3
22	−1	0	−3	−2	−15	−4	77	1	−1	−1	−2	0	−2
23	20	23	27	17	−6	0	78	−1	−2	−1	−1	−2	−2
24	12	20	20	−3	0	−7	79	0	−1	−1	−2	−2	−3

Table 5a (*cont.*)

25	9	1	30	3	12	7	80	0	−2	−2	−3	−3	−1
26	27	17	19	14	6	−1	81	−1	−1	−1	−1	−2	−3
27	19	15	8	10	3	−3	82	−1	−1	−3	−3	−1	−1
28	−2	5	7	9	3	−1	83	−1	−3	−2	−3	−2	−1
29	18	11	3	7	11	7	84	−1	−2	−3	−3	−3	−3
30	−4	5	5	6	8	2	85	−2	−2	−2	−3	−2	−4
31	7	20	5	17	12	6	86	−2	−3	−4	−4	−8	−11
32	3	8	3	2	6	5	87	−2	−2	−3	−3	0	−1
33	12	15	2	2	4	1	88	−1	−3	−3	−3	−1	0
34	8	0	9	−8	4	2	89	−2	−2	−3	−5	−1	−1
35	11	−1	6	9	−2	4	90	−2	−2	−4	−3	0	0
36	9	15	−2	2	0	−1	91	−2	−1	−3	−4	−2	−1
37	10	−5	9	5	−2	0	92	−2	−1	−3	−4	0	−1
38	2	1	6	−3	1	2	93	−2	−2	−3	−3	−1	−1
39	2	6	3	−3	5	6	94	−2	−1	−4	−5	0	−1
40	2	−3	6	2	4	7	95	−2	−1	−2	−3	−2	−1
41	2	−2	2	1	6	5	96	−2	−2	−4	−4	−1	0
42	0	−1	5	0	4	4	97	−2	−2	−2	−4	−1	−2
43	−4	−7	4	−5	−1	2	98	−1	−1	−2	−3	−1	−1
44	0	−3	2	0	0	2	99	−2	−2	−3	−4	−1	0
45	−4	−4	4	1	−3	2	100	−1	−2	−2	−2	−1	−3
46	−1	−4	1	−1	−4	−1	101	−2	−1	−2	−3	−2	0
47	−2	−5	1	−3	−3	−1	102	−1	−2	−2	−3	−1	−2
48	−3	−3	0	1	0	1	103	−1	−1		−3		−1
49	−2	−1	1	1	2	3	104	−2	−1		−1		−2
50	−2	−2	−1	1	4	3	105	0	−2		−3		
51	−1	−2	1	−1	2	1	106		−1		−2		
52	−2	−1	−4	2	1	−1	107				−1		
53	−1	0	−2	0	0	1							
54	0	1	−3	1	3	4							

7.3 From table 5a we see that E.L.T. Nos. 12 and 13 settled down to third order smoothness from the middle forties, E.L.T. 12 having second order smoothness in the upper half of the table. E.L.T. No. 10 similarly had third order smoothness over the same range, apart from the irregularity around age 86.

7.4 It does thus seem to be borne out that, particularly for the tables smoothed parametrically, the logarithms of the rates of mortality display a lower order of smoothness than the rates themselves. But this is not put forward necessarily as a reason for preferring the parametric method.

8. EXAMINATION OF OTHER TABLES

8.1 Copas and Haberman give in table 2 of their paper a number of smoothed series, lettered A to M inclusive, all derived from the same crude rates of mortality. B to H inclusive were smoothed by graphical or parametric methods. Series A gives their first smoothed series obtained by the kernel method and series J to M inclusive give the results of further smoothings by this method derived by taking different values of their constant h, apparently after considering the results of the parametric smoothings. Table 6 shows the second differences of series A, L and M. Ignoring, as before, values between -2 and $+2$, it will be seen that A and M each have 6 sign changes in the second differences but L has none. The rates given in their paper are expressed as $10^4 q_x$ and accordingly third differences can only be claimed to be shown as far as $10^3\Delta^3 q_x$, at which they are all zero for series L and all between -2 and $+2$ for A and M. Whether it is right to ignore values of -2 or $+2$ when the rate itself is only available to two significant figures is doubtful, but Copas and Haberman can certainly claim perfect third order smoothness in series L. Whether they can also claim to have made out their case for the use of the method when the smoothing is demonstrated over a range of 36 ages only, and to 4 decimal places only, is arguable, particularly as L seems to have been derived indirectly from one of the parametric series. Dr Haberman informs me that the data used are not available for a larger range of ages, and that there would be complications in providing the series to more decimal places, but the exposition of the method seems to need a smoothing of a set of data over a complete range of ages, not hitherto smoothed by any other method, and with the results shown to at least five significant figures. Until this is done, the case for the method must be regarded as not proven.

8.2 Table 7 shows $\Delta^2 l_x$ derived from Lambert's graphical smoothing of a life table. It could be argued from § 5.4 that it would have been better to operate on the reciprocal of l_x but, rightly or wrongly, it has been thought appropriate to use Lambert's figures direct, as l_x is not a divergent series, and whether or not it exceeds unity depends entirely upon the radix. The radix could have been unity, with l_x shown as a fraction throughout, and indeed this might have had the result of more significant figures being retained. It will be seen that there are two sign changes in the second differences at age 17 and in the thirties, indicating that even in the eighteenth century there was some inflexion around these ages. If it is

Table 6. *Second differences of rates of mortality from miscellaneous* tables*

Age x	C & H Series A $10^4\Delta^2 q_x$	C & H Series L $10^4\Delta^2 q_x$	C & H Series M $10^4\Delta^2 q_x$
35	−2	2	0
36	0	−1	0
37	1	2	1
38	4	−1	0
39	2	1	0
40	2	0	7
41	1	0	0
42	1	−1	1
43	−6	1	−6
44	−6	−1	−6
45	−2	1	−3
46	2	1	6
47	5	0	5
48	3	2	0
49	4	0	3
50	0	2	−4
51	2	2	7
52	−1	0	−1
53	−1	2	1
54	1	1	1
55	4	2	0
56	3	1	2
57	−3	1	2
58	−2	2	1
59	4	2	2
60	2	1	1
61	0	3	13
62	12	1	−17
63	12	3	12
64	−6	2	3
65	−15	3	2
66	0	3	4
67	13	3	2
68	4	4	5

* Taken from Table 2 of Copas and Haberman[5] referred to in headings as C & H.

thought that the run of +1's in the forties and of −2's in the fifties are too long to be ignored there are two more sign changes. Third differences are negligible after the semi-neckline, so the l_x column has third order smoothness, but if q_x is calculated it will be found that the cut-off has had the same effect as in the H^M table.

Table 7. *Second differences of Lambert's* l_x *table*

Age x	$\Delta^2 l_x$	Age x	$\Delta^2 l_x$	Age x	$\Delta^2 l_x$
0	2,000	35	0	70	3
1	270	36	−1	71	2
2	107	37	0	72	1
3	64	38	1	73	2
4	29	39	0	74	2
5	24	40	1	75	2
6	20	41	1	76	3
7	16	42	1	77	4
8	12	43	1	78	4
9	10	44	1	79	6
10	8	45	0	80	6
11	5	46	−1	81	7
12	3	47	−1	82	8
13	3	48	−1	83	10
14	2	49	−2	84	7
15	2	50	−2	85	5
16	2	51	−2	86	3
17	−3	52	−2	87	1
18	−3	53	−2	88	1
19	−4	54	−2	89	0
20	−5	55	−2	90	0
21	−6	56	−2	91	0
22	−4	57	−2	92	1
23	−4	58	−2	93	0
24	−3	59	−2	94	0
25	−2	60	−1	95	1
26	−2	61	−2	96	0
27	−1	62	−2	97	1
28	−1	63	1	98	1
29	−1	64	2	99	1
30	−2	65	3	100	2
31	−1	66	4	101	0
32	−2	67	3		
33	−2	68	4		
34	−1	69	3		

9. CONCLUSIONS

9.1 In summing up the smoothness tests made in sections 4 and 5 it is necessary to bear in mind the considerations of § 6.1 and, in particular, that the greater the number of decimal places and significant figures given in the original series, the more severe will be the tests. In the ridiculous extreme, if only one significant figure is shown the series can only go up in integral steps, there would be no real curve, and it would be impossible to test for smoothness.

9.2 No attempt has so far been made to quantify precisely how small the ratio between the *n*th difference of a probability and the probability itself ought to be for the smoothness to be accepted as being of the *n*th order. This must be a matter of personal judgement, in the same way as personal judgement determines whether statistical significance should be measured at the 5%, the 1%, or any other level of probability. If criterion (d) of § 5.8 rather than criterion (c) is being adopted, a possible suggestion might be that a ratio of $1:7^n$ should be the target for *n*th order smoothness, by the argument that 1 is small in relation to 7 or more and that the differences should become successively smaller. On that basis the assured lives' tables examined all have third order smoothness except for H^M after age 86, as do $a(m)$, $a(f)$ *and* $a(90)$ males ultimate; while $a(90)$ females ultimate and PA(90) females have fourth order smoothness. PA(90) males has third order smoothness from age 58 onwards, but below age 58 the smoothness is distorted by the blend. In the same way E.L.T. Nos. 12 and 13, males and females, all have third order smoothness after the first few years of life (apart from E.L.T. No. 13 males at ages 15–16), but E.L.T. No. 10 could only claim fourth order smoothness. There seems to have been quite a high level of consistency by the profession in its acceptance of what it believes to be smoothness in its standard tables.

9.3 The tables in Copas and Haberman are difficult to test because of the small number of decimal places (and significant figures) shown, but as far as can be judged the series L has third order smoothness. I would like to see their method used in production of series to, say, 5 or 6 decimal places, to see how it would then stand up to the smoothness tests I have suggested.

9.4 It is not claimed that there is anything original in this paper, which has aimed at bringing together a number of different ideas, at endeavouring to express them in the form of acceptable criteria of smoothness, and at testing various tables on the bases of these criteria and on a suggested quantification. But it is thought that the profession and its students need these ideas to be brought out of the purely intuitive protective cocoon in which they have lain in the past, and consolidated into something tangible.

9.5 It was stated in § 1.1 that I would avoid the use of a certain word and its derivatives. According to Chambers' Twentieth Century Dictionary 'to graduate' means to divide into regular intervals or to mark with degrees. According to the Shorter Oxford English Dictionary (the one with some 2,500 pages) it means to divide into degrees or arrange in gradations (a gradation being a series of successive steps) or to change gradually; and 'gradual' means advancing step by step. (There are other definitions, but these are the main ones.) What is not being done in the processes we have been considering is arranging the series in steps, as in the ridiculous example instanced at the end of § 9.1. What is being done, and only what is being done, is the smoothing process. That being the case, it becomes necessary to define 'smooth' and 'smoothness'. But this is where we came in.

9.6 My grateful acknowledgements go to Professor A. D. Wilkie and to Mr R.

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APPENDIX

ALGEBRAIC NOTE ON SECOND DIFFERENCES

This somewhat elementary note is appended for the benefit of those who may have forgotten just what second differences are. As the main paper has been largely concerned with rates of mortality the use of the symbol q_x will be retained.

The second difference $\Delta^2 q_x = q_{x+2} - 2q_{x+1} + q_x$

If $\Delta^2 q_x$ is positive, then $\frac{1}{2}(q_{x+2} + q_x)$ is greater than q_{x+1} and it follows that at this point the q_x curve is convex when viewed from the x axis. In the same way if $\Delta^2 q_x$ is negative the q_x curve is concave when viewed from the x axis, and if $\Delta^2 q_x$ is zero then these three values of q_x are in a straight line. It follows that a sign change in $\Delta^2 q_x$ indicates an inflexion in the q_x curve. This is why second differences are so important in assessing smoothness. If values close to zero are taken as zero then a small inflexion is being overlooked; this does not matter so long as the inflexion is no larger than could have been caused by cutting off at the last decimal place retained in the accepted values of q_x. Clearly the smaller the scale, and the fewer the decimal places retained, the rougher will the curve be, and the more important the inflexions which have to be overlooked because of the smallness of the scale.

The traditionally accepted shape of the q_x curve contains a change from convexity to concavity fairly high up the age scale. Precisely where it appears in the smoothed curve may not matter as it occurs at ages where data are usually relatively scanty. Those smoothed curves where this feature is absent have possibly had an inherent feature removed in the smoothing process (inflexion-cutting rather than wave-cutting). And those curves where it is retained have a second difference sign-change late in life. Alternatively, if the traditionally accepted shape of the q_x curve is the result of the use of Makeham or Makehamesque curves, and if the inflexion is not present in the data (but has been created for the purpose of curve fitting) then the curve fitting process has distorted the underlying curve.