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JOHANN HEINRICH LAMBERT (1728–1777)

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1. INTRODUCTION

1.1. This paper is intended as a tribute to a man who died at the age of 49 just over 200 years ago having made material contributions to what is now regarded as actuarial science, as well as to many other branches of science, but whose work has largely been overlooked.

1.2. Johann Heinrich Lambert was born in Mulhouse, Alsace on 26 August 1728 and died in Berlin on 25 September 1777. He was largely self-taught, having had to leave school at the age of 12 to help his father in his tailor's shop. At the age of 20 he became tutor to the children of a noble Swiss family; this ended 10 years later when he had taken the children on an educational tour of Europe during which he was able to meet eminent scientists of the time and go to lectures at universities and learned societies. Eventually, in 1765, he obtained a post at the Royal Academy of Sciences in Berlin where he stayed until his death.

1.3. Lambert appears to have been an original character of strange appearance and unconventional behaviour (Steck (1970) includes a portrait). Gray and Tilling (1978) say of him: "He was no genius, but a man of great intelligence and imagination with occasional sparks of brilliance; he aimed at clarifying the fundamentals of his subjects, but was never profoundly original either in his mathematics or in his science. . . . Further he was a practising scientist who achieved at least a limited degree of success in his investigations."

1.4. Lambert could write in German, French and Latin with equal ease. He made contributions to philosophy, logic, mathematics, optics, heat, magnetism, properties of matter, astronomy, meteorology, cartography, geodesy, perspective, probability, theory of errors and demographical statistics. His contributions to the last three are discussed by Sheynin (1971) who considers that in the theory of errors Lambert should be regarded as the main predecessor of Gauss and credits him with the first use of the principle of maximum likelihood. In his discussion of demographical statistics (with which we are concerned) Sheynin (1971) does not always seem to appreciate the reasons behind what Lambert does.

1.5. The purpose of this paper is to draw attention to Lambert's contributions to what we now like to regard as actuarial techniques. These seem to be almost unknown in this country, but a little better recognized on the continent, see, for example, Loewy (1927) and Eisenring (1948). The contributions with which we are concerned are in Lambert (1765–1772), which is a collection of papers and notes on various matters in pure and applied mathematics. The relevant papers are given in the References and will be referred to in this paper by the volume number, (I) or (III) followed, in the case of the latter, by the number of the section. A full bibliography of Lambert's works, and works about him, is given in Steck (1970).

2. LAMBERT'S ACTUARIAL WORK

2.1. The work we will be discussing is well described by the title of (III), "Remarks about mortality, death lists, births and marriages". Lambert quite understood that reliable mortality tables must be based on age-related enumerations of both the living and the corresponding deaths. But such data were scarce and perforce mortality tables were then being prepared from death records alone. Lambert set out clearly the conditions for this method to be satisfactory (i.e. a stationary population for about 100 years) and discusses the effect of migration between town and country when using the death records of one or other of these.

2.2. Some of his comments on the use of mortality tables have a modern ring about them. He says that financial calculations should be on the safe side, for example in widows' funds the mortality used should be inclined to be high for contributing husbands and low for surviving widows, but "one must not be deliberately unjust." (III, 1, p. 493).

2.3. *Graphs and interpolation*

2.3.1. Before Lambert's time graphs had been used to display data but hardly at all as a tool for dealing with experimental data. Lambert was the first to do this and he made extensive use of graphs in a variety of ways to analyse and draw conclusions from observational data, (e.g. III, 3). Unfortunately, this work went virtually unnoticed for at least 50 years (Tilling, 1975).

2.3.2. In (I), "Theory of reliability of observations and experiments", after discussing data of various types, he finishes with an example of the graphical graduation of a mortality table. It is worth setting out in full the principles he adopted; in the quotation (I, p. 478–9) MKH is the graduated curve and A, b, c, d, etc., are the observed points.

> We have stated that in those cases where the equation for MKH is not known, the line should be drawn freehand, so that it goes through the middle, in between the inaccurately determined points A, b, c, etc and so far as possible remains uniform. It is not that we lack a method for drawing a curved line through any number of points of which the positions are known. Newton gave a method of this type, where the problem is to find ordinates for abscissae which fall in between the given points. There is no doubt that this method is useful when the points A, b, c, d, etc are accurately known. We can take an equation which has as many terms and coefficients as there are points given, and from the points determine all the coefficients. But, as we soon find during the construction, the points A, b, c, d, etc are never as accurately determined as this, so that undoubtedly such a line would always show the small deviations of the observations and would for this reason be inaccurate. It is thus very much better to take an equation with fewer terms and coefficients, and to proceed as we described above [i.e. freehand drawing]. In this way the curved lines found are not only themselves simpler, but also come nearer the true mean of the observations. (Translation from Tilling, (1973), with the author's permission).

As a statement of principles this does not seem to leave much to add, even over 200 years later.

2.3.3. Lambert makes a graduation of the values of l_x, at (usually) decennial ages, which he had calculated from the deaths recorded in the London Bills of Mortality, 1753–58. He does not read off the graduated values of l_x at all ages from his graph but gives two methods of interpolation. The first is a graphical method assuming given two points on the curve and the tangents at those points. He describes, if I have understood him correctly, a geometrical construction for replacing the curve between any two points by (to my astonishment) "osculating parabolas" (*osculirenden Parabeln*) (I, p. 484). From these the intermediate values of l_x are presumably to be read off.

2.3.4. In the second method Lambert fitted a polynomial of the fifth degree over the age range 45 to 90. He determined the six constants so as to reproduce, (i) l_{45} (estimated, probably from his graph), (ii) the observed values of l_{60}, l_{70} and l_{90}, and (iii) the slopes of the tangents to the curve at ages 45 and 90 (estimated from his graph). In this way he obtained a polynomial to represent a section of the whole curve which was able to "hang together" with the corresponding polynomials obtained in the same way for immediately preceding and following sections of the curve. He did not calculate the values of l_x given by the polynomial he had determined.

2.3.5. In actuarial literature Sprague (1867, 1880) seems to be universally acknowledged as the inventor of osculatory interpolation; it is now clear that Lambert preceded him by over 100 years. I find that the mathematical use of the word "osculation" and its derivatives (Latin *osculare*, to kiss) in the sense of close contact between curves has a long history. The Oxford English Dictionary Vol. 7 (1933) says that this use was invented by Christiaan Huygens (1629–95) but gives no date and I have not been able to find the word in Huygens' *Oeuvres Complètes* (22 volumes). However, it is used by Leibnitz (1686) in both the title and the text of his article. Sprague (1867, 1880) does not use the word in relation to his new interpolation formulae and the first application of it to these formulae which I have found is in Karup (1899).

2.3.6. In (III, 1), "About mortality tables" p. 494, Lambert used a graph to obtain graduated values of l_x at individual ages. This was based on (usually) decennial values of l_x obtained by combining, in the proportions of one-third and two-thirds, two sets of l_x obtained respectively from records of deaths in age-groups given by Süssmilch (1761) from (i) London Bills of Mortality 1728–57, and (ii) 17 country parishes in Germany; the object was to avoid the errors due to migration mentioned above and so obtain a mortality table applicable to the whole population of a country. This mortality table was used for the calculations in (III, 7), "The mortality of smallpox in children"—see translation at the end of this paper—and is the only one of Lambert's mortality tables showing values at individual ages.

2.3.7. (III, 5), "The number of children from each marriage", contains an ingenious graphical method. Lambert wishes to correct an observed frequency distribution of the number of marriages having 0, 1, 2, . . . children, for the omission of stillbirths and deaths in infancy, which he considers amount to half the recorded births. He plots his data on a graph, graduates it, changes the scale for number of children so that 2 becomes 3, etc. He then finds the corrected distribution by reading off on the graduated curve the number of marriages having 0, 1, 2, etc., children on the revised scale.

2.4. *Expectation of life*

In Lambert's time two measures of the duration of life were in use, (i) the expectation, or average duration of life, and (ii) the probable duration of life, or the time by which l_x has been reduced by half. This had caused some confusion and argument (e.g. see Bradley, (1971), pp. 42, 68). In the mortality table of (III, 1, p. 494) Lambert gave both measures for every age. At most ages the difference did not exceed 2 years, but at birth the probable duration was about 6½ years less than the average duration. Lambert explained how and why the two measures differed, the situations in which each or neither were appropriate and proved mathematically that they would be equal at all ages only if l_x were a straight line.

2.5. *Force of mortality*

Makeham (1867) explained the term "force of mortality" which he said had recently come into use but was not found in any standard elementary works. Ogborn (1953) credits Edmonds (1832) with the introduction of the term. However, 60 years earlier Lambert (III, 1, p. 509 et seq.) considers the age variations of what he calls the "force of vitality" (*Lebenskraft*). He takes this as the length of the subtangent of the l_x curve, i.e., on a graph of l_x, the length cut off on the x-axis between its intersections with (i) the perpendicular dropped from the point l_x, and (ii) the tangent to the l_x curve at x; this length is $l_x \Big/ -\frac{dl_x}{dx}$ (see, e.g., Gibson, (1926), p. 123), or $1/\mu_x$. He points out that $1/q_x$ is a reasonable measure of the force of vitality except at the youngest age. Thus the concept of forces of vitality or mortality is rather older than is generally thought.

2.6. *Law of mortality*

2.6.1. Lambert gives (III, 1, p. 483) a formula for the l_x curve of the mortality table he obtained from the deaths of the London Bills of Mortality, 1728–1757. After correcting a printing error, the formula is

$$l_x = 10{,}000\left(1-\frac{x}{96}\right)^2 - 6{,}176\left\{\exp\left(\frac{-x}{13{\cdot}682}\right) - \exp\left(\frac{-x}{2{\cdot}43114}\right)\right\}$$

It is notable that this formula applies to all ages from 0 to 96 and the calculated values compare well with nearly all the observed values of l_x, which are (usually) at decennial ages, (except that at age 2, where the calculated value should, unfortunately, be 6,965, instead of Lambert's 6,407 which differed by only 44 in excess of the observed value!). The formula is the wrong shape to fit l_x of modern mortality tables. To do this Eisenring (1948) replaced the quadratic term by $l_0 - bx^4$.

2.6.2. Lambert does not describe how he obtained the constants in his formula, except to point out that the limiting age is 96. He says "The first term which is parabolic would specify that the human species dies off in the way in which a cylindrical vessel full of water empties itself. The other two terms resemble the warming and cooling of solid bodies..." (p. 484). The last two terms have quite a small effect after about age 50.

2.6.3. Lambert claims no scientific basis for his formula but it is worth noting that a formula like Lambert's first term gives the depth of water left in a cylinder from which it has been running out for time x. Thus Lambert's basic concept is of a similar nature to that of Gompertz (1825) of the gradual "exhaustions of a man's power to avoid death" and anticipated it by more than 50 years. Possibly this is reading too much into Lambert's formula since the first term gives values which are much too high at the younger ages and the second and third terms seem to have been chosen simply because they approximate to the shape of the correction needed.

2.7. *The double decrement table*

In Daw (1979) I have dealt with Lambert's contribution to the double decrement table, "The mortality of smallpox in children" (III, 7) and mentioned the translation kindly made of it by Mr W. W. Mehlig. This translation now follows and can be largely left to speak for itself of the essential clarity and simplicity of Lambert's writing and of the nature of his work.

3. CONCLUSION

Lambert's principal "actuarial" contributions can be summed up as:

(i) the first graphical graduation of a mortality table,
(ii) probably the first osculatory interpolation,
(iii) the introduction of the "force of vitality" (the reciprocal of the later force of mortality),
(iv) the first mathematical formula for l_x and one which had at least the semblance of a crude theoretical basis, and
(v) practical formulae for the double decrement table.

Truly a remarkable achievement for a self-taught man who seems to have written only about 140 small pages on the subject. Taking account also of his achievements in many other branches of science, we may perhaps think that Lambert was right in the curt reply he is said to have given to Frederick the Great when asked in which science he was most proficient—"All".

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(A) Library of the Institute of Actuaries.
(Sc) Science Museum Library, London.
(U) Library of University College, Gower Street, London.
(B) The British Library, London.

THE MORTALITY OF SMALLPOX IN CHILDREN

by J. H. Lambert

(From Lambert's "*Beyträge zum Gebrauche der Mathematik und deren Anwendung*" Vol. III, p. 568–599, *Berlin*, 1772)

Translated from the German by W. W. Mehlig

(and explanatory notes by R. H. Daw)

Translators' note

In making this fairly literal translation, Mr Mehlig and I have done our best to avoid destroying the flavour of the time it was written, so far as it is possible for those living in the twentieth century to deal in that way with the thought and idiom of 200 years ago. Lambert's mathematical notation has been retained unaltered apart from replacing a curious symbol unknown to us, like a large thin-lined figure seven, by n. A number of obvious errors in the German text have been corrected. These consist of some errors or omissions in the mathematical formulae, a few incorrect figures in two of the tables and nearly all of the numerical references to previous paragraphs, which are usually one out. The "Explanatory notes and comments" added at the end are indicated in the translation by number, e.g. [2]. References to other papers in the explanatory notes will be found at the end of the foregoing paper.

§ 125. Although much has been written and argued about this matter over a number of years, one has little complete experience of it. Süssmilch [1], too, does not say much other than that he has found from various death registers that out of 1,000 people dying each year, 80 are carried off by smallpox, and that out of 300 and more inoculated subjects scarcely one dies of smallpox. However, this still does not say anything about particular circumstances. Süssmilch adds that since few or none are spared from smallpox, one could assume that out of 1,000 persons who are attacked by smallpox, 80 die, that is 2 of 25. This figure is obviously too low, for out of 1,000 who die each year, there are many who have not had smallpox. If one assumes that their number $= \alpha$, then one will find among the 1,000 persons dying each year $1{,}000-\alpha-80$ who die after having recovered from smallpox, and 80 die from smallpox. Therefore, out of $1000-\alpha$ who contract smallpox, 80 die. [2]

§ 126. Now α is not a very small figure, for the 1,000 persons dying each year include 261 children under 1 year, most of whom have not had smallpox. In the second year of life, 61 die and most of these, too, have not had smallpox, because smallpox does not attack children in appreciable numbers until about the fourth year of life. [3] Thus α becomes not less than, but rather greater than, 300, and this results in the mortality of smallpox, instead of being 2/25 according to Süssmilch, to be 1/9 or even 1/8; so that out of 8 or 9 persons attacked by smallpox, one dies.

§ 127. But this is still expressed in very general terms. However, in the absence of particular experiences, Mr D. Bernoulli [4] has endeavoured to see to what extent something more could be determined by the theory. His calculation is approximately as follows. From α born let y be those who attain the age τ. Of these w have already had smallpox, but η not yet. This gives first

$$y = w + \eta$$

In the time $d\tau$, $1/n$ of η are attacked by smallpox, that is $\eta d\tau : n$ and of these the mth part die, that is $\eta d\tau : mn$. Since dy die in all, it follows that the number of those who die of other diseases in the time $d\tau$

$$= dy - \frac{\eta d\tau}{mn}$$

In order to find how many there are of those who have not had smallpox, Mr B. concludes that y is to η as $\left(dy-\frac{\eta d\tau}{mn}\right)$ is to the number to be found, and therefore in the time $d\tau$ the number

$$\frac{\eta}{y}\left(dy-\frac{\eta d\tau}{mn}\right)$$

die before smallpox. [2] Therefore, η is reduced by this many in time $d\tau$ for this reason. However, η is further reduced by all those who are attacked by smallpox in the time $d\tau$, that is by $\frac{\eta d\tau}{n}$. This thus gives the equation

$$d\eta=\frac{\eta}{y}\left(dy-\frac{\eta d\tau}{mn}\right)+\frac{\eta d\tau}{n}$$

and from this follows

$$\frac{d\eta}{\eta}-\frac{dy}{y}=\frac{d\tau}{n}\left(1-\frac{\eta}{ym}\right)$$

And since both $d\eta$ and dy must be taken as negative, it follows that

$$\frac{d(y:\eta)}{(y:\eta)\left(1-\frac{\eta}{ym}\right)}=\frac{d\tau}{n}$$

or

$$\frac{d(y:\eta)}{\frac{y}{\eta}-\frac{1}{m}}=\frac{d\tau}{n}$$

If one now puts m and n constant and for the case where $\tau = 0$, $\eta=y$, so one gets by integration

$$\log\left(\frac{y:\eta-1:m}{1-1:m}\right)=\frac{\tau}{n}$$

or

$$\log\left(\frac{ym-\eta}{\eta m-\eta}\right)=\frac{\tau}{n}$$

Since $\log e = 1$, then

$$\frac{ym-\eta}{\eta m-\eta}=e^{\tau:n}$$

therefore

$$\eta=\frac{my}{(m-1)e^{\tau:n}+1}$$

§ 128. In this way the formula, notwithstanding that it contains three variables, could now fortunately be integrated. Mr Bernoulli also achieved his purpose by showing, by means of a table calculated according to this formula, how much less harmful inoculation smallpox is than natural smallpox. For that, it was sufficient to assume a mean value for m and n and to regard it as a constant. Mr B. actually puts $m = n = 8$.

§ 129. However, neither n nor m appears to be constant. Children, who are allowed to run about from the third or fourth year, are for that reason more exposed to the danger of smallpox than younger ones who are still rocked, carried or led by the hand. From this point of view, $1:n$ appears to become greater with the years, at least up to a certain age, for n is inversely proportional to the danger of contracting smallpox. m has a different quality, for $1/m$ represents the measure of the danger of dying of smallpox, or the measure of the mortality of smallpox. m, like n, depends on the age groups, because smallpox is more dangerous or vicious and more deadly in one year than in another. However, we consider the two values only with regard to age, and since $1/m$ appears to be greater in the first few years of childhood, and the outbreak of smallpox is prevented, thus the danger is increased.

§ 130. This can actually be concluded with regard to the values of m and n. The particular determinations would have to be found from experience. Of such experience I have only the little that is contained in the third volume of the treatise of the Naturalist Society in Zürich. Dr Sulzer in Winterthur took the trouble in 1763 to count the children who died of natural smallpox or survived the disease. For most of them the age is given, and so I have been able to draw up the following little table from his list.

Age Years	Children with smallpox z	Died of it v	Survived it $z-v$
0	3	1	2
1	10	3	7
2	9	3	6
3	14	3	11
4	12	0	12
5	12	1	11
6	8	1	7
7	1	0	1
8	1	1	0
9	—	—	—
10	2	2	0
Total	72	15	57

Of course, one should have such a table for several years and a larger number of children. Meanwhile we see from it that one-third of the children under 3 years and 3/14 in the fourth year died of smallpox, but that most of those who survived the fourth, fifth and sixth year recovered, whereas most of those over 8 years died of smallpox. Thus $\frac{1}{m} = \frac{v}{z}$ was initially one-third, subsequently decreased almost to zero, and finally increased to nearly 1. Therefore, the value of m was very variable. Nothing can be directly determined from this table as regards the value of n.

§ 131. Since the figures of this table, because of the very small number of observations, are still far too irregular, I shall add another one, where in particular the column v can be adjusted. It is a list of all those who over a period of 15 years died at the Hague of smallpox in each year of age, having been published in print in that place. I shall present this in the following table.

Age	Died	Age	Died	Age	Died	Age	Died	Age	Died
1	172	11	11	21	3	31	4	41	0
2	170	12	14	22	9	32	0	42	1
3	179	13	9	23	2	33	3	43	0
4	224	14	4	24	4	34	2	44	1
5	160	15	9	25	6	35	1	45	1
6	148	16	4	26	7	36	3	46	0
7	114	17	3	27	2	37	2	47	0
8	78	18	6	28	2	38	1	48	0
9	58	19	3	29	3	39	2	49	2
10	23	20	1	30	0	40	0	+	4
Total	1,326		64		38		18		9

§ 132. The overall total is 1,455 and, therefore, almost 100 times greater than in the preceding table. One also sees at once that the figures proceed in very orderly progression, that they increase, though not much, up to the fourth year, but decrease much more from then on. Of the 1455 there are only 129 who died of smallpox after the tenth year. After the twentieth year, only 65 died, after the thirtieth year only 27, after the fortieth only 9 and after the fiftieth only 4.

§ 133. If we now assume, according to Süssmilch, that those who die of smallpox related to those who die in general are as 2 is to 25, then according to the 1,455 from the table who die of smallpox, the number of all deaths must be 18,188 and, if everything is in a state of stability, as many must be born. Therefore, of 18,188 newborn children, 1,455 will die of smallpox by and by, and the figures of the preceding table show how this happens year by year. However, how many of the 18,188 actually die and how many survive may easily be seen from the general table of mortality (§ 24). [5] The following table, therefore, results.

Age Years	Living y	Deaths generally Δy	Deaths from smallpox v	Other Deaths $\Delta y - v$	Future deaths from smallpox
0	18,188				1,455
1	13,441	4,747	172	4,575	1,283
2	12,331	1,110	170	940	1,113
3	11,713	618	179	439	934
4	11,289	424	224	200	710
5	10,982	307	160	147	550
6	10,727	255	148	107	402
7	10,516	211	114	97	288
8	10,341	175	78	97	210
9	10,196	145	58	87	152
10	10,072	124	23	101	129
20	9,319	753	64	689	65
30	8,215	1,104	38	1,066	27
40	6,799	1,416	18	1,398	9
50	5,396	1,403	5	1,398	4
	0	5,396	4	5,392	0

§ 134. In this table the first three columns are calculated from the general table of mortality (§ 24). [5] The fourth v contains the figures of the immediately preceding table or the observations made at the Hague showing how many died of smallpox each year. If these are subtracted from the total deaths Δy, there remain $\Delta y - v$ of the fifth column who die of other diseases each year. The sixth column contains the total of the figures v added from the bottom upwards, and consequently shows how many of those living at each time y will die of smallpox in the future.

§ 135. Several columns are still missing from this table which could easily be added if in the observations of § 131 those had been counted who have happily survived smallpox at each age. This could still be done if one were to assume, following Mr Bernoulli, that for every one person dying of smallpox, 7 must be reckoned to have survived the disease. For all one would have to do would be to multiply the figures of the fourth column with $7+1 = 8$, in order to find how many contract smallpox each year.

§ 136. Meanwhile we can determine from the table of § 133 the law of mortality which would apply if smallpox did not exist or at least was not fatal. At every age τ, out of the survivors y, v die of smallpox in one year, and $\Delta y - v$ of other diseases. Now it is clear that if v did not die of smallpox, at least some of them would die of other diseases during the year. We can now consider the two extreme cases here.

I. If one assumes that all the v die of smallpox right at the beginning of the year τ, there remain in all $y - v$, and of these $\Delta y - v$ then die of other diseases.

II. If one assumes, on the other hand, that the v die of smallpox at the end of the year τ, then of the total number y, $\Delta y - v$ have already died of other diseases. We must now take the mean between these two cases and, therefore, assume that of $y - \frac{1}{2}v$ who start their τth year, $\Delta y - v$ die of other diseases in the year τ; therefore, $y - \frac{1}{2}v - \Delta y + v = y - \Delta y + \frac{1}{2}v$ reach the $(\tau+1)$th year of their age. Therefore, their number decreases in the proportion of $y - \frac{1}{2}v$ to $y - \Delta y + \frac{1}{2}v$, that is from $y - \frac{1}{2}v$ to $y' + \frac{1}{2}v$. For $y - \Delta y = y'$ is the number which in the table of the preceding paragraph stands at the year $\tau + 1$. In this way one obtains the following arrangement.

$$\frac{18{,}188 - \frac{172}{2}}{13{,}441 + \frac{172}{2}} = \frac{18{,}188}{13{,}591}$$

$$\frac{13{,}441 - \frac{170}{2}}{12{,}331 + \frac{170}{2}} = \frac{13{,}591}{12{,}676}$$

$$\frac{12{,}331 - \frac{179}{2}}{11{,}713 + \frac{179}{2}} = \frac{12{,}676}{12{,}222}$$

§ 137. Thus one obtains the following table showing how the 18,188 newborn decrease in number each year, both when smallpox is included and when it is excluded.

Here the column y shows the decrease of the 18,188 newborn in the state of smallpox. On the other hand, the column r shows how they would diminish in number if smallpox either did not exist or at least was not fatal. The advantage of this latter condition becomes evident from the column $r - y$. One sees that it increases up to 10% as early as the eighth year and, thereafter, still grows a little larger because in the fiftieth year, instead of the 5,396 according to the common condition, 6,063 would remain alive if smallpox was not fatal.

Years	Living y	Deaths Δy	Living r	Deaths Δr	$r - y$
0	18,188		18,188		
1	13,441	4,747	13,591	4,597	150
2	12,331	1,110	12,676	915	345
3	11,713	618	12,222	454	509
4	11,289	424	12,012	210	723
5	10,982	307	11,769	243	787
6	10,727	255	11,577	192	850
7	10,516	211	11,472	105	956
8	10,341	175	11,366	106	1,025
9	10,196	145	11,271	95	1,075
10	10,072	124	11,158	113	1,086
20	9,319	753	10,392	766	1,073
30	8,215	1,104	9,200	1,192	985
40	6,799	1,416	7,633	1,567	834
50	5,396	1,403	6,063	1,570	667
—	0	5,396	0	6,063	0

Moreover, in this table actually

$$\frac{\Delta r}{r} = \frac{\Delta y - v}{y - \frac{1}{2}v}$$

§ 138. Much more cannot be concluded from the observations made at the Hague and mentioned earlier (§ 131); for it can easily be seen from the columns v and $\Delta y - v$ of the table of § 133 that, for instance, in the fourth, fifth, sixth and seventh years of age more children die of smallpox than of other diseases. However, one would be able to determine many more particular circumstances, if the values of n and m, or at least the ratio between the two were known for each age. However, one only knows, at least as far as I am aware, that of 7, or as Mr Bernoulli assumes, of 8, who contract smallpox, one dies, and therefore the average value of $m = 7$ or $= 8$.

§ 139. Meanwhile, one can readily see that the values of n and m are very variable. In many children smallpox originates spontaneously, but in most of them by transmission, because, once smallpox has started, most children who have not yet had it are attacked by it, unless every care is taken to prevent this. Therefore, the value of n depends to a certain extent on this care and cannot actually be determined. However, such care of children who still have to be nursed, is easier and more practicable than of those who run about and very often bring home the attack of smallpox. The value of m depends very much on the nursing of the children and the skill of the doctor. Both seem to be more difficult in the first few years of life than when the children already have more strength and are able to talk.

§ 140. If one were able to disregard such mostly external circumstances, which include the diversity of countries and modes of living, one could conclude in general that the more children are attacked by smallpox, the more die of it, and that therefore n and m are in a constant ratio. However, one cannot very well put the aforesaid circumstances aside. Since they are very variable, the values of n and m likewise cannot be confined within very narrow limits.

§ 141. This now makes it necessary that I make good, at least in the form of an example and in order to explain the method, that which is still lacking in the preceding tables, that is, to the extent that the first ten years of age are concerned. For this purpose I set out the following, in order to make the table of § 131 more complete.

τ	z	v	$z - v$
Age Years	Children who contract smallpox	Children who die of smallpox	Children who survive it
1	460	171	289
2	750	181	569
3	1,160	194	966
4	1,520	193	1,327
5	1,690	181	1,509
6	1,610	157	1,453
7	1,160	124	1,036
8	750	91	659
9	400	62	338
10	190	39	151
Total	9,690	1,393	8,297

§ 142. Now here the third column differs from the figures as given by the experience (§ 131) only to the extent that they have been made more regular. [6] I have tried to determine the figures of the second column according to different considerations. The first was that the sum of these figures should be approximately seven times greater than the sum of the third column. Then I consulted Sulzer's observations (§ 130), in order to adjust the variability of the values of m as much as possible to them, and to obtain a somewhat regular progression in the figures of the second column. The figures of the fourth column are the difference between the figures in the second and third columns, as may be seen from the heading $z - v$. This column, therefore, has the same degree of arbitrariness as the second column. However, as I shall only use it in the form of an example, the uncertain or unreliable does not matter with this intention, although I would have preferred that the calculation should be carried out when these two columns had been determined throughout and accurately from a large number of experiences. The calculation is, meanwhile, as follows.

§ 143. First we can see how many of the 18,188 newborn children still living each year have not yet had smallpox. For this purpose we shall be able to use the column $(z - v)$ of § 141 and the two columns r and Δr of § 137 as follows. At the beginning of any year τ, denote by w the number of children still living who have survived smallpox. These will die in the time of one year in the proportion of r to Δr, so that at the end of the year $w\left(1 - \frac{\Delta r}{r}\right)$ will survive.

§ 144. However, in the same year $(z - v)$ new ones are added who survive smallpox in this year. Now, in order to find a mean, one has to assume that these survive smallpox in the middle of the year; thus by the end of the year their number is reduced by other diseases in the proportion of r to $\left(r - r\frac{\Delta r}{2r}\right)$

and is, therefore, only $$(z - v)\left(1 - \frac{\Delta r}{2r}\right)$$

§ 145. If these two figures are combined into one total, it follows that at the beginning of the year $(\tau + 1)$

$$w' = w - w\frac{\Delta r}{r} + (z - v) - (z - v)\frac{\Delta r}{2r}$$

children who have already survived smallpox are still living. In this manner one can calculate from year to year the number of those still living in each year after having survived smallpox. If one subtracts it from those still living y, this leaves $y - w = \eta$, the number of those at each age who have not yet had smallpox. If one further subtracts from each figure η the next following one, there remains $\Delta\eta$, and this difference shows by how much those who have not yet had smallpox are reduced each year either by death or by contracting smallpox. If one further subtracts from these $\Delta\eta$ all those who get smallpox each year, whether they die of it or not, there remain $\Delta\eta - z$ who die before smallpox. If one finally also subtracts these from $\Delta y - v$, this leaves those who die each year after surviving smallpox, and this number will be

$$= \left(w + \frac{z - v}{2}\right)\frac{\Delta r}{r}$$

and will coincide with that which one has already had to calculate for each year, in order to find w.

§ 146. According to this information I have now calculated the following table, using the v given in § 141, in order to find $\Delta y - v$.

I	II	III	IV	V	VI	VII	VIII	IX	X
τ	y	w	$y-w=\eta$	$\Delta\eta$	z	$\Delta\eta - z$	$\Delta y - v$	$\Delta y - v - \Delta\eta + z$	v
Age	Total living	Surviving smallpox	Living before smallpox	Decrease	Contracting smallpox	Dying before smallpox	Dying of other diseases generally	Dying after smallpox	Dying of smallpox
0	18,188		18,188						
1	13,441	253	13,188	5,000	460	4,540	4,576	36	171
2	12,331	792	11,539	1,649	750	899	929	30	181
3	11,713	1,712	10,001	1,538	1,160	378	424	46	194
4	11,289	2,999	8,290	1,711	1,520	191	231	40	193
5	10,982	4,432	6,550	1,740	1,690	50	126	76	181
6	10,727	5,801	4,926	1,624	1,610	14	98	84	157
7	10,516	6,779	3,737	1,189	1,160	29	87	58	124
8	10,341	7,373	2,968	769	750	19	84	65	91
9	10,196	7,640	2,556	412	400	12	83	71	62
10	10,072	7,713	2,359	197	190	7	85	78	39

I shall now repeat once more what has already been recalled (§ 141), namely that this table should only be regarded as an example which illustrates the method of its calculation. For this reason I have not carried it further. Whoever wishes to gather the experience necessary for more accurate calculations, will easily be in a position to derive from it such a table which will have the proper reliability.

§ 147. Meanwhile, until this be done, I have calculated another such table for the first 10 years, assuming like Mr Bernoulli has done that $n = m$. This was done so as to see the difference of the result. Since, in doing so, I was further able to assume that those dying each year of smallpox v were in accordance with the experience derived in the third column of the table (§ 141), it was not necessary for me to assume an arbitrary or mean value for n and m, as Mr B. had done, but everything could be determined by means of the simple assumption that $n = m$.

§ 148. In order to explain this it will be sufficient to show, when one knows η for a given year τ, how the value of η can be found for the next year. For at the beginning of the first year $\eta = y = 18{,}188$, and thus η can then be found for every following year.

§ 149. Therefore, since in each year τ the number of those dying of smallpox equals v, one finds the number of those contracting smallpox in the same year

$$z = mv$$

and the number of those who have not yet had it at the beginning of the year

$$\eta = nz = nmv = mmv$$

therefore

$$m = \sqrt{(\eta : v)}$$

and

$$z = mv = \sqrt{(\eta v)}$$

thus z is the mean proportional number between η and v, and since v is known (§ 141), and η must be regarded as having already been calculated from the preceding year, so by this means z can also be calculated. Furthermore, since $w = y - \eta$ is also known for each year, w can thus be found for the next year according to the formula of § 145

$$w' = w + (z - v) - \left(w + \frac{z - v}{2}\right)\frac{\Delta r}{r}$$

and thus also

$$\eta' = y' - w'$$

§ 150. In this way the following table resulted, which has columns of the same kind as the preceding table.

τ	y	w	$y-w=\eta$	$\Delta\eta$	z	$\Delta\eta-z$	$\Delta y-v$	$\Delta y-v$ $-\Delta\eta+z$	v
0	18,188		18,188						
1	13,441	1,391	12,050	6,138	1,764	4,374	4,576	202	171
2	12,331	2,550	9,781	2,269	1,477	792	929	137	181
3	11,713	3,621	8,092	1,689	1,378	311	424	113	194
4	11,289	4,606	6,683	1,409	1,250	159	231	72	193
5	10,982	5,423	5,559	1,124	1,100	24	126	102	181
6	10,727	6,105	4,622	937	934	3	98	95	157
7	10,516	6,680	3,836	786	757	29	87	58	124
8	10,341	7,116	3,225	611	591	20	84	64	91
9	10,196	7,440	2,756	469	447	22	83	61	62
10	10,072	7,653	2,419	337	328	9	85	76	39

§ 151. The differences between this table and the preceding one [7] mainly derive from the figures z. Here these are greatest at the beginning, as distinct from the preceding table where they were greatest in the fourth, fifth and sixth years of age. This is also the reason why in the first few years the figures w and η differ considerably in the two tables. However, they come ever closer to each other in the following years, so that in the 10th year there is little difference between them. According to both calculations one can, therefore, assume that out of all 10-year-old children, a quarter have not yet had smallpox.

§ 152. Furthermore, since in the last table $n = m = \eta : z$

we find for

$$\tau = 1 \quad n = m = \frac{18{,}188}{1{,}764} = 10\cdot 3$$

$$2 \qquad \frac{12{,}050}{1{,}477} = 8\cdot 1$$

$$3 \qquad \frac{9{,}781}{1{,}378} = 7\cdot 1$$

$$4 \qquad \frac{8{,}092}{1{,}250} = 6\cdot 5$$

$$5 \qquad \frac{6{,}683}{1{,}100} = 6\cdot 0$$

etc.

These proportions do not agree particularly well with Sulzer's experience (§ 130) where in the first 3 years out of 22 children who were attacked by smallpox, 7 died; therefore $m = 22/7 = 3\frac{1}{7}$, instead of here, taking the mean of the first three values of m, $m = 8\frac{1}{2}$. At any rate it follows from this that we should not put $n = m$ throughout. This occurs only at those ages where the danger of being attacked by smallpox is as great as the danger of dying of it. According to the preceding table this occurs about the fourth year, and this is not at all improbable. On the other hand, it is very unlikely that, according to the last table, out of 10 children in the first year one should be down with smallpox. However, in accordance with the preceding table, it is much more likely that hardly one out of 40 contracts smallpox. In any case it is thought that the material for smallpox must first ripen in the children, at least if it is to break out spontaneously. For this purpose one or two years are rarely sufficient, and usually it does not happen until the life of the children becomes more disorderly, by eating on the sly and running about, than it is in the first two years. Finally, the fact that children, before they are able to run about, are more carefully preserved from infection, has already been noted and is easy to understand. I, therefore, conclude from all this that in the first few years the preceding table comes much closer to the truth than the latter. From the seventh to the tenth year the two do not differ much from each other. If I now also consider that the fourth, fifth and sixth years should make good the difference showing itself in the first few years of the second table, the preference always falls entirely on the former table, although even this will still need to be entirely checked by a number of experiences.

§ 153. I shall now have to produce a small proof which concerns the numbers of those dying after having had smallpox. I noted earlier that these figures are found both by means of the formula

$$(\Delta y - v) - (\Delta \eta - z)$$

and by the formula

$$\left(w + \frac{z-v}{2}\right)\frac{\Delta r}{r} \qquad (\S\ 145)$$

This becomes evident from the formula

$$w' = w - w\frac{\Delta r}{r} + (z-v) - (z-v)\frac{\Delta r}{2r}$$

(§ cit) as follows. One transforms this latter formula by an easy reduction into

$$w' - w = (z-v) - \left(w + \frac{z-v}{2}\right)\frac{\Delta r}{r}$$

Since $\qquad w = y - \eta$

so when one takes the differences $\qquad \Delta w = \Delta y - \Delta \eta$

Since $\qquad \Delta w = w' - w$

it follows that also $\qquad w' - w = \Delta y - \Delta \eta$

therefore $\qquad \Delta y - \Delta \eta = (z-v) - \left(w + \frac{z-v}{2}\right)\frac{\Delta r}{r}$

and from this follows, by a simple transposition

$$-\Delta y - v + \Delta \eta + z = \left(w + \frac{z-v}{2}\right)\frac{\Delta r}{r}$$

However, because y and η are decreasing in size, Δy and $\Delta \eta$ are negative, and therefore, if one wants to regard these differences as positive, as was done earlier, the formula should actually be

$$\Delta y - v - \Delta\eta + z = \left(w + \frac{z-v}{2}\right)\frac{\Delta r}{r}$$

All that follows from this is that z is determined through η and $\Delta\eta$ by means of the figures y, Δy, v, r and Δr, if one puts $y-\eta$ for w. Moreover, both z and v are differences, so that one may put

$$z = \Delta\zeta$$
$$v = \Delta u$$

and thus obtain the differential formula

$$\Delta y - \Delta\eta - \Delta u + \Delta\zeta = \left(y - \eta + \frac{\Delta\zeta - \Delta u}{2}\right)\frac{\Delta r}{r}$$

If one again puts $y - \eta = w$

and $\Delta\zeta - \Delta u = \Delta q$

then one can convert it into the following simpler formula

$$\Delta w = \Delta q - \left(w + \frac{\Delta q}{2}\right)\frac{\Delta r}{r}$$

But one cannot do much with the integration

§ 154. In the table (§ 146) various columns can still be compared with one another. For example, the third column states how many at each age have already survived smallpox, and the ninth column how many of these die each year. From this it could be found out of how many of these there dies one each year. However, it should be noted that the third column gives those living at the beginning of every year, but that the ninth column includes among the dying those who in the course of the year survive smallpox and die of other diseases and accidents before the end of the year. The required proportion must therefore be calculated from the table of § 137 since it equals $r:\Delta r$.

§ 155. On the other hand, in the table of § 146, the fourth column can very well be compared with the seventh and the tenth. The former states how many at the beginning of each year have not yet had smallpox. The seventh column shows how many of these die in the course of the year without contracting smallpox. The tenth, column finally shows how many of these die of smallpox. Such comparisons now provide the following table.

I	II	III	IV	V	VI	VII	VIII
τ	$\frac{y}{\Delta y}$	$\frac{r}{\Delta r}$	$\frac{\eta}{\Delta\eta - z}$	$\frac{\eta}{v}$	$\frac{\eta}{\Delta\eta - z + v}$	$\frac{\eta}{z} = n$	$\frac{z}{v} = m$
Age	Generally one dies out of	After smallpox one dies out of	Before smallpox there die of other causes one out of	There die of smallpox one out of	Before smallpox there die of all causes one out of	Smallpox is contracted by one out of	Of those who contract smallpox there die one out of
1	4	4	4	106	4	39	3
2	12	15	15	73	12	18	4
3	20	28	35	60	20	10	6
4	27	58	53	52	26	7	8
5	37	49	166	46	36	5	9
6	43	61	468	42	38	4	10
7	51	110	170	40	32	4	9
8	60	108	197	41	34	5	8
9	71	120	247	48	40	7	7
10	82	99	365	65	56	13	5

§ 156. This table, which I am likewise presenting only in the form of an example, shows the risk of smallpox and the risk of mortality both in general and with respect to smallpox in particular. The expressions placed above each column show how the figures are to be understood. However, I shall illustrate them by the example of the fifth year. Accordingly, of the children who are between 4 and 5 years old,

Col. II	there dies generally in the year 1 out of 37.
Col. III	If smallpox did not exist or at least was not fatal, 1 out of 49 would die.
Col. IV	If only those children are considered who have not yet had smallpox, then out of 166 children who start their fourth year, 1 dies of other diseases without contracting smallpox.
Col. V	However, of the same children 1 out of 46 dies of smallpox.
Col. VI	On the other hand, of the same children 1 out of 36 dies without distinction of smallpox and other diseases.
Col. VII	Of the same children, 1 out of 5 is attacked by smallpox in the fourth year.
Col. VIII	However, of those who have smallpox in the fourth year, 1 out of 9 dies of smallpox.
Col. III	And of those who survive smallpox in the fourth year, 1 out of twice 49, or 98, dies in the fourth year (§ 144).

§ 157. The figures of the table are, throughout, in inverse proportion to the risk or in straight proportion to the safety. A comparison of the third and sixth columns shows how much more secure children are from death after surviving smallpox, since the figures of the third increase much more quickly than those of the sixth. If one compares the third and fifth columns, one finds that as early as from the fourth year onwards smallpox alone is more deadly than all other diseases taken together once smallpox has been survived. It becomes evident from the fourth column that the risk of dying of other diseases before contracting smallpox very quickly decreases from the first year onwards, since the figures of this column increase the quickest. One should conclude from this that generally children who do not get smallpox in the first few years of life, either spontaneously or from others, have a better constitution, since they remain more exempt from smallpox and other fatal diseases, and that, on the other hand, children who easily contract smallpox are for the most part more susceptible to other fatal diseases. For the figures of the third column are generally much smaller than those of the fourth. If this should be due to smallpox itself, it would be of advantage to try to preserve children from smallpox as much as possible during the first six years, and at the same time it would be worth while, as in many other respects, to determine by experience whether the result is different in relation to inoculation smallpox. It appears from the fifth column that the risk of dying from smallpox is approximately equal from the fourth to the ninth year. Moreover, all these considerations require accurate adjustment from a number of more detailed experiences, and they serve our purpose only for temporary guidance. Similar experiences and considerations also apply to measles, for as far as I can see from the last table appended to Süssmilch's work, it was about a third less fatal than smallpox in 1746 in Berlin, the latter sending 186, and the former 119, children to the grave.

Explanatory notes and comments

[1] § 125. See Süssmilch (1761).

[2] § 125. In reading this paper it must be remembered that, although not specifically stated, Lambert assumes, as did Bernoulli (1766), that recovery from an attack of smallpox gives lifelong immunity. Thus phrases like "dying before smallpox" or "dying after smallpox" imply death from diseases other than smallpox.

[3] § 126. The figures 261 and 61 are the values of d_0 and d_1 with $l_0 = 1{,}000$ (in modern notation) according to the mortality table on which Lambert bases his calculations—see note [5].

[4] § 127. See Bernoulli (1766). (Translation in Bradley, 1971.)

[5] § 133–4. This is the mortality table referred to in § 2.3.6 on page 347. It is interesting to note that in this mortality table Lambert follows the present-day practice of putting d_x on the same line as l_x, but that in the various tables of this paper he puts d_x and other decrements and increments on the line of l_{x+1}.

[6] § 142. It is not the whole story to say that the only change in the second column is the smoothing of the series of smallpox deaths because in that process the total smallpox deaths up to age 10 have been increased, without explanation, from 1,326 (§ 131) to 1,393 (§ 141). This increase destroys the argument by which the initial value of y (18,188) was obtained (§ 133).

[7] § 151 and § 152 contain several references to the tables of § 146 and § 150 without ever mentioning these paragraphs. It may be helpful to note that the table of § 146 is called "the preceding" table or, once, "the former" table, while the table of § 150 is called "this", "the last", "the latter", or "the second" table.