A RESERVE BASIS FOR MATURITY GUARANTEES IN UNIT-LINKED LIFE ASSURANCE

by

W. F. SCOTT, M.A., Ph.D., F.F.A.

1. Introduction

1.1. Unit-linked policies form an established and substantial part of United Kingdom life assurance business. Many such policies carry a guarantee that if the proceeds of the units (a term we shall use whether the investment medium is an authorised unit trust or an “internal fund” of the office) are less than a fixed sum (for example, the total premiums paid) at a given date or dates, then the deficiency will be made up by the office.

It will first be assumed that the policies are of the accumulating type (Category A), and we defer discussion of Category B policies until section 8 (these terms were defined by Grant and Kingsnorth1, and the literature on the latter is discussed by Fine2.) Both single and annual premium contracts will be considered and it will be assumed that investment is in a net fund; for calculations relating to pension and other gross funds, the mean rate of growth of the units should be increased to allow for the absence of taxes, and references to capital gains tax (see section 8) should be ignored. The main purpose of this paper is to determine a basis for the calculation of “prudent” reserves for maturity guarantees at the date of issue.

1.2. In section 2 we discuss some valuation principles, arriving at the conclusion that, whereas “expected” costs, with or without adjustment, form a satisfactory basis for premium calculations, they are in general insufficient for prudent maturity guarantee reserves. For this purpose we advocate “likely maximum” costs, which are such that the probability of solvency is 1−γ, γ being a small pre-assigned value (in our examples γ = ½%). Although the probability of “ruin” is non-zero, one must remember that if unit prices fell to such low levels that insolvency occurred as a consequence of maturity guarantees, many other life offices and financial institutions could
Maturity Guarantees

well be in difficulty, though probably to a smaller extent (assuming that the units are mainly invested in a broad spread of United Kingdom equities). We therefore hope that "likely maximum" costs steer a proper course between the Scylla of inadequate reserves and the Charybdis of over-provision.

1.3. Before reserve calculations can be carried out, estimates of the distribution of the proceeds of the units are required. In the case of United Kingdom equity investment we studied the nature of stock market price movements, concluding that yearly growth rates can be represented approximately by the lognormal distribution, and that yearly stock market movements are not random, but negatively correlated. A mathematical model was then developed; the evaluation of the resulting distributions—which is, in the annual premium case, a problem of some interest in its own right—is discussed in the Appendix.

1.4. Some results are given in section 7, where it was assumed, inter alia, that the mean annual growth rate, including reinvested net income, is 7%. It should be noted that these results apply directly to broadly-based U.K. equity-linked policies only, and will probably require adjustment if investment is in, for example, a limited sector of the market, a small number of companies or properties, commodities or overseas equities.

The "likely maximum" method assumes that the proceeds will not fall below certain levels, called "minimum" proceeds (for the probability that these levels will not be achieved is γ). Thus, for example, for a 10-year annual premium policy with a guaranteed return of all office premiums paid at the maturity date, the minimum proceeds are 8.00 per unit annual premium, which is rather more than the level of proceeds actually achieved by some 10-year annual premium policies maturing at the nadir of the stock market in late 1974. The initial reserve is £1230 per £1000 annual premium, which is equivalent to 15.1% of annual premiums (cf. section 7).

1.5. On the assets side, maturity guarantee reserves should be held in fixed-interest stocks with good collective security, maturing not later than the appropriate guarantee dates. They should not, of course, be invested in the units themselves. Matching of assets and liabilities presents major problems only in the case of "open-ended" guarantees, which are discussed in section 8.

In section 8 we also discuss various other practical questions such
as the effects of mortality, withdrawals, capital gains tax, monthly premiums and the interaction with existing business.

1.6. Although not entirely germane to the present study, it may be of interest to note that if, as we argue here, stock market prices are subject to speculative movements which cause market values to fluctuate above and below "true" worth, it must be concluded that there may be circumstances when market prices are not appropriate in the valuation of the quoted equity assets of life offices, pension funds and other financial institutions: the average price over the past year, say, would probably give a closer estimate of the "true" worth than the market price at the valuation date. (An alternative approach is to use compound interest valuation techniques, but the determination of the assumptions is difficult and the results are open to dispute, unlike the yearly average of market prices.) This might help prevent undignified tampering with the actuarial basis of the liabilities when stock market prices are very depressed, as they were in late 1974! If, however, the policies being valued have guaranteed surrender values (applying now or in the near future) then the proper asset value for comparison is the current market value, no matter how low, or, in some circumstances, rather less than market value.

2. A review of some valuation principles

2.1. In the following discussion we shall make the assumption that the matching of assets and liabilities presents no problems. The present value of the liabilities of a life office (or pension fund, friendly society, etc.) may be regarded as a random variable depending on one or more factors, for example, mortality, interest and expenses, which are themselves variables, with distributions which can be estimated by the actuary. The usual "actuarial value" ascribed to the liabilities is the mean, or expected, value of the corresponding liability function, usually on the basis of conservative estimates of future mortality, interest, and so on. With the exception of offices in which all or most of the liabilities depend on the survival or otherwise of a small number of lives—and in these cases it is of course prudent to assume that all or most of these lives will soon cause claims to arise—the actuary does not normally attempt to cover the "worst possible" eventualities, such as the immediate death of all the policyholders (if they were all to die tomorrow, this would almost certainly be due to a nuclear war or some such catastrophe, and the solvency or otherwise of life offices would probably be of relatively minor interest: that is, the "we'll all go together when we go" argument applies).
Maturity Guarantees

The fluctuations about the mean of the liability function are, for a reasonably large life office, relatively small, and all but the largest fluctuations are likely to be covered by the actuary's traditional "errors on the side of caution" in determining mortality, interest and other factors. A rough measure of these fluctuations is the ratio of the standard deviation to the mean of the liability function. As a simple example we consider the liability function associated with 20,000 one-year term assurance policies each for £1000 on independent lives aged 30, when \( q_{30} = 0.001 \) and interest and expenses are ignored.

The mean liability is

\[
1000 \times 20,000 \times 0.001 = 20,000,
\]
and the standard deviation is

\[
1000 \times \sqrt{20,000 \times 0.001 \times 0.999} = 4469.90.
\]

Therefore

\[
\frac{\text{standard deviation}}{\text{mean}} = 0.22349, \text{ or } 22\%.
\]

We shall show that, according to our model, the corresponding ratio for maturity guarantees in unit-linked life assurance is very much larger than this: in our examples it varies, for annual premium policies with a "spot" guarantee of a return of all office premiums, from 428% at term 10 years to 2428% at term 30 years! These figures confirm the commonly-held view that the underwriting of maturity guarantees is a very risky activity compared with most other forms of life assurance.

2.2. It is highly probable that, as in the above examples, a small variation on the side of caution in the assumptions regarding future unit prices will not be sufficient to cover even relatively minor fluctuations from the mean: a new procedure is required, one in which a probability of "ruin" \( \gamma \) is specified in advance, from which we can calculate the reserve \( L_\gamma \) such that, with probability \( 1-\gamma \), the guarantee commitments will be met. The position is illustrated in Fig. 1 below.

(The area under the entire curve in Fig. 1 is less than 1, because there is a large probability that no payment will be required under the guarantee.)

E forms a suitable basis for the calculation of premiums for maturity guarantee provisions in unit-linked policies, but adjustments may be required to cover expenses, to simplify the presentation of policies to the public, and because of competition. (The
so-called "equivalence principle" for premium calculations, based on expected values, has been criticised by some actuaries as unsuitable for "high-risk" insurance business. It is not, of course, necessary for the public to be charged any premium for the guarantee at all, and the guarantee may be offered, for a small extra premium, as an "optional extra".

With a suitable choice of $\gamma$ (in our examples $\gamma = \frac{1}{2}\%$, or 1 in 200) $L\gamma$ forms a suitable basis for reserve calculations. Although the probability of "ruin" is non-zero, insolvency will occur only if unit prices fall so heavily that many other life offices and financial institutions will probably also be in difficulties; in other words, "we'll all go together when we go". This observation is based on the premise that the units of the office concerned are mainly invested in a broad range of U.K. equities; it may well be invalid if the units are invested in, for example, certain commodities, overseas countries, a small number of properties or companies, or a small sector of the market.

2.3. To summarise, in considering any valuation for solvency one should ask, in the words of the well-known hymn, the question:

"Will your anchor hold in the storms of life?",

where the anchor is, of course, of the actuarial variety.

We consider that the bases for reserve and premium calculations should be distinct (a view expressed by Benjamin) and, in our opinion,
expected costs are nearly always insufficient for reserve calculations for maturity guarantees.

Before $E$ and $L$, can be calculated we require estimates of the probability density function of $L$. We shall attempt to provide such estimates, in the case of units invested mainly in a broad spread of U.K. equities, by considering the performance of the stock market. The resulting figures do not apply directly to other investment media, but estimates may perhaps be obtained by making suitable adjustments.

3. The behaviour of U.K. equity prices

3.1. In order to estimate the distribution of the proceeds, in $n$ years’ time, of an investment of 1 in units we shall try to answer the following two questions:

(i) What are the distributions of the proceeds, including reinvested net income, of an investment of 1 in each of the next $n$ years, $x_1, x_2, \ldots, x_n$?

(ii) Are the variables $x_1, x_2, \ldots, x_n$ independent?

(i) Statistics of past stock market price movements, such as those of the Financial Times Ordinary and Actuaries indices, are readily available. There is, however, the following theoretical argument in support of the use of the lognormal distribution, which has probability density function

$$\frac{1}{\sqrt{2\pi\sigma x}} \exp \left( -\frac{1}{2} \log (x - \mu)^2 \right), x > 0,$$

as an approximation to the distribution of each $x_t$. (Some properties of the lognormal distribution are given by Aitchison and Brown, (chapter 1). In particular, prices cannot be negative under this distribution).

Let $P(t)$ be the price of a unit at time $t$ ($0 \leq t \leq 1$), excluding reinvested net income (we assume $P(t)$ is continuously differentiable), and let the year be divided into $n$ short periods (e.g. days, weeks). Let it be assumed that with each period there are associated variables $z_1, z_2, \ldots, z_n$, independent of each other, which change the price in such a way that

$$\frac{P \left( \frac{i}{n} \right) - P \left( \frac{i-1}{n} \right)}{P \left( \frac{i-1}{n} \right)} = k \cdot z_i, \ 1 \leq i \leq n,$$

(3.1)
where \( k \) is a constant. It follows that
\[
\sum_{i=1}^{n} \left\{ \frac{P\left(\frac{i}{n}\right) - P\left(\frac{i-1}{n}\right)}{P\left(\frac{i-1}{n}\right)} \right\} = k \sum_{i=1}^{n} z_i,
\]
and, letting \( n \to \infty \), there is, by the central limit theorem, a normally distributed variable \( z \) such that
\[
\log \left( \frac{P(1)}{P(0)} \right) = \int_0^1 \frac{1}{P(t)} \cdot \frac{dP(t)}{dt} \, dt = k \cdot z
\]
(it is also assumed that a condition of the form given by Cramer\(^5\) (17.4.3) holds). Thus \( \log \left( \frac{P(1)}{P(0)} \right) \) is normally distributed, and hence \( \frac{P(1)}{P(0)} \) is a lognormal variable. \( \frac{P(1)}{P(0)} \) is the ratio of the price at the end of the year to that at the beginning, and hence represents the "capital-only" growth rate of the units. The net income is smaller and much less variable than the capital movements, and its addition will probably have only a relatively small effect on the distribution of growth rates, although the mean will of course, be increased. Hence each \( z_i \) has, approximately, a lognormal distribution.

Equation (3.1), which is known as "the law of proportionate effect", is justifiable in the case of equity price movements if the various factors (e.g. economic indicators, earnings of the constituent companies, etc.) tend to affect prices proportionately. One of the most important factors is the latest earnings, and it is to be expected that these will increase or decrease, relative to what the market expects, by "random" proportionate factors; for example, an increase of 20\% or a decrease of 10\% in the actual or anticipated earnings of a particular company. Although this "proportionate law" may not apply exactly to all the factors influencing share prices, it is more likely to be accurate than the corresponding "arithmetic law"

\[
P\left(\frac{i}{n}\right) - P\left(\frac{i-1}{n}\right) = k \cdot z_i, \quad 1 \leq i \leq n.
\]

which gives the normal distribution rather than the lognormal.

The above theoretical argument must of course be tested against past stock market price movements. In Fig. 2 we show a histogram of the annual price movements, with net income reinvested, of the De Zoete Equity Index, 1919-20 to 1969-70, as given by Benjamin\(^3\).
Fig. 2

Histogram of De Zoete Equity Index (net income reinvested), 1919/20 to 1969/70 movements, against lognormal distribution with the same mean and standard deviation. (Mean = 1.0971, Standard Deviation = 0.1918.)
compared with the lognormal distribution with the same mean and standard deviation (the mean was 1.0971 and the standard deviation was 0.1919). The fit is reasonably good, but at the “tails” the data are insufficient for a firm conclusion. (It should be remembered, however, that the “upper tail” of the distribution of the proceeds is of no interest in maturity guarantee calculations). Other investigators have also used the lognormal distribution, although some have criticised it and suggested alternatives (cf. Wilkie\textsuperscript{a}, Kahn\textsuperscript{b}).

(ii) The independence (randomness), or otherwise of \( x_i (1 \leq i \leq n) \) is a matter of contention. Short-term (daily, weekly, etc.) stock market movements show no evidence of dependence, and we have in fact assumed as much in the above derivation of the lognormal distribution from the law of proportionate effect. It is, however, our view that yearly movements are not independent, and to justify this assertion we present the following arguments:

(a) Carrying out a serial correlation test on the annual movements of the De Zoete Equity Index, net income reinvested, 1919-20 to 1969-70, we obtained the following results:

<table>
<thead>
<tr>
<th>( k )</th>
<th>( \rho_k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0825</td>
</tr>
<tr>
<td>2</td>
<td>-0.3007</td>
</tr>
<tr>
<td>3</td>
<td>-0.1280</td>
</tr>
<tr>
<td>4</td>
<td>-0.0181</td>
</tr>
</tbody>
</table>

\( \rho_k = \text{lag-} k \text{ serial correlation coefficient} \).

From the theory of these tests (cf. Kendall and Stuart\textsuperscript{a} example 48.1) \( \rho_k \) has, on the null hypothesis that there is independence, mean 0 and standard deviation \( 1/\sqrt{51} = 0.14 \), approximately. Since \( \rho_2 = -0.3007 \) there is evidence for a negative correlation between movements two years apart (i.e., a rise two years ago makes it likely that prices will fall this year, and vice versa). The fact that a four-year cycle has not been found (\( \rho_4 \neq 0 \)) does not prove that the market is not cyclic; it could have a varying period, one of (say) four to five years, and cycles do not necessarily begin and end at the beginning of calendar years (more detailed studies are necessary), and we are of the view that chart studies of the type popular with financial journalists are of real value in detecting longer-term stock market trends. It is true that Benjamin’s statistical tests\textsuperscript{a} did not find evidence of dependence in the same set of data, but we do not consider that those tests (or ours) provide conclusive evidence either way.

(b) We now consider the “human factor” in stock market price movements.
The stock market is notoriously erratic. This statement is unlikely to cause the editors of the financial press to rush to the nearest telephone shouting "Hold the front page!", but its very familiarity demands some investigation. A recent example of erratic behaviour is shown on Fig. 3, which concerns stock market prices from 1973 to 1975. The explanation of the fact that the dividend yield of the Financial Times Ordinary Share Index was about 3% in early 1973, over 12% in late 1974 and about 5% in early 1976, with even greater variations in the earnings yield, is surely not that the real value of the shares concerned varied by as much as is indicated by these yields, but simply that the market "panicked" and then "corrected"
in Unit-linked Life Assurance

itself’. The following historical digression shows that such behaviour was not unknown in the early days of share dealings.

**The South Sea Bubble of 1720**

The stock of the South Sea Company, founded in 1711 to trade with Spanish America (mainly in slaves), was the subject of speculative mania in the year 1720. The progress of the price of the stock was as follows:

<table>
<thead>
<tr>
<th>Month</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jan.</td>
<td>128½</td>
</tr>
<tr>
<td>March</td>
<td>330</td>
</tr>
<tr>
<td>May</td>
<td>550</td>
</tr>
<tr>
<td>June</td>
<td>890</td>
</tr>
<tr>
<td>July/Aug.</td>
<td>1000</td>
</tr>
<tr>
<td>Sept.</td>
<td>175</td>
</tr>
<tr>
<td>Dec.</td>
<td>124</td>
</tr>
</tbody>
</table>

This stock was genuine, if overvalued, but as investors became carried away with the belief that stock prices would continue to rise, unscrupulous operators sold worthless stock, including that of an enterprise, supposed to be of great profit, which was so secret that no one was to be told what it was. Another feature of this episode unlikely to be repeated in modern times was the offer of South Sea stock, at a huge premium, to all the holders of the national debt (except the Bank of England and the East India Company) in exchange for their government annuities.

4. **Theory of the “true price”**

4.1. If one accepts that $x_i$, $1 \leq i \leq n$, are not independent, but are negatively correlated in some way, the mathematical model becomes so complicated as to be, in our view, of dubious practical worth. We therefore postulate the following alternative theory, which makes the mathematics more amenable to treatment.

4.2. Let us suppose that, in addition to knowing the market value, one could determine the “true price” of the units. The “true price” is defined as the value assigned by an omniscient actuary, taking into account all known factors. Since even actuaries are not omniscient this price must remain a theoretical concept, but, assuming it exists, any movements based on it must be independent (since they depend only on chance factors which have turned up since the last valuation), and are very likely to be less erratic, i.e. to have a smaller standard deviation, than the corresponding movements based on market value (because the true price is not subject to the speculative ups-and-downs that, we have argued, exist in market prices).
4.3. The proceeds of an investment in units of 1 for \( n \) years can therefore be written as

\[
\frac{A_n}{A_0} Y_n^S = A_n \frac{z_1 z_2 \ldots z_n}{A_0}
\] (4.1)

where \( z_i \) is the proportionate movement of the true price in the \( i \)th year, and \( A_t = \) the ratio of the market price to the true price at time \( t \). \( z_i, 1 \leq i \leq n, \) are independent, their standard deviations are smaller than those expected from share price movements based on market prices, but they can be expected to have the same mean.

If we consider an investment of 1 per annum for \( n \) years, the proceeds are

\[
A_n \sum_{j=1}^{n} \left( \frac{z_j z_{j+1} \ldots z_n}{A_{j-1}} \right).
\] (4.2)

If \( n \) is reasonably large we may ignore the varying denominator of (4.2), to obtain

\[
A_n Y_n^A = A_n \sum_{j=1}^{n} z_j z_{j+1} \ldots z_n.
\] (4.3)

The terms \( A_0 \) and \( A_n \) in formulae (4.1) and (4.3) are merely statements of the well-known facts that the proceeds of single-premium unit-linked policies depend greatly on the price levels at inception and at maturity, while those of annual-premium policies depend greatly on the level of prices at maturity.

5. The liability under maturity guarantees

5.1. Let us assume that \( z_i, 1 \leq i \leq n, \) are independent, identical lognormal variables, each with mean \( \alpha \) and standard deviation \( \beta \). In our examples \( \alpha = 1.07, \) i.e., there is a mean growth rate, including reinvested net income, of 7\%, and \( \beta = 0.10, \) which is (deliberately) set at a lower value than that actually experienced in the past; for example, in Benjamin's data quoted above, the standard deviation is 19\%, and Wilkie\(^6\) quotes U.K. and U.S. stock market values of 15\% to 20\%. Let secure fixed interest investments yield \( i \) net, independent of term; in our examples \( i = 0.05 \) (if a secure return of 5\% is available, one should expect the higher mean return of 7\% on a riskier investment).

5.2. We also suppose that an average proportion \( c \) of each premium is invested in units, the remainder being used for expenses, life assurance cover, and so on. The proportion \( c \) should take into account any bid/offer price spread, as this is in effect a source of margin to the
office, and expenses may be subject to "tax offset"; we took \( c = 0.90 \) in our examples. If there is an annual management charge of a percentage of income, then this percentage, net of tax, should be deducted from the mean growth rate \( \alpha \). Finally, we let \( g \) be the guaranteed sum at the maturity date, which is in \( n \) years' time. The initial liabilities per unit single or annual premium, ignoring mortality and withdrawals (see comments below on these factors), are:

\[
L = v^n \| g - c \frac{A_n}{A_0} Y_n^S \| \quad (5.1)
\]

and

\[
L = v^n \| g - c A_n Y_n^A \| \quad (5.2)
\]

respectively, where \( \| x \| = \infty \) if \( x \geq 0 \), 0 if \( x < 0 \). The distributions of \( A_0 \) and \( A_n \) are difficult to estimate and introduce further complications, so we decided that, when \( n \geq 5 \) or so, \( A_n \) could be ignored in the annual premium case (although the reserves are thereby understated), but in the single premium case the factor \( A_n/A_0 \) is so important that it should be replaced by an arbitrary constant \( k \) (in our examples \( k = 0.9 \)). We therefore obtained

\[
L = v^n \| g - k c Y_n^S \| \quad (5.3)
\]

and

\[
L = v^n \| g - c Y_n^A \| \quad (5.4)
\]

in the single and annual premium cases respectively.

5.3. The probability density functions of \( L \) can easily be deduced from those of the proceeds \( k c Y_n^S \) and \( c Y_n^A \). That of \( k c Y_n^S \) presents no difficulties, for a product of independent lognormal variables is itself lognormal, even when multiplied by a constant, but the annual premium case presents problems of some interest in their own right. So that our ideas can be presented without heavy mathematical interruptions, these calculations have been relegated to the Appendix.

6. Alternative models in maturity guarantee calculations

6.1. Whatever their limitations, the distributions used above are more realistic than some of the models of unit price movements which have been used in practice; as these are of some interest we shall briefly describe them here:

(a) a "straight growth" model in which prices increase at a constant rate per annum, including reinvested net income,

(b) a model in which there is each year a 50% probability of being on the "straight growth" line, and a 25% probability of being, say, 30% above, and 30% below, the line (cf. Squires),
Maturity Guarantees

(c) as in (b) with a four-year cycle of the form (i) on the line, (ii) 30% above, (iii) on the line, (iv) 30% below; etc.

6.2. Models (b) and (c) are preferable to model (a), which allows no fluctuations at all and will produce zero cost for all maturity guarantees when the growth line exceeds the guaranteed sum. Models (b) and (c) are more realistic, but the cost so found will be "expected" cost, and we have already criticised this as a basis for reserves (but not for premiums). It is probably impossible to produce "likely maximum" costs of the form we advocate using models (b) and (c), and although they have the advantage of simplicity we consider that our theory, though more elaborate, should be employed.

7. Results of the examples

7.1. The following values were obtained for the expected and likely maximum costs at inception per unit office premium (since annual premiums are more common in practice we give more detailed results for them).

The parameters in all cases are:

\[
\alpha = 1.07, \beta = 0.10, c = 0.90, k = 0.90, i = 0.05, \gamma = 0.005.
\]

7.2. We first tabulated the "minimum" proceeds \( P_\gamma \) which are such that the probability of a lower level of proceeds at maturity is \( \gamma \).

For the guarantee levels \( g = n \) (annual premium cases); \( g = 1 \) when \( n = 10, 15 \); \( g = 1.5 \) when \( n = 20, 25 \); \( g = 2 \) when \( n = 30 \) (single premium cases), we give the "likely maximum" costs, which are found using \( P_\gamma \) and formula (A.2), and the expected costs. The proportions of premiums required to spread the annual premium guarantee costs over the term of the policy are found by dividing the initial costs by \( \delta_n \) at rate \( i \). In practice the guarantees should be funded at inception, by the "likely maximum" cost (less the value of any extra premiums payable by the policyholder), otherwise it may not be clear where the money for the guarantee reserves is going to come from. This method of providing guarantee reserves also has the advantage of being much less sensitive to falls in future fixed-interest rates.

7.3. The annual premium results were obtained by approximate method (i) of the Appendix, and may require a small margin for error.

To avoid possible confusion, "proceeds" refers to the proceeds of unit investments at time \( n \), while "costs" are discounted to the present time.
### in Unit-linked Life Assurance

(a) **Single premium**

<table>
<thead>
<tr>
<th>(g)</th>
<th>(n = \text{term})</th>
<th>&quot;Minimum&quot; proceeds</th>
<th>&quot;Likely maximum&quot;</th>
<th>Expected cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>0.714</td>
<td>0.176</td>
<td>0.0055</td>
</tr>
<tr>
<td>1</td>
<td>15</td>
<td>0.826</td>
<td>0.084</td>
<td>0.0011</td>
</tr>
<tr>
<td>1.5</td>
<td>20</td>
<td>0.984</td>
<td>0.194</td>
<td>0.0055</td>
</tr>
<tr>
<td>1.5</td>
<td>25</td>
<td>1.186</td>
<td>0.093</td>
<td>0.0012</td>
</tr>
<tr>
<td>2</td>
<td>30</td>
<td>1.452</td>
<td>0.127</td>
<td>0.0020</td>
</tr>
</tbody>
</table>

(b) **Annual premiums (\(g = n\) in all cases)**

<table>
<thead>
<tr>
<th>(n = \text{term})</th>
<th>&quot;Minimum&quot; proceeds</th>
<th>&quot;Likely maximum&quot;</th>
<th>% of</th>
<th>Expected</th>
<th>% of</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>8.00</td>
<td>1.23</td>
<td>15.1</td>
<td>0.044</td>
<td>0.54</td>
</tr>
<tr>
<td>11</td>
<td>8.90</td>
<td>1.23</td>
<td>14.1</td>
<td>0.037</td>
<td>0.43</td>
</tr>
<tr>
<td>12</td>
<td>9.84</td>
<td>1.20</td>
<td>12.9</td>
<td>0.032</td>
<td>0.34</td>
</tr>
<tr>
<td>13</td>
<td>10.82</td>
<td>1.16</td>
<td>11.7</td>
<td>0.027</td>
<td>0.27</td>
</tr>
<tr>
<td>14</td>
<td>11.82</td>
<td>1.10</td>
<td>10.6</td>
<td>0.023</td>
<td>0.22</td>
</tr>
<tr>
<td>15</td>
<td>12.86</td>
<td>1.03</td>
<td>9.4</td>
<td>0.019</td>
<td>0.17</td>
</tr>
<tr>
<td>16</td>
<td>13.94</td>
<td>0.94</td>
<td>8.3</td>
<td>0.016</td>
<td>0.14</td>
</tr>
<tr>
<td>17</td>
<td>15.06</td>
<td>0.85</td>
<td>7.2</td>
<td>0.013</td>
<td>0.11</td>
</tr>
<tr>
<td>18</td>
<td>16.22</td>
<td>0.74</td>
<td>6.0</td>
<td>0.011</td>
<td>0.09</td>
</tr>
<tr>
<td>19</td>
<td>17.43</td>
<td>0.62</td>
<td>4.9</td>
<td>0.009</td>
<td>0.07</td>
</tr>
<tr>
<td>20</td>
<td>18.68</td>
<td>0.50</td>
<td>3.8</td>
<td>0.007</td>
<td>0.06</td>
</tr>
<tr>
<td>21</td>
<td>19.98</td>
<td>0.37</td>
<td>2.7</td>
<td>0.006</td>
<td>0.04</td>
</tr>
<tr>
<td>22</td>
<td>21.34</td>
<td>0.23</td>
<td>1.6</td>
<td>0.005</td>
<td>0.03</td>
</tr>
<tr>
<td>23</td>
<td>22.76</td>
<td>0.08</td>
<td>0.6</td>
<td>0.004</td>
<td>0.03</td>
</tr>
<tr>
<td>24</td>
<td>24.21</td>
<td>nil</td>
<td>nil</td>
<td>0.003</td>
<td>0.02</td>
</tr>
<tr>
<td>25</td>
<td>25.75</td>
<td>nil</td>
<td>nil</td>
<td>0.002</td>
<td>0.02</td>
</tr>
<tr>
<td>26</td>
<td>27.34</td>
<td>nil</td>
<td>nil</td>
<td>0.002</td>
<td>0.01</td>
</tr>
<tr>
<td>27</td>
<td>29.01</td>
<td>nil</td>
<td>nil</td>
<td>0.001</td>
<td>0.01</td>
</tr>
<tr>
<td>28</td>
<td>30.74</td>
<td>nil</td>
<td>nil</td>
<td>0.001</td>
<td>0.01</td>
</tr>
<tr>
<td>29</td>
<td>32.55</td>
<td>nil</td>
<td>nil</td>
<td>0.001</td>
<td>0.01</td>
</tr>
<tr>
<td>30</td>
<td>34.42</td>
<td>nil</td>
<td>nil</td>
<td>0.001</td>
<td>0.004</td>
</tr>
</tbody>
</table>

Remark: As explained above, we believe that while expected costs are probably suitable for premium calculations, reserve calculations should be based on "likely maximum" costs.

8. **Some practical considerations**

We now consider how the reserves should be adjusted in the following circumstances, which are quite likely to arise in practice.

8.1. **Extra premiums charged to the policyholder**

The reserves should be reduced by the present value of future extra premiums discounted at rate \(i\). If the extra premium is deducted from the amount applied to purchase units, the "minimum" level of proceeds \(P_γ\) should be multiplied by a factor of \(1 - P^*\), where \(P^*\) is the extra premium per unit office premium, and in expected cost calculations \(c\) should be replaced by \(c' = (1 - P^*)c\).
8.2. A proportion of the unit invested in suitable fixed-interest securities

If a proportion $Q$ of unit investment must be in secure fixed-interest securities maturing not later than the guarantee date, then the reserve requirements will be reduced. The “minimum” proceeds become, per unit single or annual premium,

$$P'_y = (1 - Q) P_y + cQ s^i_n$$

and the initial reserve may be calculated by formula (A.2). This argument is not applicable to managed funds and other policies in which the proportion to be invested in suitable fixed-interest securities can be varied at the discretion of the life office or the policyholder.

8.3. Monthly premiums

The greater “averaging” effect of monthly, and, to a lesser extent, half-yearly and quarterly payments for regular premium policies may be reflected in a small reduction (5%, say) in the reserve requirements as calculated above. Similarly, a small deduction should, in theory, be made if business is sold uniformly over the year.

8.4. Mortality

By ignoring mortality we have in effect allowed for a release of maturity guarantee reserves on the death of the policyholder before the guarantee matures. Since guarantees on death are usually a feature of unit-linked policies this release may be considered as being used towards meeting them. More precise calculations, taking mortality into account, may of course be carried out, but the practical effect is not likely to be large. (Our techniques are rather sophisticated for calculating mortality guarantee costs, and an *ad hoc* method is probably more suitable in practice).

8.5. Withdrawals

It is a general actuarial principle that in a solvency valuation one should not take credit in advance for releases to the office caused by voluntary actions on the part of policyholders, except perhaps in certain pension scheme and friendly society calculations. If withdrawals do occur, there will be a release of reserves which may be used for other purposes.

8.6. Capital gains tax

The effect of capital gains tax on investment returns is unlikely to be important when the proceeds are below guarantee levels, unless the guarantee levels are considerably in excess of the amount(s) invested or much of the growth in investments is from capital gains.
rather than interest income. In such cases the guarantee level may, for the purpose of reserve calculations, be set at a level such that the return after allowing for capital gains tax equals the guaranteed amount.

Another point concerns the reserves held by life offices in respect of contingent capital gains tax liability on gains made by past policyholders (the position regarding policies linked to authorised unit trusts is more complex). They may not be required if the prices of certain investments fall heavily, and there may therefore be circumstances in which capital gains tax reserves could be (partly) employed to reduce the sums required for maturity guarantee reserves. Similarly, the maturity guarantee reserves may in some circumstances be used to provide capital gains tax payments. Caution is required in applying this argument, as it could happen that, for example, an office has two funds, one of which requires maturity guarantee payments while the other requires capital gains tax payments to be made at the same time.

8.7. Reinsurance

It is of course advisable, in all branches of insurance, for offices to reinsure all or part of large risks with other insurers. Although such arrangements do exist in respect of maturity guarantees, it appears that in present circumstances reinsurance of maturity guarantees, or even unit-linked business containing them, is difficult, if not impossible, to obtain. If reinsurance could be obtained, however, the reserve requirements would in effect be passed on to the reinsurer in exchange for premiums probably based roughly on "expected" costs, which are, as we have shown, usually much lower than those required for "prudent" reserves. This means that even the largest reinsurers are limited in the amount of maturity guarantee risks they could accept, and the total sums at risk under maturity guarantees in the United Kingdom are very large.

8.8. "Open-ended" guarantees

Until now we have discussed only "spot" guarantees in which the guarantee option can be exercised by the policyholder at one specific date. When one considers "open-ended" guarantees, in which this option can be exercised at any time after a certain date or between two dates, it is clear that the reserve should be at least as much as that for a "spot" guarantee at the time when the initial liability is largest—this is often the earliest maturity date. How great the additional reserve should be is a matter of conjecture, but in view of the erratic behaviour of the stock market we consider that this
addition should be substantial, say, 50% when the sum guaranteed does not increase with term, and 75% when the sum guaranteed increases with term.

*Example.* The initial reserve per £1000 annual premium for a guarantee of a return of office premiums at term 10 years is, if the parameters of our examples hold and no extra premium is charged, £1230. If the guarantee were for a return of all office premiums payable in the first 10 years on surrender at any time after 10 years, our suggested addition is £615, and if the guarantee were for a return of all office premiums paid on surrender at any time after 10 years, the addition is £922.

A further difficulty with open-ended guarantees compared with spot guarantees is that the former present problems in the matching of assets to liabilities; the most prudent course of action would be to accumulate the assets for maturity guarantees until the earliest possible guarantee date, and then to hold them on a secure short-term basis.

Another reason for the relatively high suggested additional reserve for open-ended guarantees is that a rush of withdrawals might help to cause prices to fall, and such a rush is likely to take place when market prices are already falling.

8.9. Policies with different terms

It is quite likely in practice that each year’s new business will contain guarantees maturing at different times; it may, for example, contain 10-year, 15-year and 20-year policies. Although expected costs “add up” in the usual way, when we consider likely maximum costs the effect of such a spread of terms is to make the total reserve rather less than that found by summing the reserves for each group of policies separately.

On the basis of simulation studies, the likely maximum cost found by addition should be reduced by a factor of about 20%, if there is a good spread of terms (in our examples the terms were 10 years, 20 years and 30 years in both single and annual premium cases). Moreover, since formulae (5.3), (5.4) are simplifications of the more accurate formulae (5.1), (5.2), it is likely that the true effect of varying terms is to further reduce the reserve.

8.10. Existing business

So far all our attention has been confined to determining appropriate reserves at inception, in the absence of existing business. We now consider the following practical problems:
(i) What reserve should be used when a policy has been in force for, say, \( t \) years?

and (ii) What initial reserve should be required for new business if the office has existing business?

(i) The first point to make here is that this business must have been new at one time, and therefore an appropriate initial reserve should have been set up. The simplest approach, if not theoretically the most accurate, is merely to accumulate the initial reserve (in secure fixed-interest securities) together with any extra premiums received from policyholders; this reserve could be adjusted as the maturity date draws near.

This method ignores the actual progress of unit prices, and the more correct approach requires simulation techniques, as discussed below. Let the (bid) value of the units, per unit annual or single premium be \( U \), and let the above notation hold. Then the liability function is, in the above notation,

\[
L = v^{n-t} || g - kUY^S_{n-t} || \quad (8.1)
\]

(single premium case),

\[
L = v^{n-t} || g - kUY^S_{n-t} - cY^A_{n-t} || \quad (8.2)
\]

(annual premium case).

Remarks: (1) Formulae (8.1), (8.2) are based on simplifications of formulae of the form (5.1), (5.2), and do not hold near the maturity date—at such times an ad hoc method, with a fairly extreme “conservative” view of the market price of the units, should be employed.

(2) \( U \) should ideally be the “true” price of the units rather than the market price; a practical approximation would be the average price over the last three, six or twelve months.

In view of the complexity and approximations of this approach the simpler, but theoretically less accurate, accumulation method may be more useful in practice. An important point, however, is that the requirement to set up initial reserves on a “likely maximum” basis might deter the issue of many guarantees.

(ii) If new business is sold by an office with existing maturity guarantees on its books, the additional “likely maximum” reserve will be less than it would have been if there were no existing business, i.e., less than the value calculated above.

The correct reserve for all the office’s guarantees may be found by simulation, but the remarks (1) and (2) of (i) above should be noted.

It may also be of assistance to the office, especially in complicated
situations involving, say, open-ended guarantees, to determine the
cost involved on various pessimistic forecasts concerning unit prices.

A further complication arises if the office has several types of unit-
linked policies, each with its own investment medium (e.g., U.K.
equities, property, overseas equities). There is a case for reducing
the "likely maximum" reserves on the grounds that the various
funds' fluctuations tend to cancel each other, but this is true only to
the extent that economic conditions in various countries or sectors
of the market are independent, and it must also be remembered that
the reserve values produced above are for broadly-based U.K.
equities, and are probably smaller than would be required for more
specialised investment media.

8.11. Category B policies

These are contracts in which, in exchange for a larger allocation
of units to the policy, the income from the units accrues to the life
office. The literature on the subject is described by Fine².

These contracts will produce the same proceeds after a fixed term
as Category A (or accumulating) policies, provided the office distrib-
utes surplus to the policyholders. If the office does not distribute
surplus there is an obvious danger that large income distributions
will be made at the expense of the maturity value, and hence the
provisions for maturity guarantees on this type of policy must take
the level of distributions into account. This type of Category B
policy has been subject to criticism; for example, in the Discussion
on the Scott Committee's Report¹⁰, Berridge said: "It is difficult to
see any real advantage to the investor in the type of policy where the
life office retains the income unless one accepts that the life office will
act fairly and distribute surplus".

ACKNOWLEDGEMENTS

This paper is partly the result of work carried out for the Govern-
ment Actuary's Department, London, but the opinions expressed
are entirely our own and not those of the Department. We would also
like to thank several actuaries for their comments on an earlier
version of this paper, and Mr. I. Haddow of Heriot-Watt University
Computer Centre for carrying out some of the computations.

REFERENCES

1. A. T. Grant and G. A. Kingsnorth. Unit Trusts and Equity-Linked
XXth International Congress of Actuaries, Tokyo, 1976.
4. J. AITCHISON and J. H. C. BROWN. The Lognormal Distribution, C.U.P.
The calculation of expected and "likely maximum" costs

(a) The single premium case

The distribution of proceeds is lognormal with mean

\[ m = kv^n \]

and standard deviation

\[ s = kv\sqrt{\theta_n - \alpha^n}, \quad (\theta = \alpha^2 + \beta^2). \]

The "minimum" proceeds after \( n \) years, such that the probability of a lower value occurring is \( \gamma \), are

\[ P_\gamma = \frac{m}{\sqrt{1 + (s/m)^2}} \exp \left( x_\gamma \sqrt{\log (1 + (s/m)^2)} \right) \quad (A.1) \]

where \( \Phi(x_\gamma) = \gamma, \Phi(x) \) being the normal probability integral,

\[ \text{i.e., } \Phi(x) = \int_{-\infty}^{x} \phi(t) \, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2} \, dt. \]

The "likely maximum" reserve required for the maturity guarantee on this basis is therefore

\[ L_\gamma = v^n \| g - P_\gamma \| \quad (A.2) \]

where \( v^n \) is at rate \( i \), and \( \| x \| = x \) if \( x \geq 0, 0 \) if \( x < 0 \). The expected cost, also discounted to the present time, is

\[ E = v^n \left\{ g \Phi \left( \frac{\log \left( \frac{g}{m\sqrt{1 + (s/m)^2}} \right)}{\sqrt{\log (1 + (s/m)^2)}} \right) - m \Phi \left( \frac{\log \left( \frac{g}{m\sqrt{1 + (s/m)^2}} \right)}{\sqrt{\log (1 + (s/m)^2)}} \right) \right\} \quad (A.3) \]

(b) The annual premium case

The distribution of the proceeds \( P \) is not known exactly (when \( n > 1 \)) and approximate methods must be used. The two methods we shall employ are:

(i) a series expansion based on a knowledge of the first few moments of \( P \),

and (ii) simulation (Monte Carlo methods).

The results obtained by the analytic method (i) will be tested by method (ii), to try to determine the accuracy of the former.

The series expansion method (which is partly based on methods developed by Quensel\(^{11}\)) proceeds in the following stages.

Stage 1. The first four moments of \( P \) may be obtained by the
recursive formulae given by Wilkie\textsuperscript{6} [section 11; Wilkie acknowledges the help of Professor R. E. Beard in pointing out these results to him]. We shall denote the moments of $P$ by

$$v_j, j = 1, 2, \ldots.$$  

**Stage 2.** Let $\lambda_j, j = 1, 2, \ldots$, be the cumulants (or semi-invariants) of $\log P$ (cf. Cramer\textsuperscript{5} 15.10), and let $h(u)$ be the probability density function, p.d.f., of $\log P$.

Then, for $m \geq 1$,

$$\psi(-im)$$

$$\log v_m = \int_{-\infty}^{\infty} e^{mu} h(u) \, du$$

where $\psi(x)$ is the characteristic function of $\log P$. Using the development of Cramer\textsuperscript{5} (15.10.2), which is valid for complex as well as real $t$,

$$\lambda_j = \sum_{j=1}^{k} \frac{\lambda_j}{j!} m^j + R_k (t),$$

where

$$R_k (t) = o(|t|^k) \text{ as } |t| \to 0,$$

we obtain the identities

$$\log v_m = \sum_{j=1}^{k} \frac{\lambda_j}{j!} m^j + R_k (-im); \ m = 1, 2, \ldots.$$  

If we let $k = 4$ and ignore $R_4 (-im)$ for $m = 1, 2, 3, 4$, there follow the four linear equations

$$\log v_m = \sum_{j=1}^{4} \frac{\lambda_j}{j!} m^j; \ m = 1, 2, 3, 4,$$  

(A.4)

which may be solved for $\lambda_1, \lambda_2, \lambda_3, \lambda_4$.

**Remark:** Since it uses the first four cumulants of $\log P$, this method may be described as "four-parameter"; if one were to take $k = 2$, the method would reduce to the lognormal approximation for $P$. On the basis of calculations carried out by us, and some "two-parameter" results of Boyle\textsuperscript{12} this method is considerably less accurate than the four-parameter approach.

**Stage 3.** We now expand the p.d.f. $f(u)$ of

$$u = \frac{\log P - \lambda_1}{\sqrt{\lambda_2}}$$

in a truncated Hermite series\textsuperscript{5} (17.6.5)

$$f(u) \equiv \phi(u)\{1 + a(u^2 - 3u) + b(u^4 - 6u^2 + 3)\}$$

(A.5)

where $a = \lambda_3/6(\lambda_2)^{3/2}$, $b = \lambda_4/24(\lambda_2)^2$, and the remainder after $H_4(u) = u^4 - 6u^2 + 3$ has been ignored.
Maturity Guarantees

Let \( F(u) \) be the distribution function of \( u \); integration of (A.4) gives

\[
F(u) = \Phi(u) - \phi(u)\{a(u^2 - 1) + b(u^3 - 3u)\}.
\]  

(A.6)

[The Hermite polynomials are here defined by]

\[
H_n(x) = (-1)^n e^{ix^2} \frac{d^n}{dx^n} e^{-ix^2}; \quad n = 0, 1, 2, \ldots ,
\]

rather than the alternative

\[
(-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}; \quad n = 0, 1, 2, \ldots .
\]

Expansions of the form (A.5) are sometimes referred to as Gram-Charlier A-series or Edgeworth expansions.]

Stage 4. The expected cost, discounted to the present time, of our maturity guarantee is

\[
E = v^n \int_0^g (g - P) r(P) dP
\]

where \( r(P) \) is the p.d.f. of \( P \). Hence

\[
E = v^n \int_{-\infty}^{\log g - \lambda_1 / \sqrt{\lambda_2}} (g - e^{\lambda_1 + u\sqrt{\lambda_2}}) f(u) du.
\]

Replacing \( f(u) \) by the expression on the r.h.s. of (A.5) we obtain, after considerable manipulation and integration by parts, the formula

\[
E = v^n \left\{ g Q_1 \left( \frac{\log g - \lambda_1}{\sigma} \right) - e^{\lambda_1 + \sigma^2} Q_2 \left( \frac{\log g - \lambda_1 - \sigma^2}{\sigma} \right) \right\}
\]

where

\[
\sigma = \sqrt{\lambda_2},
\]

\[
Q_1(x) = \Phi(x) + \phi(x)\{a - 3bx - ax^2 - bx^3\},
\]

and

\[
Q_2(x) = (1 + a \sigma^2 + b \sigma^4) \Phi(x)
\]

\[
+ \phi(x)\{(a + 4b \sigma - 3a \sigma^2 - 4b \sigma^3) + (3b - 3a \sigma - 3b \sigma^2)x
\]

\[
- (a + 4b \sigma)x^2 - bx^3\}.
\]

The level of proceeds \( P_\gamma \) such that the probability of a smaller value occurring is \( \gamma \) is found by first solving the equation

\[
F(u) = \gamma.
\]

Replacing \( F(u) \) by the expression on the r.h.s. of (A.6), we obtain an equation which can be solved approximately by successive trials. If the resulting approximate solution is \( u_\gamma \), the required "minimum" proceeds are

\[
P_\gamma = e^{\lambda_1 + u_\gamma \sqrt{\lambda_2}}.
\]

The "likely maximum" reserve is then found by formula (A.2).
We now discuss method (ii), which consists of simulating, for a given set of parameters, the liability function

\[ L = v^n ||g - c \sum_{j=1}^{n} z_j z_{j+1} \ldots z_n || \]

N times.

The lognormal variables \( z_j \) were found by generating independent pairs of random numbers between 0 and 1, \( \xi_1, \xi_2 \), applying the method of Box and Muller, i.e.,

\[ \eta_1 = (-2 \log \xi_1)^{\frac{1}{2}} \cos 2\pi \xi_2, \eta_2 = (-2 \log \xi_1)^{\frac{1}{2}} \sin 2\pi \xi_2, \]

to obtain independent \( N(0, 1) \) variables \( \eta_1, \eta_2 \), and then letting

\[ z_j = e^{\mu + \sigma \eta_1} \text{ (and so on)}, \]

where

\[ \mu = \log \left( \frac{\alpha}{\sqrt{1 + (\beta/\alpha)^2}} \right), \]

\[ \sigma = \sqrt{\log (1 + (\beta/\alpha)^2)}. \]

In our main examples \( N = 5000; \ g = n; \ n = 10, 15, 20, 30; \ c = 0.90; \ \alpha = 1.07; \ \beta = 0.10; \ \gamma = 0.005; \ i = 0.05; \) the sample mean and standard deviation of \( L \) were calculated, and the 150 largest values were printed out in descending order. The results obtained by simulation, both for expected and "likely maximum" cost, are, of course, subject to random variations which decrease as \( N \) increases; we therefore consider that simulation studies should be based on as large a value of \( N \) as possible (\( N = 1000 \) is not very large for this purpose). There is also a danger in Monte Carlo procedures of "recycling" among the random numbers (which makes them non-random).

The computations were carried out on Heriot-Watt University Computer Centre's Burroughs B5700 computer.

**Analysis of the results**

The computer results were used to provide an independent check on the accuracy of the "analytical" method (i); the computer was also used to carry out simulations on data not amenable to analytical treatment (cf. 8.9 above).

To test the "expected cost" results, we let the analytical results be denoted by \( \bar{\xi} \) and the mean and standard deviations produced by the computer by \( \bar{x} \) and \( s \) respectively. We then let

\[ z = \frac{\bar{x} - \bar{\xi}}{s/\sqrt{N}} \]
and tested whether the z's were, approximately, independent N(0, 1) variables. The numerical results were as follows:

<table>
<thead>
<tr>
<th>n</th>
<th>ξ</th>
<th>\bar{z}</th>
<th>s</th>
<th>z</th>
<th>z^2</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.04364</td>
<td>0.04095</td>
<td>0.17534</td>
<td>-1.08</td>
<td>1.18</td>
</tr>
<tr>
<td>15</td>
<td>0.01888</td>
<td>0.01664</td>
<td>0.12119</td>
<td>-1.31</td>
<td>1.71</td>
</tr>
<tr>
<td>20</td>
<td>0.00725</td>
<td>0.00660</td>
<td>0.08770</td>
<td>-0.48</td>
<td>0.23</td>
</tr>
<tr>
<td>30</td>
<td>0.00068</td>
<td>0.00087</td>
<td>0.02112</td>
<td>0.64</td>
<td>0.40</td>
</tr>
</tbody>
</table>

These results, although based on only four values of z, are consistent with the hypothesis that the analytical values ξ are correct.

Turning now to the “likely maximum” reserves, we observed that the number of values of L which exceed L_{γ} in the computer simulation is binomially distributed with mean \( Nγ \) and standard deviation \( \sqrt{Nγ(1-γ)} \). Hence the average position of L_{γ} is number \( Nγ \), in descending order of magnitude, and it is very unlikely to be outside the range \( Nγ ± 2\sqrt{Nγ(1-γ)} \), in our case, 25 ± 10. We therefore noted the 15th, 25th and 35th largest values of L and compared them with L_{γ} of method (i):

<table>
<thead>
<tr>
<th>Position of value</th>
<th>Analytical method</th>
</tr>
</thead>
<tbody>
<tr>
<td>15th</td>
<td>15th</td>
</tr>
<tr>
<td>n</td>
<td>10</td>
</tr>
<tr>
<td>15th</td>
<td>1.332</td>
</tr>
<tr>
<td>25th</td>
<td>1.191</td>
</tr>
<tr>
<td>35th</td>
<td>0.937</td>
</tr>
<tr>
<td>30th</td>
<td>0.000</td>
</tr>
</tbody>
</table>

Again, the computer's figures are consistent with the hypothesis that the results of method (i) are correct.

These tests do not prove that the results of the analytical method (i) are precisely true, but it is very likely that the errors of the method are small.
SYNOPSIS

The main purpose of this paper is the determination of a basis for the calculation of "prudent" reserves for maturity guarantees in annual and single premium unit-linked life assurance policies, particularly those in which the investment medium is a broad range of U.K. equities. Estimates of the proceeds of equity investments required, and theories of stock market movements are discussed; it is concluded that yearly growth rates, including reinvested net income, may be represented approximately by the lognormal distribution and that yearly growth rates are not random, but negatively correlated. A mathematical model is then developed; the evaluation of the resulting distributions, which is a problem of some interest in its own right in the annual premium case, is discussed in the Appendix. There is a general discussion of certain valuation principles, and we give tables showing the level of reserves suggested by our methods. Various practical questions, such as the treatment of the assets and the effects on reserves of mortality, withdrawals, capital gains tax, policies with different terms and existing business, are also considered.
Dr. W. F. Scott, introducing the paper, said:—I would first like to say how great an honour it is to be asked to present a paper to the Faculty of Actuaries. As the mother whale said to the baby whale, “It’s when you go up to spout that you get harpooned” but before the harpoons begin to fly I would like to say a few words about the paper before you. Maturity guarantees and, in particular, the level of reserves required for these provisions, have been the subject of much actuarial interest in recent years. Whereas I have no criticisms of the level of premiums charged by life offices for these guarantees, indeed I think this is a matter where offices should enjoy freedom of action, I cannot say the same for reserve bases. The danger I fear this evening is not that my efforts will be criticised but that no definite alternative proposals will be forthcoming and the present state of affairs, which many regard as unsatisfactory, will be permitted to continue. Whatever its shortcomings, the level of reserves proposed in section 7.3 of this paper has, in my opinion, the merits of being fair and consistent in its treatment of the different types of maturity guarantees offered in practice, and of being relatively simple to operate. I therefore ask whether a table of the form given in section 7.3 could be used as an interim measure pending a more definite agreement, at least for all new business written after a certain date, to be decided by the actuarial profession and the supervisory authorities. I can do no more than present my proposals. It is up to the actuarial profession to decide what action, if any, should be taken.

In conclusion, may I ask speakers to say whether they consider that the reserve basis suggested in this paper errs, in the language of this article, on the side of the Scylla of inadequacy or the Charybdis of overprovision.

Mr. A. T. Grant, opening the discussion, said:—It is a privilege and a real pleasure to me to be allowed to open the discussion tonight because this is a paper in the best traditions of the Faculty, presenting mathematical and statistical research in a practical fashion on an important and very real problem. As the author states, there is now a sizeable volume of unit-linked assurance contracts embodying guarantees that the maturity value will be no less than a return of premiums paid or a proportion of premiums paid. Such guarantees tended to be given unduly freely without specific charge or at very low cost under competitive pressures and in more confident stock market conditions. The size of the appropriate reserves is therefore a matter of some importance commercially and perhaps politically and one notes that the paper is partly based on work carried out for the Government Actuary’s Department.

The basic concept of the paper is straightforward. Seek to establish a frequency distribution which will reasonably account for the annual variations in the general price level of ordinary shares. From this, derive the distribution of maturity values for a single premium policy for n years and hence the maximum likely cost of the guarantee at a significance level of one-half of one per cent. Discount back to establish the initial reserve. For annual premium policies the concept is the same although the mathematics are such that computer simulation is the best technique. At intermediate times, during the currency of the contract, it is right to take note of the experience up to date, and for this computer simulation techniques are also appropriate.

For the method to work it seems to me vital that the frequency distri-
in Unit-linked Life Assurance

bution derived for annual movements in share prices is appropriate to current and future conditions. I shall return to this point.

It is also vital that the annual movements in share prices be independently distributed. On this point the author has hit problems. His own finding is that the annual movements are not independent. In section 3 is shown the result of a test of his basic price change data for auto-correlation or serial correlation between terms in the price change sequence $k$ years apart. This shows that the data display significant negative correlation between price changes two years apart. The serial correlation of order two is $-0.3007$ and, as the author states, for full independence between successive terms, the mean would be 0 and the standard deviation 0.14 so that a value of $-0.3007$ appears fairly significant. It can be seen that the serial correlations of order 1, 3 and 4 are not significant and the last, testing for correlation with a four-year lag, is virtually zero. The author comments, rightly in my view, that this latter value does not vitiate the idea of a cyclical market with a four to five year duration and suggests some practical reasons. I suspect he has in mind the common belief that the U.K. tends to have a four-year business cycle (at least post-war) which many observers have tied, cynically, to the five-year maximum electoral cycle on the argument that the outgoing government tries to engineer improving economic conditions shortly before seeking re-election while keeping a little time to manoeuvre.

On a more technical level, I would have thought that the finding of negative correlation between terms two years apart is quite reconcilable with zero correlation for terms four years apart. For example, this could be produced by superimposing on top of a random sequence a harmonic sequence with tops and bottoms every two years. At the end of each complete harmonic cycle, up and down, every four years, the terms remaining could be quite random but the harmonic cycle itself could produce a worthwhile degree of negative correlation between terms two years apart. A cynic might suggest that the electoral cycle may involve the government seeking to create exactly such a harmonic cycle in the economy. A more hardened cynic may feel that governments do not really have so much control over events these days and that harmony is the last thing they create.

This whole point of serial correlation is well worth emphasising. The basic series used by the author is that of price changes in successive calendar years. Although this may be regarded as appropriate for balance sheets drawn up at 31st December, the more fundamental question is by how much equity prices are likely to rise or fall at any time. In other words, how large is the downside risk at any time, not just the downside risk to next 31st December? If prices have a tendency to fall for more than one year, say for 18 months, then the size of that fall is the measure of the downside risk and that risk would be larger than suggested by the author's analysis of 1st January to 1st January price changes.

The chart shown by the author (Fig. 3), although showing yields, rather than prices, illustrates this point. The fall in prices lasted over 1973 and 1974 together and while the fall in 1974 was very bad the combined fall of the two years was worse. The literature on random walk theory and associated statistical work on stock market movements, often based on American data, does tend to suggest that there is no significant serial correlation over short intervals such as a day or a week or a month. However, from some of the work carried out it does seem possible that there exist long-run fluctuations over one year, two years or longer, which could produce the sort of serial correlation the author notes.
Like the author I have regard for simple share price charts and these do give a visual impression of a tendency to cyclicality of a period of four years or so for a complete cycle. Again I refer mainly to post-war experience.

Towards the end of 3.1 (ii) (a), after demonstrating that the series showed negative correlation for price changes two years apart, the author points out that the tests carried out by Mr. Benjamin on the same data did not find evidence of dependence. He then, with disarming reasonableness, concludes that neither set of tests produces wholly conclusive evidence either way. This is a pity because the author does assert, quite strongly, earlier in section 3 and elsewhere that yearly price movements are not independent and on the whole I would like to feel that he is right in that assertion. He gives a second justification for his belief in the latter part of section 3 which is illustrative and here I have to say that I do not find this argument overpowering. In particular I find the statement that the explanation of the stock market fall of 1973/74 and the recovery of 1975 was "simply that the market 'panicked' and then 'corrected itself'" is too simple by half. There are those who believe the world-wide commodity boom of 1973, culminating in the quadrupling of oil prices by OPEC, had something to do with it. There are those who believe that Mr. Heath and Mr. Gormley, Mr. Healey and Mr. Benn had a major effect, and I have known people to assert that the dominant role was that of money supply, the credit mechanism and government financing. Whatever the merits of each possible explanation, I suggest the author's comment is something of an oversimplification.

This takes me to a less technical but most fundamental anxiety about the derivation of the frequency distribution of price changes. Is it really right to use the history of 1920 to 1970 to assess future changes in prices? Or have things changed fundamentally during and since this period? I suspect that circumstances are different enough now in Britain to urge great caution in this. Some support for this view has already been made all too plain in the history since 1970. The gyrations of 1974 and 1975 would represent very extreme points in the frequency distribution postulated by the author. Further support comes from common sense. The policies and experience of the post-war years have surely been very different from the 1920's/30's. Indeed I would argue that even shorter periods of time differ markedly in broad circumstances. For example, during the 15 years or so from 1949 to 1964, on the whole there was one exchange rate, one corporate tax system, one credit system and one Conservative government of a non-interventionist nature. Budgets even tended to be annual affairs! We tended to have balance of payments crises every second year, of course, and changes in Bank Rate but on the whole life in the stock market proceeded against a more stable background. By contrast, since 1964, we have had the doubtful blessing of interventionist politics with violent see-sawing from left to right and vice versa. We have had two new corporate tax systems, turmoil approaching chaos in our foreign exchange arrangements, violent upheavals in our credit system involving major boom and major business failures, frequent mini-Budgets, much more involvement in price and dividend controls and frighteningly rapid inflation. Against this background, one is entitled to doubt whether it is valid to use the experience of 1920 to 1970 as the base. Indeed, one notices the author is a man of his times. Twice he suggests "We will all go together when we go", and at the start of 2.3 he bursts into a hymn, although, as you will have gathered, my own preference would be for the more apposite "Abide with Me". However, if one criticises the choice of period covered by the base data, or even the concept of using the history of other times, the author is entitled
to ask what else would one use, and he has effectively said just that in his opening remarks.

Starting in paragraph 4.1, we see the author uncomfortable in case successive annual price movements are not independent. He therefore strikes out boldly in postulating the existence of true value—a value for shares which strips out over or under enthusiasm in the stock market. He then assumes that the swings in this value would be rather less pronounced and that successive yearly value changes would be independent, apparently on the argument that the value would be changed by the emergence of chance factors.

I have to say I do not find this modification convincing. Why should the chance factors not be correlated? I would on the whole prefer an unashamedly rough and ready guess of a basis with independent variables or an attempt to use time intervals different from one year.

Considering the basis actually used in paragraphs 5.1 and 5.2 for the computer simulations, the author has perhaps been too mild in assuming a standard deviation of 0.10. His mildness follows partly from his concept of "true value" fluctuation rather than actual price fluctuation. For single premium contracts, however, he offsets this by assuming a 10% fall in price at maturity over and above any downside risk from "true value" fluctuations. One might also reasonably consider 5% net too low a rate of interest for discounting back from maturity to valuation data, at least for single premium policies.

He asked in his opening remarks—Are these results about right, or how would one modify them? I think, summing up that paragraph, I have to say that these probably are not unreasonable offsetting factors, but without playing with the data it is difficult to offer this degree of comfort.

The results indicate that even when the full premium for the guarantee is charged the size of the reserve required at commencement of such policies involves a serious financing strain. Also, reassurance of the risk is difficult and limited in nature. The most practical tactic seems to be to ensure a spread of maturity dates. Even with independence of price movements, as the author points out, this cuts down the overall reserve requirement for a block of business, and if it is the case that price movements are negatively correlated, as the author suggests, then the risk must be significantly reduced for a block of business with maturities spread over the whole period of an investment cycle. Clearly more work needs to be done, as the author has said, on this area of independence of price movements.

Without spreading of maturities these guarantees represent a catastrophe risk although, curiously, a catastrophe risk whose scale can perhaps be reduced if insurance companies feel able, in the circumstances, to buy equities counter-cyclically at the time of the assumed market collapse. It is just conceivable that in thin and illiquid market conditions it might actually pay insurance companies in concert to be buyers, rather as the Japanese stock market has known the experience of buying of shares temporarily by the authorities as a stabilisation policy. Or, adapting the author's phrase—if we all go together when we go, we might not have to go at all.

Sadly, recent conditions suggest that it is precisely in conditions of collapsing stock markets that the deterioration of balance sheets tends to constrain life offices from buying equities and therefore I feel this thought is too fanciful. What is less fanciful is to say that on the basis of Dr. Scott’s excellent paper, if we all have to go together I shall be very happy to travel in Dr. Scott’s company.
I thank him for a most stimulating and enjoyable paper.

Mr. G. D. Gwilt:—The problem of determining and financing sufficient contingency reserves in life assurance business is in fact fundamental and it is one which should exercise us all, though it receives scant attention in the examination textbooks and not much more elsewhere. The author is therefore to be congratulated for bringing the problem into the open.

Now, although the class of business dealt with in the paper does not form a large proportion of the business in most offices, if indeed they transact it at all, nevertheless the general problem is of much wider application. In paragraph 7.3 the author shows premium loadings of between 0% and 15% for annual premium business and of between 8% and 20% for single premium business to cover the particular maturity guarantee discussed. Presumably a further loading would also be required in these policies to cover fluctuations in death cover where the minimum sum guaranteed is the total of premiums due in the whole period of the policy. In these cases the amount assured by the office when a death occurs will be the difference between the total premiums due and the value of the units bought so far. Since the value of the units bought so far will fluctuate, the amount paid out by the office isn’t a simple decreasing term—it can fluctuate too and no doubt extra reserves would be required for that as well as for the maturity guarantee. The reserves implied by these loadings, both for maturity and mortality guarantee, are fairly substantial and I ask the questions—What reserves are needed for conventional business? Why have actuaries virtually ignored these reserves so far, and are they right in having done so?

In paragraph 2.1, the author, quite rightly in my opinion, says that the present value of the liabilities may be regarded as a random variable depending on one or more factors. I would go further and say that the value of the liabilities not only may be regarded as a random variable but is a random variable and that we really ought to pay more attention to this.

Now if, in conventional business, we assume that expenses are under our control and that the liabilities are immunised, this effectively leaves mortality as the one remaining variable. On this assumption, theoretically we should keep a reserve to cover fluctuations in mortality. How big should this be?

In 2.1, the author gives a ratio between standard deviation and mean for a set of term policies and the standard deviation turns out to be about 22% of the premium. Most policies in an office’s portfolio are not term policies so I calculated the standard deviation for the reserve on a whole-life policy effected at age 20 and 30 years in force. The basis was 5% interest with a bonus loading of 3% and mortality A67/70; I assumed that bonuses at the rate of 3% had been added since the inception of the policy and that the premium was based on 2% interest. The standard deviation came out to be about 30% of the reserve. Now this means that a contingency reserve at the 3σ level which means a 99.9% chance of not incurring ruin would thus, for one policy, be about 90% of the reserve—quite large!

However, for 10,000 policies the contingency reserve needed is only 1% of the reserve and for a million policies it is only one-tenth per cent. This was a with-profits policy. The reserve on a non-profit basis, i.e. reserving for no future bonus, would have been roughly half and this can be considered a minimum reserve. This implies that the with-profit reserves contain a margin for contingencies of about 100%.

Now for a portfolio of 10,000 policies this implies that the conventional
in Unit-linked Life Assurance

reserves contain a contingency reserve at the level of 300%. In ordinary language this means that with such reserves ruin is impossible. This then, I think, is one reason why actuaries have ignored additional contingency reserves—they're just too small to be noticeable. But we have seen that if interest fluctuations affect the liabilities, as in the case of maturity guarantees, then the contingency reserves needed are far greater. This is because a change in interest rate affects all reserves in the same direction. The standard deviation of the reserve for the liability does not fall as the number of policies increases. But need this worry those of us who are responsible for portfolios of conventional policies? As I have already said, the contingency reserve is small if it is independent of movements of interest rates, but is it really true that the reserve is independent of interest? Might not alterations in market level affect the actual outcome of policies? Does matching and immunisation really work?

Now in J. I. A., Vol. 103, page 205, J. H. Pollard indicates premium loadings for non-profit policies to cover fluctuations in mortality and interest. As was to be expected, he found that the effect of mortality fluctuations on reserves was small compared with that due to interest fluctuations.

It would appear from that note that if Pollard's basis is correct, however large the portfolio, a contingency reserve of around 20-25% of the actuarial reserve is needed to cover interest fluctuations. This is not consistent with my earlier assumption that liabilities can be immunised absolutely. Nevertheless this reserve, which is intended in fact to be at the 3σ level, is still likely to be covered by the margins in a reserve allowing for future bonus. Hence, actuaries may still be safe to ignore the contingency reserves for mortality and interest, at least for with-profit conventional business. Nevertheless, I suggest that the time has come for a theoretical investigation into the size of contingency reserves needed for adverse interest fluctuations in conventional business, assumed matched and immunised as may be, just as the author has investigated the reserves for maturity guarantees.

Mr. R. J. Squires:—Mr. President, my name is Squires. I'm a visitor to the Faculty and I would like to thank you for your invitation to attend the meeting and your invitation to speak. I hope, when I've finished doing so, the members will not say, "Forgive him, Lord, he knew not what he did".

I must start by confessing that I do have a practical concern with this problem. I'm encouraged to speak at this point of the meeting, although I hadn't intended to, by the fact that I think that I can be as close to the subject of this paper as Mr. Gwilt. Incidentally, I would support everything he said.

I would plead in mitigation to my confession that this was a situation that I inherited: an existing portfolio containing maturity guarantees. Unfortunately, I did not inherit either a note of the premium basis or the intended valuation basis. Plead as one will, the time comes when one has to decide what to do and to start with, of course, I took the easy option of accumulating the actual premiums which were being charged. I have recorded elsewhere my gratitude to Mr. Sidney Benjamin for shaking me out of that particular complacency. I then decided to go for a trend-line approach on the basis of the highly simplified model that Dr. Scott has described briefly in his paper, assuming that the distribution could be described by three values only—a 50% chance of having the middle value, a 25% chance of having one 30% greater and a 25% chance of having one
398 Maturity Guarantees

30% less. I am gratified to find, looking at Fig. 2 and drawing in the lines where I think the middle 50% of the distribution lies, that it produces answers which are not too bad at all. It seems to me that the centre of this distribution is around 105%; in other words, we are looking at something like 5% underlying growth with a fluctuation about it and 50% of the distribution appears to lie between 90 and 125, so my plus or minus 30 is not too far out. However, the question then is, how to use this model. In my paper to the Institute I described a method of using the model on the same basis for valuation as for premium calculation and I now freely admit the error of my ways. I am now persuaded that indeed that is not the right basis for valuation. What is required is not, if I may deem it, the sort of general hypothesis that I proposed and that Dr. Scott has proposed in a very much more detailed form but one specific hypothesis which may be considered to be a special case of this general hypothesis; which for a single generation of business is that the price is at the maximum level until the year of maturity and at the minimum level in that year. If you will excuse me for reverting to my own model, because it is simpler to describe, I’m saying that the price is 130% till maturity and 70% at maturity, relative to the general trend-line. If you take that position relative to a single block of business, you come out with the sort of initial reserves that are described in this paper. But for a block of business I don’t think that’s right. I think for a block of business what one must do is find the particular hypothesis from this set which represents the least favourable case taken overall. It’s very interesting to look at the implication of that statement, to the problem of the spot guarantee versus the continuing guarantee. I have instinctively felt that the continuing guarantee must be very much more expensive than the guarantee that applies at a single point of time, and I was concerned with the fact that papers such as this seem to show that the continuing guarantee is only marginally more expensive. If you accept my hypothesis that one must look at the single case—the single set of values—that give the worst overall result, then it shows very clearly why the continuing guarantee is much more expensive than the spot guarantee. With the spot guarantee if you consider the case where prices are at a maximum until 1986, shall we say, with a minimum in 1986, it follows that, as you pass the option date of all policies for which that occurs before 1986, you can disregard them for the rest of the calculation but with a continuing guarantee you can’t. With a continuing guarantee, as you go on, you merely accumulate more and more business to be considered.

Another interesting comparison is that between annual and monthly premiums and again I instinctively feel that it is right that the cost of the guarantee on a monthly premium case should be less than the cost of the guarantee on an annual premium case but, of course if you take this particular hypothesis that prices are going to follow a smooth trend-line it doesn’t make any difference. That worried me a little but I think it is something that I can accept. I think it is simply to be seen as something that follows from the choice of this specific hypothesis for the purpose of reserves. It does not invalidate the suggestion that in calculating the premium the basis can be less onerous for a monthly premium than for an annual premium.

Finally, I would like to mention the problem of applying this hypothesis in practice. I have quite recently tried to do this for the business of my own office and am intending to use it for my next valuation. When I first considered the possibility, I thought one can surely simplify by saying that the worst case must be one where prices are either at a
maximum or a minimum. I cannot conceive that the worst case could be achieved with a price that is not at one extreme or the other. Nevertheless, if you are going to consider 20 years of business, for each of those 20 years you have two choices and it seemed therefore that you would have 2^{20} cases to enumerate and consider, which is rather a daunting thought. So I started off by simply scheduling the data, looking at them and to the answer was almost obvious. For, having simply scheduled the data, the amounts guaranteed year by year, the numbers of units attributed to each generation of policies and the amount of continuing premium on those policies, it was immediately apparent where the problem was going to be if at all—that was—1986. It was then a simple matter to enumerate the cost of those guarantees if prices stayed at the maximum until 1985, and dropped in 1986. It was immediately apparent that if prices were still at the maximum in 1986 the cost of the guarantee on subsequent maturities came down so sharply that they didn't have to be considered. So the question was—Was there a pattern which gave a larger overall cost as a result of payments in respect of earlier maturities? and the first stage was to look at a drop in 1985. The answer was, Yes, a drop in 1985 involved payments under maturity guarantees which were greater than the savings on policies in 1986—a saving because the last year's premium had bought units at 70 instead of buying units at 130.

Taking the argument back one stage further to 1984 produced the reverse of this situation and again it was quite clear that an earlier drop in price needn't be considered because the saving on all subsequent maturities would be greater than the additional cost in that year. So my 2^{20} cases came down to three in practice.

Now I accept that that is on the basis of my highly simplified model. I haven't tried to consider how Dr. Scott's hypothesis, which doesn't allow for one single price representing the worst possibility, should be applied in practice, but I do think it must be possible to apply the model along these lines, and I personally believe that this is the way that the solution should be sought.

Mr. J. C. Fagan:—First of all, I would like to express my gratitude to your President for inviting me to speak tonight. Not only do I belong to your sister body, the Institute, but I am also a member of the sub-species of Associate which I understand does not even exist in this part of the world!

Regular actuarial investigations can be looked on in a way as a series of high jump competitions. At each investigation the height of the bar is set by the controlling authority at the market value of the assets at that time. The aim in this competition is to ensure that the "height" of the liabilities is less than that of the assets. At least this is the position as it applies to the valuation of liabilities for conventional business.

With regard to the valuation of liabilities for maturity guarantees, the author claims that the bar should be set, not at the unit price current at the valuation date, but at an imaginary level calculated by reference to the unit price at the time the contracts were issued originally and based on what expectations of growth rates and variance were at that time. Not only that, but the imaginary height of the bar will differ for groups of contracts taken out at different times in the past according as the unit price differed at their respective dates of issue. In other words, the reserves have a prospective justification at the date of issue only and are unaffected by subsequent price changes.

Any logical system of reserving for maturity guarantees must recognise the fact that liabilities should increase if the unit price falls between
successive valuation dates and will normally reduce if the price rises between successive valuation dates. It is unfortunate that the author relegates discussion of this vital aspect of the problem to a few lines in paragraph 8.10.

The other aspect of the paper on which I would like to comment is the use of the lognormal distribution to predict future price movements. Some work has been done in our office on the probabilities resulting from the adoption of this distribution and the results may be of interest.

Assuming a mean growth rate of 5½% per annum and a standard deviation of 20% per annum the chances are 99-9% that the price in one year's time will not be less than 58% of the current price. There is a probability of 99% that the unit price after one year will not be less than 67% of the current price. However, if one assumes that the same mean growth rate and standard deviation will apply for each of the next forty years, there is a 1 in 1000 chance that the price at that time will be less than 11% of the current price and a 1 in 100 chance that the price will be less than 27% of the current price. This is despite the fact that the expected unit price after forty years is \((1.055)^{40}\) which equals 8½ times the current price.

This fanning out of the distribution—the expanding funnel of doubt, as Mr. Redington called it—is a direct result of the assumption of independence of successive probabilities. It would seem that the one-year results underestimate the true probability of adverse price movements while the forty-year results tend to overstate (we hope!) the probabilities of very low prices occurring. Added weight is therefore given to the argument that there is in fact correlation between successive price movements, although the dangers inherent in trying to speculate on the "true" price are obvious. Because of these dangers, there is no alternative to assuming independence of probabilities.

Mr. C. B. Russell:—May I first congratulate the author on covering so difficult a subject in so few pages, and particularly thank him for putting at least some of the mathematics in an appendix. There used to be a widely held view outside the profession that it required a mathematician to be an actuary. If that was ever true the advent of computers has made it less so since actuarial techniques as used on computers are normally simply complex and lengthy arithmetic. It is curious then that those taking up the profession are generally better qualified nowadays and somewhat alarming, at least to me from a personal point of view, that papers concerned with statistical theory, and attempts to fit mathematical formulae to mortality rates and stock market prices, have become more sophisticated.

Fortunately this paper, I find, deals with two distinct subjects. The first is the creation of a model and the second is the use of a model, and it is mainly in creating the model that mathematical concepts beyond my comprehension are used. Having for several years used a model of the Squires type referred to in 6.1, although it must be said a rather more detailed one with a rather larger range of probabilities, I see little advantage in using a mathematical distribution which requires parameters to be deduced from experience rather than using several levels of dispersion from a trend-line as applying with probabilities drawn from experience. An alternative approach I have used is to establish a range of growth rates with given probabilities rather than to establish a range of dispersions from a single growth rate. The former produces a widening funnel of doubt going further into the future and avoids the very low cost of guarantee figures brought out by the Squires method at long terms.
The methods of the author will not, I think, prove readily expressible in legislative form should it be required, though I now gather from his opening comments that he possibly has in mind that his figures, rather than his methods, should be enforced by legislation. If so, I hope that he has in mind that figures will be tabulated for every conceivable case because there is no way I could deduce those which are not tabulated.

The more important subject is the use of the model and in particular of the "likely maximum" method as opposed to the traditional "conservative expected" method. As Mr. Gwilt says, the author justifies his method in paragraph 2.1 by showing that reserving for maturity guarantees calls for greater conservatism in the provision of adverse fluctuation reserves than does reserving for one-year term assurance. However, is it not common knowledge that virtually all forms of assurance, both long-term and general, are subject to greater fluctuations than a one-year mortality risk? Whatever rate of interest is chosen for valuing non-profit endowment policies of a certain term, the assumption is as to a single future event, namely the total yield during that time, and there must be a probability that that rate will not be achieved. Nevertheless, traditional methods have stood the test of time and the reason I believe is, as other speakers have said, the existence in a normal portfolio of business of differing durations and terms resulting from a continuous flow of new business. The author treats the fact that business is in fact of varying durations as a mathematical inconvenience worthy only of one paragraph whereas I view a flow of new business of varying types and terms as a basic cause of stability in life assurance.

If, indeed, the experience of a generation is near the half per cent worst of the probability range, then either the next generation who are buying the units cheaper will require a much lower reserve or the trend will be downwards for ever in which case we will, as the author suggests, all go together. The total reserve required must, therefore, be much less than the total of the reserves for each generation at the outset.

In short, I find no case proved to abandon traditional techniques using conservative assumptions as to future experience based on observations of the past and, in particular, I do not accept that a method of reserving for a single generation at outset is necessarily even a good guide to appropriate figures for a continuing portfolio.

The author seeks views on the level of the figures in 7.3. I must confess that I read the paper as a description of a method and had assumed that the figures were a mere example of how that method could be used rather than actual suggestions. I have not therefore studied the figures in detail but my initial feeling is that, at least if I worked for a mutual office in Scotland, I would regard one-half per cent probability of ruin as altogether too high. I conclude that, as I must assume that Dr. Scott's methods of doing the arithmetic and the mathematics are correct, the figures will be too low for a single generation at outset, but I suggest that they are altogether too high for a continuing portfolio.

Mr. J. Plymen:—If annual price changes of the equity index are random, surely investment policy decisions regarding equity shares can logically be replaced by tossing a coin.

I maintain that the vital question of the independence between the yearly movements of the roll-up price index is not soluble by the statistician's significance tests. All we need is to look at a long-term chart of the equity price index. Traditionally, as Mr. Grant has pointed out, the equity price index shows a cyclical formation like a sort of sine wave with something like a four-yearly cycle between the peaks. This, of course, explains
402 Maturity Guarantees

Dr. Scott’s two-year negative correlation. If we accept this cyclical pattern, which has been evident for the whole history of share indices and is clearly linked to developing economic conditions, and if we work on the four-year basis, isn’t it likely that we’ll find something like this? Starting off from the trough of the cycle the price goes up for two years. There is surely positive correlation between the price movements between the first and the second year. Then you come to the peak and you start going down and then you get negative correlation between the second year and the third and positive correlation between the third and the fourth, and so on. It seems to me that these general considerations make it perfectly clear that successive movements are correlated, but sometimes they’re positively correlated and sometimes they’re negatively correlated, and this alternation completely fools the statistician’s tests. But, at the same time, the correlation is surely very much there, and I find it very very difficult to have any sympathy with, or to regard with any validity, this elaborate mathematics which is happily based on the assumption that there isn’t any correlation at all and Dr. Scott, as pointed out already, does display remarkable inconsistency here in producing the two-yearly correlation at one stage and then later saying “Well, for the purpose of the mathematics I’m going to ignore it”. That’s the easy way out, isn’t it?

Whilst this erroneous assumption of independence largely invalidates a good deal of the mathematical treatment, I think it’s particularly misleading when we’re seeking the maximum loss for reserve purposes. Presumably the maximum loss revealed by mathematical process over, say, a ten-year term arises when the random sampling puts together a long run of significantly negative values. The de Zoete and Gorton roll-up index published in Benjamin’s paper and referred to by the author does in fact have nineteen negative values and thirty-one positive ones. Well, obviously if you put all the negative ones together you do get a very poor result and a very large loss at the end of the time. Admittedly conditions can be envisaged when the equity market falls for ten years without much favourable reaction in between. Such conditions would, of course, be associated with a continuation of inflation coupled with historic accounting and taxation and with an eventual communist society. In these circumstances, of course, reserves wouldn’t be much good anyhow. In addition, on the way down new equity-linked business would disappear and a good deal of what there was on the books already would be surrendered, reducing the subsequent reserve problem.

Outside such extreme political conditions a ten-year run of negative values is certainly not in accordance with equity price history. If the mathematical method, as I suggest, is misleading, what do we put in place of it? I would not want to use the statistics of this fifty-year index because they are far too favourable. As Mr. Grant has suggested, over this period the equity market has put up a performance rather like an advertisement for an equity-linked policy in that the roll-up index has gained 53 times in 51 years—a gross rate of 10% per annum. From some hurried calculations I think that if one worked out the result of ten-year policies taken out each year over 41 years, I don’t think really any of them would have produced a loss at the end of the time. It’s just possible that if one started off from 1922 and finished in 1932 there might be a marginal loss but, at any rate, I think these statistics would suggest that a ten-year contract in the past would not have run into much trouble. That is not, of course, correct for the future because, for a proper assessment, we should look at more recent experience and here we have got a magnificent example of an adverse situation in the 1974 market. I did some calculations between
1964 and 1974. This is of course an extremely testing time because the F.T.-Actuaries All Share index started off at around 97 in '64, shot up to 218 in 1972 and at the end of '74 was back at 66. This is the absolute nightmare of a situation for this guarantee because you get the price going up in between so that there are a lot of situations where the money has been put in at the top of the market and is going to depreciate heavily.

Well, if one works on yearly averages of the index which correspond to policies being issued steadily throughout the year, even this situation does not produce too bad a result. The yearly average of the index in 1964 is 105 and in 1974 is 104, and I reckon that the ten-year policy, £100 a year for ten years, would have a maturity value of about £850 at the end of '74. I must admit I did this first on the annual figures, which of course is quite unrealistic, reckoning that all the business was issued in December '64 and matured in December '74. No insurance company operates like that doing all its new business in December in one lump. On that basis you might have a loss of about £400, but I think we can ignore that.

Strangely enough, my hurried calculations seemed to produce an answer something like Dr. Scott's—15% £150 loss in ten years. I would reckon that the conditions I've envisaged, the '64-'74 experience, should be regarded as particularly savage and that this loss of 15% is high rather than low.

The next question comes as to how the guarantee scales down as the term increases. Well, what it means is, if you have a 20-year policy, what is the prospect of the performance in the first ten years being sufficiently good to provide some £150 investment appreciation reserve for the possible adverse experience in the second ten years. Successive 10-year periods, with results as adverse as that of 1964/74 are surely so unlikely as to be ignored. I would think that tests made on general experience would show that under most circumstances 20-year and 30-year contracts would be expected to produce sufficient profits, so to speak, in their first ten and twenty years so that the liability for a 30-year contract is relatively small, as is shown by Dr. Scott. Altogether there is no doubt that one must take the 10-year situation quite seriously and these 10-year guarantees should obviously not be given on a wide front for all equity contracts. Naturally for a managed fund the risk is very much less as it can be immunised.

Mr. A. C. Stalker:—I must congratulate Dr. Scott on his paper, which I find most fascinating and I just hope I will be able to construct some remarks worthy of his paper from my rather disjointed notes.

I'd like to make one or two remarks about his model first of all. The de Zoete index which he uses is a geometric one—I think Mr. Grant should know that—and a portfolio behaves as an arithmetic index. Now, this means that there is a slight bias on the low side in the geometric index. This feature was discussed quite deeply in an issue of the Financial Times in December 1971 and it was concluded there that the difference in growth rate of a portfolio over a geometric index, portfolio-type index, was about 0.8%.

The next feature of the paper I would like to comment on is this serial correlation on the annual movements of the index, 3.1 (ii) (a). Well, this has been done before and it appears in a paper by Coen, Gomme and Kendall, Life Relationships in Economic Forecasting in the Journal of the Royal Statistical Society in January 1969 and what they did was a quarterly regression of the Financial Times 30 Share Index on itself. The coefficient correlation started off at +1, naturally enough, and then
it went through 0 out to \(-0.4\) after 2 years and this seems to accord very closely with Dr. Scott's finding of \(-0.30\). Then, after a further two years, the coefficient correlation went back through 0 to about \(+0.4\) after 4\(\frac{1}{2}\) years and thereafter it faded out to 0. They also, in their paper, offered a "detrended" curve of the same coefficient of correlation and I don't think they specified exactly how they had detrended the curve, but on the detrended curve the coefficient went down to about 0 after one year and then faded out, but the detrended version never went negative.

This leads me to believe that the stock exchange index is a time series index with a decaying relationship but with a sharp, fairly short-term series of damped oscillations between greed and disappointment superimposed on top of the long-term trend. I think this negative correlation after two years is a quite important feature in this type of business because it means that there is a sharp cut-off point when an office goes through a period of claims experience because, if the index goes down, you know that for some time the fat it has built up in the accumulation of the equity values will ensure that no claims are made under the minimum guarantee for the first part of the low period of stock exchange values, and it's only towards the end of the low period that claims will occur and the office can probably look forward to a rise in the market which will lift its policy values out of the minimum guarantee claims level.

However, I do agree with the opener that guarantees should not be lightly given and I think that any office contemplating giving such a guarantee should look very closely at two things. One is the quality and ability of the investment management where the guarantee is being given, and the second one is the likelihood that there will be continuity of such management over a very long period.

Of course, the random walk theorists tend to deny that investment management does exist and this is very difficult to prove or disprove. I did some counting from the latest issue of *Money Management* of the various unit trusts, which I think are all authorised ones in the section that I counted, as compared with the *FT*-Actuaries All Share Index with net income reinvested and this comparison, as you probably know, goes over five years. Now, the total number of trusts multiplied by periods came to 1,422 and of these 1,422 trust periods only 687 trusts equalled or bettered the *Financial Times*-Actuaries All Share Index, which is not very encouraging, and of course when one looks at the range of values achieved by these trusts even in a comparatively short time one wonders just what sort of pattern and dispersion is the correct one to use. After only one year the best performing trust produced a result which was over two times that of the worst performing trust. After two years the best was more than seven times the worst and after three years about five times, and after four and five years it was about six to six and a half times, so that there is a fantastic degree of dispersion possible in allegedly responsible management of equity portfolios. Admittedly many of these trusts were not investing wholly in the British market. In fact all the best performers weren't, with one exception.

However, I do believe that if the statistics were investigated more thoroughly it would indicate that certain management companies had been more successful than others.

I would like now to turn to some of the practical implications of this business. The author rejects the idea that allowance should be made for withdrawals in calculating the reserves necessary for minimum guarantees. Well, unit-linked policyholders are an entirely different body of people from conventional life policyholders. They are much more active investors.
and my impression is that the portfolio of unit-linked business will be two
times more volatile in withdrawals than a portfolio of conventional
business sold through the same type of intermediary. Also, it is highly
probable that if an office were to be so unfortunate as Mr. Plymen
suggested and suffer declining unit trust prices for a period of ten years one
would expect the policyholders to encash their policies, many of them not
for minimum guaranteed values but at durations before these started.
In other words they would be looking for a more profitable investment for
their savings.

Lastly, I would like to say one or two things on the design of contracts—
the spot guarantee and the continuing guarantee. The trouble about the
spot guarantee is that it is rather vulnerable to another feature of unit-
linked business and that is that a particular type of contract or a particular
management link may be very fashionable one moment and easily saleable,
large quantities of it be sold, and it may be completely out of fashion the
next moment, so that if one has a spot guarantee attached to that link
the cost of that link, or rather the exposure of policies to the guarantee
attached to that link, may vary very considerably between one year and
another and if, to take Mr. Plymen's example, one office had been so
unfortunate as to sell a lot of ten-year unit-linked business in 1964 and
very little in 1965 and 1966, then it would have had a very bad experience
indeed. On the other hand, a complete continuing and continuous
guarantee gives the whole body of policyholders an accumulating option
against the office and that is not a satisfactory solution, even although the
office collects a more stable flow of risk premium income for the minimum
guarantee. I think the best design is to have a series of spot guarantees,
say one month in every five years, so that at any time only one-sixtieth
of the business which has come into the guarantee duration is exposed to a
risk of the guarantee being called and the office is all the time collecting
risk premiums for the minimum guarantee from sixty months of business
written.

One final word—a few remarks have been made about the relevance
of the past. There is one feature which I think the two great stock
exchange slumps of 1929/1933 and 1974/75 share, and that is that the
preceding boom was built on shares which were financial in character.
In fact, this remark was made by one financial commentator round about
1972 or 1973 and he remarked that the 1928/29 boom had been built on
the same very insecure foundations and that therefore the slump would
be catastrophic. I think perhaps that in managing a portfolio of this business
one has to look at what the basis for the market is and, if the basis is
insecure, then one has to reserve very substantially indeed.

Professor J. J. McCutcheon:—The author denotes by At the ratio
between the market price and the true price at time t. In the single
premium case (5.2) it is said that the ratio between A_n and A_o is important,
but at the same time this factor is replaced by an arbitrary constant k—
which is taken as 0.9. Can Dr Scott give some indication of the justifi-
cation for this figure, because it does seem quite important?

In 7.2 there is reference to spreading the reserve for the guarantee cost
over the term of the policy by dividing by the annuity factor ä at rate of
interest i. If the amounts inserted are themselves liable to greatly
fluctuating values, why is this the appropriate rate of interest to use?

Finally may I refer to the table (b), Annual premiums, in 7.31. The
"percentage of premiums" factor, which I shall call z, is shown as an
addition to the premium to cover the guarantee reserve. One point which
Maturity Guarantees

occurs to me is that, if the guarantee relates to the office premiums, any loading to these premiums increases the amount of the guarantee itself. Should, therefore, the factor for the reserve loading be of the form $\frac{\alpha}{1-\alpha}$ rather than the simple $\alpha$ indicated?

Mr. D. H. Loades:—I am another visitor and this is the first meeting of the Faculty I have been able to attend. I would like to say that I am very pleased to be here. I am also extremely thankful for the opportunity to speak. I ought to explain that I am here for several reasons. Firstly, because I am interested in this subject and was closely connected with Dr. Scott while he was writing his paper, and secondly I am here as the official representative of the Government Actuary's Department.

The question of reserves for maturity guarantees for equity-linked contracts has been under consideration for several years. Several papers have now been published and their authors have used different stock market models and different approaches to calculate the possible costs of these guarantees.

Mr. Russell hinted at the possibility of regulations in this area. It will not come as any surprise that eventually there will be regulations specifying that an appropriate level of reserves must be held. In due course the Department of Trade will make known what level it will accept as appropriate, but in the meantime it may be helpful if I outline the current, and I stress current, thinking of the Government Actuary’s Department and the tenor of the advice likely to be given to the Department of Trade.

Tonight’s paper, although prepared under the aegis of the Government Actuary’s Department, is Dr. Scott’s personal contribution to the debate and is only one piece of the evidence the GAD must consider when formulating its view. That view is not fixed nor final, but will take into account views expressed tonight and those made later in March at the proposed Institute of Actuaries’ discussion.

It is common ground that, although expected costs are low, reserves required to give proper security to policyholders must be much higher. How much higher varies considerably between authors. For example, if a single block of ten-year contracts is considered, where the guaranteed maturity level is equal to the investment content of the premiums, the estimates of the sum required at maturity as an additional reserve vary from 10% to 50% of the guaranteed maturity level. Therefore something of the order of one-third of the guaranteed maturity level suggests itself as being a suitable figure. On that basis, the initial reserve required at the issue of a contract would be one-third of the guaranteed maturity level, discounted at an appropriate rate, less the present value of any specific charge for the guarantee made to the policyholder. These high costs question the wisdom of issuing contracts with maturity guarantees, particularly if those guarantees exceed the investment content of the premium.

The case for reserving a lower percentage of the guaranteed maturity level where the terms of the contract are higher seems well substantiated—at least no one has challenged that view tonight.

On the question of the level of reserves required for contracts in mid-term the tentative Government Actuary Department view is that something simple should be used. It is suggested that the initial reserve plus a specific charge to policyholders should be accumulated. If the level of reserve is adequate to begin with, it should usually be neither necessary nor appropriate in practice to be influenced by changes in unit prices,
except perhaps near the end of the term, if it seems clear that the reserve is likely to be inadequate.

Mr. Russell also touched upon the continuing situation. Given a proper spread of maturity dates it seems to be justified to hold lower total reserves. This could be expressed either as a lower percentage applied to each cohort or as a full percentage applied to a limited number of cohorts. The total reserve, however, must be adequate to meet the claims which might arise if the guarantee is called on when a particular cohort matures. This implies that if the reserves are drawn on they must be replenished to cover later cohorts. I put this in general terms which I hope will give a useful indication of the direction the Government Actuary Department views are taking. Obviously many detailed points will arise when these principles come to be applied, but it would not be appropriate to deal with them tonight.

Professor J. R. Gray: There are several different points I would like to raise by way of comment. I think, first of all, I must express my admiration of Dr. Scott's ingenuity and courage in speaking out and producing mathematical models in a completely new area.

The fact that other members of the audience have already criticised certain of his assumptions, certain of his ingredients, the form of certain of his results, should not detract from the effort that he has made. It is only by pioneering something, putting forward some ideas where they can be discussed by other people that weaknesses and possible improvements can be pinpointed.

I have one or two criticisms myself, matters of detail perhaps, but if we start with the justification of the lognormal distribution it is clear that what is put forward is a plausibility argument, not a mathematical justification. Coming down to details, I am not clear that I fully understand the role of $k$ that appears in formula 3.1, and I think the limiting process mentioned requires a fuller presentation which should make underlying assumptions clearer. Presumably mean values and variances of the $z$ must be regarded as being of the order of magnitude of $\frac{1}{n}$ otherwise there may be problems about infinite means and infinite standard deviations.

Coming next to the serial correlations in 3.1 (ii) (a) I would make two comments. There are dangers in interpreting the level of significance attaching to the largest individual result of a set of results. If one had set out to calculate only the two-year lag correlation and nothing else, then one can argue its significance as has been done in the paper, but if one calculates a whole string of serial correlations and then focuses on the numerically largest one, the significance level is not the same as in the pre-decided issue. Nevertheless I think an interesting result worthy of fuller investigation has been discovered.

As far as the range of data used to calculate the serial correlations is concerned, I probably agree with both Mr. Grant and Dr. Scott. I think Dr. Scott has done the sensible thing in using data drawn from an extended time interval to get broad pictures emerging, but would suggest that when broad pictures emerge one has then to look at things in greater detail. The broad picture indicates the possible presence of some feature deserving fuller investigation. Once this stage is reached questions of the general homogeneity, the up-to-dateness and the relevance of the data become important. I would suggest that perhaps one should look at sub-sections of the data to see how stable these correlation coefficients have been over different periods of time. One has more confidence in interpreting obser-
408 Maturity Guarantees

vations if one sees some sort of stability pattern. The general question of homogeneity of data is very relevant and may be very difficult.

I would like just to finish by again stressing that when mathematical models are applied to this new area the fact that certain deficiencies appear and certain criticisms arise does not constitute a case for concluding that mathematical models and statistical models can do nothing for actuaries. It should simply provoke critics into looking at things more closely. Dr. Scott has taken very useful initial steps here in developing models which attempt to explain underlying structures and relationships. I hope others will follow up in detail some of his ideas, diverge where they think fit, and look at some alternative relevant mathematical models.

Mr. A. T. Grant:—I have known Mr. Stalker long enough to know that it is extremely unwise to disagree with any of his assertions, but I believe it to be the case that the particular basis of that particular index used as the base for the author, that is with income reinvested, is on a weighted average by capitalisation of 30 stocks and is therefore a portfolio basis, not geometric.

Mr. A. F. Wilson:—My main concern is with models being employed. The name is Wilson; I'm a visitor from the south.

Dr. Scott divides his $x_t$, which are the proceeds of an investment over one year, into $z$ and a change in $A$. In 5.2 he states that it is very difficult to find the distribution of $A$. If one starts from the idea that $z$ times the change in unit time of $A$ is equal to $x$ which has a lognormal distribution, then the distribution of $A_n/A_0$ is also a lognormal distribution.

I think we've got two models by Mr. Squires and by Dr. Scott which are very close together. They both, if you like, include a random walk element, except that in the case of Mr. Squires the standard deviation of this random walk is 0, in other words it is simply a trend line with no fluctuation about it. In Dr. Scott's model, on the other hand, the random walk has a very large variability and I personally believe that the true answer is somewhere in between.

The second part of the model is a variability about the trend line. In the case of Dr. Scott, he assumes that for annual premium cases this fluctuation is so small that it may be ignored. In the case of Mr. Squires it is of paramount importance. Again I feel that the truth lies somewhere in between.

I feel that the first part, the random walk, we should regard as having a trend line which increases through the actual reinvested income received plus the change in the rate of dividend being declared plus the change, if any, in what one might regard as the expected value of the long-term rate of interest. Now, all of these three rates do not vary very much so that whilst one may have a mean of 7% as the growth rate, it is likely the standard deviation about that mean is more of the order of 2 or 3% rather than 10%. Under the random walk theory, if you do deviate from the trend line, you are effectively, at any point of time, starting up a new trend line from where you are: it seems to me that only the changes mentioned can have this property.

Coming to the second part of the model, I suggest that here we have a Markov chain process, where the change in any period depends on how far from the trend line you are at the start of the period. The further you are from the trend line, the more likely you are to return towards it, and I feel modelling on this line would be much more useful than the present
in Unit-linked Life Assurance

model, where I think the increasing funnel of doubt implicitly used increases to whirlpool proportions—a veritable Charybdis.

One last point, a small one. In paragraph 8.9, if you take one policy for each of the durations 10 to 20, each of £1000 annual premium, and you allow for the suggested 20% reduction in reserve because of the spread, you get a reserve required of £8500 for a spot guarantee. Now, if you take out just one policy with a continuous guarantee between years 10 and 20 the reserve that Dr. Scott is suggesting is £2152. Admittedly, under the one policy with a continuous guarantee you will only have to pay out once, if you have to pay out at all, but I cannot believe that one policy for each of 10 to 20 years with spot guarantees really require a reserve of four times that for a single policy with a continuous guarantee.

Mr. A. D. Wilkie, closing the discussion, said:—Let me first say that, whatever I may say in criticism of it in detail subsequently, I am 100% behind Dr. Scott's general method as described in the early part of the paper, of setting up an L likely maximum liability. There are obviously some problems in paragraph 2.2 of the paper—what should you choose? Is 1/200 at the right level anyway? Dr. Scott says in 2.2 that E, the expected value of the guarantee liability, forms a suitable basis for the calculation of premiums for maturity guarantee provisions in unit linked policies. I don’t agree with that at all and I shall explain that later.

There is quite a bit of discussion in section 3 about the random behaviour of U.K. equity prices; but there is an awful lot of literature, mostly American but also quite a lot of British literature, about the movements of prices of ordinary shares. There has been some criticism of using the period 1919-1970, or indeed any past history, to forecast the future. Now, I would suggest that this is equivalent to saying that it is a very bad idea using past experience for mortality rates; what we should do is take a guess at what mortality rates are going to be and then, as Dr. Scott has done, should just halve them to allow for forecasting!

In paragraph 3.1 Dr. Scott quotes from Sidney Benjamin's recent Tokyo paper that the mean of the distribution of annual rates of return on the de Zoete and Bevan index was 1-0971 with a standard deviation of 0-1919. You will note that the values only go up to 1970. If you add the next seven values you will find, looking at Figure 2, that one of them, 1974 of course, comes just above 0-50, another one, 1975, comes at 2-5 rather off the page to the right and the others are intermediate in the middle. The mean of 58 values is pretty much the same at 1-105, but the standard deviation goes up to 0-285. So Dr. Scott thinks that it is a better idea to say that since we’ve had some bad mortality experience we’ll take one-third of our present experience for the future!

In the same section, under (a) about serial correlations: I may have got it wrong and I may not understand Kendall & Stuart which I carefully looked at, but I think they assume for this particular null hypothesis that the random variables you are auto-correlating are normally distributed. But it is quite clear that these are not normally distributed—they are not even lognormal, but are “fatter-tailed” than that. So you need even bigger reserves than even a lognormal assumption would produce, and also you are likely to get even bigger auto-correlations. Even so, setting up the regression model, as it were, \( x_t = \alpha + \beta x_{t-1} + \epsilon_t \) where \( \epsilon \) is a residual random variable with zero mean and some standard deviation, one finds that the regression roughly accounts for a proportion \( \rho^2 \) of the original variance. A correlation of about 0-3 “explains” about 10% of the original variance leaving about 90% of the variance unexplained,
giving a standard deviation of the residual, \( \varepsilon \), of about 0.95 of the original standard deviation. This is another reason for using only half!

I shall now refer to the "true price" of section 4. Do any of the companies that issue these policies and pay out claims buy and sell units at true prices? I thought that they bought units at market price and sold units at market price and what true prices—whatever they may be—have got to do with this I don’t know. If you are running ordinary conventional with-profit business you can pay what you think is a "true price" because you don’t need to declare all the profits in the form of bonus—that’s how you can pay the losses if they occur. But in the case of unit-linked business you are working on market values or unit values, not on "true prices".

I refer to paragraph 6.1 (b)—Squires’ model: I am very glad to hear that Dick Squires has recanted. The straight growth model implies a standard deviation of zero. There is one policy I was looking at in one company’s schedule 4: a £100 premium buys a single premium whole-life assurance invested in units: on death in the first ten years the sum assured is the greater of £100 and the value of the units; on death between terms ten and twenty it is the greater of £150 and the value of the units; on death after twenty years the greater of £225 and the value of the units. There is an option to surrender on some of the policies—I’m not sure if it is only at term ten or between terms ten and twenty—of the greater of £150 and the value of the units and after (or at) term 20 the greater of £225 and the value of the units. The valuation basis is to assume that share prices go up at 7% per annum compound including reinvested income, so the value of maturity guarantees is nil—that seems to me a remarkably optimistic basis.

In paragraph 6.2 Dr. Scott says models (b) and (c) are more realistic but the cost so found will be “expected cost”. It won’t be anything of the sort. If we turn back to Figure 1 we see that models (b) and (c) both assume that the distribution is curtailed somewhere to the left of \( L_o \) or even to the left of zero so these models are of no use whatever in calculating the expected costs. The problem here is in estimating the tail of the distribution.

Dr. Scott then gives examples using a method of which I approve in general and a basis which I think is wrong. Wrong because he uses a \( \beta \) of 0.10 where I would use a \( \beta \) of 0.20. In a basis for setting up reserves, particularly on a basis for statutory reserves, I think you must be conservative. Nor can you take any credit at all for the state of the market at the time, otherwise you would find that in 1974 you might have very low guarantee reserves because the market was going to go up and in 1975 you might need larger reserves because the market was now going to go down; you would also find that policies for a 12- or 16- or 20-year term should have different premium rates from those for 14- or 18- or 22-year terms. We are going to have a bad year for mortality in 1977 with all this cold weather. Is anybody going to weaken their mortality basis because deaths in 1978 will be lighter; or reduce premium rates for one-year term assurances?

Mr. Plymen referred to some calculations he had made about the results of hypothetical policies maturing in 1974. I have also been looking at some particular units, and investing monthly over various periods up to a maturity date in December 1974 I find that the actual proceeds ended up at about 55% of the guaranteed sum assured. That of course is about a 1 in 10,000, or whatever it may be, probability level on Dr. Scott’s assumptions. So the only conclusion that I can come to is that some people think 1974 did not occur. It doesn’t much matter what unit trust you look at, the standard deviation of monthly price changes for equity type investment is typically about 0.05 or maybe sometimes higher.
Property units seem to have a lower standard deviation of 0.03 per month. Multiplying by \( \sqrt{12} \) produces standard deviations per year of about 0.17 and 0.10 respectively for equities and property. 0.17 is a suitable number but if you bring in the experience of 1974 and 1975 it should go up a bit. (I don’t think it actually matters whether you use a geometric or an arithmetic index, because the all-important thing is not the mean but the standard deviation anyway.) Anyway, these I think are the items of the basis which one could argue about just as one can argue about what is the proper rate of interest to use in a net premium valuation. What we have to agree on in the first place is that the net premium valuation is the right thing to use.

Someone very kindly referred to papers by Benjamin and myself which were presented at Tokyo. I should explain that both of us use simulation methods, but that Benjamin takes numbers at random from the actual 51 recorded in the de Zoete and Bevan index, whereas I have used a variety of different assumptions for what Dr. Scott calls \( \alpha \) and \( \beta \) (I use \( \mu \) and \( \sigma \) as in Dr. Scott’s Appendix, which are different from but similar to \( \alpha \) and \( \beta \)). Let us look at Dr. Scott’s results for annual premium policies in paragraph 7.3. The minimum proceeds, \( P \), for term 10 are 8.00 out of 10 as it were—a shortage of 2 out of 10 or some 20%. The minimum proceeds for term 20 are 18.68 out of 20, i.e. a shortfall of 1.32 per 20 or roughly 6%. Using the same assumptions as Dr. Scott I get the same sort of results with my simulations. But using what in effect are Sidney Benjamin’s assumptions, as in the de Zoete and Bevan index up to 1970, I find that out of 200 simulations the worst deficit is 37% of the sum assured for 10 years, 50% for 20 years and 35% for 30 years. When using parameters based on the de Zoete and Bevan index updated to 1976 I get a 49% deficit for 10 years—which is plausible in view of what actually happened in 1974—67% for 20 years and 55% for 30 years. I know that to estimate the 1/2 percentile point of a distribution it is not satisfactory just to take the worst out of 200 simulations but it does give you some idea of the way the figures are going. The moral of that is that lengthening the term doesn’t make it any better at all if the standard deviation is big enough—it makes it worse. You have an “expanding funnel of doubt” which is in fact a parabola shape with the axis pointing upwards. The horizontal axis represents time and the vertical (on a log scale) price. The parabola represents the line of equal probabilities—say at a 1/200 level. The problem is the lower limb; where it falls below the axis it represents guarantee claims. If the standard deviation is small, the claims are also small. If the mean is increased, the parabola tilts upwards and claims are reduced. But if the standard deviation is increased, the lower limb of the parabola falls, and the claims increase.

Dr. Scott refers to monthly premiums in section 8.3: there is no “averaging” effect of monthly premiums; it is possible to calculate the moments of the appropriate distributions for premiums either yearly in advance, monthly in advance, monthly in arrears, or yearly in arrears. Monthly in advance premiums produce a slightly lower standard deviation than yearly in advance simply because they are exposed for slightly less long—they have on average half a year less in which to fall.

In paragraph 8.7 Dr. Scott says “the largest reinsurers are limited in the amount of maturity guarantee risks they could accept, and the total sums at risk under maturity guarantees in the United Kingdom are very large”. At a rough guess I could find over £2000 million. I think these companies are short of around £1000 million of reserves. Where are they going to come from? It seems to me that the right way of looking at this is also
not the way that Dr. Scott has done in discounting backwards and then spreading forwards again. We get all too confused by learning "life contingencies" at an early age and never seem able to forget it! The company must have these reserves to start with; the policyholders cannot build them up at all; they must be supplied from some outside source of capital and the policyholders must be charged a rate for the use of them. It seems to me that in order to "fund" this satisfactorily a company wanting to do this sort of business either has to have shareholders that are willing to take a fixed interest type of rate of return with chances only of loss, or has to issue a type of fixed interest "deferred guarantee fund stock", with a lower capital priority than an unsecured loan stock. You put up your £1 million at the beginning; this allows you to write £2 million of business, or maybe £4 million if the Department of Trade is going to be generous. You need to set this aside in a separate fund with assets invested in cash or suitably matching Government stock or secure fixed interest securities. You need to pay the suppliers of this capital the going rate of interest on gilts plus a surplus, an extra of x%. You need to charge the policyholders enough to cover x, whatever x may be. Or roughly, since I think you need 50% of the sum assured, you need to charge the policyholders \( \frac{x}{2} \) supplementary premium per annum per £100 sum assured. If any claims arise under the guarantees they come from this reserve fund and after a suitable period the redemption amount is decided, which might be 100% and might be a lot less. One might need to do this in tranches for each separate block of business. I was trying to persuade my investment department this afternoon to take up some of this stock when it becomes issued and their first reaction was "Not on your life we won't". Somebody else's thought was that one would need at least 4% over gilts; half of 4% is 2%; that means a premium of 2% of the sum assured per annum for the guarantee; it would be cheaper writing whole-life policies!

There is a very serious and practical problem about all this in that it seems to be perfectly clear that this is a totally uninsurable sort of risk and the sooner that companies stop writing it the better: and really stop writing it; it doesn't matter that the guarantee is on death and therefore it is only paid on a proportion of the cases or at different times. Sidney Benjamin's Tokyo paper shows that if you do take model portfolios with maturity dates spread over suitable periods you still get almost as big reserve requirements. I think more work needs to be done on this. There is a practical problem of how to put on to a satisfactory basis the business that has already been written and that is a very serious problem for the Department of Trade. It's also a very serious problem for the other life assurance companies because, unless the benefits are declared to be "excessive benefits", they are going to foot the bill when the companies writing this business—if they go bust—do go bust. So it is up to all life offices as well to think about how it should be done.

I welcome the fact that Dr. Scott has started discussion on this subject at the Faculty, though it is not the first time it has been discussed in Britain. Sidney Benjamin's ideas were discussed four or five years ago at a meeting sponsored by the research committee of the Institute but remarkably little has been done since then. It seems to me very important that this business should not be allowed to proliferate, and also very important that the actuarial profession should decide on what is a satisfactory basis for reserving on these policies. Perhaps we should also learn the lesson in future that if there are doubts about the basis of reserving for some type of business then it is not sufficient to go along with the most optimistic and say that since the profession is in two minds we can afford to throw
in Unit-linked Life Assurance

caution to the winds; I think on the whole we should take the view of the most cautious person around rather than the least cautious.

Dr. W. F. Scott, replying to the discussion, said—I would like to thank sincerely all the speakers for taking the trouble to be present and to make such interesting and constructive remarks, to which I hope to reply more fully in writing. In reply to Mr. Grant about the changes in the stock market, all I can say is that human nature hasn't changed all that much: the stock market has always been a risky place. In reply to Mr. Gwilt, you may notice that guarantees on death are approximately covered in the proportion of premiums not required to buy units; about the solvency of conventional business, this is of course a subject on its own which we don't want to get into, but, as a policyholder, I trust that the likelihood of insolvency of well-run conventional life offices is extremely small, and this paper is an attempt to reduce the probability of insolvency for unit-linked offices to something the same order of magnitude. I am pleased to hear of Mr. Squires' conversion from this previous adherence to expected costs in valuations and would be very pleased to study his new methods of valuation. Mr. Fagan rightly points out that there are difficulties in valuing maturity guarantee risks as the term goes on but I would like to ask him how many of these guarantees would have been issued in the first place, by some offices, if my bases were accepted. I don't agree with Mr. Russell's defence of the conservative expected costs in unit-linked business; traditional methods have stood the test of time in conventional business, but I don't think they had a chance to stand the test of time in unit-linked business and I don't think they are adequate. I agree with Mr. Plymen's point that long-term stock market movements are not random and have acted upon this assumption in the paper. I also agree with him that to take the experience of December 1964 to December 1974 would be too drastic but the yearly averages from 1964 to 1974 appear to agree roughly with my figures, and I think this supports my reserves as being of the right order. I'd like to thank Mr. Stalker for his very interesting remarks and for pointing out references of which I was unaware. As for withdrawals, I don't think that the authorities are prepared to accept an allowance for them in life assurance calculations, and in any case if there are withdrawals the release can be used to provide for such matters as expense escalation, which is the other big trouble in unit-linked life assurance valuations. Regarding Professor McCutcheon's first point—justification of the value \( k \) in the single premiums—I don't have any but I thought that something of this order of magnitude was required. The other two points I think I can explain to him tomorrow when I see him in the office. I'd like to thank Mr. Loades for explaining the current thinking of the Government Actuary's Department, which I hope is not just to take the arithmetic mean of the reserve produced by several papers! I agree with Professor Gray's points regarding the justification of the lognormal distribution and convergence questions; the latter are referred to in Cramér's book; I also am pleased that he pointed out that significance tests are altered when you calculate lots of correlation coefficients instead of just one of them. To Mr. Wilkie I reply that the value of \( \gamma \) is of course arbitrary but I have some results about varying \( \gamma \) and would be pleased to show him them. I think \( \gamma = \frac{1}{2} \% \) is more or less fair. I also now agree with him that premium calculations should not be based on expected costs but should be loaded to provide expected profits, because of the risky nature of the contract; I don't however think that premium bases need be supervised anyway; the premiums used in practice are fair and adequate.
414 Maturity Guarantees

The standard deviation $\sigma$ is deliberately reduced to allow for the non-random nature of yearly stock market movements. I agree that $x_1$ in the serial correlation tests is normal, and this introduces an approximation; finally, the true prices mentioned in the paper are meant to be a mathematical fiction and a means to an end; they are not intended to have any real identity. To conclude, I'd like to thank again the other speakers for their very interesting comments.

The President (Mr. M. D. Thornton):—Thank you, Dr. Scott, for a most comprehensive—and compact—reply to the discussion. It is a tribute both to the qualities of the paper before us and to the importance of the subject that we have had such a full discussion: it is a tribute to Dr. Scott's mastery of the subject that in the time available we have had so full a reply. On both counts I ask you now to express to Dr. Scott our warm appreciation for bringing before us in so stimulating a form the important subject of the reserves required for maturity guarantees.

Professor Gray wrote:—A more detailed investigation of the movement of stock exchange prices must be carried out if a clearer, more convincing account of interrelationships is to emerge. It is difficult to reconcile Dr. Scott's idea of a short-term independence with his assertion that there is evidence of a systematic negative relationship between prices two years apart. The explanation may be that this is only an apparent relationship which is not significant; alternatively the assumption of short-term independence may not be strictly correct. One should analyse the more detailed time series of prices extending over the same total period but using quarterly or even monthly prices to provide more data. It would be interesting to discover whether there was any real evidence of eight-period lag correlation in the quarterly series. With more data it should be possible to use alternative, more sophisticated methods of time series analysis. Care would have to be taken to confirm stability of behaviour throughout the total time interval considered. Any conclusions based on non-homogeneous data could be misleading.

Dr. Scott subsequently wrote:—I would first like to repeat my thanks to all those who took the trouble to comment on the paper. In March 1977 a meeting on the same topic was held by the Institute of Actuaries and I have had the benefit of reading Mr. F. B. Corby's discussion note and being present at the meeting. I accept Mr. Wilkie's comments that a reduction on the standard deviation $\beta$ to 10% in paragraph 5.1 leads to rather smaller reserves than are produced by simulations with auto-correlation, although it is recognised in paragraph 5.2 that reserves are understated by ignoring the factors $A_n/A_0$ in the annual premium case. The method put forward here, however, has the merit of giving reserve values based on an analytical method rather than entirely on simulation studies, which produce slightly different answers on different occasions. On a statistical point, it would be better to calculate the serial correlation coefficients of paragraph 3.1 from the logarithms of the movements, which are more nearly normal than the movements themselves, but this has only a minor effect on the serial coefficients, as is shown by Mr. Wilkie's calculations. Professor Gray is rightly concerned that the case against randomness of longer-term stock market movements has not been demonstrated with overwhelming conviction, and that the yearly interval is too long to enable firm conclusions to be drawn. I have recently begun a study of past stock market movements using monthly figures, and hope to expand upon this subject.
in future, but in written communications to me and to the March meeting of the Institute of Actuaries Mr. R. E. Beard stated that his studies revealed a cyclical trend with a period of about 4½ years. An important point which emerged at the March meeting of the Institute concerned a possible allowance for withdrawals: if the supervisory authorities were prepared to permit such an allowance, reserve requirements for maturity guarantees could of course be considerably reduced. But who would decide the magnitude of the withdrawal rates? Having conceded the principle in the case of certain unit-linked assurance contracts, would the authorities be able to refuse permission for an allowance for withdrawals in other forms of business? What would be the effect of withdrawals on the future maintenance expenses of the remaining policies? The question of withdrawals must therefore be related to the other aspects of assurance business and not employed merely as a means of reducing the reserve requirements for maturity guarantees.