ACTUARIAL NOTE ON ANALYSIS OF SURPLUS

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INTRODUCTION

A system of analysis of surplus has two main objects, viz.,
(i) to analyse the surplus according to its various sources, and
(ii) to provide a check on the accuracy of the valuation reserves.

Many methods have been devised for analysing surplus, and in all of them a vital link is the calculation of the expected death strain. Normally this is done by an approximate method e.g. by multiplying the sums assured and bonuses less reserves at the beginning and end of the valuation year by the valuation rates of mortality, and then taking certain proportions of the results. When the reserves do not build up i.e. when there is a sizeable untraced balance, this may be due either to an arithmetical error or to some cause not fully taken into account. As it is not possible at a glance to determine the reason for the untraced balance, considerable work may have to be done before the balance can be reduced to a size that can be ignored.

This note sets out a simple method by which an exact calculation of the expected death strain can be made, and consequently an exact growth of reserve can be arrived at. If the reserves do not build up the cause is automatically limited to an arithmetical error, which can readily be traced by a process of elimination. For example, the values of sums assured, bonuses and premiums can be built up separately to determine in which category the error occurred. Then broad age-groups can be built up, and finally each age-group within the broad age-group in which the error was located. In practice it is found that, with valuation factors limited to five significant figures, reserves amounting to millions of pounds build up with discrepancies, due entirely to the rounding off of significant figures, of under a hundred pounds, and in many cases of less than ten pounds. Since the method involves little more work than other methods, the advantages will be obvious. Nowadays electronic computers are
increasingly being used to perform actuarial valuations. Practical experience has shown the need to have overall checks on computer systems, mainly to guard against errors made by the operators. It is considered that this method of exact reserve build-up, which is a powerful check on the results produced by the computer system, has therefore become even more valuable than before.

While the method may be known to individual actuaries it does not appear to have been published in any of the actuarial journals. To my knowledge the original development, on the net premium valuation basis, was the work of Messrs. J. A. Carson and T. A. Murray, to whom due acknowledgment is gladly made.

Valuation Basis

The development is based on a particular valuation basis and arrangement of the valuation data. Initially the case is chosen of a net premium valuation of whole-life policies with premiums payable throughout life. The extension to bonus reserve valuations and to the valuation of endowment assurances will be dealt with at a later stage. The valuation data is grouped according to office year of birth, which is taken as the office year of entry less the age at entry. We shall consider the group of policies where the office year of valuation less the office year of birth is $y$, and trace the growth of the reserve in that valuation year.

The valuation assumes that all premiums are payable yearly in advance, and that claims are paid at the end of the policy year of death. It is further assumed that a proportion $k$ of the policies enter at the beginning of an office year, and that a proportion $(1-k)$ of the policies enter at the end of an office year, all policy anniversaries thus being assumed to coincide with a valuation date.

The following notation is used:—

- $i$ is the valuation rate of interest
- $s_0^y$ are the sums assured and bonuses at the beginning of the year
- $s_1^y$ are the sums assured and bonuses at the end of the year
- $\pi_0^y$ are the net premiums at the beginning of the year
- $\pi_1^y$ are the net premiums at the end of the year
- $R_0^y$ is the reserve at the beginning of the year
- $R_1^y$ is the reserve at the end of the year.

Notes—(i) The difference between the bonuses at the beginning and end of the year is due to causes such as surrenders, lapses, etc. and not to a bonus declaration.
(ii) The suffix $y$ for the net premiums does not imply that all the policies entered in the same year. The net premiums are the total net premiums for the valuation group, irrespective of the entry dates of the individual policies.

The valuation formula for the reserve at the end of the valuation year for the group of policies in question is therefore:—

$$R_y^1 = k\{S_yA_{y+1} - \nu_yA_{y+1}\} + (1-k)\{S_yA_y - \nu_yA_y\}$$

since the proportion $k$ that entered at the beginning of an office year are a year older than the proportion $(1-k)$ that entered at the end of an office year. Also, in respect of the proportion $k$ a premium is due, whereas in respect of the proportion $(1-k)$ a premium has just been paid.

**Development of Growth of Reserve Formula**

We start from the basic identity

$$(A_x - \pi\bar{a}_x + \pi)(1 + i)$$

$$= q_x(1 - \nu_x + A_{x+1} - \pi\bar{a}_{x+1}) + A_{x+1} - \pi\bar{a}_{x+1}$$

and the obvious transformation

$$(A_x - \pi\bar{a}_x)(1 + i) + \pi$$

$$= q_x(1 - \nu_x + A_{x+1} - \pi\bar{a}_{x+1}) + A_{x+1} - \pi\bar{a}_{x+1}$$

The reserve at the beginning of the valuation year, corresponding to formula (1) for the reserve at the end of the valuation year, is

$$R_y^0 = k\{S_yA_y - \nu_yA_y\} + (1-k)\{S_yA_{y-1} - \nu_yA_{y-1}\}$$

and we must show how this reserve grows, with interest and mortality and with due allowance for on and off movements, to the reserve at the end of the year.

We make the further valuation assumption that all movements take place on the policy anniversary, just before the payment of the premium.

Consider first the proportion $k$ whose policy anniversary is the beginning of the valuation year. The on and off movements therefore take place at the beginning of the valuation year. The growth of reserve formula thus requires the existing reserve to get a full year’s interest, the net movement reserve (on less off) to get a full year’s interest, and a year’s premiums in respect of the existing plus the net movements to get a full year’s interest. This will then provide the expected death strain plus the existing reserve at the end of the valuation year in respect of the proportion $k$ of the business.
In symbols we have

\[ k(S_y^0 A_y - \tau^0_y a_y)(1+i) \]  
\[ + k[(S_y^1 - S_y^0) A_y - (\tau^1_y - \tau^0_y) a_y](1+i) \]  
\[ + k\tau^0_y(1+i) \]  
\[ + k(\tau^1_y - \tau^0_y)(1+i) \]  
\[ = k(S_y^1 A_y - \tau^1_y a_y)(1+i) + k\tau^0_y(1+i) \]  

which by applying formula (2) is

\[ kq_y[S_y^0 - (S_y^1 A_{y+1} - \tau^1_y a_{y+1})] \]  
\[ + k(S_y^1 A_{y+1} - \tau^1_y a_{y+1}) \]  

equals

expected death strain plus

existing reserve at end of year.

Consider now the proportion \((1-k)\) whose policy anniversary is the end of the valuation year. The on and off movements take place at the end of the valuation year and so does the payment of the year's premium. The growth of reserve formula thus requires the existing reserve to get a full year's interest, and this plus the net movement reserve (on less off), plus a year's premium without interest in respect of the existing plus the net movement (on less off), will provide the expected death strain plus the existing reserve at the end of the year in respect of the proportion \((1-k)\) of the business.

In symbols we have

\[ (1-k)[S_y^0 A_{y-1} - \tau^0_y a_{y-1}](1+i) \]  
\[ + (1-k)\tau^0_y \]  
\[ + (1-k)[(S_y^1 - S_y^0) A_y - (\tau^1_y - \tau^0_y) a_y] \]  

which by applying formula (3) is

\[ (1-k)q_{y-1}[S_y^0 - (S_y^0 A_y - \tau^0_y a_y)] \]  
\[ + (1-k)(S_y^0 A_y - \tau^0_y a_y) \]  
\[ + (1-k)[(S_y^1 - S_y^0) A_y - (\tau^1_y - \tau^0_y) a_y] \]  
\[ + (1-k)(\tau^1_y - \tau^0_y) \]  
\[ = (1-k)q_{y-1}[\tau^0_y - (S_y^0 A_y - \tau^0_y a_y)] \]  
\[ + (1-k)(S_y^1 A_y - \tau^1_y a_y) \]  

expected death strain plus

existing reserve at end of year.
On combining the proportions $k$ and $(1-k)$ we have

$$[S^0_y A_{y-1+k} - \pi_y^0 (a_{y-1+k} + k)](1+i)$$

existing reserve at beginning of year with interest plus

$$+ [(S^1_y - S^0_y) A_y - (\pi_y^1 - \pi_y^0) \bar{a}_y](1+ki)$$

net movement reserves with interest plus

$$+ \pi_y^1 (1+ki)$$

year's premiums with interest

equals

$$= kq_y [S^1_y - (S^1_y A_{y+1} - \pi_y^1 \bar{a}_{y+1})]$$

expected death strain plus

$$+ (1-k)q_{y-1} [S^0_y - (S^0_y A_y - \pi_y^0 \bar{a}_y)]$$

$$+ S^1_y A_{y+k} - \pi_y (a_{y+k} + k)$$

existing reserve at end of year

. . . . . . . . . . (5)

This formula is a mathematically exact build-up of the growth of the reserve, but clearly the formula for the expected death strain is not in a suitable form for calculating the expected death strain directly from the valuation results. The next section of this note, which is the whole basis of the method, shows how the formula for the expected death strain may be transformed into a suitable form.

In passing it may be said that while the build-up is mathematically exact the valuation assumptions, for example that movements occur just before a policy anniversary, may distort the sources of profit, but bulk adjustments can be made to correct this. The assumption that movements occur on the policy anniversary before the payment of the premium implies that for off movements such as surrenders and death claims the full premium is received up to the anniversary and no premium after the anniversary falling in the valuation year. For surrenders this may be a reasonable assumption since some surrenders will in practice be before and some after the anniversary, and there may be a complete offset between the premiums not received through surrenders before the anniversary and the premiums received through surrenders after the anniversary. For death claims where the office charges instalment premiums this offset will not occur, as the full premium will be received both for claims that occur before and after the anniversary that falls in the valuation year, and a bulk adjustment of a year's premiums in respect of death claims after the anniversary is necessary. In practice all movements would be investigated and bulk premium adjustments made where necessary.
Transformation of Expected Death Strain

We have

\[(1+i)R_y^0 = k(S_y^0A_{y-1} - \pi_y^0\tilde{a}_y)(1+i) + (1-k)(S_y^0A_{y-1} - \pi_y^0\tilde{a}_y-1)(1+i)\]
\[= k(S_y^0A_{y-1} - \pi_y^0\tilde{a}_y)(1+i) + (1-k)(S_y^0A_{y-1} - \pi_y^0\tilde{a}_y-1 + \pi_y^0)(1+i)\]
\[= k(S_y^0A_{y-1} - \pi_y^0\tilde{a}_y)(1+i) + (1-k)(S_y^0A_y - \pi_y^0\tilde{a}_y)\]
\[+ (1-k)q_y^{-1}[S_y^0 - (S_y^0A_y - \pi_y^0\tilde{a}_y)] \quad \text{.. .. from (2)}\]
\[= (1 + ki)(S_y^0A_y - \pi_y^0\tilde{a}_y)\]
\[+ (1-k)q_y^{-1}[S_y^0 - (S_y^0A_y - \pi_y^0\tilde{a}_y)]\]

Hence we have

\[S_y^0A_y - \pi_y^0\tilde{a}_y = \frac{(1+i)R_y^0 - (1-k)q_y^{-1}S_y^0}{(1+ki) - (1-k)q_y^{-1}}\]

Thus

\[(1-k)q_y^{-1}[S_y^0 - (S_y^0A_y - \pi_y^0\tilde{a}_y)]\]
\[= (1-k)q_y^{-1}\left[ S_y^0 - \frac{(1+i)R_y^0 - (1-k)q_y^{-1}S_y^0}{(1+ki) - (1-k)q_y^{-1}} \right]\]
\[= (1-k)q_y^{-1}\left[ S_y^0 - \frac{(1+i)R_y^0}{1+ki} \right]\]
\[= (1-k)q_y^{-1}(S_y^0 - mR_y^0) \quad . \quad . \quad . \quad . \quad \text{(6)}\]

where

\[q_y^{-1} = \frac{q_y^{-1}}{1 - (1-k)q_y^{-1}}\]

and

\[m = \frac{1+i}{1+ki}\]

Also we have

\[(1+i)R_y^1 = k(S_y^1A_{y+1} - \pi_y^1\tilde{a}_{y+1})(1+i) + (1-k)(S_y^1A_{y} - \pi_y^1\tilde{a}_{y})(1+i)\]
\[= k(S_y^1A_{y+1} - \pi_y^1\tilde{a}_{y+1})(1+i) + (1-k)(S_y^1A_{y} - \pi_y^1\tilde{a}_{y+1})\]
\[+ (1-k)q_y[S_y^1 - (S_y^1A_{y+1} - \pi_y^1\tilde{a}_{y+1})] \quad \text{.. .. using (2)}\]

Whence,

\[S_y^1A_{y+1} - \pi_y^1\tilde{a}_{y+1} = \frac{(1+i)R_y^1 - (1-k)q_yS_y^1}{(1+ki) - (1-k)q_y}\]

and, therefore, proceeding as before we have that

\[kq_y[S_y^1 - (S_y^1A_{y+1} - \pi_y^1\tilde{a}_{y+1})]\]
\[= kq_y^1(S_y^1 - mR_y^0) \quad . \quad . \quad . \quad \text{(7)}\]
Combining (6) and (7) it will be seen that the expected death strain is equal to

\[(1-k)q_y^1(S_y^0 - mR_y^0) + kq_y^1(S_y^0 - mR_y^1) \quad \ldots \quad (8)\]

This is a simple form, and all figures are immediately available from the valuation. A table of \(q_y^1\) is calculated and is used as long as the mortality table and interest rate in the valuation remain unchanged, while \(k\) and \(m\) are constants and can therefore be applied to table totals, and not necessarily per valuation group. Multiplication by only one value of \(q_y^1\) is required at each valuation. The value of \(q_y^1(S_y^0 - mR_y^1)\) will be used for the expected death strain in the next year as well, adjusted this time by the constant \((1-k)\). However, if a bonus declaration is made, it is necessary to multiply \(q_y^1\) into the bonus and the value of bonus and to make the appropriate adjustment to the expected death strain.

The method involves little more work than an approximate method, and has the virtue of extreme exactitude, thus rendering any form of check valuation unnecessary.

**Application to Bonus Reserve Valuation**

As the main additional problem posed by the bonus reserve method of valuation is the specific reserve for future bonuses, we shall confine ourselves to single premium policies and not make any specific reserve for future expenses. Again we take the whole life with profits class and as before \(S_y^0\) denotes the sums assured and declared bonuses at the beginning of the year. It will be assumed that the office declares an annual compound bonus so that \(S_y^0\) includes the bonus just declared.

The reserve at the beginning of the year is

\[R_y^0 = kS_y^0A_y^j + (1-k)S_y^0 \frac{A_y^{j-1}}{1+b} \quad \ldots \quad (9)\]

and at the end of the year, before the bonus declaration, is

\[R_y^1 = kS_y^1A_y^{j+1} + (1-k)S_y^1 \frac{A_y^j}{1+b} \quad \ldots \quad (10)\]

where \(b\) is the annual rate of compound bonus reserved for and \(j\) is defined by the relationship

\[1+j = \frac{1+i}{1+b}\]

Now

\[A_y^j(1+i) - A_y^j(1+j)(1+b)\]

\[= (1+b)[A_y^{j+1} + q_y(1-A_y^{j+1})] \quad \ldots \quad (11)\]
Following the lines of the previous development and using formula (11) we arrive at the following growth of reserve formula:

\[
S'_y \left[ k A^y + (1 - k) \frac{A^y_{y-1}}{(1 + b)} \right] (1 + i) + (S^1_y - S^0_y) A^y (1 + ki) = k(1 + b)q^y_y (S^1_y - S^0_y A^y_{y+1}) + (1 - k)q^y_{y-1} (S^0_y - S^0_y A^y_y) + S^1_y \left[ k A^y_{y+1} + (1 - k) \frac{A^y_y}{(1 + b)} \right] + bS^1_y \left[ k A^y_{y+1} + (1 - k) \frac{A^y_y}{(1 + b)} \right]
\]

The expected death strain is thus in a very simple form for calculation purposes. If premiums had been taken into account in the reserve the factor \((1 + b)\) in (12) would not be multiplied into the whole reserve, but only into the value of sums assured and existing bonuses.

**Application to Endowment Assurances**

Where endowment assurances are valued by the Karup method the treatment follows that for whole-life policies with the addition of one or more functions of the form \(C\) in the reserve, where \(C\) is a constant. Now

\[
\frac{C}{D^y_y} (1 + i) = \frac{C}{D^y_{y+1}} (1 - q^y_y)
\]

and following the usual development it will be found that the expected death strain is of exactly the same form as before, with the appropriate \(\frac{C}{D^y_y}\) functions included in the reserves \(R^0_y\) and \(R^1_y\).

Where endowment assurances are valued on a net premium basis with reference to a fixed maturity age, say 60, no special points arise in regard to the development of the growth of reserve formula, which follows the same lines as for the whole life class.
A valuation of endowment assurances by Lidstone's Z method calls for some additional comment. A growth of reserve formula on similar lines to that for the whole life class can be developed, the group of policies being those policies that mature in the same office year. Because of the effect of movements the average maturity age per group will not necessarily be the same at the beginning and end of the valuation year and unless an adjustment is made where the maturity ages differ, the build-up will not be correct. A suitable method of dealing with this problem is as follows. All movements and also the reserve at the year end are calculated on the maturity age that was used for valuing the group at the beginning of the year, and the growth of reserve worked out accordingly, the expected death strain also being calculated on the year-end reserve based on the maturity age used at the beginning of the year. The reserves thus build up exactly to this year-end reserve. The year-end reserve is then recalculated on the true maturity ages and the corresponding expected death strains worked out. The differences in the year-end reserves and the expected death strains caused by the change in the average maturity ages are then treated as being special valuation releases or strains, the year-end reserves and the expected death strain accordingly being adjusted by these amounts. The special merit of the reserve build-up described in this paper, i.e. the principle of automatic verification of the reserve, would appear to be somewhat diminished in this case, but on the other hand experience shows that the magnitude of the adjustments is predictable, and that an eye check will readily establish the reasonableness of the results.