MINIMUM SOLVENCY MARGIN OF A GENERAL INSURANCE COMPANY: PROPOSALS AND CURIOSITIES

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Minimum solvency margin of a general insurance company: proposals and curiosities

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Summary

An analytical model is presented for the determination of the minimum solvency margin of a general insurance company. The technical risk proportional to the standard deviation of the aggregate claim amount and the financial risk represented by a multiplying factor are both considered. Further, the ruin probability criterion and the zero expected utility approach starting from a simple solvency condition are compared.
1 Model description

Denote by

\( U \) the required solvency margin,

\( P \) the risk premium income net of reinsurance,

\( \lambda \) the aggregate safety loading coefficient,

\( X \) the aggregate claim amount net of reinsurance.

\( j \) the rate of return on investment.

The solvency condition relative to a certain accounting period \([0,1]\) is represented by the following inequality:

\[
U(1 + j) + P(1 + \lambda)(1 + j) - X > 0.
\]

In this way, it is assumed that the premiums \( P \) are collected at time 0 and invested at the random rate \( j \), together with the solvency margin \( U \), in order to match the random aggregate claim amount \( X \) to be settled at time 1 or to be put into reserve for outstanding claims.

According to the ruin probability criterion, we choose as the minimum solvency margin \( U_{MIN} \) the minimum \( U \) satisfying the identity

\[
\Pr\{U(1 + j) + P(1 + \lambda)(1 + j) - X > 0\} = 1 - \epsilon,
\]

which is equivalent to

\[
\Pr\{U + P(1 + \lambda) \leq X - (U + P(1 + \lambda))j\} = \epsilon.
\]

Once the normal approximation for the independent random variables \( X \) and \( j \) (and then for the difference \( X - (U + P(1 + \lambda))j \)) has been assumed, we get

\[
\Pr\{U + P(1 + \lambda) \leq E(X) - (U + P(1 + \lambda))E(j) + \sigma [X - (U + P(1 + \lambda))j] Z\} = \epsilon,
\]

where \( Z \) is a normally distributed random variable with mean zero and standard deviation one.
If we denote by \( i \) the deterministic rate of inflation, and we let \( E(X) = P(1 + i) \), the identity (1) is equivalent to

\[
\frac{(1 + E(j))(U + P(1 + \lambda)) - P(1 + i)}{\sigma(X - (U + P(1 + \lambda))j)} = z_\epsilon, \tag{2}
\]

with \( z_\epsilon \) percentile of \( Z \) corresponding to the \( \epsilon \) ruin probability.

In order to determine \( U \), it is convenient to consider the property

\[
\sigma(X - (U + P(1 + \lambda))j) = \alpha [\sigma(X) + (U + P(1 + \lambda))\sigma(j)], \tag{3}
\]

with \( \sqrt{0.5} \leq \alpha \leq 1 \) (see appendix 1).

Putting (3) into (2), we finally obtain

\[
U = \frac{1}{1 + E(j) - \alpha z_\epsilon \sigma(j)} [\alpha z_\epsilon \sigma(X) + P(1 + i)] - P(1 + \lambda), \tag{4}
\]

and we choose

\[
U_{MIN} = \frac{1}{1 + E(j) - \sqrt{0.5} z_\epsilon \sigma(j)} \left[ \sqrt{0.5} z_\epsilon \sigma(X) + P(1 + i) \right] - P(1 + \lambda) \tag{5}
\]

as the minimum\(^1\) solvency margin (i.e., the minimum safety reserve).

In particular, given \( \epsilon = 0.2\% \), we have

\[
U_{MIN} = \frac{1}{1 + E(j) - 2\sigma(j)} [2\sigma(X) + P(1 + i)] - P(1 + \lambda). \tag{6}
\]

In the sequel, \( c(j) \) will stand for the risk coefficient \( \frac{1}{1 + E(j) - 2\sigma(j)} \).

We note that it is reasonable to assume \( c(j) > 0 \). In fact, only a very risky investment can lead to \( 2\sigma(j) - E(j) > 1 \).

From (6), we can observe that

\[
a) \text{ if } E(j) > 2\sigma(j) \text{ (riskless investment }\circ c(j) < 1) \quad U_{MIN} < [2\sigma(X) + P(i - \lambda)]
\]

\[
b) \text{ if } E(j) < 2\sigma(j) \text{ (risky investment }\circ c(j) > 1) \quad U_{MIN} > [2\sigma(X) + P(i - \lambda)]
\]

\(^1\)It is easy to prove that \( U \) is an increasing function of \( \alpha \).
c) if \( E(j) = 2\sigma(j) \) (neutral investment \( \circ c(j) = 1 \))

\[
U_{MIN} = [2\sigma(X) + P(i - \lambda)].
\]

In case b), for example, \( U_{MIN} \) should cover

1. the technical risk \( 2\sigma(X) \),
2. the amount \( P(i - \lambda) \) (if \( i > \lambda \)),
3. the financial risk (measured by the multiplying factor \( c(j) \)).

In order to have some practical applications\(^2\) of this model, let us consider figure 1.

**Figure 1**

\[
\begin{align*}
P &= 84.42 \\
\lambda &= 3\% \\
i &= 3\% \\
\gamma &= 15\% \\
P_N &= 100 \\
\sigma(X) &= 9
\end{align*}
\]

<table>
<thead>
<tr>
<th>investment</th>
<th>( j )</th>
<th>( E(j) )%</th>
<th>( \sigma(j) )%</th>
<th>( c(j) )</th>
<th>( U_{MIN} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>REAL ESTATE ASSETS</td>
<td>( j_1 )</td>
<td>6</td>
<td>3</td>
<td>1</td>
<td>18</td>
</tr>
<tr>
<td>BONDS</td>
<td>( j_2 )</td>
<td>5</td>
<td>10</td>
<td>1.17</td>
<td>35.85</td>
</tr>
<tr>
<td>EQUITIES</td>
<td>( j_3 )</td>
<td>20</td>
<td>25</td>
<td>1.43</td>
<td>63.15</td>
</tr>
</tbody>
</table>

In the last column you can find the minimum solvency margin, expressed as percentage of \( P_N \) (premium income net of reinsurance), corresponding to three different kinds of investment.

\(^2\gamma \) is the expenses loading coefficient, and the assumption \( \sigma(X) = 9 \) allows us to compare \( U_{MIN} \) with the minimum solvency margin required by EC regulation ('73). For a practical estimation of \( \sigma(j) \) and \( \sigma(X) \), see Daris [3], Daykin, Pentikainen and Pesonen [4], and Rantala [5].
Figure 2 considers the more realistic case of mixed investments\(^3\).

\begin{table}
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
\text{REA } & \alpha_1 & \% & \text{BON } & \alpha_2 & \% & \text{EQ } & \alpha_3 & \% & \text{E}(j) & \% & \sigma(j) & \% & c(j) & & U_{MIN} \\
\hline
10 & 80 & 10 & 6.6 & 8.38 & 1.11 & 29.55 & \\
10 & 65 & 25 & 8.85 & 8.92 & 1.09 & 27.45 & \\
20 & 40 & 40 & 11.2 & 10.78 & 1.11 & 29.55 & \\
0 & 80 & 20 & 8 & 9.43 & 1.12 & 30.60 & \\
0 & 70 & 30 & 9.5 & 10.25 & 1.12 & 30.60 & \\
0 & 60 & 40 & 11 & 11.66 & 1.14 & 32.70 & \\
0 & 50 & 50 & 12.5 & 13.46 & 1.16 & 34.80 & \\
\hline
\end{tabular}
\end{table}

We conclude our considerations about the previous model observing that (6) can be rewritten as follows:

\[ U + \lambda P = 2c(j)\sigma(X) + \left[ c(j) - \frac{1}{1+i}\right] E(X). \]  

(7)

Observe that the aggregate safety amount \( U + \lambda P \), which is necessary to guarantee a solvency situation with probability 0.2\%, is a linear combination of \( \sigma(X) \) and \( E(X) \) with coefficients\(^4\) 2c(j) and \( c(j) - \frac{1}{1+i} \).

\(^3\)Once the independence of \( j_1, j_2 \) e \( j_3 \) has been assumed, \( \sigma(j) = \sqrt{\alpha_1^2\sigma^2(j_1) + \alpha_2^2\sigma^2(j_2) + \alpha_3^2\sigma^2(j_3)} \) holds. Since \( j_2 \) e \( j_3 \) are positively correlated in practice, it should be noted that \( U_{MIN} \) is underevaluated in the latter four cases.

\(^4\)Identity (7) generalizes \( U + \lambda P = z\sigma(X) \) in the case when the solvency condition \( U + P(1 + \lambda) - X > 0 \) is adopted (see Beard, Pentikainen and Pesonen [1]).
2 Expected utility approach

It may be interesting to compare the ruin probability criterion and the zero expected utility approach when the solvency condition is simply

\[ U + P - X > 0. \]

In the first case, it is well known that, if a normal approximation for \( X \) is adopted, and \( P = E(X) \), then the condition \( \text{prob}\{U + P - X > 0\} = 1 - \epsilon \) leads to

\[ U_{\text{MIN}} = z_\epsilon \sigma(X). \] (8)

On the other hand, if we consider, for example, the exponential utility function \( u(x) = B \left(1 - e^{-\frac{x}{B}}\right) \), the solvency margin \( U \) can be determined as the amount satisfying the following zero expected utility condition:

\[ E \left( B \left(1 - e^{-\frac{U+P-X}{B}}\right)\right) = 0. \]

Under the previous assumptions, we can easily find

\[ U = B \ln E \left( e^{-\frac{X}{B}} Z \right). \]

Once a second degree approximation for the cumulant generating function of \( \frac{X}{B} \) has been used, we may choose as the minimum solvency margin

\[ U_{\text{MIN}} = \frac{1}{B} \frac{\sigma^2(X)}{2}, \] (9)

where \( \frac{1}{B} \) (equal to \( \frac{-u''(x)}{u'(x)} \)) is the well known risk aversion coefficient\(^5\).

The comparison of (8) and (9) yields to the following relation between \( \frac{1}{B} \) and \( z_\epsilon \):

\[ \frac{1}{B} = \frac{2z_\epsilon}{\sigma(X)}. \] (10)

Therefore, if we assume a ruin probability equal to 0.3% and a standard deviation \( \sigma(X) \) equal to 6.5% of the premium income (net of reinsurance)

\(^5\)Even if we use a quadratic utility function, the same expression of \( U_{\text{MIN}} \) is obtained (see appendix 2).
$P_N$, or equivalently (from (8)), $U_{MIN} = 0.18P_N$, just like in EC regulation (see Campagne [2]), it is somewhat surprising that
\[
\frac{1}{B} \simeq 117P_N.
\]

**Appendix 1**

Let us show that, given two independent random variables $X$ and $Y$, the following inequality holds:

\[
\sqrt{\frac{1}{2}} (\sigma(X) + \sigma(Y)) \leq \sigma(X + Y) \leq \sigma(X) + \sigma(Y). \tag{11}
\]

Since
\[
\sigma(X + Y) = \sqrt{\sigma^2(X + Y)} = \sqrt{\sigma^2(X) + \sigma^2(Y)}, \tag{12}
\]
and
\[
\sigma(X) + \sigma(Y) = \sqrt{(\sigma(X) + \sigma(Y))^2 - \sqrt{\sigma^2(X) + \sigma^2(Y) + 2\sigma(X)\sigma(Y)}}, \tag{13}
\]
we note that
\[
\sigma(X + Y) \leq \sigma(X) + \sigma(Y) \tag{14}
\]
(the equality holds only if $\sigma(X)$ and/or $\sigma(Y)$ are zero).

If $\sigma(X)$ and $\sigma(Y)$ are not zero, and we consider both (12) and (13), we obtain
\[
\sigma(X + Y) = \sqrt{k (\sigma(X) + \sigma(Y))}, \tag{15}
\]
with
\[
k = \frac{\sigma^2(X) + \sigma^2(Y)}{\sigma^2(X) + \sigma^2(Y) + 2\sigma(X)\sigma(Y)} \tag{16}
\]

By letting $h = \frac{\sigma(Y)}{\sigma(X)}$, we finally have
\[
\sqrt{k} = \sqrt{\frac{1 + h^2}{(1 + h)^2}}.
\]

---

It is the same if $h = \frac{\sigma(X)}{\sigma(Y)}$. 
The function \( \sqrt{\frac{1+\delta^2}{(1+h)^2}} \), which is defined for \( h > 0 \), takes its minimum value \( \sqrt{\frac{1}{2}} \) for \( h = 1 \) (i.e., \( \sigma(X) = \sigma(Y) \)). Further, it tends to one as \( h \) diverges (i.e., \( |\sigma(X) - \sigma(Y)| \to +\infty \)).

Appendix 2

Given a quadratic utility function \( u(x) = x - \frac{x^2}{2B} \), which is defined for \( 0 \leq x \leq B \), we look for the solvency margin \( U \) satisfying

\[
E \left( (U + P - X) - \frac{(U + P - X)^2}{2B} \right) = 0
\]

The approximation \( X \approx P + \sigma(X)Z \), together with straightforward computations, leads to

\[
U^2 - 2BU + \sigma^2(X) = 0,
\]

with roots

\[
U_2 = B \left[ 1 + \sqrt{1 - \left( \frac{\sigma(X)}{B} \right)^2} \right] \approx \frac{1}{B} \frac{\sigma^2(X)}{2}
\]

Hence, we choose the positive root as the minimum solvency margin.

References


The square root has been approximated by \( 1 - \frac{1}{2} \left( \frac{\sigma(X)}{B} \right)^2 \).