A GENERALISATION OF G. F. HARDY'S FORMULA FOR THE YIELD ON A FUND

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Synopsis. Let A, B be the values placed on the funds of a life office, pension fund, investment trust or other financial organisation at the beginning and end respectively of an accounting year, and let I be the interest and dividend income received during the year. There is a well-known approximate formula for the effective yield per annum (i) on the funds during the year, viz.,

\[ i = \frac{I}{\frac{1}{2}(A + B - I)}. \] (1)

This formula was first given by G. F. Hardy in an article in the Transactions of the Actuarial Society of Edinburgh, December 1890, reprinted in T.F.A., 8, pp. 61-62, and is derived by D. W. A. Donald in Compound Interest and Annuities-Certain, second edition, 1970, C.U.P., example 11.7.

The above formula is used in the official form F.40 for the valuation of a friendly society.

We shall show that Hardy's formula (1) is a measure of the growth rate of the funds only if there are no capital gains or losses to be considered. In present inflationary conditions this formula may give an incomplete picture of the progress of the funds. We shall show that the growth rate during the year is the sum of the rate of growth due to interest (j), plus the rate of growth due to capital appreciation or depreciation (k). The approximate formulae for j, k are

\[ j = \frac{I}{\frac{1}{2}(A + B - I - C)} \] (2)

and

\[ k = \frac{C}{\frac{1}{2}(A + B - I - C)} \] (3)

where A, B, I are as above and C is the capital gain or loss brought into account during the year. Clearly, if C = 0 then k = 0 and formula (2) reduces to Hardy's formula (1).

We consider the revenue account of a fund or financial institution for an accounting year. A, B, I are defined as above, and we define
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\( M = \) "new money" received during the year, i.e., the excess of income over outgo excluding the proceeds of investments. In the case of a life office,

\[ M = \text{premium income} - \text{claims paid} - \text{expenses} - \text{taxation}; \]

for a pension fund,

\[ M = \text{contribution income} - \text{benefit payments} - \text{expenses (if any)} - \text{taxation (if any)}. \]

We also let \( C = \) capital appreciation or depreciation brought into account during the year.

It follows that

\[ B = A + M + I + C. \]  

Let \( i \) be the effective annual rate of growth of the funds. If "new money" is received on average at time \( r \) from the beginning of the year, we have the approximate relationship

\[ A(1+i) + M(1+(1-r)i) = B. \]

If "new money" is received uniformly over the year we have

\[ A(1+i) + Mr \frac{i}{1} = B \]

and the approximation \( \frac{i}{1} = 1 + \frac{1}{2}i \) shows that in this case we obtain equation (5) with \( r = \frac{1}{2} \).

Subtracting (4) from (5) gives

\[ Ai + (1-r)Mi = I + C \]

so that

\[ i = \frac{I+C}{A+(1-r)M} = \frac{I+C}{rA+(1-r)(B-I-C)}. \]

If we define \( j, k \) by

\[ j = \frac{I}{rA+(1-r)(B-I-C)} \]

and

\[ k = \frac{C}{rA+(1-r)(B-I-C)} \]

then the rate of growth per annum, \( i \), is the sum of the rate of growth \( j \) due to interest, and the rate of growth \( k \) due to capital appreciation or depreciation. It may normally be assumed that \( r = \frac{1}{2} \), in which case formulae (6), (7) reduce to formulae (2), (3).

The amounts of interest income \( I \) and capital gains \( C \) should, in theory, be those appropriate to the capital invested. If, for example, new money is invested in securities bearing annual dividends, the
interest income received this year will be nil, which does not correspond to the capital invested. The amounts of any capital gains or losses during the year depend on the methods of valuing the assets, including the rate at which redeemable fixed-interest securities are written up or down to their redemption or sale prices. These points will not arise if interest and capital gains accrue continuously from each investment.

It is unnecessary to assume that all interest and capital gains are received at the end of the year, for they may be assumed to be reinvested immediately on receipt, which has the effect of bringing them into account at the end of the year.

It may be the practice of a life office to take $A, B$ at "book value", i.e., cost price with any adjustments to date. If there are no adjustments to book values during the year, then

$$B = A + M + I,$$

so that $C = 0$ and our formulae reduce to Hardy's. It may, however, be desired to value $A, B$ at other values, at least for internal purposes, in which case $C$ may be non-zero. We illustrate our formulae by means of a hypothetical life office which has the following revenue account:

\[
\begin{array}{ll}
\text{Funds at 1 January} & £m \\
\text{Premium income} & 15 \\
\text{Interest income} & 6 \\
\text{Capital appreciation} & 3 \\
\hline
 & £m \\
\text{Claims paid} & 8 \\
\text{Expenses} & 2 \\
\text{Taxation} & 1 \\
\text{Funds at 31 December} & 113 \\
\hline
 & £m \\
 & 124 \\
 & 124
\end{array}
\]

In the above example, $A = 100$, $B = 113$, $I = 6$, $C = 3$ and $M = 4$. If "new money" is received uniformly over the year we may assume $r = \frac{1}{2}$ so that, from formulae (2), (3),

$$j = \frac{6}{102} = 5.88\%$$

and

$$k = \frac{3}{102} = 2.94\%.$$

Hence the rate of growth of the office's funds in the year was $8.82\%$ per annum, which was made up of $5.88\%$ per annum interest income and $2.94\%$ per annum capital appreciation. Taxation has been regarded as an "expense", so these growth rates are gross. Any sum transferred to reserve to cover a contingent liability to capital gains tax, or for other reasons, may also be regarded as an "expense".
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An alternative approach. Instead of finding the annual rates of interest and capital growth we could instead consider the forces of interest and capital growth. We shall do this along the lines of the first method in Hardy's original paper, but to clarify the argument we shall derive our formulae from first principles.

Let $A$, $B$, $I$, $C$ and $M$ have the meanings assigned to them above, and let $F(t)$ be the value of the funds at time $t$, $0 \leq t \leq 1$. We shall assume that there is a constant force of growth $\delta$ throughout the year. It follows that if $M(t)$ is the amount of "new money" (defined above) received during the year up to time $t$, then

$$\frac{dF(t)}{dt} = \delta F(t) + \frac{dM(t)}{dt}.$$ 

Integrating from 0 to 1 we obtain

$$[F(t) - M(t)]_{t=0}^{t=1} = \delta \int_0^1 F(t)dt;$$

that is,

$$B - A - M = I + C = \delta \int_0^1 F(t)dt. \quad (8)$$

If we now assume that $F(t)$ is linear for $0 \leq t \leq 1$, then

$$F(t) = A + t(B - A) \text{ for } 0 \leq t \leq 1,$$

and

$$\int_0^1 F(t)dt = \frac{(A + B)}{2}.$$ 

Hence, from (8),

$$\delta = \frac{I + C}{\frac{1}{2} (A + B)}. \quad (9)$$

We observe that the forces of interest and capital appreciation $\delta_j$, $\delta_k$ are such that $\delta = \delta_j + \delta_k$, and

$$\delta_j = \frac{I}{\frac{1}{2} (A + B)} \quad (10)$$

and

$$\delta_k = \frac{C}{\frac{1}{2} (A + B)}. \quad (11)$$

The familiar approximation $i \approx \frac{\delta}{1 - \frac{1}{2} \delta}$ shows that the annual rate of growth is approximately equal to

$$i = \frac{I + C}{\frac{1}{2} (A + B - I - C)},$$

and we may define $j$, $k$ as above.
Several of the points made here were made by P. F. Hooker in the discussion following the paper "Pension Fund Valuations in Modern Conditions" by Heywood and Lander (J.I.A., 87, pp. 314-370). In particular, the authors and Hooker referred to the fact that Hardy's formula gives only the running or interest yield. We hope that our formulae will be of value in determining the rate of capital appreciation, and hence the total growth rate, of a fund.