JIA 113 (1986) 509-520

ON THE RELATIONSHIP BETWEEN GROSS AND NET YIELDS TO REDEMPTION—PRACTICAL VERSUS THEORETICAL APPROXIMATIONS

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INTRODUCTION

In the market for fixed-interest securities, several variants of the yield to redemption (YTR) concept are used. The YTR of a stock is simply the internal rate of return which the holder can expect to receive if he/she holds the stock until maturity. Since income tax has to be paid by some, if not all investors on the interest received, and in some cases on the capital gain as well, a distinction is made between the gross YTR and the net YTR.

In practice, surrogates for the gross YTR and net YTR are used by investors and analysts. An estimate of net YTR is often calculated by adjusting the gross YTR for taxation. This method is attractive as the one calculation of gross YTR can be made and subsequently adjusted for investors in different taxation situations. Another estimation is sometimes used when the net YTR for an investor on a particular tax rate is grossed-up to give an estimate for gross YTR. The objective of this work is to explore the efficacy of these surrogate calculations.

Gross YTR

If we assume for simplicity that the bonds are purchased on issue, thus eliminating the problem of accrued interest, the gross YTR is defined as r_G in the equation:

$$P = \frac{I}{1+r_G} + \frac{I}{(1+r_G)^2} + \ldots + \frac{I}{(1+r_G)^n} + \frac{P+Q}{(1+r_G)^n}, \qquad (1)$$

where P is the purchase price of the security (issue price), Q is the capital gain ($Q > 0$) or capital loss ($Q < 0$), I is the interest per period and n the number of periods to maturity. In most cases the periods of interest will be annual but it is not necessary to assume this. Note that $P + Q$ corresponds to the maturity value, so that $P + Q > 0$. Equation (1) reduces to:

$$P = \frac{I}{r_G} + \frac{Q}{(1+r_G)^n - 1}. \qquad (2)$$

Part of this paper was written while John Rickard was Visiting Professor of Finance at the London Business School.

A necessary and sufficient condition for the existence and uniqueness of a positive r_G is provided by $nI + P + Q > P$, or $nI > -Q$, which requires that the total dividend payments exceed the capital loss. This condition emanates from the requirement that total receipts exceed purchase price, given that there is, of necessity, only one sign change in the cash flow sequence.

Net YTR

We shall now make a number of additional simplifying assumptions. First, assume that the rate of personal income tax is the same as the rate of capital gains tax. Second, assume that both taxes are payable without delay. Third, assume that capital losses earn a rebate at the capital gains tax rate, also without delay. Under the above assumptions the net YTR is defined as r_N in the equation:

$$P = \frac{I(1-t)}{1+r_N} + \frac{I(1-t)}{(1+r_N)^2} + \ldots + \frac{I(1-t)}{(1+r_N)^n} + \frac{P+Q(1-t)}{(1+r_N)^n}, \qquad (3)$$

where t is the investor's marginal tax rate. Equation (3) reduces to:

$$P = \frac{I(1-t)}{r_N} + \frac{Q(1-t)}{(1+r_N)^n - 1}. \qquad (4)$$

As in the case of r_G, a necessary and sufficient condition for the existence of a unique positive r_N is provided by $nI > -Q$. This means that whenever a unique positive r_G exists then a unique positive r_N exists, and vice-versa.

Surrogates

As outlined in the introduction, surrogates for gross and net YTR are created by applying the tax rate to each. An approximate net YTR, R_N, is created by multiplying the gross YTR by the factor $(1-t)$, and an approximate gross YTR, R_G, is created by dividing the net YTR by the factor $(1-t)$. That is:

$$R_N = (1-t)r_G \qquad (5a)$$

and

$$R_G = \frac{r_N}{1-t}. \qquad (5b)$$

R_G is commonly called grossed-up net YTR and R_N could be called netted-down gross YTR.

The approximations in (5a) and (5b) can be rewritten as:

$$r_G\,(1-t) = R_N$$

and

$$R_G\,(1-t) = r_N.$$

These can be more conveniently expressed as:

$$\frac{R_G}{r_G} = \frac{r_N}{R_N} \quad \text{or} \quad R_G R_N = r_G r_N, \qquad (6)$$

which emphasises that an over-estimated R_G for r_G corresponds to an under-estimated R_N for r_N, and obviously vice-versa.

Grossed-up Net YTR

In circumstances where capital gains are not taxed at the same rate as the investor's income (or capital losses are not rebated at the same rate) one should not expect the grossed-up net YTR to equal the gross YTR. Thus $R_G \neq r_G$ seems reasonable since the grossing up process implicitly treats capital gains or losses as income for tax purposes. Consequently where there is a tax-free capital gain, $R_G > r_G$, and where there is a capital loss which earns no tax rebate, $R_G < r_G$.

However, where capital gains are taxed at the same rate as the investor's income (or capital losses are rebated at the same rate), it might appear plausible to expect that $R_G = r_G$. Yet it turns out that even in this case $R_G \neq r_G$ when $t > 0$ and $n > 1$. The situation here is the same as in the tax-free case, that is where there is a capital gain $R_G > r_G$, and where there is a capital loss $R_G < r_G$.

This inequality is basically due to the fact that the internal rate of return is a type of average and the grossing up process 'spreads' the capital gain/loss over the component periods involved in a manner which does not coincide with the actual timing of the gain/loss.

Illustration

To illustrate the point, assume an investor buys a \$100 bond for \$95 with a coupon of 16% and redemption at par three years hence. The gross YTR is r_G in the equation:

$$95 = \frac{16}{1 + r_G} + \frac{16}{(1 + r_G)^2} + \frac{116}{(1 + r_G)^3},$$

which is 18·311% p.a.

If the investor's marginal income tax rate is 32% p.a. and if the capital gain of \$5 (i.e. \$100 − \$95) is also taxed at this rate, the net YTR is r_N in the equation

$$95 = \frac{16(1 - \cdot 32)}{1 + r_N} + \frac{16(1 - \cdot 32)}{(1 + r_N)^2} + \frac{(16 + 5)(1 - \cdot 32) + 95}{(1 + r_N)^3},$$

which is 12·508% p.a.

The grossed-up net YTR, R_G, is 12·508/(1 − ·32) = 18·394% p.a. which is slightly higher than the gross YTR of 18·311%. The reason for this inequality is that the capital gain features only at the end of the period and is not distributed across the three years in a manner which would result in $R_G = r_G$. (Notice that the netted-down gross YTR, R_N, is 18·311 (1 − ·32) = 12·451% p.a. which is slightly lower than the net YTR, r_N, of 12·508% p.a.)

When $n = 1$ the equations defining r_G and r_N, namely (2) and (4), reduce to:

$$P = \frac{I+Q}{r_G}$$

and

$$P = \frac{(I+Q)(1-t)}{r_N},$$

respectively. It follows immediately that:

$$r_G = \frac{r_N}{1-t} = R_G$$

and

$$r_N = r_G(1-t) = R_N \qquad \text{when } n = 1.$$

In other words, the gross YTR is the same as the grossed-up net YTR and the net YTR is the same as the netted-down gross YTR when the life of the instrument is equal to the interest period.

Again, for illustration, suppose the debenture in the previous example had only one year to run to maturity ($P = 95$, $I = 16$, $Q = 5$, $t = \cdot 32$). It readily follows that $r_G = R_G = 22{\cdot}105\%$ and $r_N = R_N = 15{\cdot}032\%$.

Both the gross and the net YTR are internal rates of return, and the internal rate of return is just one of several possible averages. Grossing up the net YTR by dividing by $(1-t)$ does not give the gross YTR except where $t = 0$ or $n = 1$ because the capital gain component is, as stated earlier, spread over the component interest periods in a manner which does not coincide with the actual timing of the capital gain.

In an analogous manner, one could demonstrate the difference between the net YTR, r_N, and its approximation R_N.

The Efficacy of $\mathrm{R_N}$

In this section we wish to examine the circumstances under which the net YTR, i.e. r_N, and its approximation used by practitioners, i.e. R_N, are approximately equal. Recall that, as shown in equation (5a):

$$R_N = (1-t)r_G.$$

To discuss r_N or r_G, it is convenient to introduce the function $f(t, r)$ given by:

$$f(t, r) = P - I(1-t)/r - Q(1-t)/((1+r)^n - 1). \tag{7}$$

Then r_G is the solution of:

$$f(0, r_G) = 0 \tag{8}$$

and r_N is the solution of:

$$f(t, r_N) = 0. \tag{9}$$

It is then relatively easy to demonstrate (see Appendix for further details) that given $nI + Q > 0$ and $0 < t < 1$, the following relationships hold

$$\text{(a) } Q > 0, R_N < r_N \tag{10a}$$

$$\text{(b) } Q < 0, R_N > r_N \tag{10b}$$

$$\text{(c) } Q = 0, R_N = r_N. \tag{10c}$$

Thus for the not uncommon case of the bond price being below the redemption price (implying a capital gain), the netted-down gross YTR (R_N) is always a biased-low estimate of r_N. Conversely, in a capital loss situation, this estimate of the net YTR is always biased-high.

From the point of view of estimation, it is unsatisfactory to have an estimator that always has an inbuilt bias. The question then arises as to whether an estimate can be developed that does not have this inbuilt bias property. To this end, it is convenient to develop r_N as an expression in r_G as follows:

$$r_N = (1-t)r_G\,[1 + \alpha(r_G)]. \tag{11}$$

Clearly, using the approximation formula (5a) is equivalent to asserting that the term $\alpha(r_G)$ is identically zero. Using the fact that r_G and r_N are solutions to equations (2) and (4) respectively, an asymptotic expansion (see for example, Copson, Erdélyi or Sirovich) was developed for $\alpha(r_G)$ in terms of powers of r_G, that is:

$$\alpha(r_G) = \alpha_1 r_G + \alpha_2 r_G^2 + 0(r_G^3). \tag{12}$$

It can then be shown (see Appendix for further details) that:

$$\alpha_1 = \tfrac{1}{2}(n-1)\beta \tag{13}$$

and

$$\alpha_2 = \tfrac{1}{6}(n-1)(n-2)\beta(2-t) - \tfrac{1}{4}(n-1)^2\beta(2-t-\beta) \tag{14}$$

$$\text{where } \beta = \frac{tQ}{nI + Q},$$

provided $nI \neq -Q$. The latter was ruled out in order to guarantee a unique positive r_G and r_N.

This approximation has been evaluated over a range of prices, capital gains, dividend payments, period lengths and marginal tax rates. The first term approximation

$$(1-t)r_G[1 + \alpha_1 r_G]$$

is always better than $R_N = (1-t)r_G$, though the error is of the opposite sign to that of R_N and the improvement sometimes is only marginal. However the two term approximation

$$(1-t)r_G[1 + \alpha_1 r_G + \alpha_2 r_G^2]$$

is always much better than $R_N = (1-t)r_G$. In the great majority of cases computed, the two term estimate of r_N and the actual r_N differed by no more than the second decimal place, considering rates as percentages (or the fourth decimal place considering rates as decimals).

The Efficacy of R_G

Due to the inverse nature of the over- and under-estimation of R_N and R_G, the following result is clear.

If $nI + Q > 0$ and $0 < t < 1$, then

$$\text{(a)}\ Q > 0,\ R_G > r_G$$

$$\text{(b)}\ Q < 0,\ R_G < r_G$$

$$\text{(c)}\ Q = 0,\ R_G = r_G.$$

Thus for the not uncommon case of a (positive) capital gain, the grossed-up net YTR is always optimistic with respect to the actual gross YTR. Conversely, in a capital loss situation, this estimate of gross YTR is always pessimistic.

Again, this implies that it is a poor estimator. A more accurate one is given by the expansion (see Appendix for further details):

$$r_G = (1-t)^{-1}r_N[1 + \gamma_1 r_N + \gamma_2 r_N^2 + 0(r_N^3)] \tag{15}$$

where

$$\gamma_1 = -\tfrac{1}{2}(1-t)^{-1}(n-1)\beta$$

and

$$\gamma_2 = \tfrac{1}{4}(1-t)^{-2}(n-1)^2[2-t+\beta]\beta - \tfrac{1}{6}(1-t)^{-2}(n-1)(n-2)(2-t)\beta$$

with

$$\beta = \frac{tQ}{nI+Q};\ nI + Q \neq 0.$$

The two term expansion has been computed to give much more accurate estimates than the grossed-up net YTR, as given by equation (5b), namely:

$$R_G = (1-t)^{-1}\, r_N.$$

The estimation qualities of the asymptotic expansions for r_N and r_G given in equations (11)–(14) and (15) are best illustrated by means of several examples. Table 1 gives the successive approximations $R_N = (1-t)r_G$, $R_N(1 + \alpha_1 r_G)$ and $R_N(1 + \alpha_1 r_G + \alpha_2 r_G^2)$ to r_N for a range of values of P (and hence Q), I, n (which, together, determine r_G) and also t. The relative percentage error at each level of approximation is also given. Any apparent discrepancies in the error calculations are due solely to the rounding-off to two decimal places of those values reported in the tables.

Similarly, Table 2 gives the three successive approximations $R_G = (1-t)^{-1}r_N$,

$R_G(1 + \gamma_1 r_N)$ and $R_G(1 + \gamma_1 r_N + \gamma_2 r_N^2)$ to r_G for a range of values of P, I, n and t (which, together, determine r_N).

The computer program used to produce the results in this paper is available from the authors upon request.

CONCLUDING REMARKS

The approach taken by practitioners when adjusting bond yields to either include or exclude the effect of taxation rates, always produces a biased estimation, except when the life of the bond is equal to the interest period. This is borne out by the tabulated results. The nature of the bias depends on whether a capital loss or capital gain is involved. This error in estimation always increases with increasing rates of taxation. In times of reasonably stable interest rates, this error will consistently appear in the yield estimates calculated by practitioners. While the relevant yield estimate will be consistently biased, it will nevertheless be a reasonable guide to the behaviour of actual yields. However in periods of fluctuating interest rates, the prices of bonds will correspondingly fluctuate and capital gains/losses will commensurately fluctuate. The error in the calculation of estimated yield will then itself fluctuate. This calculation error will confound the fluctuating behaviour of reported yields.

The approximation developed by way of expansions in this work serve two purposes. Firstly they include the practitioners' approximation as the leading term. The remaining terms thus serve the purpose of explaining most of the difference between the actual yields and those used by practitioners. Secondly, they can be used as more accurate estimators of actual yields. When calculating the adjusted yields, one needs to be aware of the nature of the bias in their simple estimation process. As this will vary with circumstances, one needs to be wary of the presence of such biased results.

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Sirovich, L. (1971) *Techniques of Asymptotic Analysis*. Springer-Verlag, New York.

Table 1. *Approximations to NYR given GYR and various bond parameters (price, coupon rate and duration) for various marginal tax rates*

(i) $P=80$, $Q=20$, $I=10$, $n=4$: $r_G=17{\cdot}34\%$

Tax rate i	NYR $r_N\%$	Netted-down GYR% $R_N=(1-i)r_G$	% Error	First approximation $R_N(1+\alpha_1 r_G)$	% Error	Second approximation $R_N(1+\alpha_1 r_G+\alpha_2 r_G^2)$	% Error
·32	12·05	11·79	−2·18	12·12	·53	12·05	−·05
·46	9·67	9·36	−3·20	9·74	·66	9·67	−·05
·60	7·24	6·94	−4·26	7·30	·72	7·24	−·02

(ii) $P=80$, $Q=20$, $I=10$, $n=10$: $r_G=13{\cdot}81\%$

Tax rate i	NYR $r_N\%$	Netted-down GYR% $R_N=(1-i)r_G$	% Error	First approximation $R_N(1+\alpha_1 r_G)$	% Error	Second approximation $R_N(1+\alpha_1 r_G+\alpha_2 r_G^2)$	% Error
·32	9·59	9·39	−2·09	9·70	1·15	9·58	−·12
·46	7·70	7·45	−3·13	7·81	1·49	7·69	−·09
·60	5·77	5·52	−4·25	5·87	1·70	5·76	−·04

(iii) $P=80$, $Q=20$, $I=20$, $n=4$: $r_G=29{\cdot}09\%$

Tax rate i	NYR $r_N\%$	Netted-down GYR% $R_N=(1-i)r_G$	% Error	First approximation $R_N(1+\alpha_1 r_G)$	% Error	Second approximation $R_N(1+\alpha_1 r_G+\alpha_2 r_G^2)$	% Error
·32	20·16	19·78	−1·87	20·34	·87	20·13	−·17
·46	16·16	15·71	−2·78	16·34	1·13	16·13	−·17
·60	12·09	11·64	−3·74	12·25	1·30	12·07	−·15

(iv) $P=80$, $Q=20$, $I=20$, $n=10$: $r_G=25{\cdot}73\%$

Tax rate i	NYR $r_N\%$	Netted-down GYR% $R_N=(1-i)r_G$	% Error	First approximation $R_N(1+\alpha_1 r_G)$	% Error	Second approximation $R_N(1+\alpha_1 r_G+\alpha_2 r_G^2)$	% Error
·32	17·73	17·49	−1·35	18·08	1·97	17·64	−·55
·46	14·19	13·89	−2·11	14·56	2·62	14·11	−·59
·60	10·61	10·29	−3·01	10·94	3·12	10·55	−·54

(v) $P=120$, $Q=-20$, $I=10$, $n=4$: $r_G=4{\cdot}43\%$

Tax rate i	NYR $r_N\%$	Netted-down GYR% $R_N=(1-i)r_G$	% Error	First approximation $R_N(1+\alpha_1 r_G)$	% Error	Second approximation $R_N(1+\alpha_1 r_G+\alpha_2 r_G^2)$	% Error
·32	2·96	3·01	2·00	2·95	−·17	2·96	·01
·46	2·33	2·39	2·89	2·32	−·26	2·33	·02
·60	1·71	1·77	3·79	1·70	−·35	1·71	·03

Table 1 (*continued*)

(vi) $P=120$, $Q=-20$, $I=10$, $n=10$: $r_G=7{\cdot}13\%$

Tax rate i	NYR $r_N\%$	Netted-down GYR% $R_N=(1-i)r_G$	% Error	First approximation $R_N(1+\alpha_1 r_G)$	% Error	Second approximation $R_N(1+\alpha_1 r_G+\alpha_2 r_G^2)$	% Error
·32	4·76	4·85	2·03	4·73	−·59	4·76	·05
·46	3·74	3·85	2·98	3·71	−·82	3·74	·08
·60	2·74	2·85	3·97	2·72	−1·04	2·75	·12

(vii) $P=120$, $Q=-20$, $I=20$, $n=4$: $r_G=13{\cdot}24\%$

Tax rate i	NYR $r_N\%$	Netted-down GYR% $R_N=(1-i)r_G$	% Error	First approximation $R_N(1+\alpha_1 r_G)$	% Error	Second approximation $R_N(1+\alpha_1 r_G+\alpha_2 r_G^2)$	% Error
·32	8·85	9·00	1·75	8·81	−·40	8·85	·04
·46	6·97	7·15	2·56	6·93	−·56	6·98	·07
·60	5·12	5·30	3·40	5·09	−·71	5·13	·09

(viii) $P=120$, $Q=-20$, $I=20$, $n=10$: $r_G=15{\cdot}88\%$

Tax rate i	NYR $r_N\%$	Netted-down GYR% $R_N=(1-i)r_G$	% Error	First approximation $R_N(1+\alpha_1 r_G)$	% Error	Second approximation $R_N(1+\alpha_1 r_G+\alpha_2 r_G^2)$	% Error
·32	10·64	10·80	1·45	10·52	−1·12	10·67	·20
·46	8·39	8·58	2·21	8·26	−1·53	8·41	·29
·60	6·16	6·35	3·04	6·05	−1·87	6·19	·37

Table 2. *Approximations to GYR given various bond parameters (price, coupon rate and duration) and marginal tax rates*

(i) $P=80$. $Q=20$, $I=10$, $n=4$: $r_G=17{\cdot}34\%$

Tax rate i	NYR $r_N\%$	Grossed-up NYR% $R_G=(1-i)^{-1}r_N$	% Error	First approximation $R_G(1+\gamma_1 r_N)$	% Error	Second approximation $R_G(1+\gamma_1 r_N+\gamma_2 r_N^2)$	% Error
·32	12·05	17·73	2·23	17·22	−·67	17·36	·14
·46	9·67	17·91	3·31	17·17	−·95	17·37	·20
·60	7·24	18·11	4·45	17·13	−1·23	17·39	·28

(ii) $P=80$, $Q=20$, $I=10$, $n=10$: $r_G=13{\cdot}81\%$

Tax rate i	NYR $r_N\%$	Grossed-up NYR% $R_G=(1-i)^{-1}r_N$	% Error	First approximation $R_G(1+\gamma_1 r_N)$	% Error	Second approximation $R_G(1+\gamma_1 r_N+\gamma_2 r_N^2)$	% Error
·32	9·59	14·10	2·13	13·62	−1·32	13·85	·30
·46	7·70	14·25	3·23	13·55	−1·85	13·87	·44
·60	5·77	14·42	4·44	13·48	−2·34	13·89	·61

Table 2 (*continued*)

(iii) $P=80$, $Q=20$, $I=20$, $n=4$: $r_G=29{\cdot}09\%$

Tax rate i	NYR $r_N\%$	Grossed-up NYR% $R_G=(1-i)^{-1}r_N$	% Error	First approximation $R_G(1+\gamma_1 r_N)$	% Error	Second approximation $R_G(1+\gamma_1 r_N+\gamma_2 r_N^2)$	% Error
·32	20·16	29·65	1·90	28·80	−1·00	29·18	·29
·46	16·16	29·92	2·85	28·69	−1·39	29·21	·41
·60	12·09	30·22	3·89	28·58	−1·76	29·25	·54

(iv) $P=80$, $Q=20$, $I=20$, $n=10$: $r_G=25{\cdot}73\%$

Tax rate i	NYR $r_N\%$	Grossed-up NYR% $R_G=(1-i)^{-1}r_N$	% Error	First approximation $R_G(1+\gamma_1 r_N)$	% Error	Second approximation $R_G(1+\gamma_1 r_N+\gamma_2 r_N^2)$	% Error
·32	17·73	26·08	1·37	25·19	−2·09	25·93	·80
·46	14·19	26·28	2·16	24·98	−2·89	26·01	1·11
·60	10·61	26·52	3·10	24·80	−3·61	26·08	1·40

(v) $P=120$, $Q=-20$, $I=10$, $n=4$: $r_G=4{\cdot}43\%$

Tax rate i	NYR $r_N\%$	Grossed-up NYR% $R_G=(1-i)^{-1}r_N$	% Error	First approximation $R_G(1+\gamma_1 r_N)$	% Error	Second approximation $R_G(1+\gamma_1 r_N+\gamma_2 r_N^2)$	% Error
·32	2·96	4·35	−1·96	4·44	·09	4·43	·01
·46	2·33	4·31	−2·81	4·44	·08	4·43	·01
·60	1·71	4·27	−3·65	4·44	·05	4·43	·01

(vi) $P=120$, $Q=-20$, $I=10$, $n=10$: $r_G=7{\cdot}13\%$

Tax rate i	NYR $r_N\%$	Grossed-up NYR% $R_G=(1-i)^{-1}r_N$	% Error	First approximation $R_G(1+\gamma_1 r_N)$	% Error	Second approximation $R_G(1+\gamma_1 r_N+\gamma_2 r_N^2)$	% Error
·32	4·76	6·99	−1·99	7·17	·48	7·14	·01
·46	3·74	6·93	−2·89	7·18	·59	7·14	·03
·60	2·74	6·86	−3·82	7·18	·64	7·14	·06

(vii) $P=120$, $Q=-20$, $I=20$, $n=4$: $r_G=13{\cdot}24\%$

Tax rate i	NYR $r_N\%$	Grossed-up NYR% $R_G=(1-i)^{-1}r_N$	% Error	First approximation $R_G(1+\gamma_1 r_N)$	% Error	Second approximation $R_G(1+\gamma_1 r_N+\gamma_2 r_N^2)$	% Error
·32	8·85	13·01	−1·72	13·29	·32	13·24	−·01
·46	6·97	12·91	−2·50	13·29	·40	13·24	·00
·60	5·12	12·81	−3·29	13·30	·43	13·24	·02

(viii) $P=120$, $Q=-20$, $I=20$, $n=10$: $r_G=15{\cdot}88\%$

Tax rate i	NYR $r_N\%$	Grossed-up NYR% $R_G=(1-i)^{-1}r_N$	% Error	First approximation $R_G(1+\gamma_1 r_N)$	% Error	Second approximation $R_G(1+\gamma_1 r_N+\gamma_2 r_N^2)$	% Error
·32	10·64	15·65	−1·43	16·04	1·03	15·87	−·09
·46	8·39	15·54	−2·16	16·09	1·34	15·87	−·07
·60	6·16	15·41	−2·95	16·12	1·54	15·88	−·03

APPENDIX

Lemma A

Suppose $I > 0$, $0 \leqq t < 1$ and $nI + Q > 0$. Then

(i) $Q > 0$ implies $R_N < r_N$
(ii) $Q < 0$ implies $R_N > r_N$
(iii) $Q = 0$ implies $R_N = r_N$.

Proof

It follows from equations (2) and (5a) that

$$P = (1-t)IR_N^{-1} + Q((1 + R_N(1-t)^{-1})^n - 1)^{-1}. \tag{A1}$$

By expanding $(1 + R_N(1-t)^{-1})^n$ it is readily established that

$$(1 + R_N(1-t)^{-1})^n - 1 \geqq (1-t)^{-1}((1 + R_N)^n - 1). \tag{A2}$$

The result now follows from equations (4) and (A1) after using (A2) and observing that r^{-1} and $((1 + r)^n - 1)^{-1}$ are both decreasing functions of r.

Lemma B

If r_N is expressed in the form

$$r_N = (1-t)r_G[1 + \alpha_1 r_G + \alpha_2 r_G^2 + 0(r_G^3)], \tag{A3}$$

then

$$\alpha_1 = \tfrac{1}{2}(n-1)\beta$$

and

$$\alpha_2 = \tfrac{1}{6}(n-1)(n-2)\beta(2-t) - \tfrac{1}{4}(n-1)^2\beta(2-t-\beta),$$

with

$$\beta = tQ(nI + Q)^{-1}, \text{ provided } nI + Q \neq 0.$$

Proof

It follows from equations (2) and (4) that

$$P - \frac{I}{r_G} - \frac{Q}{(1 + r_G)^n - 1} = P - \frac{I(1-t)}{r_N} - \frac{Q(1-t)}{(1 + r_N)^n - 1},$$

which implies that

$$\begin{aligned} &Ir_N H(r_N)H(r_G) + Qr_N H(r_N) \\ &= I(1-t)r_G H(r_N)H(r_G) + Q(1-t)r_G H(r_G), \end{aligned} \tag{A4}$$

where

$$H(r) = \sum_{k=0}^{n-1} {}^nC_k\, r^k.$$

Expressing r_N in the form (A3), carrying out the formal expansion of the power series in equation (A4) and comparing coefficients yields the result.

Lemma C

If r_G is expressed in the form

$$r_G = (1-t)^{-1}\, r_N[1 + \gamma_1 r_N + \gamma_2 r_N^2 + 0(r_N^3)],$$

then

$$\gamma_1 = -\tfrac{1}{2}(n-1)(1-t)^{-1}\beta$$

and

$$\gamma_2 = \tfrac{1}{4}(n-1)^2(1-t)^{-2}\,[2-t+\beta]\beta - \tfrac{1}{6}(n-1)(n-2)(1-t)^{-2}\,(2-t)\beta,$$

with

$$\beta = tQ(nI+Q)^{-1}; \text{ provided } nI + Q \neq 0.$$

Proof

Proceeding in a similar manner to the proof of Lemma B or by inverting the power-series obtained therein, the result is readily established.