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DYNAMIC RESPONSE OF INSURANCE SYSTEMS WITH DELAYED PROFIT/LOSS-SHARING FEEDBACK TO ISOLATED UNPREDICTED CLAIMS

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Summary

A mathematical model of the dynamic behaviour of an insurance system with delayed profit/loss sharing feedback is developed. The model is then subjected to a disturbance input consisting of an isolated group of unpredicted claims and the dynamic responses of cash flow and accumulated cash flow determined. Increasing delays are seen to lead first to undesirable oscillatory responses and eventually to instability, where the responses become unbounded. Such behaviour is noted to be independent of the type of business and to be a property of the feedback mechanism and not related to the type of disturbance input.

1. LIMITED PROBLEM AND ACTUARIAL SOLUTION

Introduction

1.1. The present paper is the result of an approach by the second author to the first to see if certain problems arising from profit-sharing 'group' business could be placed in a wider theoretical context, which would allow greater insight into the fundamental mechanisms at work in the system. Although the problems arise from the control of profit-sharing group schemes they are similar to problems arising in the control of any insurance portfolio.

1.2. The wider theoretical context of Control or Dynamic Systems Theory will be unfamiliar to most readers. Hence, in this part of the paper, we take the unusual step of first presenting the limited problem as first seen and its solution via an approach which will be familiar to actuaries.

Having quickly assimilated the essential features of the problem the reader will then be able to progress more confidently to the next part, where the wider theoretical framework is presented. There, the standard control theory notation is used not for its own sake but because it is essential for progressing to later topics, especially those which are beyond the scope of this first introductory paper.

Formulation and solution

1.3. Define C_t = claims incurred in year t

P_t = premium paid in year t

k such that $(1-k)P_t$ = margin for expenses; typically $k = \cdot 8$

$z_t = kP_t - C_t$ = surplus for year t (1)

$$G_t = z_t + z_{t-1} + \ldots + z_1 = \text{accumulated surplus to year } t$$
$$= z_t + G_{t-1} \tag{2}$$

B_t = 'base' premium for year t.

In the present example a base premium B_t is adjusted by subtracting a proportion h of the surplus accumulated to date. Typically h might be ·5. Because there is a delay between occurrence and settlement of claims G_{t-l} is used where l is the time delay. Consequently,

$$P_t = B_t - hG_{t-l} \tag{3}$$

and, hence, from equations (1), (2) and (3)

$$G_t - G_{t-1} + hkG_{t-l} = kB_t - C_t \tag{4}$$

which is a difference equation in the unknown G_t, the accumulated surplus. Ordinary finite difference methods lead to the following solutions:

For $l = 1$, equation (4) becomes a first-order difference equation with solution

$$G_t = k(B_t + fB_{t-1} + \ldots + f^{t-1}B_1) - (C_t + fC_{t-1} + \ldots + f^{t-1}C_1) \tag{5}$$

where $f = 1 - kh = 1 - \cdot 8 \times \cdot 5 = \cdot 6$ typically.

For $l = 2$, equation (4) becomes a second-order difference equation. If $4hk > 1$, as is the case for $h = \cdot 5$ and $k = \cdot 8$, then the solution is

$$G_t = \frac{k}{\sin\theta}(B_t \sin\theta + B_{t-1}\, r \sin 2\theta + \ldots + B_1 r^{t-1} \sin t\theta) - \frac{1}{\sin\theta}(C_t \sin\theta + C_{t-1}\, r \sin 2\theta + \ldots + C_1\, r^{t-1} \sin t\theta) \tag{6}$$

where

$$r = \sqrt{hk} = \cdot 6325 \text{ (here)}$$
$$\theta = \tan^{-1}\sqrt{4hk - 1} = \cdot 659 \text{ radians (here).}$$

1.4 Some insight can be gained by observing the reaction of accumulated surplus to a single pulse of incurred claims C_t. Consequently, put $B_t = 0$ and $C_2 = C_3 = \ldots = 0$ while leaving C_1 non-zero. Under these conditions and for $l = 1$

$$G_t = -f^{t-1}\, C_1 = -\cdot 6^{t-1}\, C_1 \tag{7}$$

which is a simple decay factor of ·6 p.a. For $l = 2$

$$G_t = -r^{t-1}\left(\frac{\sin t\theta}{\sin\theta}\right)C_1 = \frac{-\cdot 6325^{t-1} \sin \cdot 659t}{\cdot 612} \tag{8}$$

which is an oscillatory result with period $t' = 2\pi/\theta = 9{\cdot}5$ years and a decay factor of ·6325 p.a.

1.5. Results for larger values of l are most easily obtained directly by successive application of equation (4), as was done initially to obtain an extended set of numerical values.

1.6. The oscillatory nature of the results for $l \geqslant 2$, as shown in Figure 4 (page 528) are disturbing. This concern led to the initial question of whether these results could be set in a wider theoretical context which would give greater insight into the fundamental mechanisms at work.

2. CONTROL THEORETIC APPROACH

Introduction

2.1. As will be seen more clearly later, a negative feedback control mechanism is in operation in the insurance system under consideration. Furthermore, the system is intrinsically dynamic since differential or, in this case, difference equations rather than algebraic ones are needed to describe the way in which the variables (premiums, claims, etc.) vary with time. An extensive body of knowledge known as control theory or dynamic systems theory is available for analysing and predicting the dynamic behaviour of such systems. The purpose of this paper is to carry out some introductory analyses on an insurance system with delayed profit/loss-sharing feedback using some elementary control theoretic techniques. Some very basic and fundamental features will be seen to emerge.

2.2 *Nomenclature*

$b(k)$	Level of business written in period k
$c_i(k)$	Claims incurred from business written in period k
$c_p(k)$	Claims paid in period k
$c_u(k)$	Unpredicted claims paid in period k
$f(k)$	Cash flow for period k
$f_a(k)$	Accumulated cash flow at end of period k
$g(k)$	Accumulated surplus at end of period k
k	Integer indicating financial period
k_c	Costs and profit factor
k_p	Proportion of estimated surplus fed back
l	Number of delay periods
$p(k)$	Premiums paid in period k
$p_b(k)$	Base premiums for period k
$p_f(k)$	Share of estimated accumulated surplus fed back
$p_n(k)$	Net premium income for period k
T	Length of financial time period
z	Transform parameter (complex variable)
$\delta(k-i)$	Kronecker delta at $k-i$
v	Frequency of sinusoidal response.
$\hat{}$	Predicted value of variable.

Modelling philosophy and perspective

2.3. The starting point of any control investigation is the development of a mathematical model of the dynamic system to be studied. An understanding of the concept of mathematical modelling is very important and will be elaborated upon briefly. A mathematical model must not be considered as some inviolate law of nature. It is simply a set of equations which describes the behaviour of the system with sufficient accuracy for the purpose at hand. What is sufficient one day may be too inaccurate or unnecessarily elaborate another.

2.4. An example may help. Newton's law of motion, force equals mass times acceleration, might be though to be a law of nature. However, if applied thoughtlessly to an aircraft it would predict continued acceleration to infinite speeds. One of the limiting phenomena present in the real situation but not modelled is wind resistance. Even with the latter included the model would still not be perfectly accurate. The simple $f=ma$ model is only true if the mass is constant. For some types of calculations an aircraft might be considered to have constant mass. For others the weight reduction due to the continuous burning of fuel is important. Carried to the extreme, as it might be for space travel, effects arising from the special theory of relativity (such as mass increase with speed) could be included.

2.5. Our approach here has been to use a relatively simple model which intentionally excludes various aspects of fine detail. Many such details will be taken up in later papers after the reader has come to terms with an approach which will be unfamiliar to most. Even with the chosen model some quite basic results can be demonstrated.

2.6. Most real systems can be modelled in continuous time using differential equations, in discrete time using difference equations or as sampled-data or hybrid systems, in which the process to be controlled is acknowledged to be continuous but controlled on the basis of samples taken at discrete instants in time. In this paper a discrete time model is developed. If the time interval between successive values of the variables is one year, $x(k)$ would denote the value of the variable x at the end of the kth financial year. It should be emphasized that the results presented here are equally valid for any time period and not just for annual figures. Although there are many introductory texts dealing with discrete-time systems theory, Cadzow (1973) is one of the few written with an interdisciplinary or non-engineering audience in mind. Texts with a greater control theory content are referred to later.

2.7. Two approaches to the analysis are possible. Classical control theory, which dates from the second world war and is still in widespread industrial use, is based on z-transform techniques and transfer functions (see later). State space (sometimes called modern) control theory, which dates from the 1960s, operates directly on the state equation (a vector difference equation which describes the system) with time domain techniques. The latter is more powerful but also more difficult for the neophyte. The former allows many important results to be demonstrated and is used in this introductory paper. Attention is also restricted to linear systems with constant parameters (the coefficients in the equations).

A structural model

2.8. Figure 1 shows quite a general model of the structure of the system under consideration. It is similar to, but not identical with, the type of block diagrams drawn by control theorists. The latter would normally label the lines indicating the flow of information with the z-transforms of the variables concerned and then show in the boxes, the transfer functions relating these transformed variables. To facilitate comprehension we avoid these minor technicalities.

2.9. The business written $b(k)$ in period k flows in two directions; namely, into the section of the model describing premiums and into that describing claims. Consider the premium section first. The box labelled 'Incurred claims predictor' produces an estimate $\hat{c}_I(k)$ of the incurred claims. It represents the total of all claims which are likely to arise at any time from business written in period k. On the basis of this estimate the base premium $p_b(k)$ for this kth period is then determined. From this, a profit-sharing feedback $p_f(k)$ is subtracted, leaving the premium $p(k)$ to be paid. After the margin for costs and profit has been allowed the net premium income $p_n(k)$ is available. The cash flow $f(k)$ for the period k is then simply $p_n(k)$ minus the actual claims paid $c_p(k)$ in that period. These are accumulated in a pool or 'kitty' to give the accumulated cash flow $f_a(k)$ at the end of period k.

2.10. From the accumulated cash flow $f_a(k)$, an estimate $\hat{g}(k)$ of the accumulated surplus at the end of the kth period is made. Note that $\hat{g}(k)$ is an estimate of the surplus after allowing for all claims incurred to period k, including those which have not been paid or even received. How much of this surplus is fed back as profit/loss sharing feedback $p_f(k)$ is determined by the profit-sharing scheme.

2.11. In the claims section of the model a prediction of the claims paid $\hat{c}_p(k)$ as a function of time is made from the business written. To this are added the unpredicted claims $c_u(k)$ for the period k, giving the claims $c_p(k)$ actually paid in that period.

2.12. The points where addition or subtraction of variables (signals in the jargon) takes place are called summing points. The signal take off points where a signal divides into two are easily misunderstood. An analogy might be to think of them as electricity supply lines where, no matter what one might connect, the 240 V supply voltage does not change. They are certainly not analogous to water pipes where the flow divides between two branches.

2.13. Whilst an actuarial reader might think of premiums as input and claims as output, a control theorist would classify unpredicted claims $c_u(k)$ as a disturbance input to the system because he has no control over its behaviour. He would design a control strategy to minimize its effects. Base premium $p_b(k)$ and profit-sharing feedback $p_f(k)$ would be classed as control inputs and used to obtain or approach some specification of desired behaviour for the system. Here two controllers are present, namely the Base Premium Calculator and the Profit-

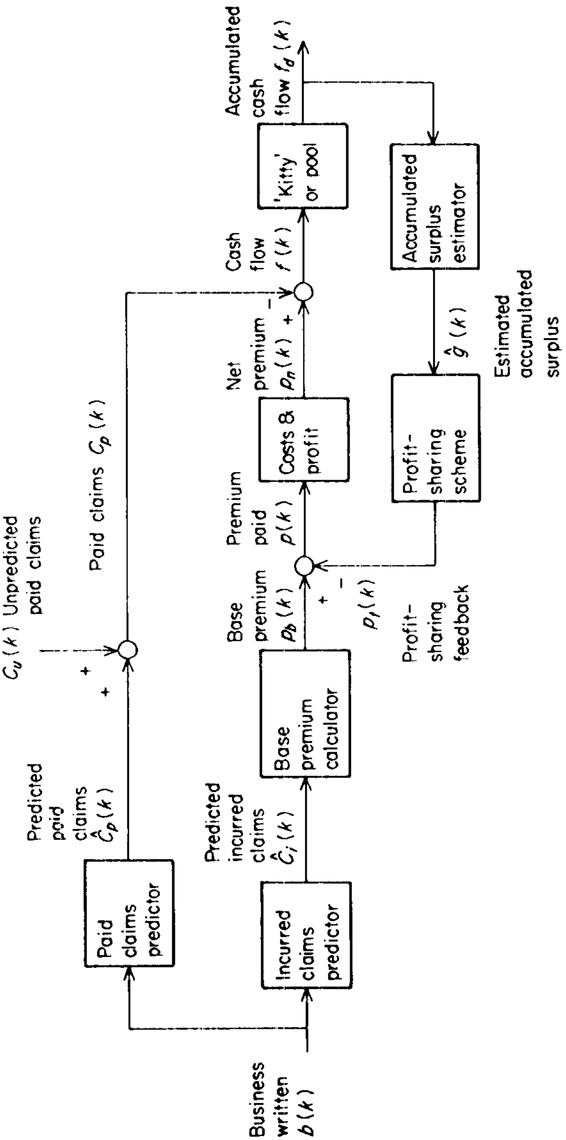

Figure 1. Structure of insurance system model with profit-sharing feedback.

Sharing Scheme. Control via $p_f(k)$ is termed negative feedback control while that using the path from $b(k)$ through to $p_b(k)$ is termed feedforward control. In general, negative feedback has the advantages of reducing the error between the actual value of a variable and its desired value following a disturbance, reducing the sensitivity of the system to changes in its parameters or constants, improving its speed of response and is 'self-correcting' in a sense. Feedforward control is generally less desirable as it is very sensitive to errors in the assumptions upon which it is based. It has no self-correcting mechanism if its predictions are wrong. This is not to say that it should not be used, clearly it is essential above, just that care should be taken in its use and more reliance placed on the feedback control mechanism.

2.14. In classical control theory the block diagram is usually arranged so that the output is in fact the quantity to be controlled. This has not been done in Figure 1. Its output is accumulated cash flow $f_a(k)$, while the accumulated surplus $g(k)$ is to be controlled. Unfortunately, $g(k)$ cannot be measured because future claims remain unknown. The best that can be done is to use an estimate $\hat{g}(k)$ of $g(k)$ for control purposes. Consequently, it is preferable to show the measurable quantity $f_a(k)$ as the output.

2.15. A normal commercial percentage profit to the insurer is allowed in the 'Costs & profit' subsystem, hence the desired value of the accumulated surplus $g(k)$ is zero.

2.16. For further information on introductory control systems theory the reader is referred to one of the multitude of texts, for example Dorf (1970) or Ogata (1970).

Subsystem mathematical models

2.17. In classical control theory a system is described mathematically by its transfer function. For the discrete time system with input sequence $x(k)$ and output sequence $y(k)$ for $k=0,1,2,\ldots$, the discrete transfer function, $H(z)$ say, is defined as the ratio of the z-transform $Y(z)$ of the output to the z-transform $X(z)$ of the input

$$H(z) = \frac{Y(z)}{X(z)} = \frac{Z\{y(k)\}}{Z\{x(k)\}} \tag{11}$$

where $Z\{\ \}$ denotes the z-transform, which is defined by

$$Z\{x(k)\} = \sum_{k=0}^{\infty} x(k)z^{-k} = x(0)+\frac{x(1)}{z}+\frac{x(2)}{z^2}+\ldots \tag{12}$$

It is assumed that $x(k)$ is zero for all negative time

$$x(k) = 0 \text{ for } k = -1, -2, -3, \ldots \tag{13}$$

Mathematically, the z-transform is a rule by which a sequence of numbers is converted into a function of the complex variable z. The real power of the

z-transform is that it will convert a difference equation into an algebraic equation thus simplifying its solution. For further details, see for example Cadzow (1973).

2.18. z^{-1} is often referred to as the backward shift operator because if $y(k)$ is the sequence obtained by delaying the sequence $x(k)$ by l time periods then it is simple to show that

$$y(k) = x(k-l) \Rightarrow Y(z) = z^{-l}X(z). \tag{14}$$

The modelling of systems with pure time delays is thus greatly simplified by the use of the z-transform.

2.19. The final general point to be made is most important. For certain technical reasons related to the z-transform the initial value of a variable is always assumed to be zero. Hence the numerical value of a variable at the kth time instant represents the change in that variable from its initial value at $k=0$ and not its absolute value.

2.20. The remainder of this section presents the mathematical models used for only four subsystems, namely Costs and Profit, 'Kitty' or Pool, Accumulated Surplus Estimator and the Profit-Sharing Scheme. The subsystems related to claims and base premium are intentionally left unspecified at this stage to emphasize that what follows is totally independent of them; another significant point.

Costs and profit model

2.20.1. The simplest model for costs and profits is that they are a fixed percentage of premium paid. Consequently, the net premiums $p_n(k)$ for the financial period k after costs and profit are

$$P_n(k) = k_c\, p(k) \tag{15}$$

where the constant k_c will be in the range $0<k_c<1$. The numerical value used later is $k_c = \cdot 8$. The transfer function for this subsystem is then simply

$$\frac{P_n(z)}{P(z)} = k_c \tag{16}$$

'Kitty' or pool

2.20.2 The 'kitty' or pool is the fund in which all the cash flows $f(k)$ up to and including period k are accumulated. The accumulated cash flow $f_a(k)$ at the end of period k is

$$f_a(k) = \sum_{i=0}^{k} f(i) = f_a(k-1)+f(k). \tag{17}$$

Taking its z-transform and using equation (14) gives

$$F_a(z) = z^{-1}F_a(z)+F(z).$$

Hence the transfer function is

$$\frac{F_a(z)}{F(z)} = \frac{z}{z-1} \tag{18}$$

provided that the initial conditions are zero. In this paper the effect of interest earned (or payable) on a positive (or negative) accumulated cash flow is neglected.

Profit-Sharing scheme

2.20.3 The intuitively obvious approach is to feedback a simple proportion k_p of the estimated accumulated surplus $\hat{g}(k)$. The control theorist would call this proportional action and would suspect that such a simple approach may not provide all of the desirable features for which he might normally aim. He would expect to have to at least consider the addition of integral and possibly derivative actions. However, for the moment assume that the profit-sharing feedback is given by

$$p_f(k) = k_p \hat{g}(k) \tag{19}$$

whence

$$\frac{P_f(z)}{\hat{G}(z)} = k_p \tag{20}$$

Since the true value of $g(k)$ is uncertain a suitable arrangement might be to feedback 50% of what the surplus is currently estimated to be. This figure is used in the later numerical examples.

Accumulated surplus estimator

2.20.4.1. Herein lies one of the intrinsic difficulties associated with the dynamic control of an insurance system. Claims often take a long time to come in and some take a long time to be settled. In some classes of business, 25% of the incurred claims may be unreported and/or unpaid after two years, while with 'long tail' business the situation may be even worse.

2.20.4.2. Several approaches are obviously possible. In this paper we begin to analyse a strategy which might colloquially be termed a 'wait and see' approach. In practice most companies would experience difficulties in having final figures from year $(k-1)$ available for use in year k. In any case there may be a sufficient number of unpaid claims to render those figures undesirable for profit-sharing feedback. The 'wait and see' or time-delayed approach amounts to saying that for l time periods after the premium is paid the accumulated cash flow is an unreliable predictor of accumulated surplus and hence at any time k, the value from the $(k-l)$th period, should be used for profit-sharing feedback. Essentially, then,

$$p_f(k) = k_p f_a(k-l), \qquad 0 < k_p < 1 \tag{21}$$

and hence

$$P_f(z) = k_p z^{-l} F_a(z). \tag{22}$$

It should be noted that, by implication, the estimated accumulated surplus $\hat{g}(k)$ is

$$\hat{g}(k) = f_a(k-l) \tag{23}$$

Response to an isolated group of unpredicted claims

2.21. In the remainder of the paper attention is restricted to considering the response of the system to an isolated group or 'spike' of unexpected paid claims $c_u(k)$. A control theorist would normally also look at other patterns of unpredicted claims, such as 'step' and 'ramp' inputs. During the following analysis the level of business written will be assumed to remain unchanged. Consequently, from the discussion on zero initial conditions in § 2.19 it is clear that $b(k)$ is zero for all k. Under this condition the relationship between unpredicted paid claims and either cash flow $f(k)$ or accumulated cash flow $f_a(k)$, as shown in the block diagrams in Figures 2(a) and (b) respectively, can be extracted directly from Figure 1.

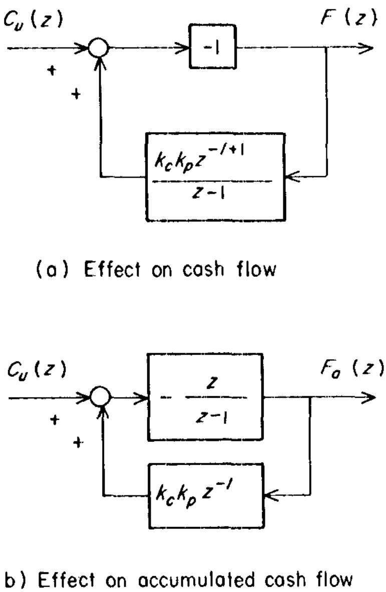

Figure 2. Effect of unpredicted claims.

2.22. By the simple technique of block diagram reduction (see any of the previously mentioned texts) the transfer functions become

$$\frac{F(z)}{C_u(z)} = \frac{z^{l-1}(z-1)}{z^l - z^{l-1} + k_c k_p} \tag{24}$$

and

$$\frac{F_a(z)}{C_u(z)} = -\frac{z^l}{z^l - z^{l-1} + k_c k_p}. \tag{25}$$

What happens if there is a group of unpredicted claims in one isolated year or period? For convenience make that period $k=0$ and let the unpredicted paid claims total one unit (which could be millions of pounds or some other convenient quantity). Mathematically,

$$c_u(k) = \delta(k) \tag{26}$$

where $\delta(k)$ is the Kronecker delta

$$\delta(k) = \begin{cases} 1, k = 0 \\ 0, k = \pm 1, \pm 2, \pm 3, \ldots \end{cases}$$

which leads to the convenient result

$$C_u(z) = 1. \tag{27}$$

Consequently, from equations (24) and (25) the z-transforms of the cash flow and the accumulated cash flow are

$$F(z) = \frac{-z^{l-1}(z-1)}{z^l - z^{l-1} + k_c k_p} \times 1 \tag{28}$$

and

$$F_a(z) = \frac{-z^l}{z^l - z^{l-1} + k_c k_p} \times 1. \tag{29}$$

2.23. Important conclusions can now be drawn about the steady state values, to which $f(k)$ and $f_a(k)$ will settle down as k increases and approaches infinity. Use is made of the Final Value Theorem which follows directly from the definition in equation (12) which states that the final value $x(\infty)$ of a sequence $x(k)$ as $k \to \infty$ is given by

$$x(\infty) = \lim_{z \to 1} (z-1) X(z) \tag{30}$$

provided that $(z-1)X(z)$ is analytic for $|z| \geqslant 1$. Here, the final values of the cash flow $f(\infty)$ and the accumulated cash flow $f_a(\infty)$ are

$$f(\infty) = \lim_{z \to 1} \frac{-z^{l-1}(z-1)^2}{z^l - z^{l-1} + k_c k_p} = 0 \tag{31}$$

$$f_a(\infty) = \lim_{z\to 1} \frac{-z^l(z-1)}{z^l - z^{l-1} + k_c k_p} = 0 \tag{32}$$

provided that $k_c k_p \neq 0$ and that $(z-1)F(z)$ and $(z-1)F_a(z)$ are analytic for all $|z| \geqslant 1$. Consequently, the cash flow and the accumulated cash flow will eventually reach the desired value of zero after isolated unpredicted claims, provided that certain conditions are met. These mathematical conditions are equivalent to the control system being stable. As will be seen briefly later and more fully in a subsequent publication, there is a limiting value of l beyond which the system is unstable.

2.24. The time histories of the responses are obtained by inverse z-transformation. For example when $l=1$

$$F(z) = \frac{-(z-1)}{z-1+k_c k_p} = \frac{-1}{1-k_c k_p} + \frac{k_c k_p}{1-k_c k_p}\frac{z}{z-1+k_c k_p}$$

and

$$F_a(z) = \frac{-z}{z-1+k_c k_p}$$

from which

$$f(k) = \frac{-\delta(k)}{1-k_c k_p} + \frac{k_c k_p}{1-k_c k_p}(1-k_c k_p)^k \tag{33}$$

and

$$f_a(k) = -(1-k_c k_p)^k \tag{34}$$

which can also be seen in this case by inspecting the series expansions in terms of z^{-1}. For $k_c = \cdot 8$ and $k_p = \cdot 5$, equations (33) and (34) become

$$f(k) = -1{\cdot}667\,\delta(k) + {\cdot}667\,({\cdot}6)^k$$
$$f_a(k) = -({\cdot}6)^k$$

which are shown in Figures 3 and 4, together with results for time delays of $l=2$ and 5.

2.25. The oscillatory results for $l=2$ arise as follows. From equations (28) and (29) for $l=2$

$$F(z) = \frac{-z(z-1)}{z^2 - z + k_c k_p}$$

and

$$F_a(z) = \frac{-z^2}{z^2 - z + k_c k_p}$$

2.26. The roots of $z^2 - z + k_c k_p = 0$ will be a complex conjugate pair whenever $k_c k_p > \cdot 25$. Under such conditions, which would seem to be normal for the insurance system considered, the response will be oscillatory rather than monotonic. Inverse transformation leads to relatively complicated expressions which will not be included here. Essentially, they are simply a sinusoidal waveform of frequency

$$\nu = \frac{\cos^{-1}(\cdot 5/\sqrt{k_c k_p})}{2\pi T} \tag{35}$$

multiplied by a decaying envelope generated by $(\sqrt{k_c k_p})^k$. For $T = 1$ year (annual figures) and for $k_c k_p = \cdot 4$, the frequency is $\nu = \cdot 105$ cycles per year which corresponds to a period of 9·54 years. The envelope is $(\cdot 6325)^k$, which is only a slightly slower decay than that seen for $l = 1$. The numerical results for $l = 2$ and $l = 5$ were obtained by the direct division technique, which avoids inversion.

Comments on responses and conclusions

2.27. For a delay of one period the dynamic response of both cash flow and accumulated cash flow are seen to be fairly satisfactory. Remember, however, that we are dealing with the limited situation involving only one isolated group of unpredicted claims and of no change in the level of business written. The effects of the disturbance still take approximately seven periods to be eliminated (for $k_c k_p = \cdot 4$).

2.28. When the delay is increased to two periods, the responses become oscillatory and overshoot. No recovery of the loss is attempted for two periods. Then it is over-collected in the next four periods resulting in the insurer having to repay some of it in following periods. This is not a situation with which either party would be happy. The overall settling time is extended by about one or two periods.

2.29. As a general principle, the introduction of time delays into a feedback loop is known to have a destabilizing effect. This is clearly demonstrated using $l = 5$, where the system becomes completely unstable with ever-increasing oscillations in both $f(k)$ and $f_a(k)$. The first positive peak in $f_a(k)$ is already one third greater than the initial disturbance. Mathematically, both will diverge to infinity! In practice, one or other of the parties would withdraw from the scheme well before that point.

2.30. It is important to note that $l = 5$ does not imply a delay of 5 years. The use of quarterly feedback and a delay time of only 2 years results in $l = 8$, which is clearly completely unstable. In fact, for $k_c k_p = \cdot 4$, the first positive peak in $f_a(k)$ is $f_a(17) = 2\cdot 52$, which is already two and a half times the initial isolated disturbance after only $4\frac{1}{4}$ years. Progressive negative and positive swings increase without bound.

2.31. It is equally important to realize that the above results are independent of the type of business, its claims model and base premiums. Furthermore, stability and instability are properties of the system itself and not related in any way to the nature of the disturbance input sequence $c_u(k)$.

2.32. The preceding results arose from a positive spike of unexpected claims. The insured may well be more interested in the effect of lower than expected claims. This can be seen by simply inverting the time histories in Figures 3 and 4.

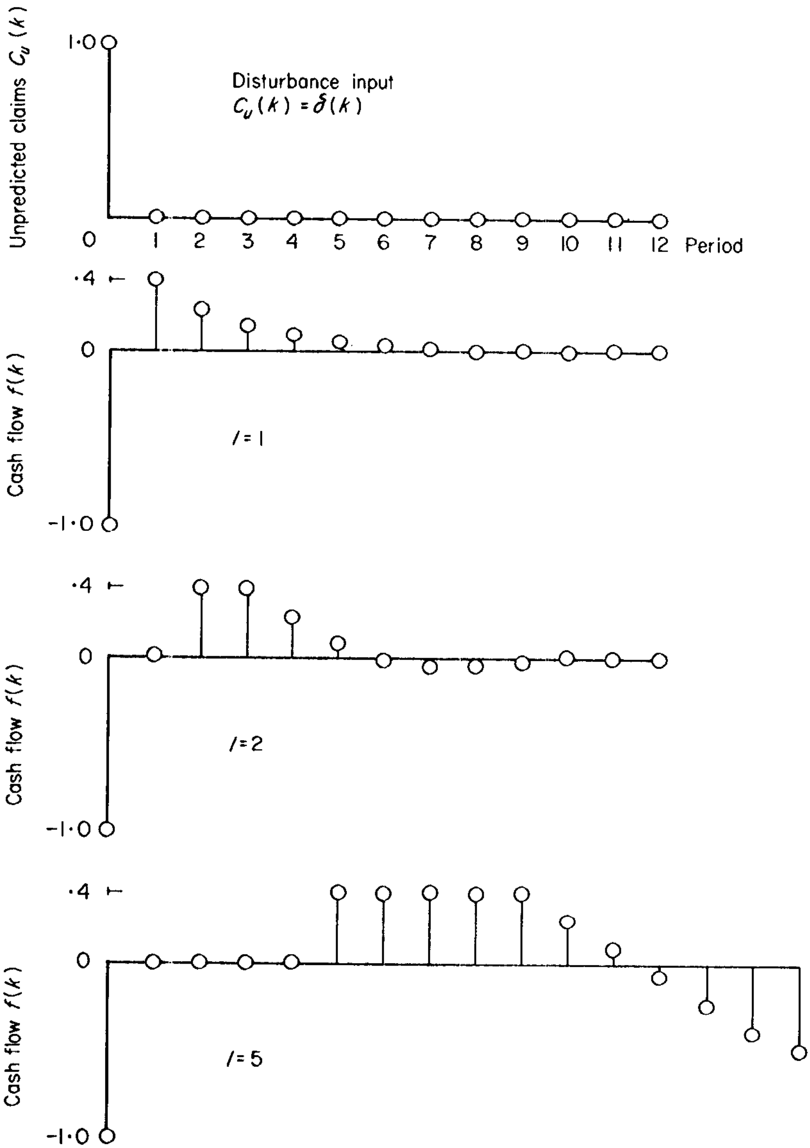

Figure 3. Effect of isolated unpredicted claims on cash flow.

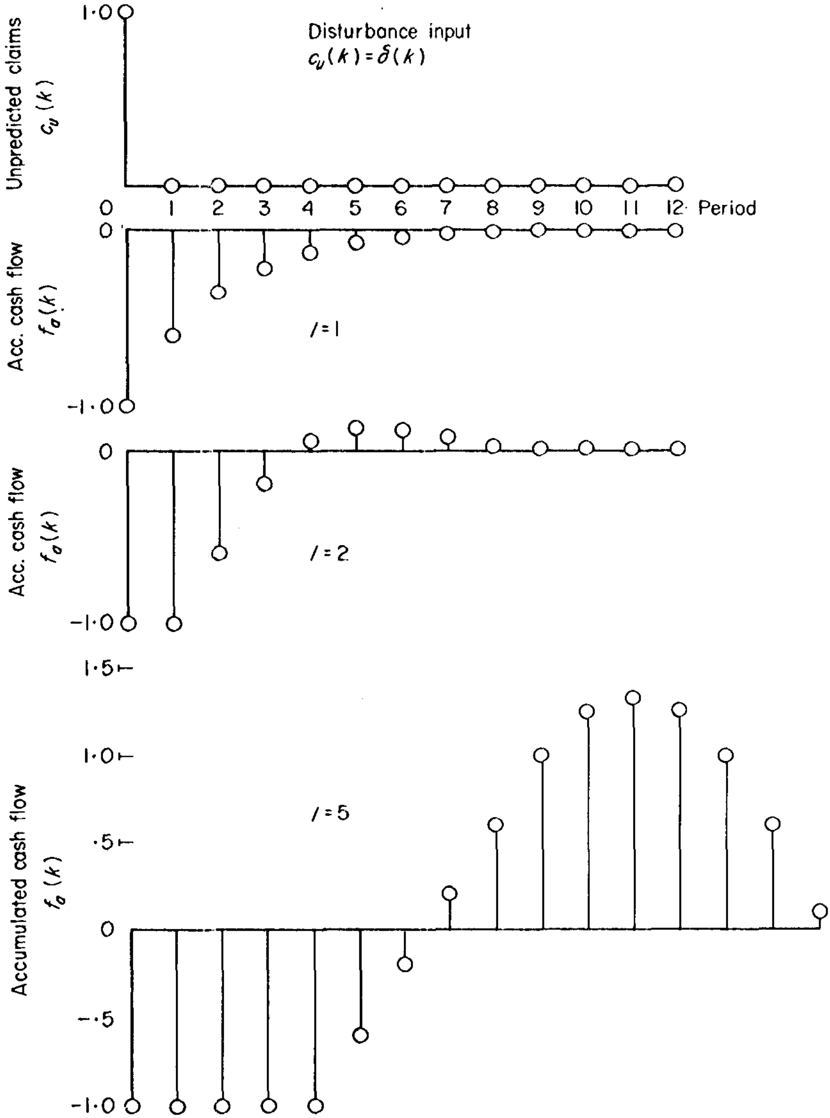

Figure 4. Effect of isolated unpredicted claims on accumulated cash flow

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