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GENERALIZED ANNUITIES AND ASSURANCES, AND THEIR INTER-RELATIONSHIPS

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ABSTRACT

By the definition of generalized assurances and annuities, the relation $1 = \bar{A}_s + \delta\bar{a}_s$ is shown to be the simplest member of a family of relations in which the present value of the capital and interest repayments is equated to the original amount of a loan. When the force of interest is constant over time, the relation takes the simple form above, but when the force is a polynomial in time, the interest payments become a series of annuities, the payments on which increase over time at different rates.

1. INTRODUCTION

In the equation

$$1 = \bar{A}_s + \delta\bar{a}_s, \tag{1.1}$$

the 'status' function $s(t)$ has generally been considered a step function, falling from unity to zero upon the occurrence of some event (e.g. in the usual notation s can be x, $\overline{n}|$, $x:\overline{n}|$, xy, etc.); and δ is the constant force of interest.

The equation (1.1) is in fact the simplest representative of a family of equations relating the capital repayments and the interest repayments on a loan.

Consider a loan advanced at time 0. There can later be further loans, or some repayments, and the timing and amounts of these extra loans or repayments can be stochastic. The random variable $s(t)$ represents the capital outstanding at time t. We define generalized assurances, under which payments are proportional to changes in the function $s(t)$, and generalized annuities, the payments on which are proportional to the function $s(t)$. The relationship between the expected present values of the capital repayments (the generalized assurances) and interest payments (a sum of generalized annuities) provides the generalization of (1.1). Only if the force is constant do we obtain the simple formula (1.1).

The well-known formula $1 = A_s + d\ddot{a}_s$ can only be derived for constant δ, but the formula remains valid for general functions $s(t)$ by suitable definitions of A_s and $\ddot{a}_s$.

2. OVERVIEW

The principal result (equation 2.5) is derived in this section. Proofs are detailed in section 3 and the Appendices, whilst section 4 provides some simple examples.

The force of interest at time t is denoted by $\delta(t)$. We then define

$$\Delta(t) = \int_0^t \delta(\tau)d\tau$$

and $f(t) = \exp(-\Delta(t))$, so that $f(t)$ is the discounting factor from time t to time 0. Finally, $s(t)$ is the amount of capital outstanding at time t on a putative loan advanced at time 0.

Integration by parts yields

$$f(t)s(t) - f(0)s(0) = \int_0^t f(\tau)ds(\tau) + \int_0^t s(\tau)df(\tau) \tag{2.1}$$

Here $df(\tau) = -\delta(\tau)\cdot f(\tau)\cdot d\tau$, so that (2.1) becomes

$$f(t)s(t) - f(0)s(0) = \int_0^t f(\tau)ds(\tau) - \int_0^t s(\tau)\delta(\tau)f(\tau)d\tau$$

Assuming that $f(t)s(t) \rightarrow 0$ as $t \rightarrow \infty$ (i.e. the discounted value of the outstanding capital $s(t)$ tends to zero as the time t tends to infinity), and that the integrals converge,

$$-\int_0^\infty ds(t)e^{-\Delta(t)} + \int_0^\infty s(t)\delta(t)e^{-\Delta(t)}dt = s(0) \tag{2.2}$$

$$\text{i.e. } \mathrm{PV}\bar{\mathrm{A}}_s + \mathrm{PVINT}_s = s(0) \tag{2.3}$$

The first term on the left side of (2.3), $\mathrm{PV}\bar{\mathrm{A}}_s$, represents the present value of the capital repayments*, and the second term, PVINT_s, the present value of the continuously payable interest payments. The term on the right side is the initial amount of the loan.

For brevity we define the following quantities. For our purposes j can be restricted to the non-negative integers. The notation for the expected values is consistent with conventional actuarial notation when $j = 0$ or 1 (Neill, 1977).

$$\begin{aligned} \mathrm{PV}\bar{\mathrm{I}}^j\bar{\mathrm{A}}_s &= -\int_0^\infty ds(t)\cdot t^j\cdot e^{-\Delta(t)}; \\ \mathrm{PV}\bar{\mathrm{I}}^j\bar{a}_s &= \int_0^\infty s(t)t^j e^{-\Delta(t)}dt; \\ \mathrm{E}(\mathrm{PV}\bar{\mathrm{I}}^j\bar{\mathrm{A}}_s) &= (\bar{\mathrm{I}}^j\bar{\mathrm{A}})_s; \\ \mathrm{E}(\mathrm{PV}\bar{\mathrm{I}}^j\bar{a}_s) &= (\bar{\mathrm{I}}^j\bar{a})_s\dagger \end{aligned}$$

* Usually the function $s(t)$ will be a step function, in which case let ds_i be the jump in s at t_i. Then

$$\mathrm{PVA}_s = -\sum ds_i\cdot e^{-\Delta(t_i)}.$$

For a whole life assurance on the life of (x), for example, PVA_x is the present value of a payment of 1 upon death, with expected value $\bar{\mathrm{A}}_x$.

† $(\bar{\mathrm{I}}^j\bar{a})_s$ would normally be calculated as

$$\int_0^\infty \mathrm{E}(s(t))\cdot t^j\cdot f(t)\cdot dt$$

which, when $s(t)$ is $\{0,1\}$ valued, reduces to

$$\int_0^\infty tPs\cdot e^{-\Delta(t)}dt, \text{ where } \quad tPs = \mathrm{Prob}(s(t) = 1).$$

The general validity of taking the expectation operator under the integral sign in this way is shown in Appendix 1.

When j = 0, we simply write

$$\text{PV}\bar{\text{A}}_s = -\int_0^\infty ds(t)\cdot e^{-\Delta(t)};$$
$$\text{PV}\bar{a}_s = \int_0^\infty s(t)\cdot e^{-\Delta(t)}dt;$$
$$\text{E}(\text{PV}\bar{\text{A}}_s) = \bar{\text{A}}_s;\ \text{E}(\text{PV}\bar{a}_s) = \bar{a}_s.$$

Assuming δ(t) to have the Taylor expansion $\sum\delta_j t^j$, (2.2) becomes

$$\text{PV}\bar{\text{A}}_s + \sum_{j=0}^{\infty}\delta_j\cdot\text{PV}\bar{\text{I}}^j a_s = s(0) \tag{2.4}$$

Taking expected values (over possible realizations of $s(t)$) then yields

$$\bar{\text{A}}_s + \sum_{j=0}^{\infty}\delta_j(\bar{\text{I}}^j\bar{a})_s = \text{E}(s(0)). \tag{2.5}$$

For constant force of interest over time and $s(0)$ fixed at unity this equation reduces to (1.1).

The force of interest is assumed deterministic in this paper. In general, though, both $\delta(t)$ and $s(t)$ will be stochastic, although they should be stochastic in such a way that the integrals above exist with probability one, and preferably in such a way that their expected values exist. It is quite feasible that $s(t)$ and $\delta(t)$ are not independent: for instance, a decision to repay capital, or borrow more, will depend upon changes in the interest rate charged on the outstanding loan, viz. $\delta(t)$.

SECTION 3

For most actuarial applications, the upper limit of the integrals in (2.2) will be finite, and independent of the particular realization $s(t)$. An example is that of an assurance on an individual's life: $s(t) = 1$ as long as the person lives, and falls to zero upon his death, so that $s(t)$ will certainly vanish after the end of the life table used. Let the upper limit of the integrals be T; precisely, let T be such that

$$s(t) = 0 \qquad \Delta t \geqslant \text{T, almost certainly,}$$

where almost certainly means with probability one, and will be abbreviated to $a\cdot c$. If no such T exists, we shall say that 'T is infinite'.

3.1. *Finite T*

For finite T, sufficient conditions for (2.2) to be true are that $s(t)$ be of bounded variation on [0,T], and $\delta(t)$ be integrable on [0,T]. For then $f(t)$ is continuous on [0,T] (provided δ is not allowed to assume infinite values on a non-null set), the integrals in (2.2) exist, and relation (2.2) is true (Moran, 1968 p. 218). In order that finite expected values exist for these integrals, however, the following stronger condition is imposed:

C.1. There is a K such that, with probability one, the total variation of $s(t)$ on [0,T] is not greater than K; i.e.,

$$\sup\sum|s(t_{i+1}) - s(t_i)| \leqslant \text{K a.c.,}$$

where the supremum is taken over all finite sets of points t_i such that

$$0 \leqslant t_l < t_2 < \ldots < t_m \leqslant \mathrm{T}, \tag{3.1}$$

for any positive integral m.

This is stronger than requiring $s(t)$ to be of bounded variation on [O,T] almost certainly, for then K could vary with the realization of $s(t)$. Now $f(t)$ being continuous on the closed interval [O,T] implies that it is bounded on that interval, say $|f(t)| \leqslant \mathrm{M}$. Then

$$|\mathrm{PV\bar{A}}_s| = \left|-\int_0^T ds(t)f(t)\right| \leqslant \mathrm{M.K, a.c.}$$

and this bound is independent of the realization $s(t)$, so that $\bar{\mathrm{A}}_s$ exists and is finite.

It can be similarly shown that $\mathrm{PV}\bar{\mathrm{I}}^j\bar{a}_s$ has a finite expected value—this is done in Appendix 2.

Suppose now that the force of interest $\delta(t)$ has a Taylor expansion $\sum\delta_j t^j$ which includes the real interval [O,T] in the interior of its circle of convergence (regarding t as a complex variable).* Then the Taylor series is integrable on [0,T], and converges uniformly to the function $\delta(t)$ on [O,T], so that we can interchange the order of integration and summation in $\mathrm{PVINT_s}$ in (2.2) to yield

$$\mathrm{PV\bar{A}}_s + \sum\delta_j \cdot \mathrm{PV}\bar{\mathrm{I}}^j\bar{a}_s = s(0) \tag{2.4}$$

(Brand, 1955 pps 416, 405).

Taking expectations in (2.4) again yields

$$\bar{\mathrm{A}}_s + \sum\delta_j(\bar{\mathrm{I}}^j\bar{a})_s = \mathrm{E}(s(0)). \tag{2.5}$$

Convergence of the infinite series in this last expression is also shown in Appendix 2.

3.2. *Infinite T*

In actuarial work this situation would normally only arise in the consideration of contingent assurances. If the lives die in the wrong order, nothing is paid on the assurance, and the status never falls to zero.

The force of interest is assumed to have a positive infimum for sufficiently large t:

C.2. There is a number δ^0 such that

$$\delta(t) \geqslant \delta^0 > 0 \qquad \forall t \text{ sufficiently large.}$$

‡ This assumption is stronger than it appears, because in fact it requires that the interval [−T,T] lie within the circle of convergence. It could be weakened by centring the Taylor series at T/2, or equivalently by changing the time origin so that the original loan were granted at time −T/2, and $s(t)$ vanished on [T/2,∞); then the radius of convergence of the Taylor series would only need to exceed T/2, not T. The algebra would proceed as above.

The condition is trivially satisfied when δ is a finite polynomial, the radius of convergence of which is infinite.

We further assume the analogue of the condition imposed upon $s(t)$ when T was finite:

C.3. There is a number L_1 such that the total variation of s over $[0,\infty)$ is less than L_1, almost certainly. The expression (3.1) given in C.1 remains valid, save for the omission of the final inequality $t_m \leqslant T$.

Finally, the following weak assumption is imposed to ensure that $s(t)$ has a bound which does not depend on the realization $s(t)$:

C.4. There exist numbers L_2 and t_0 such that

$$|s(t_0)| \leqslant L_2, \text{ a.c.}$$

That $PV\bar{A}_s$ and $PV\bar{I}^j\bar{a}_s$ are convergent and have well-defined expected values under these assumptions is shown in Appendix 3.

In particular, if δ is a finite polynomial in t (with the coefficient of the highest power of t positive to ensure that C.2 holds), the expressions (2.4) and (2.5) are valid.

SECTION 4

4.1. *Example* 1

Consider the annuity increasing continuously over time, payable continuously to (x) while he lives, and ceasing upon his death. The payment at time t is $t \cdot dt$, and the function $s(t)=t$ until death, whereupon it drops to zero. The corresponding generalized assurance consists of an annuity of -1 p.a. payable continuously until death, followed by a lump sum payment of t_0 paid upon death at time t_0.

$$\begin{aligned} PV\bar{A}_s &= -\int_0^T ds(t)\cdot f(t) \text{ if T is the end of the life table used,*} \\ &= -\int_0^{t_0} dt\cdot f(t) + t_0 \cdot f(t_0) \\ &= -PV\bar{a}_x + PV\bar{I}\bar{A}_x \end{aligned}$$

where the status function $x(t)$ is unity while (x) lives, and zero thereafter.

$$\begin{aligned} PVINT_s &= \int_0^T s(t)\left(\textstyle\sum \delta_j t^j\right)\cdot f(t)\cdot dt \\ &= \textstyle\sum \delta_j \cdot PV\bar{I}^{j+1}\bar{a}_x, \end{aligned}$$

assuming that the Taylor expansion has the interval [O,T] inside its circle of convergence. Thus (2.3) becomes

$$s(0) = -PV\bar{a}_x + PV\bar{I}\bar{A}_x + \textstyle\sum \delta_j \cdot PV\bar{I}^{j+1}\bar{a}_x.$$

Taking expectations of this last equation, we obtain

$$0 = -\bar{a}_x + (\bar{I}\bar{A})_x + \textstyle\sum \delta_j (\bar{I}^{j+1}\bar{a})_x.$$

* If $ds(T)$ has a non-zero probability of being finite, $s(t)$ needs to be continuous from the right at T.

If δ is constant this reduces to the well-known equation $\bar{a}_x = (\bar{\mathrm{I}}\bar{\mathrm{A}})_x + \delta(\bar{\mathrm{I}}\bar{a})_x$ (See Neill, 1977 p. 102).

4.2. *Example* 2

When δ is a constant, the relation

$$1 = \mathrm{A}_s + d\ddot{a}_s$$

is obtained by constraining $s(t)$ to jump or fall at integral values of t. Formally, we need to define a new 'status' s^* thus:

$$ds^*(t) = s(t) - s(t-1) \qquad \forall \text{ integral } t > 0$$

($s(t)$ must be continuous from the right if we use this definition, so that if s falls at an integral time t, $s^*(t)$ also falls at t)

$$ds^*(t) = 0 \text{ otherwise.}$$

$$\begin{aligned} \text{Then } 1 &= \bar{\mathrm{A}}_s{}^* + \delta \cdot \bar{a}_s{}^* \\ &= \mathrm{A}_s + \mathrm{d} \cdot \ddot{a}_s \end{aligned}$$

upon defining $\ddot{a}_s = (\delta/d)\bar{a}_s{}^*$.

There are of course analogues of this expression when $s(t)$ is constrained to fall or jump at intervals other than annual.

APPENDIX 1

We show that

$$E(\int_0^\infty s(t)\cdot f(t)\cdot t^j\cdot dt) = \int_0^\infty E(s(t))\cdot t^j\cdot f(t)\cdot dt \qquad (A1.1)$$

when $s(t)$ is a step function with a finite limit n to the possible number of steps. We infer the validity of (A1.1) for most functions $s(t)$ of interest in actuarial science.

We appeal to a result from Doob (1967). Let $\omega_1, \omega_2, \ldots, \omega_n$ denote the times of the first, second, . . . , nth steps of $s(t)$, and let ω be the vector of these step times. The function $s(t,\omega)$ is then the value of $s(t)$ when step times occur at times ω_i (we should strictly use different symbols for the two functions $s(t)$ and $s(t,\omega)$). The symbols t and ω correspond to those used in Doob. The imposition of a finite maximum on the number of steps is a convenience to simplify the proof, but represents no real restriction for actuarial work. A simple example will illustrate before we proceed with the proof.

Example A1.1. Consider an annuity of 2 p.a. payable continuously while a group of four people all remain alive, reducing to 1 p.a. while three of them survive, and zero thereafter. Then

$$s(t,\omega) = \begin{cases} 2 \text{ if } t < \omega_1, \\ 1 \text{ if } \omega_1 < t < \omega_2, \text{ and} \\ 0 \text{ otherwise,} \end{cases}$$

where ω_1 and ω_2 are respectively the times of the first and second deaths.

The question of the measurability of the function $s(t,\omega)$ reduces to consideration of the measurability of the union of finitely many sets of the form $\{(t,\omega): s(t,\omega) = \text{constant}\}$, relative to the measure which is the product of the Lebesque measure on the t axis, and the measure induced on the ω space by the joint probability distribution of the ω_i. Any such set is measurable because it is the union of finitely many sets of the form $\{(t,\omega): t < \omega_1 < \omega_2 < \ldots < \omega_n\}$, $\{(t,\omega): \omega_1 < t < \omega_2 < \ldots < \omega_n\}$, etc., each of which is measurable.

The function $s(t,\omega)$ is then measurable, whence $f(t)\cdot s(t,\omega)\cdot t^j$ is also measurable, the two product functions being continuous, which means that the stochastic process

$$\{f(t)\cdot s(t,\omega)\cdot t^j,\ t\varepsilon[0,\infty)\}$$

is measurable (Doob, 1967 p. 62; $f(t)\cdot s(t,\omega)\cdot t^j$ corresponds to Doob's $x_t(\omega)$). Conditions C.3 (which is equivalent to C.1 for finite T) and C.4 ensure that the bound on $s(t,\omega)$ applies independently of the realization $s(t)$ (see Appendix 3), so that the conditions for Doob's Theorem 2.7 are satisfied, and the interchange of the order of the expectation and integration operators is justified.

The proof hinges on the measurability of the function $s(t,\omega)$. In actuarial work, this function will usually take the form of a step function; even in more complicated cases, it will often assume the form of a product of a measurable function and a step function, which is itself measurable (for instance, $s(t)$ is the product of a step function and the function $g(t)=t$ in example 4.1). It is difficult to imagine a situation arising in actuarial work to which Doob's result would not apply.

APPENDIX 2

T is finite throughout this appendix, and any relations are to be interpreted as being almost certainly true.

We first show that $PV\bar{I}^j\bar{a}_s$ has a finite expected value.

From condition C.1, choosing the partition $\{0,t,T\}$, we have

$$\begin{aligned} |s(t)| &= |s(t)-s(T)| \\ &\leqslant |s(t)-s(0)|+|s(T)-s(t)| \\ &\leqslant K. \end{aligned}$$

Since $|f(t)|\leqslant M$,

$$\begin{aligned} |PV\bar{I}^j\bar{a}_s| &= \left| \int_0^T s(t)\cdot f(t)\cdot t^j\cdot dt\right| \\ &\leqslant K.M.T^{j+1}. \end{aligned}$$

This bound applies regardless of the realization $s(t)$, so that $(\bar{I}^j\bar{a})_s$ exists and is finite.

Next, we show that $\sum\delta_j(\bar{I}^j\bar{a})_s$ is convergent.

$\delta(t)$ is continuous on the closed interval [O,T], because this interval lies within the circle of convergence, and therefore $\delta(t)$ is bounded on [O,T]. Now

$$PVINT_s = \int_0^T s(t)\cdot\delta(t)\cdot f(t)\cdot dt$$

so that a similar argument to that above shows that $PVINT_s$ is bounded, and the bound does not depend on the realization $s(t)$. Thus the expected value of $PVINT_s$ exists and is finite.

$$PVINT_s = \sum_{j=0}^{N-1}\delta_j\cdot PV\bar{I}^j\bar{a}_s + \sum_{j=N}^{\infty}\delta_j\cdot PV\bar{I}^j\bar{a}_s. \tag{A2.1}$$

The magnitude of the last term in (A2.1) can be made arbitrarily small by choosing N sufficiently large, by the uniform convergence of the Taylor series to $\delta(t)$. The expected value of this term can likewise be made arbitrarily small, so that

$$E(PVINT_s) = \sum_{j=0}^{\infty}\delta_j\,(\bar{I}^j\bar{a})_s$$

and the right side of this equation converges.

APPENDIX 3

T is infinite in this appendix, and again any relation in this Appendix is true almost certainly. We show first that $PV\bar{A}_s$ exists and has finite expected value.

From C2 there is an $r \geqslant 0$ such that

$$\delta(t) \geqslant \delta^0 > 0 \qquad \Delta t \geqslant r.$$

Then, for $u \geqslant r$,

$$\begin{aligned} f(u) &= f(r)\cdot\exp(-(\Delta(u)-\Delta(r))) \\ &\leqslant f(r)\cdot\exp(-\delta^0(u-r)). \end{aligned}$$

From C.3,

$$\left|\int_u^v ds(t)\cdot f(t)\right| \leqslant L_1\cdot f(r)\cdot\exp(-\delta^0(u-r))\rightarrow 0 \textit{ as } u\rightarrow\infty. \tag{A3.1}$$

Thus the integral defining $PV\bar{A}_s$ converges almost certainly, by the analogue for integrals of Cauchy's convergence criterion for series (see, i.a., Brand (1955).

Concerning the expected value $\bar{A}_s$, first consider

$$\int_r^\infty ds(t)\cdot f(t) = PV\bar{A}_2, \text{ say}$$

$$\text{Now } |PV\bar{A}_2| \leqslant L_1\cdot f(r)\cdot\exp(\delta^0\cdot r),$$

and this bound applies independently of the realization $s(t)$, so that $PV\bar{A}_2$ has a finite expected value.

Similarly, let

$$PV\bar{A}_1 = \int_0^r ds(t)\cdot f(t).$$

The function $f(t)$ is bounded on $[0,r]$, being continuous on that closed interval; say

$$|f(t)| \leqslant Q \;\Delta t \;\varepsilon[0,r].$$

Then $|PV\bar{A}_1| \leqslant L_1\cdot Q$, independently of the sample function $s(t)$, so that again the expected value exists and is finite. Thus $\bar{A}_s = E(PV\bar{A}_1 + PV\bar{A}_2)$ exists and is finite.

The proof that $PV\bar{I}/\bar{a}_s$ exists and has finite expectation is similar, but we need the following preliminary result.

For t_0 satisfying C.4,

$$\begin{aligned} |s(t)| &= |s(t)-s(t_0)+s(t_0)| \\ &\leqslant |s(t)-s(t_0)|+|s(t_0)| \\ &\leqslant L_1+L_2, \text{ by C.3 and C.4,} \\ &= L', \text{ say.} \end{aligned}$$

Then

$$
\begin{aligned}
&\left|\int_u^v s(t)\cdot t^j\cdot f(t)\cdot dt\right| \\
&\leqslant \int_u^v \mathrm{L}'\cdot t^j\cdot f(r)\cdot\exp(-\delta^0(t-r))\cdot dt \\
&\leqslant \int_u^v \mathrm{L}'\cdot f(r)\cdot\exp(-\delta^0(t-r)/2)\cdot dt \text{ for sufficiently large } u; \\
&\leqslant \int_u^\infty \mathrm{L}'\cdot f(r)\cdot\exp(-\delta^0(t-r)/2)dt \\
&= \mathrm{N}.\exp(-\delta^0\cdot u/2) \text{ for constant N,} \\
&\rightarrow 0 \text{ as } n\rightarrow\infty.
\end{aligned}
$$

This, again, shows the almost certain convergence of $PV\bar{I}^j\bar{a}_s$, by the integral analogue of Cauchy's convergence criterion for series.

Analogously to the reasoning above for $PV\bar{A}_s$, let

$PV\bar{I}^j\bar{a}_1 = \int_0^r s(t)\cdot t^j\cdot f(t)\cdot dt$,
and $PV\bar{I}^j\bar{a}_2 = PV\bar{I}^j\bar{a}_{\overline{s}|}-PV\bar{I}^j\bar{a}_1$.
Then $|PV\bar{I}^j\bar{a}_1| \leqslant L'\cdot r^{j+1}$. Q,
and $|PV\bar{I}^j\bar{a}_2| \leqslant N.\exp(-\delta^0\cdot r/2)$,
so that $(\bar{I}^j\bar{a})_s$ exists and is finite.

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